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A robust confidence interval of historical Value-at-Risk for small sample

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Abstract

Finiteness of sample, as one major sources of uncertainty, has been ignored by the regulators and risk managers domains such as portfolio management, credit risk modelling and finance (or insurance) regulatory capital calculations. To capture this uncertainty, we provide a robust confidence interval (CI) of historical Value-at-Risk ($hVaR$) for different length of sample. We compute this CI from a saddlepoint approximation of the distribution of $hVaR$ using a bisection search approach. We also suggest a Spectral Stress Value-at-Risk measure based on the CI, as an alternative risk measure for both financial and insurance industries. Finally we perform a stress testing application for the SSVaR. [1]

Keywords: Value-at-Risk, Small sample, Uncertainty, Asymptotic normality approximation, Saddlepoint approximation, Bisection search approach, Spectral Stress VaR, Stress testing
JEL: C14, D81, G28, G32

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1. Introduction

Nowadays, Value-at-Risk (VaR) is widely used in both financial and insurance industries. As an internal model to compute the required regulatory capital, it has become a standard risk measure for large banks (Jorion (2006) [14]). In statistical terms, the VaR is a quantile, often using the p^{th} percentile of the loss distribution ($0 < p < 1$). Typically the VaR is not known with certainty and needs to be estimated from sample of relevant observations. Pérignon and Smith (2010) [20] find that historical VaR ($hVaR$) is one of the most popular VaR methods, as 73% of the banks report their VaR estimation methodologies using $hVaR$. But Bignozzi and Tsanakas (2015) [3] point out that the observations are often very small creating statistical error, which means that the values of sample estimators can diverge substantially from the true values [2]. Jorion (1996) [13] refers to it as the risk in Value-at-Risk itself.

To integrate the uncertainty contained in the $hVaR$, we propose a robust confidence interval (CI) for it using small sample. We consider the saddlepoint (SP) method to obtain an approximation of the distribution of $hVaR$. The inversion of this approximation can be found by performing the bisection search approach, which allows us to build the CI of $hVaR$. SP method generally provides accurate approximation whenever it is applicable (Wang (1995) [25]). Because it provides a good approximation both in the center and tail of the probability density function (pdf), even if sample size is small (Easton and Ronchetti (1986) [6]). More precisely, Damiels' pioneering paper (Damiels (1954) [5]) shows that the relative error of SP approximation is of order $O(n^{-1})$ [3] uniformly. Consequently, when the sample is small or when we consider the tail confidence level of $hVaR$, the CI from SP approximation is robust [4]. Thus in practice, we can use it to

[2]Note that this true value is unknown.

[3]The notation $O(n^{-t})$ denotes a function that satisfies $\lim_{n\to\infty} n^t O(n^{-t}) = constant$.

[4]Notice that the relative error of the widely used asymptotic normality approximation is of order $O(n^{-\frac{1}{2}})$.

build the Spectral Stress VaR (SSVaR) introduced by Guégan et al. (2016) [11].

Some papers discussed already the confidence interval of the $hVaR$. For example, Pritsker (1997) [21] computes a nonparametric CI to evaluate the accuracy of different VaR approaches. Christoffersen and Gonçalves (2005) [4] assess the precision of the VaR forecast by using a bootstrap prediction intervals. However, these approaches rely on simulations and then the results can be unstable. Jorion (1996) [13] provides the asymptotic standard error and confidence bands for $hVaR$, assuming the loss distribution is known and the asymptotic distribution of $hVaR$ is Gaussian. Since the loss distribution is always unknown in practice, Guégan et al. (2016) provide a parametric CI based on the asymptotic normality (AN) approximation without assuming the loss distribution is known. Although CI based on AN is straightforward and widely used, it is often inaccurate, especially for small sample size and tail $hVaR$. Indeed in these cases, the distribution of $hVaR$ can have asymmetric or fat-tailed behaviours. But the AN approximation always provides a Gaussian approximation, which is always symmetrical and thin-tailed (see Guégan et al. (2015) [10] and Jorion (1996) [13]). In this paper, we propose a robust parametric CI from SP approximation, which can model the asymmetric and fat-tailed behaviours of $hVaR$ for small samples.

In practice, in order to verify the AN approximation, some risk managers tend to perform Monte-Carlo simulation with a sufficient large sample size artificially. We argue that this may lead ignoring the uncertainty in the $hVaR$ and consequently it is biased. In fact, the finite size is a crucial criteria to assess the information in the sample. Thus, we suggest using the length of the observations (or the length of a stress scenario) to compute the CI of $hVaR$.

For statistician, the $hVaR$ is an order statistic (or sample quantile, or empirical quantile). The problem of approximating the distribution of an order statistic is an important one in statistical theory and in practice. The AN (see Rao (2002) [22]), the Edgeworth (see Reiss (1976) [24]) and the SP approximations (see Ma

(1998) [17]) are the three most commonly used methods. Hall and Sheather (1988) [12] give an Edgeworth expansion approximation for the distribution of a studentized sample quantile, without assuming the underlying distribution is known. It is more accurate than an AN approximation since it contains high-order terms that are otherwise ignored. Kaplan (2015) [15] proposes a test for the optimal choice of a smoothing parameter, which is crucial in the Edgeworth expansion for the studentized sample quantile. While the Edgeworth expansion usually improves over the Gaussian approximation, their numerical accuracy is still often questionable. Even worse, they have some undesirable properties, such as negative tail probabilities (see Wang (1995) [25] and Easton and Ronchetti (1986) [6]).

On the other hand, the SP approximation, can generate accurate probabilities in the distribution tails (without the problem of negative tail probabilities), even for small sample size. It can generally be obtained for a statistic that admits a cumulant generating function. In contrast to the AN approximation, the SP approximation holds without any assumption on the derivatives of population cumulative distribution function (cdf) (Ma and Robinson (1998) [17]). It has been used with success by many authors, for example: Ma and Robinson (1999) [18] propose SP approximations for the difference of order statistics and studentized sample quantiles. Easton and Ronchetti (1986) [6] provide the general SP approximations with applications to L-statistics, which are the general case of the order statistics. Wang (1995) [25] proposes two simple one-step methods to compute the inversion of SP approximation numerically. The bisection search approach is also suggested. Thus, it is possible to obtain the CI of $hVaR$ from SP approximation. See Goutis and Casella (1999) [9], Reid (1988) [23] and Field and Ronchetti (1990) [7] for general reviews of the background and development of SP methods.

