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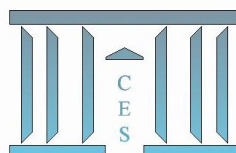
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Equilibrium of a production economy with unbounded attainable allocations set*

Senda Ounaies[†], Jean-Marc Bonnisseau[‡] and Souhail Chebbi[§]

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Abstract

In this paper, we consider a production economy with an unbounded attainable set where the consumers may have non-complete non-transitive preferences. To get the existence of an equilibrium, we provide an asymptotic property on preferences for the attainable consumptions. We show that this condition holds true if the set of attainable allocations is compact or, when preferences are representable by utility functions, if the set of attainable individually rational utility levels is compact. So we extend the previous existence results with unbounded attainable sets in two ways by adding a production sector and considering general preferences.

Keywords: production economy, unbounded attainable allocations, quasi-equilibrium, non complete non transitive preferences.

JEL Codes: C62, D11, D51

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1 Introduction

Since the seventies, with the exception of the seminal paper of Mas-Colell [13] and a first paper of Shafer-Sonnenschein [16], equilibrium for a finite dimensional classical economy is commonly proved using explicitly or implicitly equilibrium existence for the associated abstract economy (see [2], [8], [7], [11], [14], [15]) in which agents are the consumers, the producers and an hypothetic additional agent, the Walrasian auctioneer. Moreover, in exchange economies, it is well-known that the existence of equilibrium with consumption sets that are unbounded from below requires some non-arbitrage conditions (see [12], [18], [4], [3], [5], [6], [1]). In [6], it is shown that these conditions imply the compactness of the individually rational utility level set, which is clearly weaker than assuming the compactness of the attainable allocation, and proved an existence result of an equilibrium under this last condition.

The purpose of our paper is to extend this result to finite dimensional production economies with non-complete, non-transitive preferences, which may not be representable by a utility function. Furthermore, we also allow preferences to be other regarding in the sense that the preferred set of an agent depends on the consumption of the other consumers. We posit the standard assumptions about the closedness, the convexity and the continuity on the consumption side as well as on the production side of the economy like in Florenzano 2003 and a survival assumption.

Our main contribution is to provide a sufficient condition to replace the standard compactness of the attainable allocation set, which is suitably written to deal with general preferences. More precisely, we assume that for each sequence of attainable consumptions, there exists an attainable consumption where the preferred sets are asymptotically close to the preferred sets of the elements of the sequence. Actually, we also restrict our attention to the attainable allocation, which are individually rational, in a sense adapted to the fact that preferences may not be transitive.

We prove that our condition is satisfied when the attainable set is compact and when preferences are represented by utility functions and the set of attainable individually rational utility levels is compact. So, our result extends the previous ones in the literature.

We also remark that our condition deals only with feasible consumptions and not with the associated productions. So, our condition can be identically stated for an exchange economy or for a production economy. This means that even, if there exists unbounded feasible productions, an equilibrium still

exists if the attainable consumption set remains compact. In other words, the key problem comes from the behavior of the preferences for large consumptions and not from the geometry of the productions sets at infinity.

To prove the existence of a quasi-equilibrium, we use several tricks borrowed from various authors. Using a truncated economy in order to apply a fixed point theorem to an artificial compact economy is an old trick as in the first equilibrium proofs. The definition of the "augmented preferences" due to Gale and Mas-Collel (see [9],[10]) is slightly modified by using the convex hull of preferences correspondences since, in our setting, we used non-convex preferences. This definition allows to have the local insatiability of consumers at any point of their attainable consumption set. Further, we restrict prices to be in the closed unit-ball of \mathbb{R}^L , the commodity space, and we used modified budget sets, which are reduced to the original ones when prices belong to the unit-sphere using Bergstrom's trick (see [2]). In this way, it is possible to avoid the problem of discontinuity at the origin of some correspondences. Finally, by applying our assumption on the asymptotic behavior of preferences for attainable allocations to the sequence of allocations in growing associated compact economies, we prove that the attainable consumption having preferred sets close to the ones of the consumptions of the compact economies is a quasi-equilibrium consumption of the original economy. The originality of the proof is mainly contained in this last section.

The reminder of this paper is organized as follows. Section 2 describes the model, gives the definition of a quasi-equilibrium and studies the minimality of the assumptions. Further, we provide a simple numerical example where the set of the attainable allocations is not bounded and preferences are not representable by utility functions but the preferences satisfy our asymptotic condition. In Section 3, we first introduce the definition of modified "augmented preferences" and we prove some properties that are transmitted from equivalent properties of the preferences correspondences like convexity, lower semicontinuity and irreflexivity. Then, we define a new economy \mathcal{E}' where the production sets are the closed convex hull of the initial production sets. We show that the hypothesis made on the original economy \mathcal{E} still hold for \mathcal{E}' . Last, we prove that one easily deduces a quasi-equilibrium of \mathcal{E} from a quasi-equilibrium of \mathcal{E}' . Section 4 concludes the present work by giving a proof of the existence of the quasi-equilibrium of the production economy \mathcal{E}' in two steps, a fixed point like argument and an asymptotic argument.

2 The Model

In this paper, we consider the private ownership economy:

$$\mathcal{E} = (\mathbb{R}^L, (X_i, P_i, \omega_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{(i,j)})$$

where L is a finite set of goods, so that \mathbb{R}^L is the commodity space and the price space of the model. I is a finite set of consumers, each consumer i has a consumption set $X_i \subset \mathbb{R}^L$ and an initial endowment $\omega_i \in \mathbb{R}^L$. The tastes of this consumer are described by a preference correspondence $P_i : \prod_{k \in I} X_k \rightarrow X_i$. It represents the set of strictly preferred consumption of $x_i \in X_i$ given the consumption $(x_k)_{k \neq i}$ of the other consumers. J is a finite set of producers and $Y_j \subset \mathbb{R}^L$ is the set of possible productions of firm $j \in J$. For each i and j , θ_{ij} is the portfolio of shares of the consumer i on the profit of the producer j . The θ_{ij} are nonnegative and for every $j \in J$, $\sum_{i \in I} \theta_{ij} = 1$. These shares together with their initial endowment determine the wealth of each consumer.

