Reflections upon the Presentation of Parallel Algorithms Across the Astral and Mathematical Sciences in First-Millennium China

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I'd like to thank the organisers *again* for giving me this opportunity, this is my first time to give a talk in China as a *historian of astronomy*; you've made me feel very *welcome*. What I'm going to present today is more-or-less what many of you already heard in March, but I've added some things and I've moved some other things around... so that it looks new.

So, I'm now a historian of *li*, or 'mathematical astronomy', but I only came to this topic in 2010, at Chicago, and like Li Jimin, I didn't have a teacher, so I had to teach myself. I tried different things: I tried starting with the primary sources, I tried reading secondary scholarship in Chinese and English, but I never really made any progress until I discovered Christopher Cullen's translation of a Han-era procedure text into Excel.

By way of explanation: The way a *li* procedure text works, is that you input a single variable—*the year*—and the subsequent steps take you through everything you could think to calculate for said year. And what Cullen did was translate the Chinese into code so that the process was automated and you could thus see how each procedure worked, and what result it's supposed to give.

So, I decided that I would teach myself how to read these texts from Cullen's spreadsheet, but that didn't really work. It didn't work because it was *just* as difficult for me to read code and classical Chinese side-by-side as it was to to do the same with symbolic algebra. Instead, I decided that the only way to *really* learn was to create my own spreadsheet so as to turn text into performance and see where I failed to perform.

So, what *I* did was start over: I worked procedure-by-procedure through a book by Liú Hóngtāo, which explains things in symbolic algebra and prose, even giving you the odd sample problem, and I translated the Chinese of each procedure into English and spreadsheet code, referencing Cullen only near the end to see if I had understood. This took *forever*, one of the biggest problems being that there was a huge ambiguity between the operations as written in classical Chinese and the modern operations as identified by Cullen and Liú—you had *one word* that was being identified with different operations, you had *one operation* represented with different words... Confused, I would sometimes turn to Karine and Guō Shūchūn's glossary at the end of the *Nine Chapters*, but *there*, I wouldn't find the definition that I was looking for *but yet another definition!*

OK, the thing with my procedure texts is that you know the constants and you know the sort of answer you should get so you

can work backwards from there to the operation. So, after two weeks of hell, I gave up on the idea that I could could understand the text by reading it, and I gave up on Karine's glossary, and I just went case-by-case through the procedures until they worked and about a year later the texts just started making sense to me, and I stopped thinking so much about their actual language.

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That's how I learned the astronomical corpus. Later, I came to Paris to work with Karine and we had all all of these awkward moments of miscommunication: 'your texts say *that*?', 'that's funny, we only see that in manuscripts', etc. And after a while it was clear that we were kind of in different worlds.

The Rule of Three: Astronomy vs. Mathematics

To give you a sense of the sort of difference we were looking at, let's look at how the two *genres* write the rule of three. In the Triple Concordance *li* of *circa* 5 CE, we have the typical algorithm for finding the number of months elapsed since the coincidence of new moon and winter solstice at midnight:

> Mount the years entered into current concordance—*y*—by 235. Overflowing 19, get one. Name this 'accumulated months', and name that which does not overflow 'intercalary remainder'.

'Mount' simply means multiply. The expression for division is somewhat less apparent to the modern reader: 'Overflowing 19, get one', we can understand to mean something like:

> 'Count one for each time 235 *y* is greater than 19'.

Now, turning to the *Nine Chapters*, we see the rule of three written this way:

> Procedure: to find milled grain from unhulled grain, THREE it, FIVE-then-one.

Same algorithm, different words. Again, 'THREE it' is pretty easy to understand as multiplication. 'FIVE then one' is a little vague, but it's the same idea as 'overflowing *x*, get one', that is, 'count one for every *x*'.

It's much more complicated than that, of course, because you see considerable variation within a single genre—even in the exact same procedure. So in the Triple Concordance *li*, for example, we saw 'mount *y* by 235, overflowing 19, get one'. Then we see 'filling 19, get one' and we see 'NINETEEN-*then*-one' like in the *Nine Chapters*, and we see 'then one per NINETEEN'. And we're only getting started, because we also have 'make accumulated months per NINETEEN' and 'NINETEEN is like one'… and the list goes on.