This paper is organised as follows. Section 2 describes the AN and SP approximations for the asymptotic distribution of $hVaR$. Section 3 compares the performance of these two approximations by simulation. Section 4 computes the SSVaR using these two approximations and bisection search approach with different data sets. In the end we compare the results by a stress testing application. Section 5 concludes.

2. Asymptotic distribution of $hVaR$

Consider a random variable (r.v.) X (for example the return of a portfolio, the return of a risk factor or an operational loss), with a cdf $F_{\boldsymbol{\theta}}$ ($f_{\boldsymbol{\theta}}$ is the associated probability density function (pdf) and $\boldsymbol{\theta}$ are the parameters). Let $X_1, ..., X_n$ be the information set of X with length n. We assume they are independent and identically distributed (i.i.d) [5]. We sort them and obtain $X_{(1)} \leq ... \leq X_{(n)}$. Given $0 < p < 1$, we define the $hVaR$ as $X_{(m)}$, where $m = np$ if np is an integer and $m = [np] + 1$ otherwise [6]. Rao (2002) [22] provides an AN approximation for the distribution of $X_{(m)}$

Theorem 1 (Asymptotic normality approximation (Rao (2002))). *Assume $F_{\boldsymbol{\theta}}$ is continuous and differentiable and $f_{\boldsymbol{\theta}}$ is strictly positive at $F_{\boldsymbol{\theta}}^{-1}(p)$, then*

$$\sqrt{n}(X_{(m)} - F_{\boldsymbol{\theta}}^{-1}(p)) \rightarrow_{(d)} N(0, V), \quad as \quad n \longrightarrow \infty \tag{1}$$

where $\rightarrow_{(d)}$ means convergence in distribution, $V = \frac{p(1-p)}{f_{\boldsymbol{\theta}}(F_{\boldsymbol{\theta}}^{-1}(p))^2 n}$. $N(F_{\boldsymbol{\theta}}^{-1}(p), V)$ represents the Gaussian distribution with mean $F_{\boldsymbol{\theta}}^{-1}(p)$ and variance V.

Notice that expression (1) depends on the values of $F_{\boldsymbol{\theta}}$ and $f_{\boldsymbol{\theta}}$, which are unknown in most cases. Therefore density estimation is necessary. One possible way is to use the Siddiqui-Bloch-Gastwirth estimator, whose construction depends crucially on the choice of a smoothing parameter (see Hall and Sheather

[5] Or if they are not, we assume that we can transform them to an i.i.d set by filtering

[6] $[x]$ denotes the largest integer less than or equal to x.

(1988) [12]). Instead of using such nonparametric estimator which suffers difficulty of smoothing parameter choice, we fit a panel of distributions using $X_1, ..., X_n$ to compute the estimators of $\boldsymbol{\theta}$, denoted $\hat{\boldsymbol{\theta}}$. Then $F_{\hat{\boldsymbol{\theta}}}$ and $f_{\hat{\boldsymbol{\theta}}}$ are the estimators of $F_{\boldsymbol{\theta}}$ and $f_{\boldsymbol{\theta}}$. By plugging $F_{\hat{\boldsymbol{\theta}}}$ and $f_{\hat{\boldsymbol{\theta}}}$ in expression (1), we provide a corollary of Theorem 1

Corollary 1 (Plug-in AN approximation). *Assume $F_{\boldsymbol{\theta}}$ and $f_{\boldsymbol{\theta}}$ are continuous functions with respect to (w.r.t) $\boldsymbol{\theta}$, and $\hat{\boldsymbol{\theta}}$ is an asymptotically consistent estimator of $\boldsymbol{\theta}$ [7]. Then we have*

$$\sqrt{n}(X_{(m)} - F^{-1}_{\hat{\boldsymbol{\theta}}}(p)) \rightarrow_{(d)} N(0, \widehat{V}), \quad as \quad n \longrightarrow \infty \tag{2}$$

where $\widehat{V} = \frac{p(1-p)}{[f_{\hat{\boldsymbol{\theta}}}(F^{-1}_{\hat{\boldsymbol{\theta}}}(p))]^2 n}$. The proof is presented in Appendix A.

Based on the integral representation of the binomial distribution and Barndorff-Nielsen formula (see Barndorff-Nielsen (1991)[1] and Ma (1998) [17]), Zhu and Zhou (2009) [26] derives a SP approximation for $X_{(m)}$

Theorem 2 (Saddlepoint approximation (Zhu and Zhou (2009))). *For $\forall \epsilon > 0$ and $\forall x$ in the domain of $F_{\boldsymbol{\theta}}$, we assume $\epsilon < p < 1 - \epsilon$. Let $r_0 = \frac{m}{n}$, $F_{\boldsymbol{\theta}}(x) = t$. Then for $t \neq p$*

$$X_{(m)} \rightarrow_{(d)} 1 - \Phi(\sqrt{n}\omega^{\sharp}) \quad as \quad n \longrightarrow \infty \tag{3}$$

where the convergence speed is $O(\frac{1}{n})$ uniformly w.r.t x. Φ denotes the cdf of

[7] Asymptotically consistent estimator means $\hat{\boldsymbol{\theta}} \rightarrow_{(P)} \boldsymbol{\theta}$, where $\rightarrow_{(P)}$ represents convergence in probability

standard Gaussian distribution $(N(0,1))$

$$\begin{aligned}
&\omega^{\sharp} = \omega + \frac{1}{n\omega} ln \frac{1}{\psi(-\omega)} \\
&\psi(-\omega) = \frac{\omega(t-1)}{t-r_0}\left(\frac{r_0}{1-r_0}\right)^{\frac{1}{2}} \\
&\omega = -\sqrt{2h(t)}\, sign(t-r_0) \\
&h(t) = r_0 ln \frac{r_0}{t} + (1-r_0) ln \frac{1-r_0}{t}
\end{aligned} \tag{4}$$

where $sign(x) = 1$ *if* $x \geq 0$ *and* $sign(x) = -1$ *otherwise. For* $t = p$

$$X_{(m)} \rightarrow_{(d)} \frac{1}{2} + \frac{1}{\sqrt{2\pi n}} \frac{1+r_0}{3r_0}\left(\frac{r_0}{1-r_0}\right)^{\frac{1}{2}} \quad as \quad n \longrightarrow \infty \tag{5}$$