Definition 2.1. An allocation $(x, y) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$ is called attainable if:

$$\sum_{i \in I} x_i = \sum_{j \in J} y_j + \sum_{i \in I} \omega_i.$$

We denote by $\mathcal{A}(\mathcal{E})$ the set of attainable allocations.

Definition 2.2. A quasi-equilibrium of the private ownership economy is a pair of an allocation $((\bar{x}_i)_{i \in I}, (\bar{y}_j)_{j \in J}) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$ and a non-zero price vector $\bar{p} \neq 0$, such that:

- (a) (Profit maximization): For every $j \in J$, for every $y_j \in Y_j$, $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{y}_j$,
- (b) (Quasi-demand): For each $i \in I$, $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot \omega_i + \bar{p} \cdot (\sum_{j \in J} \theta_{ij} \bar{y}_j)$ and $x_i \in P_i(\bar{x}) \Rightarrow \bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i$
- (c) (attainability): $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} \omega_i + \sum_{j \in J} \bar{y}_j$.

Notice that, in view of condition (c), condition (b) can be rephrased as for every $i \in I$, $\bar{p} \cdot \bar{x}_i = \bar{p} \cdot \omega_i + \bar{p} \cdot (\sum_{j \in J} \theta_{ij} \bar{y}_j)$ and $[x_i \in P_i(\bar{x}) \Rightarrow \bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i]$

Before stating the assumptions considered on \mathcal{E} , let us introduce some notations:

- $\omega = \sum_{i \in I} \omega_i$ is the total initial endowment;
- $Y = \sum_{j \in J} Y_j$ is the total production set;
- $\hat{X} = \{x \in \prod_{i \in I} X_i : \exists y \in Y : \sum_{i \in I} x_i = \omega + y\}$ is the set of all attainable consumption allocations;
- $\hat{Y} = \{y \in Y : \exists x \in \prod_{i \in I} X_i : \sum_{i \in I} x_i = \omega + y\}$ is the attainable total production set.

In this paper, we consider the following hypothesis:

Assumption (H1) For every $i \in I$

- (a) X_i is a non-empty closed, convex subset of \mathbb{R}^L ;
- (b) [irreflexivity] $\forall x \in \prod_{i \in I} X_i$, $x_i \notin \text{co}P_i(x)$ (the convex hull of $P_i(x)$);
- (c) [lower semicontinuous] $P_i : \prod_{k \in I} X_k \rightarrow X_i$ is lower semicontinuous;
- (d) $\omega_i \in X_i - \sum_{j \in J} \theta_{i,j} Y_j$, i.e. there exists $(\underline{x}_i, (\underline{y}_{i,j})) \in X_i \times \prod_{j \in J} Y_j$ such that $\underline{x}_i = \omega_i + \sum_{j \in J} \theta_{i,j} \underline{y}_{i,j}$;
- (e) For each $x \in \hat{X}$, one has $P_i(x) \neq \emptyset$.

Assumption (H2) Y is a non-empty, closed and convex subset of \mathbb{R}^L .

Assumption (H3) For all sequence $((x_i^\nu))$ of \hat{X} such that for all i , $\underline{x}_i \in \overline{P_i(x^\nu)^c}$, there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$ and $(\bar{x}_i) \in \hat{X}$ such that for all i , for all $\xi_i \in P_i(\bar{x}_i)$, there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i with $\xi_i^{\varphi(\nu)} \in P_i(x^{\varphi(\nu)})$, for all $\nu \geq \nu_1$.

Closedness and convexity are classical assumptions on consumptions and productions sets. They imply in particular that commodities are perfectly divisible. Assumption (H1)(c) is a weaker continuity assumption on preferences. Assumption (H1)(b), i.e. the irreflexivity, is made on the sets $\text{co}P_i(x)$ to avoid to assume the convexity of the preference correspondences P_i . Assumption (H1)(d) implies that using his own shares in the productive system, consumer i can survive without participating in any exchange. This implies no trader will be allowed to starve no matter what the prices are. It also insures that the set $\mathcal{A}(\mathcal{E})$ is nonempty. Usually, in exchange economy, this

assumption is merely written as $\omega_i \in X_i$, which corresponds to $\omega_i = \underline{x}_i$ and $y_{i,j} = 0$ for all j . Assumption (H1)(e) assumes, for every i the insatiability of the i th consumer at any point of his attainable consumption set.

Assumption (H3) is an attempt to weaken the compactness assumption on the global attainable set $\mathcal{A}(\mathcal{E})$ by considering what is called a non-arbitrage condition in several papers (see for example [1], [3], [5], [6]). In the exchange economy, when the preferences are represented by utility functions, it corresponds to a compactness assumption on the attainable individually rational utility levels set. Actually, we do not suppose that the set of attainable allocations for the economy is compact but we assume that for each sequence of attainable consumptions there exists an attainable consumption where the preferred sets are asymptotically close to the ones of the elements of the sequence. Indeed, the element \bar{x} of \hat{X} is not necessarily a cluster point of the sequence (x^ν) but the preferred sets at this point are close to the ones at a point $(x^{\varphi(\nu)})$ for ν large enough.

Note that the productions are not considered in Assumption (H3). So, only the total production set matters since it determines the attainable consumptions. The fact that some unbounded sequences of individual productions can be attainable does not prevent the existence of an equilibrium as long as the total production set is not modified.