The Rule of Three: Mathematics vs. Mathematics

Naturally, you get variation within the *suàn*-mathematics corpus as well, so where chapter two of the *Nine Chapters* gave us 'THREE it, FIVE-*then*-one', the *Suànshùshù* manuscript gives us 'FIVE it, THREE

becomes one', and chapter six of the *Nine Chapters* gives us 'mount it by SIX, then *eliminate* it by TWENTY-FIVE'.

The Rule of Three: Division

All-in-all, the language of multiplication is consistent across instances of the rule of three: you have 'mount two by three'; and you have 'take two and three it' or 'triple it'. The language of division is much more eclectic, but it too comes down to two basic patterns.

On the one hand, you have variations on the expression 'then get one per...' You have 'for *x*, get one per *y*', which makes sense, and you have what I suspect are abbreviations of this expression: 'For *x*, then... one per *y*', 'For *x*... *y*... get one', 'For *x*... *y*... then one', 'For *x*... one-per *y*'. You then have variations on each of the auxiliary words in this expression: '*X* per *y* makes *z*', 'For *x*... overflowing *y* get one', and 'For *x*... filling *y* get one'.

OK, separate from this you have *chú*—eliminate—which is quite simple, you have 'eliminate *x* by *y*', and that is all.

'Elimination' (*chu* 除) in Mathematical Astronomy

Chú is a weird one though, because when you look at how it's used in astronomical procedure texts, the expression stays the same, but it stands for different operations. Let's look at the *very first* procedure from the *very first* procedure text: the Triple Concordance system's procedure to 'calculate the luni-solar origin and concordance'. Now, what this procedure does is tell us what '*concordance*' cycle we're in and how many years we are into it. Let me explain.

So, we begin by:

> Setting out the number of years elapsed since grand-culmen high-origin, excluding the year sought.

The year twenty-fifteen, for example, falls one-hundred and forty-five thousand, two-hundred and forty-five years after 'high origin', counting 'exclusively' from the *end* of 2014.

That's a big number, and we don't really need it. You see, high origin begins with a coincidence of winter solstice, syzygy, midnight, and the sexagenary day-count. *But...* those conditions repeat *every origin*—every 4617 years. Therefore, we can toss all the previous origins out:

> If overflowing the origin divisor, eliminate it.

So, for TWENTY-FIFTEEN, we would toss out THIRTY-ONE origins, leaving us 2118 years into... the present one. So, here... 'eliminate' means *modulo*.

Now, an origin is comprised of three 'concordances' each concordance beginning with a coincidence of winter solstice, syzygy and midnight, but on a different sexagenary day. Therefore:

> Any remainder that does not overflow a concordance is... 'the number of years since heaven concordance' at day *jiǎ-zǐ*.

> *But...* If it overflows a concordance, you eliminate it, and the remainder is 'the number of years since Earth concordance' at day *jiǎ-chén*,

and so on.

So, we're 2118 years in, which is greater than 1589, so we '*eliminate*' that value placing us 579 years into the second concordance. Now, *we* stop *there* for 2015, but were the result more than 1589, we would '*eliminate*' again, placing us so many years into the third concordance. So, here, in the same procedure, 'eliminate' also means *subtract.*

So, 'eliminate' is used for subtraction and for the *repeated* subtraction of a given number, but it's also used for repeated subtraction of a sequence of *different* numbers. Let's take this procedure from the Quarter-remainder *li* for example. After calculating the number of *du* travelled by the moon from winter solstice to syzygy, and '*eliminating*' full cycles of 365¼, one then converts this number into a zodiacal lodge and the number of *du* entered therein. So you...

> add the accumulated *du* to the 21 *du* of Dipper and add 235 fractional parts.

You do this because that is the position of winter solstice. And then you...

> Sequentially eliminate it by lodges, and that which does not fill the last lodge is the star and *du* position of the sun and moon at syzygy.

So, let's say that you... (improv.)

Sources

So, you get a sense of how *confusing* this operational vocabulary is, and what makes matters worse is that *each of the four operations that 'eliminate' performs are performed by other words as well.* What Karine suggested we do is is to compare language between procedure texts in astronomical and mathematical corpora, looking specifically at how *lì* procedures might resonate with usages we see in early manuscripts and later commentary. Our goal would be to see if we could find a pattern to word-choice, and what would be *great* is if this pattern could conclusively demonstrate the existence of distinct mathematical cultures within a civilisation that's often treated as a unity. *OK...* So this is what we've got in terms of sources.