The proof of Theorem 2 is provided by Zhu and Zhou (2009) [26]. We introduce a corollary of Theorem 2, by plugging $F_{\hat{\boldsymbol{\theta}}}$ and $f_{\hat{\boldsymbol{\theta}}}$ in expression (3) and (4)

Corollary 2 (Plug-in SP approximation). *Assume* $F_{\boldsymbol{\theta}}$ *and* $f_{\boldsymbol{\theta}}$ *are continuous functions with respect to (w.r.t)* $\boldsymbol{\theta}$*, and* $\hat{\boldsymbol{\theta}}$ *is an consistent estimator of* $\boldsymbol{\theta}$*. For* $\forall \epsilon > 0$ *and* $\forall x$ *in the domain of* $F_{\boldsymbol{\theta}}$*, we assume* $\epsilon < p < 1-\epsilon$*. Let* $r_0 = \frac{m}{n}$*,* $F_{\hat{\boldsymbol{\theta}}}(x) = t$*. Then for* $t \neq p$

$$X_{(m)} \rightarrow_{(d)} 1 - \Phi(\sqrt{n}\widehat{\omega}^{\sharp}) \quad as \quad n \longrightarrow \infty \tag{6}$$

where the convergence speed is $O(\frac{1}{n})$ *uniformly w.r.t* x*.*

$$\begin{aligned}
&\widehat{\omega}^{\sharp} = \widehat{\omega} + \frac{1}{n\widehat{\omega}} ln \frac{1}{\psi(-\widehat{\omega})} \\
&\psi(-\widehat{\omega}) = \frac{\widehat{\omega}(t-1)}{t-r_0}\left(\frac{r_0}{1-r_0}\right)^{\frac{1}{2}} \\
&\widehat{\omega} = -\sqrt{2h(t)}\, sign(t-r_0) \\
&h(t) = r_0 ln \frac{r_0}{t} + (1-r_0) ln \frac{1-r_0}{t}
\end{aligned} \tag{7}$$

For $t = p$

$$X_{(m)} \rightarrow_{(d)} \frac{1}{2} + \frac{1}{\sqrt{2\pi n}} \frac{1+r_0}{3r_0}\left(\frac{r_0}{1-r_0}\right)^{\frac{1}{2}} \tag{8}$$

The proof of Corollary 2 is similar as of Corollary 1. Comparing Theorem 1 and 2, we find that the convergence speed of SP approximation ($O(\frac{1}{n})$) is faster than the speed of AN approximation ($O(\frac{1}{\sqrt{n}})$). It means the SP approximation can be more accurate, especially when we have a small sample. Consequently we suggest to use SP approach in practice, in order to provide a robust approximation for the distribution of the $hVaR$.

3. Compare the AN and SP approximations by simulations

In this section, we compare the performance of Theorem 1 and Theorem 2 by simulation, using $N(0,1)$ distribution, NIG_0 (Normal-inverse Gaussian, see Godin (2012) [8]) distribution [8] and GEV (Generalized extreme value, see Longin (2000) [16]) distribution [9]. These distributions belong respectively to elliptical distribution family, Generalised hyperbolic distribution family and extreme value distribution family.

3.1. Graphical comparison

First we compare the AN and SP approximation graphically. Preliminarily we recall the definition of empirical cdf (ecdf). For $X_1, ..., X_n$, its ecdf is defined as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{X_i \leq x\}} \tag{9}$$

where $1_{\{X_i \leq x\}} = 1$ if $X_i \leq x$ and 0 otherwise.

Let $n = 241$ [10], we generate $241 * 1000$ random numbers from one of the previous distributions. Then given $0 < p < 1$, we take the realizations of the $hVaR$

[8] The tail parameter parameter equals to 0.3250, skewness parameter equals to $5.9248e-04$, location parameter equals to $-1.6125e-04$ and scale parameter equals to 0.0972.

[9] The shape parameter equals to 0.8876698, scale parameter equals to 2049.7625278 and location parameter equals to 245.7930751.

[10] 241 is around the number of one year trading days.

for every 241 points to have 1000 realisations. We plot the ecdf of $hVaR$ as a benchmark.

Given $p = 0.975$ with $N(0,1)$, on Figure 1 we plot: the solid line, which is the ecdf of $hVaR$ and benchmark. The dot-dash line is the AN approximation from Theorem 1. The dash line is the SP approximation from Theorem 2. We observe that the SP approximation is nearly on the ecdf. But the AN approximation is away from the ecdf.

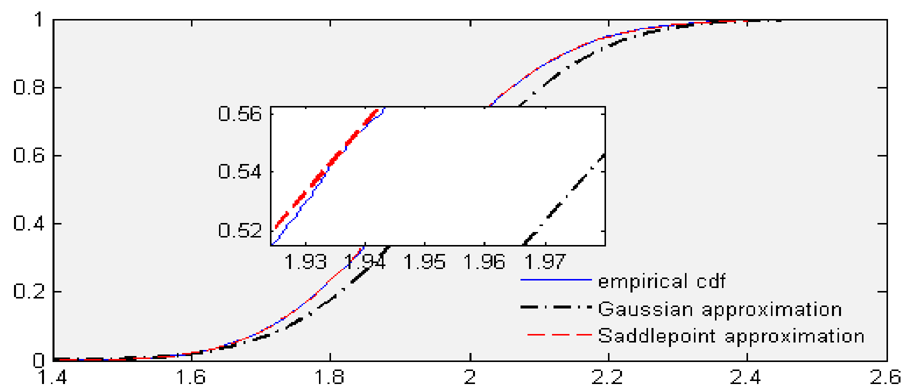

Figure 1: Given $p = 0.975$ and $n = 241$, with $N(0,1)$, in Figure 1 the solid line is the the ecdf of $hVaR$. The dot-dash line is the AN approximation from Theorem 1. The dash line is the SP approximation from Theorem 2.

Also with NIG_0 and $p = 0.01$ (left tail), we plot the ecdf of $hVaR$, the AN and SP approximation in the left graph of Figure 2. In the right graph with GEV and $p = 0.995$ (right tail), we plot the ecdf of $hVaR$, the AN and SP approximation. In Figure 2, the solid line is the ecdf. The dot-dash line is the AN approximation. The dash line is the SP approximation.