The following proposition shows that Assumption (H3) is weaker than the compactness of $\mathcal{A}(\mathcal{E})$.

Proposition 2.1. *Under assumption (H1),*

(i) *If $\mathcal{A}(\mathcal{E})$ is compact, then (H3) is satisfied.*

(ii) *If P_i is represented by a lower semicontinuous utility function u_i , i.e.:*

$$P_i(x) = \{x'_i \in X_i : u_i(x'_i) > u_i(x_i)\}$$

and if the set:

$$U = \{(v_1, v_2, \dots, v_m) \in \mathbb{R}_+^I : \exists x \in \hat{X} \text{ s.t. } u_i(\underline{x}_i) \leq v_i \leq u_i(x_i)\},$$

is compact, then (H3) is satisfied.

In exchange economy, if the survival assumption is $\omega_i \in X_i$, then the set U in Condition (ii) above is just the set of individually rational attainable consumptions, so we recover exactly the compactness assumption of [6].

Proof.

(i) Let $((x_i^\nu))$ be a sequence in \hat{X} . From the definition of \hat{X} , there exists a sequence $((y_j^\nu))$ of $\prod_{j \in J} Y_j$ such that $((x_i^\nu), (y_j^\nu)) \in \mathcal{A}(\mathcal{E})$. Since $\mathcal{A}(\mathcal{E})$ is compact, there exists a subsequence $((x_i^{\varphi(\nu)}), (y_j^{\varphi(\nu)}))$ convergent to $((\bar{x}_i), (\bar{y}_j)) \in \mathcal{A}(\mathcal{E})$. Let $i \in I$ and $\xi_i \in P_i(\bar{x})$. For all integer $k \geq 1$, we set $V_k = \{x \in \prod_{i \in I} X_i \mid B(\xi_i, \frac{1}{k}) \cap P_i(x) \neq \emptyset\}$. Since P_i is lower semi-continuous, V_k is an open neighborhood containing \bar{x} . Since $(x^{\varphi(\nu)})$ converges to \bar{x} , there exists an integer $\bar{\nu}(k)$ such that for all $\nu \geq \bar{\nu}(k)$, $x^{\varphi(\nu)} \in V_k$. We can assume without loss of generality that for all $k \geq 1$, the sequence $(\bar{\nu}(k))$ is strictly increasing, which implies that for all $\nu \geq \bar{\nu}(1)$, there exists a unique integer $\kappa(\nu)$ such that $\bar{\nu}(\kappa(\nu)) \leq \varphi(\nu) < \bar{\nu}(\kappa(\nu) + 1)$. Hence, we have $x^{\varphi(\nu)} \in V_{\kappa(\nu)}$ and there exists $(\xi_i^{\varphi(\nu)})$ such that $\xi_i^{\varphi(\nu)} \in B(\xi_i, \frac{1}{\kappa(\nu)}) \cap P_i(x^{\varphi(\nu)})$. Now, since $\varphi(\nu)$ goes to infinity and for all $k \geq 1$, $(\bar{\nu}(k))$ is strictly increasing, then $\kappa(\nu)$ goes to infinity. Hence, the sequence $(\xi_i^{\varphi(\nu)})$ converges to ξ_i and for all $\varphi(\nu) \geq \bar{\nu}(1)$, $\xi_i^{\varphi(\nu)} \in P_i(x^{\varphi(\nu)})$. So, Assumption (H3) holds true.

(ii) Let $((x_i^\nu))$ be a sequence in \hat{X} such that for all i , $\underline{x}_i \in \overline{P_i(x^\nu)^c}$. Since $P_i(x) = \{x'_i \in X_i : u_i(x'_i) > u_i(x_i)\}$ and $\underline{x}_i \in \overline{P_i(x^\nu)^c}, \forall i \in I$, the lower semi-continuity of u_i implies that $u_i(\underline{x}_i) \leq u_i(x_i^\nu)$. Let now consider the sequence (v_i^ν) defined by

$$(v_i^\nu) = (u_i(x_i^\nu))$$

For all $i \in I$, we have $u_i(\underline{x}_i) \leq v_i^\nu = u_i(x_i^\nu)$ so $(v_i^\nu) \in U$. Since U is compact, there exists a subsequence $(v_i^{\varphi(\nu)}) = (u_i(x_i^{\varphi(\nu)}))$ convergent to $\bar{v}_i \in U$. By definition of U , there exists $(\bar{x}_i) \in \hat{X}$ such that $\bar{v}_i \leq u_i(\bar{x}_i)$ for all $i \in I$. Let $i \in I$ and $\xi_i \in P_i(\bar{x})$, i.e. $u_i(\bar{x}_i) < u_i(\xi_i)$, consequently $\bar{v}_i \leq u_i(\bar{x}_i) < u_i(\xi_i)$ for all $i \in I$. Since $(u_i(x_i^{\varphi(\nu)}))$ converges to \bar{v}_i , there exists ν_1 such that for all $\nu \geq \nu_1$, $u_i(x_i^{\varphi(\nu)}) < u_i(\xi_i)$, hence $\xi_i \in P_i(x^{\varphi(\nu)})$. Then, the constant sequence (ξ_i) satisfies $\xi_i \in P_i(x^{\varphi(\nu)})$ for all $\nu \geq \nu_1$ and converges to ξ_i . \square

Example 2.1. We present an example of an exchange economy where Assumption (H3) is satisfied while the attainable set is not bounded and the preference correspondences are not representable by utility functions. Let us consider an exchange economy with two commodities A and B and two consumers.