- *First of all*, I personally stop at the eighth century. My training as a sinologist began with the *second millennium BCE*, so... give me a couple years to catch up.

- Now, on the top, here, we've got THIRTEEN procedure texts in *li* extant from this period. all of these are preserved in the monographs of the standard histories, and they read very similar to one another, so it's safe to treat them as an integral *genre* of text.

- Now, on the *suan*—mathematics—side of thing, you have even more literary production, and you have different kinds of text. You have, *first of all*, the canon—in orange—codified and

commentated by Lǐ Chúnfēng's team in the mid-seventh century.

- Lǐ Chúnfēng's canon was comprised of ten classics, nine of which survive.

- And in addition to the seventh-century commentary of Lǐ Chúnfēng's team, the canon also preserves a number of *other* commentaries to the aforementioned classics.

- In addition to *that*, you also have a growing number of anonymous manuscripts excavated from the early Qín and Hàn period.

- And in addition to *that*, you have this weird little text called the *Shùshù jìyí* which is *not* a classic, but survives anyway. And this thing has a commentary too!

OK, so we've got a good idea, and we have THIRTY-SOMETHING texts in FOUR or FIVE *genre* to work with. The question, *then*, is how do you sort out the sort of synchronic variation that Karine suggest we explore from diachronic variation across each *genre* over time. The *answer*, I think, is that we need to look at *every single instance* of a given term in *every single text*. I'm not very clever about such things... so this is kind of how I like to go about a problem.

Methodology

OK, so this is what I did. First... I gathered-slash-created digital versions of every text, I marked every instance of every term with 'find-and-replace' and I identified the operation intended by the word on a case-by-case basis from secondary studies. Second, I used these files to tabulate the distribution of each word's intended usage and the word choice behind each operation. Third, I then plotted this data onto graphs to visualise synchronic differences and diachronic trends.

Now, if we can return to our time line.... This is what I had finished in March. And this is what I finished last week—so... the math part is new. All-right, let's look at the results!

Distribution

So here we have division in astronomical procedure texts—and you're looking at the percentage use of different words for this operation. As you see, the distribution here is incredibly varied, but we do see a couple of trends within astronomical texts.

First, *yíng*—'overflow'—is no longer really used for division by 85 CE, whereabouts it is switched with *mǎn*—'fill'—. Second, 'Eliminate' appears in 85 CE in the context of division, but it only ever constitutes about 12% of cases, and the word disappears from the last source of *circa* 600.

Now, when you look at the mathematical texts, you see that the distribution there is much, much neater (note that I am combining *ér yī* and *dé yī* for these texts though). Anyway, you basically have two expressions for division: 'then-one' and 'eliminate', and 'then-one' is much more common, except in the *Gnomon of Zhou*. Now, by comparison with the astronomical texts, we find that 'eliminate' is used for division in math texts a good three centuries earlier, *and*... it sees *more common* use as division in math than it does in astronomy.

Now, let's look at subtraction as we see it represented in astronomical texts. Here, we only really see three ways to express subtraction: *chú*—eliminate, *jiǎn*—diminish, and *qù*—remove. *Over time*, we see instances of subtraction by 'diminish' overtake those by 'eliminate' in popularity. Part of the reason why 'eliminate' loses popularity, *I suspect*, is that it begins to share this function with 'remove' around 200, which actually *replaces* it in the 600 text.

Now, what's interesting when we turn to math texts is that we don't see 'diminish' used as an operation *at all* in the early manuscripts; it only first appears in the *Gnomon of Zhou*, after which it quickly rises to about 80% of all cases of subtraction. *Now...* we don't know exactly when the *Gnomon of Zhou* and *Nine Chapters* were authored, but it is interesting to note that it was around the time that 'diminish' began gaining popularity in astronomical texts as well.

OK, now we turn to *modulo* in astronomical texts. We have three words used for *modulo*: 'eliminate', 'remove', and the hybrid 'remove by elimination'. First, 'remove by elimination' fails to take off, disappearing by 200. Second, we see 'remove' appear in 85 CE, after which it quickly comes to replace 'eliminate'. 'Eliminate' then no longer means *modulo* by the mid fifth century.