We observe that in both graphs of Figure 2, the dash line is always closer to the solid line than the dot-dash line. That means for fat-tailed $F_{\boldsymbol{\theta}}$, the SP approximation is still more accurate than the AN. Specially, the $hVaR$ realisa-

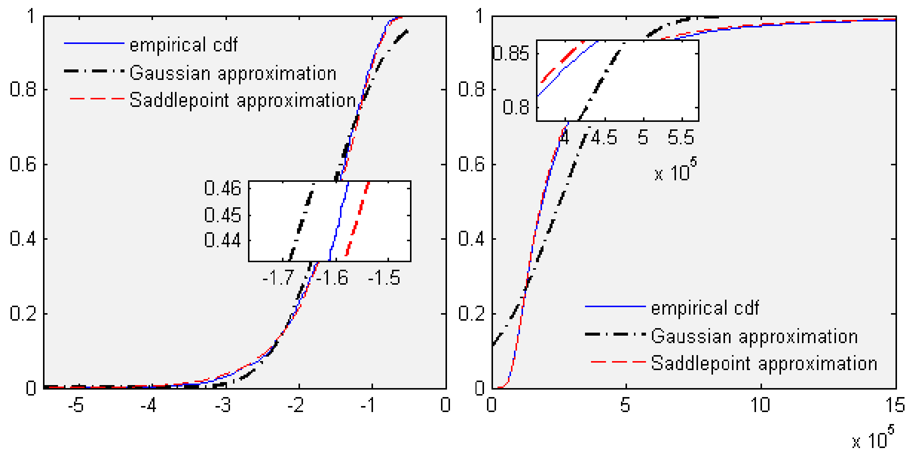

Figure 2: With NIG_0 and $p = 0.01$, we plot the ecdf of $hVaR$, the AN and SP approximation in the left graph of Figure 2. In the right graph with GEV and $p = 0.995$, we plot the ecdf of $hVaR$, the AN and SP approximation. In figure 2, The solid line is the ecdf. The dot-dash line is the AN approximation. The dash line is the SP approximation.

tions simulated from GEV is asymmetric ($skewness = 9.8760$) and leptokurtic ($kurtosis = 176.0057$). Apparently, the AN approximation can not model these behaviours accurately, since it is symmetric and its tail is thin. Consequently, it is reasonable to use SP approximation in this case.

From Figures 1 and 2, we observe that no matter we consider the left or right tail $hVaR$, no matter the $F_{\boldsymbol{\theta}}$ is symmetric or asymmetric, fat-tailed or thin-tailed, the SP approximation is always more accurate. Besides, it is always more precise than the AN approximation, which is symmetric and thin-tailed. Consequently, we suggest the risk manager and regulator to use the SP approximation to model the uncertainty in $hVaR$, especially when the sample size is small (for example one year daily data), the sample has asymmetric and leptokurtic properties and the tail $hVaR$ needs to be computed.

3.2. Quantitative comparison

For robustness purpose, we also compare AN and SP approximations quantitatively. Preliminarily we recall the definitions of both Kolmogorov–Smirnov (K-S) statistic and Anderson–Darling (A-D) statistic. For an ecdf $F_n(x)$ and a cdf F: the Kolmogorov–Smirnov statistic is

$$DKS_n = sup_x \mid F_n(x) - F(x) \mid \tag{10}$$

The Anderson–Darling statistic is

$$DAD_n = n \int_{-\infty}^{\infty} \frac{(F_n(x) - F(x))^2}{F(x)(1 - F(x))} dF(x) \tag{11}$$

In this paper, we use the two-sample A-D statistic provided by Pettitt (1976) [19].

First we use the same simulation process as in section 3.1, for different n and p [11]. Then, we compute the K-S statistic and A-D statistic between the AN (or SP) approximation and the ecdf of $hVaR$. The results of K-S statistic are provided in Table 2 and the results of A-D statistic are provided in Table 3.

In Table 2 and 3, the results provided by SP are always smaller than those provided by AN, for all the three distributions and whatever n and p. Consequently, the SP approximation always performs better than the AN approximation. More precisely, when p is closer to 0 or 1, the difference of accuracy between these two approximations is more obvious. That means the SP provides more precise approximation for the tail $hVaR$ than the AN. Also, when n is small, the SP can provide robust approximation but AN cannot. Nevertheless, we observe when n is large, the AN approximation is acceptable too.

[11] Here, $n = 11, 121, 241, 501, 1001, 10001, 30001$ and $p = 0.05, 0.01, 0.005, 0.95, 0.99, 0.995$.

After checking the performance of AN and SP approximations graphically and quantitatively, we conclude that the SP approximation always provide robust approximation, while AN can only do that when n is large. Thus, we suggest risk managers and regulators in financial institutions to use the SP approximation when they measure VaR uncertainty, especially when sample sizes are small.

4. Application: SSVaR with saddlepoint approximation and stress testing

In assessing the SP approximation of $hVaR$ in practice, we consider a fictive financial institution. This one holds four market portfolios (that is, the same stock components and weights as a benchmark index of a stock market): the Standard Poor's 500 (S&P 500) in the U.S., the CAC 40 in France, the Hang Seng Index (HSI) and the Shanghai Composite Index (SHCOMP) in China. The S&P 500 represents a developed stock market in the U.S.; the CAC 40 represents a developed market in Europe; the HSI represents a developed market in Asia and the SHCOMP represents an emerging stock market. We consider the daily returns computed using daily closing prices of the portfolio. More precisely, eight data sets are considered for different period [12].

[12]The data sets are: the daily return of S&P 500 over the period from 01/10/1985 to 12/08/1991 with 1483 observations (denoted SP1); the daily return of S&P 500 over the period from 02/01/1987 to 31/12/1987 with 253 observations (denoted SP2); the daily return of CAC 40 over the period from 02/01/2008 to 26/02/2016 with 2089 observations (denoted CAC1); the daily return of CAC 40 over the period from 02/01/2008 to 31/12/2008 with 256 observations (denoted CAC2); the daily return of HSI over the period from 01/10/1985 to 12/08/1991 with 1452 observations (denoted HSI1); the daily return of HSI over the period from 02/01/1987 to 31/12/1987 with 246 observations (denoted HSI2); the daily return of SHCOMP over the period from 06/04/2005 to 25/04/2011 with 1471 observations (denoted SH1); the daily return of SHCOMP over the period from 26/02/2015 to 26/02/2016 with 246 observations (denoted SH2). All the data sets have been obtained from Bloomberg.