The consumption sets are given by

$$X_1 = X_2 = \{(a, b) \in \mathbb{R}^2 \mid a + b \geq 0\}$$

The attainable allocations set $\mathcal{A}(\mathcal{E})$ of the economy is then

$$\mathcal{A}(\mathcal{E}) = \{((a, b), (\omega_A - a, \omega_B - b)) \mid 0 \leq a + b \leq \omega_A + \omega_B\}$$

where (ω_A, ω_B) with $\omega_A + \omega_B > 0$ denotes the global endowment. The set $\mathcal{A}(\mathcal{E})$ is clearly unbounded.

We consider the following continuous function $\Pi : X_i \rightarrow \mathbb{R}^2$ defined by:

$$\Pi(a, b) = \left(\frac{1}{2} + \frac{a - b}{(|a - b| + 1)(a^2 + b^2 + 2)}, \frac{1}{2} + \frac{b - a}{(|a - b| + 1)(a^2 + b^2 + 2)} \right).$$

The preference correspondence is the same for the two consumers and it is defined by $P_i : X_1 \times X_2 \rightarrow X_i$

$$P_i((a_1, b_1), (a_2, b_2)) = \{(\alpha, \beta) \in X_i \mid \Pi(a_i, b_i) \cdot (\alpha, \beta) > \Pi(a_i, b_i) \cdot (a_i, b_i)\}$$

One easily checks that Assumption (H1) is satisfied by the preference relations since Π is continuous so P_i has an open graph and $\Pi(a, b) \gg (0, 0)$ so that the local non-satiation holds true everywhere.

We remark that if (a_i^ν, b_i^ν) is a sequence of X_i such that $\|(a_i^\nu, b_i^\nu)\|$ converges to $+\infty$ and $a_i^\nu + b_i^\nu$ converges to a finite limit c , then $\Pi(a_i^\nu, b_i^\nu)$ converges to $(1/2, 1/2)$ and $\Pi(a_i^\nu, b_i^\nu) \cdot (a_i^\nu, b_i^\nu)$ converges to $\lim_\nu (1/2)(a_i^\nu + b_i^\nu) = \frac{c}{2}$.

Let $((a_1^\nu, b_1^\nu), (a_2^\nu, b_2^\nu))$ be a sequence of $\mathcal{A}(\mathcal{E})$. If it has a bounded subsequence, then this subsequence has a cluster point $((\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_2))$. Then the desired property of Assumption (H3) holds true thanks to the fact that the preference correspondences have an open graph. See the proof of Proposition 2.1 (i).

If the sequence is unbounded, we remark that the sequences $(a_1^\nu + b_1^\nu)$ and $(a_2^\nu + b_2^\nu)$ belongs to $[0, \omega_A + \omega_B]$ and for all ν , $a_1^\nu + b_1^\nu + a_2^\nu + b_2^\nu = \omega_A + \omega_B$. So, there exists a subsequence $((a_1^{\varphi(\nu)}, b_1^{\varphi(\nu)}), (a_2^{\varphi(\nu)}, b_2^{\varphi(\nu)}))$ such that the sequences $(a_1^{\varphi(\nu)} + b_1^{\varphi(\nu)})$ and $(a_2^{\varphi(\nu)} + b_2^{\varphi(\nu)})$ converges respectively to $c \in [0, \omega_A + \omega_B]$ and to $\omega_A + \omega_B - c$. Let us consider the attainable allocation $((\bar{a}_1 = c/2, \bar{b}_1 = c/2), (\bar{a}_2 = (\omega_A + \omega_B - c)/2, \bar{b}_2 = (\omega_A + \omega_B - c)/2))$. We remark that $\Pi(\bar{a}_1, \bar{b}_1) = \Pi(\bar{a}_2, \bar{b}_2) = (1/2, 1/2)$ and $\Pi(\bar{a}_1, \bar{b}_1) \cdot (\bar{a}_1, \bar{b}_1) = (1/2)(\bar{a}_1 + \bar{b}_1) = c/2$ and $\Pi(\bar{a}_2, \bar{b}_2) \cdot (\bar{a}_2, \bar{b}_2) = (1/2)(\bar{a}_2 + \bar{b}_2) = (\omega_A + \omega_B - c)/2$. Let $i = 1, 2$ and $(a_i, b_i) \in X_i$ such that $(a_i, b_i) \in P_i((\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_2))$. From the definition of P_i , one deduces that $(1/2)(a_i + b_i) > (1/2)(\bar{a}_i + \bar{b}_i) = (1/2) \lim_{\nu \rightarrow \infty} (a_i^{\varphi(\nu)} + b_i^{\varphi(\nu)}) = \lim_{\nu \rightarrow \infty} \Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})$. Furthermore, since $\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})$ converges to $(1/2, 1/2)$, $(1/2)(a_i + b_i) = \lim_{\nu \rightarrow \infty} \Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i, b_i)$. Consequently, for ν large enough, $\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i, b_i) > \lim_{\nu \rightarrow \infty} \Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})$, which means that $(a_i, b_i) \in P_i((a_1^{\varphi(\nu)}, b_1^{\varphi(\nu)}), (a_2^{\varphi(\nu)}, b_2^{\varphi(\nu)}))$, so the desired property in Assumption (H3) holds true. \square

The main result of this paper is the following existence theorem of a quasi-equilibrium for a production economy.

Theorem 2.1. *Assume (H1), (H2) and (H3), then there exists a quasi-equilibrium of the economy \mathcal{E} .*

3 Preliminary results

In this section, we introduce the definition of "augmented preferences" borrowed from Gale and Mas-Colell ([9], [10]). This definition allows to replace the local insatiability by the insatiability of consumers assumed at any point of their attainable consumption set.

$$\hat{P}_i(x) = \{x'_i \in X_i \mid x'_i = \lambda x_i + (1 - \lambda)x''_i, 0 \leq \lambda < 1, x''_i \in coP_i(x)\},$$

where $coP_i(x)$ denotes the convex hull of $P_i(x)$.