Now, I would love to show you the graph for *modulo* in mathematical texts, but the fact of the matter is that *mathematical texts don't use modulo.*

When it comes to sequence subtraction, in astronomical texts, we mainly see 'sequentially eliminate', but every so often a new contender emerges. What's interesting, is that beginning in the sixth century, the word *cì*—sequentially—is increasingly dropped, so 'eliminate' alone begins to mean '*sequentially* eliminate'. Here again, 'sequential subtraction' is not an operation that occurs in math texts, so we don't have anything to compare.

All-right, now let's look at all this from the other direction, moving from simple to complex.

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Now, before we go on, I want everyone to note that we've changed now from percentage to absolute number. So if we remember, early texts exclusively use 'eliminate' for subtraction, around the first century, however, we see *jiǎn*—'diminish'—appear in this role in both mathematical and astronomical texts *then*, it begins to quickly replace 'eliminate'. OK, so if we look at 'diminish' in astronomical texts, we notice that it is used almost exclusively for subtraction.

The same thing is true for our math texts.

OK, when it comes to the whole 'and then get one' complex, it is also pretty exclusive—it only really means divide. Here... what you're looking at is instances of 'get one' in astronomical texts, but for the sake of time I'll just ask you to believe me that it's the same for all variations on the expression, and that its all the same as in math texts.

For 'overflow' things get more interesting. We'll note that the word makes a clean break in the first century, prior to which it's used to mean 'greater than' and, thus, as an auxiliary and abbreviation for division; *but...* it re-appears in the third century, where it is used exclusively in the rule of false position.

Now, if we remember, we saw 'fill' replace 'overflow' in instances of division earlier on, and that's what we see when we look at the word alone—it replaces the exact functions of

'overflow', while 'overflow' moves to the rule of false position. The reason for this is really simple: Yíng—'overflow'—was the personal name of an early Han emperor, which became a political taboo upon his death, so people swapped 'overflow' with the synonym 'fill', and this new convention stuck, well beyond the fall of the dynasty, leaving the word 'overflow' free to be used for new concepts. I didn't have time to make comparative graphs for math texts, but believe me when I tell you that we see the same phenomenon there as well.

Finally, this brings us back to 'eliminate', which is the weirdest of all our operational vocabulary. In astronomical texts, we see 'eliminate' stand for one of four operations: subtraction, repeated subtraction, or *modulo*, sequence subtraction, and division. And then, we see, *I think*, two trends with its use over time. First, we see it used less-and-less over time to mean *modulo*. And if we remember, its the synonym 'remove' that takes over that function. Second, 'eliminate' suddenly gets used for division in 85 CE, and that usage remains more popular than almost any other for the remainder of our period.

Interestingly, we see a similar trend in mathematical texts, where *this* meaning of 'eliminate', which appears centuries earlier than in astronomy, only really begins to take off at around the same time.

<center>***</center>

In March, I reached the following conclusions from this analysis: First, this analysis shows that the operational vocabulary of *li* procedure texts is a mess: one word describes multiple operations, and one operation is described by multiple words. *And now....* I can say that that math texts exhibit the same ambiguity, *but they experience less of it*. Second, *amidst ambiguity*, this analysis reveals *trends* in historical usage: in terms of both actors' and observers' categories, usages emerge, disappear and get redistributed over time.

And now... I can add that the same thing happens in math, and some of the trends there correspond, i.e.: the emergence of 'diminish' for subtraction, the shift of 'eliminate' to division and the replacement of 'overflow' with 'fill'. Furthermore, some of these trends happen *first* in mathematics, which suggests that the astronomical corpus is importing things from mathematics; and if we go further forward in time, maybe we can find importations running the other way.

Lastly, in March I had evidence of *diachronic* diversity in Chinese mathematical cultures. *And now...* by comparing the two corpora, we can see *synchronic* diversity as well, and we haven't even begun distinguishing the sub-genres within the mathematical corpus.

Confusion!