In Table 1, we provide the first four empirical moments and the number of observations of these eight data sets. Among them, SP1, CAC1, HSI1 and SH1 are asymmetric and leptokurtic, so we fit a NIG distribution on them [13]. Given the confidence level $q = 0.01, 0.001$ of the CI of $hVaR$, for $n = 241$, we compute the lower bound of CI using AN and SP approximation with $0.001 \leq p \leq 0.05$ as the lower bound of the SSVaR. The CI are computed from Corollary 1 and 2 using the fit. For the CI from AN, we have the closed form. For the CI from SP, we use bisection approach to compute numerically. The upper bound of the SSVaR is the quantile of the fit (For details of SSVaR, see Guégan et al. (2016) [11]). To compare the SSVaR from AN (AN-SSVaR) and the SSVaR from SP (SP-SSVaR), we compute the ecdf of SP2, CAC2, HSI2 and SH2 as a stress testing application (BCBS (2005) [2]).

		Empirical moments			
	points	mean	variance	skewness	kurtosis
SP1 (01/10/1985-12/08/1991)	1483	0.0006	0.0001	-3.6096	65.8124
SP2 (02/01/1987-31/12/1987)	253	0.0003	0.0004	-4.0440	45.5834
CAC1 (02/01/2008-26/02/2016)	2089	0.0000	0.0003	0.2442	8.5171
CAC2 (02/01/2008-31/12/2008)	256	-0.0018	0.0007	0.5586	7.2127
HSI1 (01/10/1985-12/08/1991)	1452	0.0008	0.0003	-6.6479	119.2823
HSI2 (02/01/1987-31/12/1987)	246	0.0000	0.0008	-6.7209	78.8165
SH1 (06/04/2005-25/04/2011)	1471	0.0008	0.0004	-0.2487	5.6722
SH2 (26/02/2015-26/02/2016)	246	-0.0003	0.0007	-0.8473	4.2512

Table 1: In Table 1, we provide the first four empirical moments of these eight data sets and the number of observations.

In Figure 3, we plot our results with two panels. In panel (A), with $0.001 \leq p \leq 0.05$ and $q = 0.01$, we use SP1 fitting a NIG to build the SSVaR. We plot

[13] We use the moment method to get the estimates of NIG parameters. The estimates are provided in Appendix B

the ecdf of SP2 for the stress testing. In panel (B), with $0.015 \leq p \leq 0.05$ and $q = 0.001$, we use CAC1 fitting a NIG to build the SSVaR. We plot the ecdf of CAC2 for the stress testing. The solid line is the upper bound of the SSVaR. The dash line is the lower bound of the AN-SSVaR. The dash-dot line is the lower bound of the SP-SSVaR. And the dash line with points is the ecdf.

In panel (A) of Figure 3, the ecdf is inside the AN-SSVaR and SP-SSVaR. Thus, the SSVaR controls the risk. Furthermore, the SP-SSVaR is inside the AN-SSVaR. It means the AN-SSVaR is more conservative. But it seems to overestimate the risk in this case. In panel (B) of Figure 3, the ecdf is totally outside the AN-SSVaR but it is almost inside [14] the SP-SSVaR. So in this case, the SP-SSVaR controls the risk but AN-SSVaR seems to underestimates the risk.

In order to check the performance of SSVaR from SP approximation comprehensively, we also consider other two portfolios from Asia stock markets. In Figure 4, panel (C) is the same plot as panel (A) of Figure 3, but we change the data set SP 1 to HSI 1 and SP2 to HSI2. Panel (D) is the same plot as panel (B) of Figure 3, but we change the data set CAC 1 to SHCOMP 1 and CAC2 to SH2. We find the result of panel (C) in Figure 4 is similar as of panel (A) in Figure 3 and the result of panel (D) in Figure 4 is similar as of panel (B) in Figure 3.

We conclude the SP-SSVaR is an improvement risk measure of $hVaR$ in practice. It permits integrating the uncertainty robustly from different sample sizes. Thus, risk managers can use it to evaluate a capital buffer to cover the risk embedded in the data set and the risk of measurement uncertainty.

[14] There are some outliers in the left part.

5. Conclusion

In this paper, we suggest using the SSVaR considering a saddlepoint approximation as an efficient risk measure, especially measuring the risks from a small finite sample. Indeed, theoretically the SP method can approximate the distribution of $hVaR$ more accurately than the AN method. Consequently it provides more robust CI to build the SSVaR. To understand the performance of SP approximation comprehensively, we compared SP and AN approximation graphically and quantitatively. The simulation results are consistent with the theoretical result. Finally, we built the SSVaR from AN and SP approximations and perform a stress testing application with data sets from different stock markets. We find that compared to the SSVaR from AN approximations, the SSVaR computed from SP approximations can always capture the risk more efficiently, neither underestimates nor overestimates the risks. Therefore, in practice, we suggest risk managers and regulators to use the SSVaR considering a SP approximation to integrate the risk measurement uncertainty, particularly when sample sizes are small.

References

[1] O. E. Barndorff-Nielsen. Modified signed log likelihood ratio. *Biometrika*, 78(3):557–563, 1991.

[2] Basel Committee on Banking Supervision. Amendment to the Capital Accord to incorporate market risks. Working Paper , 2005.

[3] V. Bignozzi and A. Tsanakas. Parameter uncertainty and residual estimation risk. *Journal of Risk and Insurance*, page Forthcoming. , 2015.

[4] P. Christoffersen and S. Gonçalves. Estimation risk in financial risk management. *Journal of Risk*, 7(3):1–28, 2005.

[5] H. E. Daniels. Saddlepoint approximations in statistics. *The Annals of Mathematical Statistics*, 25(4):631–650, 1954.

[6] G. S. Easton and E. Ronchetti. General Saddlepoint Approximations with Applications to L Statistics. *Journal of the American Statistical Association*, 81(394):420–430, 1986.