First, we show that some properties of the preference correspondences P_i are still true for \hat{P}_i .

Proposition 3.1. *Assume that for all i , X_i is convex*

- (i) *If P_i is lower semicontinuous on $\prod_{i \in I} X_i$, then the same is true for \hat{P}_i .*
- (ii) *$\hat{P}_i(x)$ has convex values. Furthermore, if for all $x_i \in X_i$, $x_i \notin coP_i(x)$, then $x_i \notin \hat{P}_i(x)$.*

Proof.

- (i) Let $x \in \prod_{i \in I} X_i$ and V an open subset of X_i such that

$$V \cap \hat{P}_i(x) \neq \emptyset.$$

Then, there exists $\xi_i \in \hat{P}_i(x) \cap V$, which means that $\xi_i = \lambda x_i + (1 - \lambda)\zeta_i$ for some $\lambda \in [0, 1[$, $\zeta_i \in coP_i(x)$. Let $\epsilon > 0$ such that $B(\xi_i, \epsilon) \subset V$. Since the correspondence P_i is lower semicontinuous, then coP_i is lower semicontinuous (see [8], page 154). Consequently, there exists a neighborhood W of x in $\prod_{i \in I} X_i$ such that

$$x' \in W \Rightarrow coP_i(x') \cap B(\zeta_i, \epsilon) \neq \emptyset.$$

Thus, for all $x' \in W$, there exists $\zeta'_i \in \text{co}P_i(x') \cap B(\zeta_i, \epsilon)$. Let W' such that

$$W' = \{x' \in W \mid \|x'_i - x_i\| < \epsilon\}.$$

Let $x' \in W'$ and $\xi'_i = \lambda x'_i + (1 - \lambda)\zeta'_i$, then $\xi'_i \in \hat{P}_i(x')$

$$\|\xi'_i - \xi_i\| \leq \lambda \|x'_i - x_i\| + (1 - \lambda) \|\zeta'_i - \zeta_i\| < \epsilon.$$

Then, one gets $\xi'_i \in B(\xi_i, \epsilon) \subset V$. Hence, $\xi'_i \in \hat{P}_i(x') \cap V$, which proves the lower semi-continuity of \hat{P}_i .

- (ii) Let $x \in \prod_{i \in I} X_i$ and $z_i, z'_i \in \hat{P}_i(x)$ such that $z_i = x_i + \lambda(\xi_i - x_i)$ and $z'_i = x_i + \beta(\xi'_i - x_i)$ for some $\lambda, \beta \in]0, 1]$ and $\xi_i, \xi'_i \in \text{co}P_i(x)$. For $\alpha \in]0, 1[$, we have:

$$\begin{aligned} \alpha z_i + (1 - \alpha)z'_i &= x_i + \alpha\lambda\xi_i + (1 - \alpha)\beta\xi'_i - [\alpha\lambda x_i + (1 - \alpha)\beta x_i] \\ &= x_i + \alpha\lambda\xi_i + (1 - \alpha)\beta\xi'_i - [\alpha\lambda + (1 - \alpha)\beta]x_i \\ &= x_i + \gamma(\xi''_i - x_i). \end{aligned}$$

where $\gamma = \alpha\lambda + (1 - \alpha)\beta$ and $\xi''_i = \frac{\alpha\lambda}{\gamma}\xi_i + \frac{(1 - \alpha)\beta}{\gamma}\xi'_i$. One easily checks that $\gamma \in]0, 1]$ since $\lambda, \beta \in]0, 1]$ and $\xi''_i \in \text{co}P_i(x)$. Then, $\alpha z_i + (1 - \alpha)z'_i \in \hat{P}_i(x)$ which means that \hat{P}_i has convex values.

We prove by contraposition the irreflexivity. Let us suppose that $x_i \in \hat{P}_i(x)$ for some i , then $x_i = \lambda x_i + (1 - \lambda)x'_i$ with $\lambda \in [0, 1[$ and $x'_i \in \text{co}P_i(x)$. Hence, for all i , we have $x_i = x'_i \in \text{co}P_i(x)$ which contradicts our assumption. \square

Now, we consider the following economy

$$\mathcal{E}' = (\mathbb{R}^L, (X_i, \hat{P}_i, \omega_i)_{i \in I}, (Y'_j)_{j \in J}, (\theta_{ij})_{(i,j)})$$

where for each j , $Y'_j = \overline{\text{co}}Y_j$.

Lemma 3.1. *Under assumption (H2), the economies \mathcal{E} and \mathcal{E}' have the same total production set.*

Proof. Let $Y = \sum_{j \in J} Y_j$.

It is clear that $Y \subset Y'$. Conversely, $Y' = \sum_{j \in J} \overline{\text{co}}Y_j \subset \text{cl}(\sum_{j \in J} \text{co}Y_j)$, see [17] (Corollary 6.6, page 48). Since the convex hull of a sum is the sum of the convex hulls, one gets

$$Y' = \sum_{j \in J} \overline{\text{co}}Y_j \subset \text{cl}(\sum_{j \in J} \text{co}Y_j) = \text{cl}(\text{co}(\sum_{j \in J} Y_j)) = \overline{\text{co}}Y.$$

Since Y is a non-empty closed, convex subset of \mathbb{R}^L , then $\overline{co}Y = Y$. Hence $Y = Y'$. \square

Proposition 3.2. *If $((\bar{x}_i), (\bar{\zeta}_j), \bar{p})$ is a quasi-equilibrium of \mathcal{E}' , then there exists $\bar{y} \in \prod_{j \in J} Y_j$ such that $((\bar{x}_i), (\bar{y}_j), \bar{p})$ is a quasi-equilibrium of \mathcal{E} .*

Proof. Let $((\bar{x}_i), (\bar{\zeta}_j), \bar{p})$ be a quasi-equilibrium of \mathcal{E}' . So, $\sum_{j \in J} \bar{\zeta}_j \in \sum_{j \in J} Y'_j$. By Lemma 3.1, $\sum_{j \in J} Y'_j = Y$. Consequently, there exists $\bar{y} \in \prod_{j \in J} Y_j$ such that $\sum_{j \in J} \bar{\zeta}_j = \sum_{j \in J} \bar{y}_j$. Hence $\sum_{i \in I} \bar{x}_i = \omega + \sum_{j \in J} \bar{y}_j$. In other words, Condition (c) of Definition 2.2 is satisfied.