When I began the work of analysing operational vocabulary, this is *kind of* what I wanted to find. I wanted to find solid evidence of diversity, but I ended up with *so much* that this diversity began feeling like chaos, and I had a hard time trying to discern any order to it. It's not just me who has trouble with this vocabulary though: The Supernal Emblem *li*, for example, feels it necessary to define these words:

Any talk of 'as per', 'overflow', 'simplify' or 'fill' all refers to elimination to seek the integer quotient; 'remove' as well as 'eliminate it' refer to elimination to take the remainder.

Now, what's fun about this definition is not only that it defines 'elimination' by itself but that the definition doesn't reflect how the text actually uses these terms. Anyways, in math texts, we see something kind of similar: as you see here, you see the one way to express division used *together* with the other:

Eliminate and get one *bu* of length per the divisor.

And:

Eliminate it and get one *bu* per the divisor.

Linguistically, this is completely unnecessary, but I think what we're seeing is an example of glossing to differentiate the one use of 'eliminate' from the other. and it's interesting you don't see this in the *Nine Chapters*.

Algorithm as Vector of Change

That's where I left things in March. For this meeting, however, the topic was 'algorithms', so I thought it would be appropriate to find examples of identical *li* algorithms where the same operation is represented with each of every possible choice.

That I couldn't really find. Instead, the example I chose revealed something much more interesting. The expressions you see here are all part of a procedure you see throughout *li* procedure manuals. Simply put, when you calculate a position, especially a mean position, you can simply add the mean period of the sun, moon planet or whatever to get the subsequent one, right? As you do that, you have to 'eliminate' the zodiacal lodges as you go, which are integers, but when you pass winter solstice, you have to subtract the fractional part at the end of the circuit. In the Quarter-remainder and Supernal Image systems, we see this expressed as:

If passing through Dipper, eliminate *something* parts.

It's the same thing in 237, but when we get to the Epochal Excellence *li* of 443, we suddenly have 'remove'. What's interesting, is that the change in word choice corresponds with a change in the lodge of winter solstice—here, it's no longer Dipper but Hall—as if, linguistically, the one change triggered the other...

The other interesting thing, as we see in the last line, is that *the instructions surrounding this phrase use a different word— diminish—to describe the act of subtraction.*

Moving forward, people keep using 'remove' in this expression, even though the lodge of winter solstice changes from one system to the next. So, this is like how *mǎn* replaces *yíng...*

'Eliminate' pops back up from time to time, but it basically disappears after the fifth century. *So, what it looks like is that the procedure or the sub-procedure is the vector of change for operational terminology*

Now, if we go back to our graph on subtraction, it's much clearer what's going on: By about 200 CE, *jiǎn*—diminish—replaces 'eliminate' in most cases of subtraction *but not entirely*. also,

'remove'--in green--appears at the same time in this function. Something like a third of the cases where 'eliminate' and 'remove' appear as subtraction after TWO-HUNDRED appear in the *the same algorithm*. So... some—*if not all*—of the variety that we see on this chart is due to the survival of old algorithms, while the new algorithms and the explanations of old algorithms use new vocabulary. Now, what I'm thinking that we can do with this information is this: I think that we can sort out cases like 'passing through dipper' to eliminate noise from these graphs; *then*, we can identify the exact points at which a procedure changes or a new procedure is introduced; *then*, we can look to see how these insertions relate to what's happening concurrently in other mathematical *genres*, and *maybe*, we could even extend this to maybe look for procedures that have parallels with Sanskrit sources.

Now, that said, there are other things going on as well....

Conclusion

This is the part where I repeat myself. If you analyse operational vocabulary in mathematical and astronomical procedure texts, you find a mess, but the mess of usages is distributed differently across corpora, and it redistributes over time, which adumbrate distinct idioms or cultures of mathematical practice. These are *distinct*, but they exist in communication, and it looks like certain trends either occur at the same time or feed one into the other.

This is all a bit chaotic at the level of the individual word and the individual operation, but when we focus at the procedure or the sub-procedure level, we see that one of the culprits for the relative chaos of astronomical usages is probably old algorithms that get repeated over-and-over, so, in this sense, I suggested that we look at algorithms as vectors of change (or constancy)—I've only been working on this for a week, but I think if we focus on algorithmic and operational language side-by-side, we might be able to really easily identify how mathematical practices *move* across time and across genres.