[7] C. Field and E. Ronchetti. *Small sample asymptotics*. Institute of Mathematical Statistics, Hayward, CA, 1990.

[8] F. Godin, S. Mayoral, and M. Morales. Contingent claim pricing using a normal inverse gaussian probability distortion operator. *The Journal of Risk and Insurance*, 79(3):841–866, 2012.

[9] C. Goutis and G. Casella. Explaining the saddlepoint approximation. *The American Statistician*, 53(3):216–224, 1999.

[10] D. Guégan, B. Hassani, and K. Li. The Spectral Stress VaR (SSVaR). Centre d'Économie de la Sorbonne Working Paper, 2015.

[11] D. Guégan, B. Hassani, and K. Li. Uncertainty in historical Value-at-Risk: an alternative quantile-based risk measure. Centre d'Économie de la Sorbonne Working Paper, 2016.

[12] P. Hall and S. J. Sheather. On the distribution of a studentized quantile. *Journal of the Royal Statistical Society: Series B*, 50(3):381–391, 1988.

[13] P. Jorion. Risk2: Measuring the risk in value at risk. *Financial Analysts Journal*, 52(6):47–56, 1996.

[14] P. Jorion. *Value at Risk: The New Benchmark for Managing Financial Risk*. McGraw-Hill, New York, 3 edition, 2006.

[15] D. M. Kaplan. Improved quantile inference via fixed-smoothing asymptotics and edgeworth expansion. *Journal of Econometrics*, 185(1):20–32, 2015.

[16] F. M. Longin. From value at risk to stress testing: The extreme value approach. *Journal of Banking & Finance*, 24(7):1097–1130, 2000.

[17] C. Ma and J. Robinson. Saddlepoint approximation for sample and bootstrap quantiles. *Australian & New Zealand Journal of Statistics*, 40(4):479–486, 1998.

[18] C. Ma and J. Robinson. Saddlepoint approximations for the difference of order statistics and studentized sample quantiles. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 61(3):563–577,, 1999.

[19] A. N. Pettitt. A two-sample anderson-darling rank statistic. *Biometrika*, 63(1):161–168, 1976.

[20] C. Pérignon and D.R. Smith. The level and quality of value-at-risk disclosure by commercial banks. *Journal of Banking & Finance*, 34(2):362–377, 2010.

[21] M. Pritsker. Evaluating value at risk methodologies: Accuracy versus computational time. *Journal of Financial Services Research*, 12(2–3):201–242, 1997.

[22] C. R. Rao. *Linear statistical inference and its applications*. John Wiley & Sons, Hoboken, 2 edition, 2002.

[23] N. Reid. Saddlepoint methods and statistical inference. *Statistical Science*, 3(2):213–227,, 1988.

[24] R. D. Reiss. Asymptotic expansions for sample quantiles. *The Annals of Probability*, 4(2):249–258, 1976.

[25] S. Wang. One-step saddlepoint approximations for quantiles. *Computational Statistics & Data Analysis*, 20(1):65–74, 1995.

[26] J. Zhu and W. Zhou. Saddlepoint approximation for sample quantiles with some applications. *Communications in Statistics - Theory and Methods*, 38(13):2241–2250, 2009.

	$N(0,1)$		NIG_0			GEV	
	AN	SP	AN	SP		AN	SP
$p = 0.05$					$p = 0.95$		
$n = 11$	0.2183	0.0361	0.1933	0.0260	$n = 241$	0.0847	0.0265
$n = 121$	0.0767	0.0192	0.0876	0.0358	$n = 501$	0.0503	0.0265
$n = 241$	0.0509	0.0274	0.0621	0.0197	$n = 1001$	0.0756	0.0449
$n = 1001$	0.0435	0.0248	0.0276	0.0228	$n = 10001$	0.0371	0.0363
$n = 10001$	0.0322	0.0259	0.0208	0.0207	$n = 30001$	0.0368	0.0327
$p = 0.01$					$p = 0.99$		
$n = 11$	0.0757	0.0211	0.1963	0.0191	$n = 241$	0.1155	0.0205
$n = 121$	0.0847	0.0318	0.1072	0.0222	$n = 501$	0.1409	0.0370
$n = 241$	0.0948	0.0330	0.0564	0.0265	$n = 1001$	0.0988	0.0266
$n = 1001$	0.0456	0.0191	0.0546	0.0213	$n = 10001$	0.0459	0.0238
$n = 10001$	0.0204	0.0195	0.0227	0.0187	$n = 30001$	0.0327	0.0238
$p = 0.005$					$p = 0.995$		
$n = 11$	0.3930	0.0245	0.3959	0.0138	$n = 241$	0.1584	0.0278
$n = 121$	0.1711	0.0213	0.1418	0.0278	$n = 501$	0.1213	0.0253
$n = 241$	0.0653	0.0225	0.0744	0.0266	$n = 1001$	0.1461	0.0311
$n = 1001$	0.1071	0.0296	0.0976	0.0308	$n = 10001$	0.0515	0.0149
$n = 10001$	0.0367	0.0139	0.0359	0.0163	$n = 30001$	0.0392	0.0227

Table 2: For $N(0,1)$, NIG_0 and GEV, for $n = 11, 121, 241, 501, 1001, 10001, 30001$ and $p = 0.05, 0.01, 0.005, 0.95, 0.99, 0.995$, we compute the K-S statistic between the AN (or SP) approximation and the ecdf of $hVaR$. The results are provided in Table 2.