Moreover, one can remark that $\bar{y}_j \in Y'_j$ for every j . Consequently, $\bar{p} \cdot \bar{y}_j \leq \bar{p} \cdot \bar{\zeta}_j$. But since $\sum_{j \in J} \bar{\zeta}_j = \sum_{j \in J} \bar{y}_j$, one gets $\bar{p} \cdot \bar{y}_j = \bar{p} \cdot \bar{\zeta}_j$.

We now show that condition (a) is satisfied. Let $j \in J$ and $y_j \in Y_j$. Then, $y_j \in Y'_j$, so, $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{\zeta}_j = \bar{p} \cdot \bar{y}_j$. Hence, $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{y}_j$ and Condition (a) of Definition 2.2 is satisfied.

Last, we show that condition (b) is satisfied. Since $\bar{p} \cdot \bar{\zeta}_j = \bar{p} \cdot \bar{y}_j$ for all $j \in J$, we have, $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j$ for all i . Now, let $i \in I$ and $x_i \in X_i$ such that $x_i \in \hat{P}_i(\bar{x})$. Since $P_i(\bar{x}) \subset \hat{P}_i(\bar{x})$, $\bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i$. \square

Proposition 3.3. *If the economy \mathcal{E} satisfies assumption (H3), then the same still holds for the economy \mathcal{E}' .*

Proof. Since the total production sets of the economies \mathcal{E} and \mathcal{E}' are equal, then the set of attainable consumption are also the same. Let $(x^\nu) \in \hat{X}$ such that $\underline{x}_i \in \hat{P}_i(x^\nu)^c$ for all ν . Since $P_i(x^\nu) \subset \hat{P}_i(x^\nu)$, then $\hat{P}_i(x^\nu)^c \subset \overline{P_i(x^\nu)^c}$; so $\underline{x}_i \in P_i(x^\nu)^c$ for all ν .

Applying assumption (H3), there exists a subsequence $(x^{\varphi(\nu)})$ of \hat{X} and $\bar{x} \in \hat{X}$ such that for all i , for all $\xi_i \in P_i(\bar{x})$, there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ with $\xi_i^{\varphi(\nu)} \in P_i(x^{\varphi(\nu)})$, for all $\nu \geq \nu_1$. Now, let $i \in I$, $\hat{\xi}_i \in \hat{P}_i(\bar{x})$. By definition of the augmented preference correspondences \hat{P}_i , $\hat{\xi}_i = \lambda \bar{x}_i + (1 - \lambda) \xi_i$ for some $\lambda \in]0, 1]$ and $\xi_i \in coP_i(\bar{x})$. Then there exists $\xi'_{i,1}, \xi'_{i,2}, \dots, \xi'_{i,m} \in P_i(x)$ such that $\xi_i = \sum_{k=1}^m \mu_{i,k} \xi'_{i,k}$ with $\sum_{k=1}^m \mu_{i,k} = 1$ and $\mu_{i,k} \in [0, 1]$.

By assumption (H3), for each k , there exists ν_k such that $\forall \nu \geq \nu_k$, there exists a subsequence $\xi'_{i,k}{}^{\varphi(\nu)}$ convergent to $\xi'_{i,k}$ with $\xi'_{i,k}{}^{\varphi(\nu)} \in P_i(x^{\varphi(\nu)})$. Let us consider the sequence $\hat{\xi}_i^{\varphi(\nu)} = \lambda \bar{x}_i + (1 - \lambda) \sum_{k=1}^m \mu_{i,k} \xi'_{i,k}{}^{\varphi(\nu)}$. Hence, $\hat{\xi}_i^{\varphi(\nu)}$ converges to $\hat{\xi}_i$ and since $\xi'_{i,k}{}^{\varphi(\nu)} \in P_i(x^{\varphi(\nu)})$ for all $\nu \geq \nu_k$, we have $\hat{\xi}_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$, for all $\nu \geq \max_{k=1, \dots, m} \{\nu_k\}$. \square

4 Existence of quasi-equilibria

In this section we consider the economy \mathcal{E}'

$$\mathcal{E}' = (\mathbb{R}^L, (X_i, \hat{P}_i, \omega_i)_{i \in I}, (Y'_j)_{j \in J}, (\theta_{ij})_{(i,j)})$$

where the preference correspondences are replaced by the augmented preference correspondences and the production sets are replaced by their closed convex hull. We have seen in the previous section that we can deduce the existence of a quasi-equilibrium of \mathcal{E} from a quasi-equilibrium of \mathcal{E}'

In what follow, we will consider Assumptions (H1)', (H2)' and (H3)' whose correspond to (H1), (H2), (H3) but adapted to \mathcal{E}' . In the previous section, we have shown that these assumptions are satisfied by \mathcal{E}' if Assumptions (H1), (H2) and (H3) are satisfied by \mathcal{E} .