	$N(0,1)$		NIG_0			GEV	
	AN	SP	AN	SP		AN	SP
$p = 0.05$					$p = 0.95$		
$n = 11$	46.5478	0.3701	45.4794	0.5787	$n = 241$	4.1887	0.3820
$n = 121$	8.1637	0.4610	5.6806	0.3378	$n = 501$	2.9653	0.2765
$n = 241$	2.6894	0.2122	1.8976	1.0557	$n = 1001$	1.7426	0.3486
$n = 1001$	1.7430	0.3728	0.6408	0.7989	$n = 10001$	0.6837	0.6725
$n = 10001$	0.4598	0.2603	0.2827	0.1647	$n = 30001$	0.7047	0.5905
$p = 0.01$					$p = 0.99$		
$n = 11$	5.9214	0.1463	35.3727	0.3105	$n = 241$	16.2251	0.5430
$n = 121$	9.5424	0.8465	9.8631	0.2396	$n = 501$	17.6468	0.6223
$n = 241$	7.2525	0.3538	5.1074	0.2027	$n = 1001$	5.1793	0.6358
$n = 1001$	2.2158	0.4727	1.0760	0.4856	$n = 10001$	1.1022	0.0926
$n = 10001$	0.1742	0.1876	0.5491	0.2616	$n = 30001$	0.9970	0.4869
$p = 0.005$					$p = 0.995$		
$n = 11$	181.0830	0.2873	180.4938	0.1900	$n = 241$	43.3808	0.4520
$n = 121$	27.1190	0.3710	27.9483	0.1365	$n = 501$	20.2579	2.0985
$n = 241$	4.6398	0.1650	10.7707	0.5890	$n = 1001$	17.7949	1.2266
$n = 1001$	10.0369	0.2362	9.6725	0.4381	$n = 10001$	5.4230	1.6109
$n = 10001$	1.1836	0.2042	1.8859	0.3874	$n = 30001$	2.1239	0.9011

Table 3: For $N(0,1)$, NIG_0 and GEV, for $n = 11, 121, 241, 501, 1001, 10001, 30001$ and $p = 0.05, 0.01, 0.005, 0.95, 0.99, 0.995$, we compute the A-D statistic between the AN (or SP) approximation and the ecdf of $hVaR$. The results are provided in Table 3.

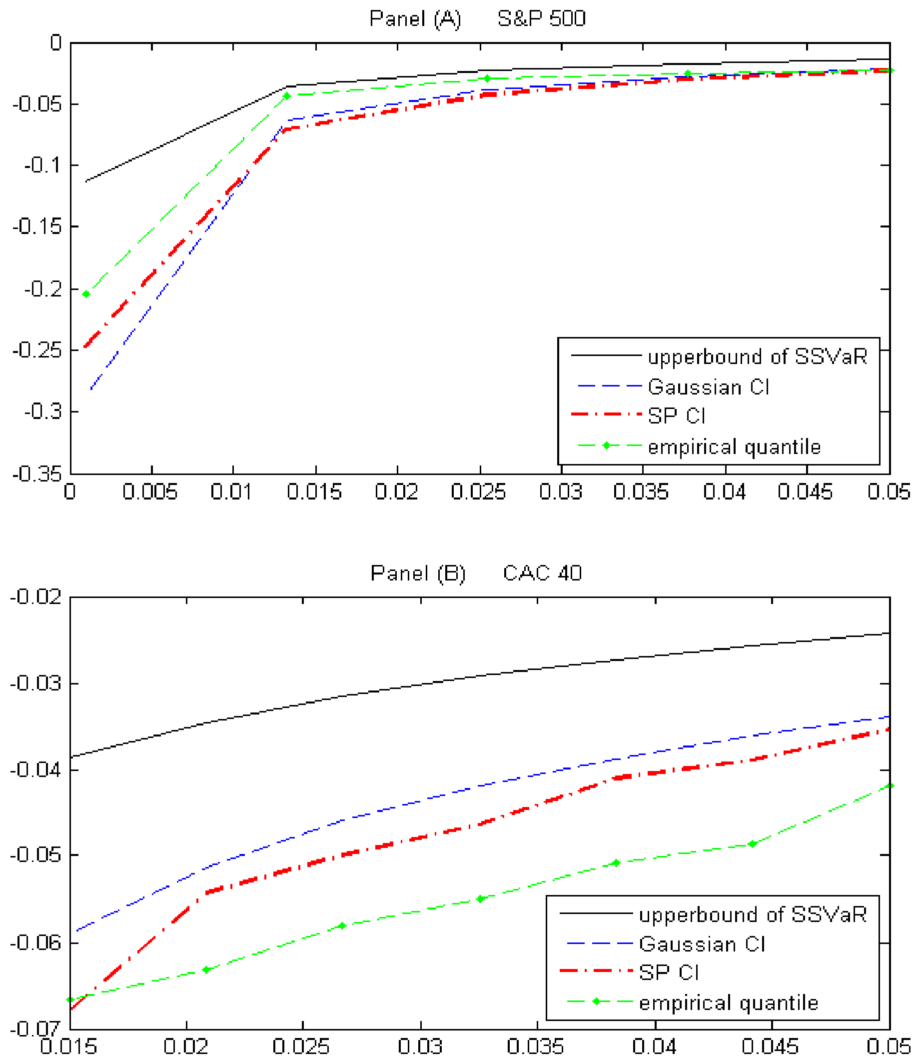

Figure 3: We plot our results with two panels. In panel (A) we use SP1 with NIG fit to build the SSVaR (from AN and SP). We plot the ecdf of SP2 for the stress testing. In panel (B) we use CAC1 with NIG fit to build the SSVaR (from AN and SP). We plot the ecdf of CAC2 for the stress testing. The solid line is the upper bound of the SSVaR. The dash line is the lower bound of the SSVaR from AN approximation. The dash-dot line is the lower bound of the SSVaR from SP approximation. And the dash line with points is the ecdf.