Assumption (H1)' For every $i \in I$

- (a) X_i is a non-empty closed, convex subset of \mathbb{R}^L ;
- (b) [irreflexivity] $\forall x \in \prod_{i \in I} X_i$, $x_i \notin \hat{P}_i(x)$;
- (c) [lower semicontinuous] $\hat{P}_i : \prod_{k \in I} X_k \rightarrow X_i$ is lower semicontinuous and convex valued;
- (d) $\omega_i \in X_i - \sum_{j \in J} \theta_{i,j} Y'_j$, i.e. there exists $(\underline{x}_i, (\underline{y}_{i,j})) \in X_i \times \prod_{j \in J} Y'_j$ such that $\underline{x}_i = \omega_i + \sum_{j \in J} \theta_{i,j} \underline{y}_{i,j}$;
- (e) For each $x \in \hat{X}$, one has $\hat{P}_i(x) \neq \emptyset$ and for all $\xi_i \in \hat{P}_i(x)$, for all $t \in]0, 1]$, $t\xi_i + (1-t)x_i \in \hat{P}_i(x)$.

Assumption (H2)' For each $j \in J$, Y'_j is a closed, convex subset of \mathbb{R}^L .

Assumption (H3)' For all sequence $((x_i^\nu))$ of \hat{X} such that for all i , $\underline{x}_i \in \hat{P}_i(x^\nu)^c$, there exists a subsequence $((x_i^{\varphi(\nu)}))$ of \hat{X} and $((\bar{x}_i)) \in \hat{X}$ such that for all i , for all $\xi_i \in \hat{P}_i(\bar{x})$, there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i with $\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$, $\forall \nu \geq \nu_1$.

We now state the existence result of a quasiequilibrium for a finite private ownership economy satisfying Assumptions (H1)', (H2)' and (H3)'.

Theorem 4.1. *If Assumptions (H1)', (H2)' and (H3)' are satisfied, then there exists a quasiequilibrium of the economy \mathcal{E}' .*

The idea of the proof is as follows: we first truncate consumption and production sets with the closed ball of radius ν large enough; following an idea of Bergstrom [2], we modify the budget sets in such a way that it will coincide with the original ones when the price belongs to the unit sphere; then, by applying the well known result of Gale and Mas-Colell - Bergstrom about the existence of maximal elements to an intermediary family of lower semi-continuous correspondences, we obtain a sequence $((x^\nu), (y^\nu), p^\nu)$ such that $((x^\nu), (y^\nu))$ is an attainable allocation of the economy $\mathcal{A}(\mathcal{E}')$, p^ν belongs to the unit ball of \mathbb{R}^L , the domain of admissible prices, the producers maximize the profit over the truncated production sets and the consumers maximize preferences on the truncated consumption sets but with a relaxed budget constraint; from Assumption $(H3)'$ and the compactness of the price set, we obtain a subsequence $(x^{\varphi(\nu)}, y^{\varphi(\nu)}, p^{\varphi(\nu)})$ and an element $(\bar{x}, \bar{y}, \bar{p})$ such that the preferences at this point are close to the preferences at $x^{\varphi(\nu)}$ for ν large enough and $p^{\varphi(\nu)}$ converges to \bar{p} ; finally, we prove that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of \mathcal{E}' . Note that the difficulty of the limit argument comes from the fact that (\bar{x}, \bar{y}) is not necessarily the limit of $(x^{\varphi(\nu)}, y^{\varphi(\nu)})$.

4.1 The fixed-point argument

From assumption $(H1)'(d)$, let us fix $\underline{x}_i \in X_i$ and $\underline{y}_{i,j} \in Y_j'$ such that $\underline{x}_i = \omega_i + \sum_{j \in J} \theta_{i,j} \underline{y}_{i,j}$ for every $i \in I$. Let \bar{B}^ν be the closed ball with center 0 and radius ν with ν large enough so that $\underline{x}_i, \underline{y}_{i,j}$ belong to B^ν , the interior of \bar{B}^ν for all i, j . We consider the truncated economy obtained by replacing agent's consumption and production sets respectively by $X_i^\nu = X_i \cap \bar{B}^\nu$, $Y_j^\nu = Y_j' \cap \bar{B}^\nu$ and the augmented preference correspondences $\hat{P}_i^\nu = \hat{P}_i \cap B^\nu$. The closed unit ball $\bar{B} = \{x \in \mathbb{R}^L : \|x\| \leq 1\}$ will be the price set.

We now consider the compact economy

$$\mathcal{E}^\nu = \left(\mathbb{R}^L, (X_i^\nu, \hat{P}_i^\nu, \omega_i)_{i \in I}, (Y_j^\nu)_{j \in J}, (\theta_{i,j})_{(i \in I, j \in J)} \right)$$

Remark 4.1. The correspondence \hat{P}_i^ν is lower semi-continuous. Indeed, let $x \in \prod_{i \in I} X_i^\nu$ and V be an open set such that:

$$V \cap \hat{P}_i^\nu(x) = V \cap B^\nu \cap \hat{P}_i(x) \neq \emptyset.$$

Since $V \cap B^\nu$ is an open set and \hat{P}_i is lower semi-continuous, there exists a neighborhood U of x such that for all $x' \in U$,

$$V \cap B^\nu \cap \hat{P}_i(x') \neq \emptyset,$$

which means that $V \cap \hat{P}_i^\nu(x') \neq \emptyset$, hence \hat{P}_i^ν is lower semicontinuous.

Remark 4.2. With the above remark and since \bar{B}^ν is convex and closed, note that the compact economy \mathcal{E}^ν satisfies Assumption (H1)' but the non satiation of preferences at attainable allocations and Assumption (H2)'. Furthermore, Y_j^ν is now compact.