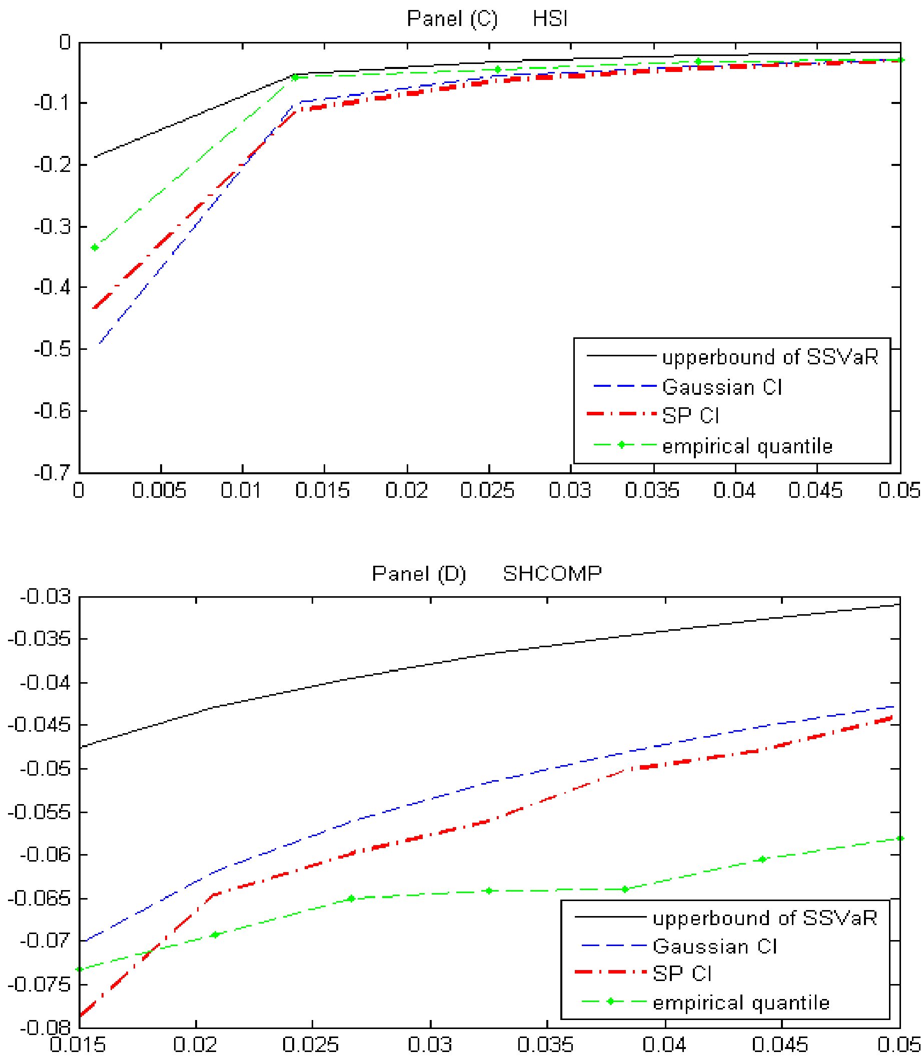

Figure 4: We plot our results with two panels. In panel (C) we use HSI1 with NIG fit to build the SSVaR (from AN and SP). We plot the ecdf of HSI2 for the stress testing. In panel (D) we use SH1 with NIG fit to build the SSVaR (from AN and SP). We plot the ecdf of SH2 for the stress testing. The solid line is the upper bound of the SSVaR. The dash line is the lower bound of the SSVaR from AN approximation. The dash-dot line is the lower bound of the SSVaR from SP approximation. And the dash line with points is the ecdf.

Appendix A. Proof of Theorem 2

At first, we introduce the Slutsky's theorem

Theorem 3 (Slutsky's theorem). *Let $\{X_n\}$, $\{Y_n\}$ be sequences of r.v., If $\{X_n\}$ converges in distribution ($\rightarrow_{(d)}$) to a r.v. X and $\{Y_n\}$ converges in probability to a constant c ($\rightarrow_{(p)}$), then*

$$X_n Y_n \rightarrow_{(d)} cX \tag{A.1}$$

Proof 1. *To prove Theorem 2, we begin with*

$$\begin{aligned}
\frac{X_{(m)} - F_{\hat{\boldsymbol{\theta}}}^{-1}(p)}{\sqrt{\widehat{V}}} &= \frac{X_{(m)} - F_{\boldsymbol{\theta}}^{-1}(p)}{\sqrt{\widehat{V}}} + \frac{F_{\boldsymbol{\theta}}^{-1}(p) - F_{\hat{\boldsymbol{\theta}}}^{-1}(p)}{\sqrt{\widehat{V}}} \\
&= \sqrt{\frac{V}{\widehat{V}}}\frac{X_{(m)} - F_{\boldsymbol{\theta}}^{-1}(p)}{\sqrt{V}} + \frac{F_{\boldsymbol{\theta}}^{-1}(p) - F_{\hat{\boldsymbol{\theta}}}^{-1}(p)}{\sqrt{\widehat{V}}} \\
&= \frac{f_{\hat{\boldsymbol{\theta}}}(F_{\hat{\boldsymbol{\theta}}}^{-1}(p))}{f_{\boldsymbol{\theta}}(F_{\boldsymbol{\theta}}^{-1}(p)}\frac{X_{(m)} - F_{\boldsymbol{\theta}}^{-1}(p))}{\sqrt{V}} + \frac{F_{\boldsymbol{\theta}}^{-1}(p) - F_{\hat{\boldsymbol{\theta}}}^{-1}(p)}{\sqrt{\widehat{V}}}
\end{aligned} \tag{A.2}$$

Since convergence in probability is preserved under continuous transformations, from $\hat{\boldsymbol{\theta}} \rightarrow_{(P)} \boldsymbol{\theta}$ we have

$$F_{\boldsymbol{\theta}}^{-1}(p) - F_{\hat{\boldsymbol{\theta}}}^{-1}(p) \rightarrow_{(P)} 0 \tag{A.3}$$

$$f_{\hat{\boldsymbol{\theta}}}(F_{\hat{\boldsymbol{\theta}}}^{-1})(p) - f_{\boldsymbol{\theta}}(F_{\boldsymbol{\theta}}^{-1}(p)) \rightarrow_{(P)} 0 \tag{A.4}$$

From Theorem 1 we know that $\frac{X_{(m)} - F_{\boldsymbol{\theta}}^{-1}(p))}{\sqrt{V}} \rightarrow_{(d)} N(0,1)$, then

$$\frac{X_{(m)} - F_{\hat{\boldsymbol{\theta}}}^{-1}(p)}{\sqrt{\widehat{V}}} \rightarrow_{(d)} N(0,1) \tag{A.5}$$

Appendix B. Fit the data sets in Table 1 with NIG distribution

We fit the data sets SP1, CAC1, HSI1 and SH1 with NIG distribution. The estimates are provided in Table B.4.

	Fitted parameters (NIG)			
	tail	skewness	location	scale
SP1	23.7081	-7.3294	0.0015	0.0029
CAC1	46.4099	2.8063	-0.0007	0.0119
HSI1	17.7947	-9.0184	0.0029	0.0034
SH1	56.0441	-4.9995	0.0027	0.0208

Table B.4: In Table B.4, we provide the fitted parameters (NIG) for data sets SP1, CAC1, HSI1 and SH1.