Since each Y_j^ν is compact, we can define for every $p \in \bar{B}$ the profit function

$$\pi_j^\nu(p) = \sup p \cdot Y_j^\nu = \sup \{p \cdot y_j : y_j \in Y_j^\nu\}$$

and the wealth of consumer i is defined by:

$$\gamma_i^\nu(p) = p \cdot \omega_i + \sum_{j \in J} \theta_{ij} \pi_j^\nu(p).$$

Note that the function $\pi_j^\nu : \bar{B} \rightarrow \mathbb{R}$ is continuous since it is finite and convex.

In what follows, we will use the following notations for simplicity

- $Z^\nu = \prod_{i \in I} X_i^\nu \times \prod_{j \in J} Y_j^\nu \times \bar{B}$
- $\hat{\gamma}_i^\nu(z) = \gamma_i^\nu(p) + \frac{1 - \|p\|}{\#I}$
- $\tilde{\gamma}_i^\nu(z) = \max\{\hat{\gamma}_i^\nu(z), \frac{1}{2}[\hat{\gamma}_i^\nu(p) + p \cdot x_i]\}$

Remark 4.3. Note that $p \cdot x_i > \tilde{\gamma}_i^\nu(z) > \hat{\gamma}_i^\nu(z)$ when $p \cdot x_i > \hat{\gamma}_i^\nu(z)$ and $\tilde{\gamma}_i^\nu(z) = \hat{\gamma}_i^\nu(z)$ when $p \cdot x_i \leq \hat{\gamma}_i^\nu(z)$

Let now $N = I \cup J \cup \{0\}$ be the set of consumers I indexed by i , the set of producers J indexed by j and an additional agent 0 whose function is to react with prices to a given excess total demand.

For all $i \in I$, we define the correspondences $\alpha_i^\nu : Z^\nu \rightarrow X_i^\nu$ and $\tilde{\beta}_i^\nu : Z^\nu \rightarrow X_i^\nu$ as follows.

$$\begin{aligned} \alpha_i^\nu(z) &= \{\xi_i \in X_i^\nu : p \cdot \xi_i \leq \hat{\gamma}_i^\nu(z)\} \\ \tilde{\beta}_i^\nu(z) &= \{\xi_i \in X_i^\nu : p \cdot \xi_i < \tilde{\gamma}_i^\nu(z)\} \end{aligned}$$

From the construction of the extended budget set, one checks that for all i , the consumption \underline{x}_i belongs to $\tilde{\beta}_i^\nu(z)$ if $x_i \notin \alpha_i^\nu(z)$. Indeed, from (H1)'(d),

$$\underline{x}_i = \omega_i + \sum_{j \in J} \theta_{i,j} \underline{y}_{i,j}$$

since $x_i \notin \alpha_i^\nu(z)$, $p \cdot x_i > \hat{\gamma}_i^\nu(z)$ and $\tilde{\gamma}_i^\nu(z) > \hat{\gamma}_i^\nu(z)$. Furthermore

$$\begin{aligned} p \cdot \underline{x}_i &= p \cdot \omega_i + \sum_{j \in J} \theta_{i,j} p \cdot \underline{y}_{i,j} \\ &\leq p \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \pi_j^\nu(p) \\ &= \hat{\gamma}_i^\nu(z) \\ &< \tilde{\gamma}_i^\nu(z) \end{aligned}$$

which means that \underline{x}_i belongs to $\tilde{\beta}_i^\nu(z)$. Furthermore, since $\tilde{\gamma}_i^\nu$ is continuous, the correspondence $\tilde{\beta}_i^\nu$ has an open graph in $Z^\nu \times X_i^\nu$.

Now, for $i \in I$, we consider the mapping ϕ_i^ν defined from Z^ν to X_i^ν by:

$$\phi_i^\nu(z) = \begin{cases} \tilde{\beta}_i^\nu(z) & \text{if } x_i \notin \alpha_i^\nu(z) \\ \tilde{\beta}_i^\nu(z) \cap \hat{P}_i^\nu(x) & \text{if } x_i \in \alpha_i^\nu(z) \end{cases}$$

For $j \in J$, we define ϕ_j^ν from Z^ν to Y_j^ν by:

$$\phi_j^\nu(z) = \{y'_j \in Y_j^\nu \mid p \cdot y_j < p \cdot y'_j\},$$

and the mapping ϕ_0^ν from Z^ν to \bar{B} is defined by:

$$\phi_0^\nu(z) = \{q \in \bar{B} \mid (q - p) \cdot (\sum_{i \in I} x_i - \omega - \sum_{j \in J} y_j) > 0\}$$

Now we will apply to Z and the correspondences $(\phi_i)_{i \in I}, (\phi_j)_{j \in J}, \phi_0$ the well known theorem of Gale and Mas-Colell [10]. We will actually use the Bergstrom version of this theorem in [2], which is more adapted to our setting.

Theorem 4.2. (*Gale and Mas-Colell - Bergstrom*) For each $k = 1, \dots, \bar{k}$, let Z_k be a nonempty, compact, convex subset of some finite dimensional Euclidean space. Given $Z = \prod_{k=1}^{\bar{k}} Z_k$, let for each k , $\phi_k : Z \rightarrow Z_k$ be a lower semicontinuous correspondences satisfying for all $z \in Z$, $z_k \notin \text{co}\phi_k(z)$. Then there exists $\bar{z} \in Z$ such that for each $k = 1, \dots, \bar{k}$:

$$\phi_k(\bar{z}) = \emptyset \tag{1}$$

For the correspondences $(\phi_j^\nu)_{j \in J}$ and ϕ_0^ν , one easily checks that they are convex valued, irreflexive and lower semi-continuous since they have an open graph.

We now check that for all $i \in N$, the correspondence ϕ_i^ν satisfies the assumption of Theorem 4.2. We first remark that ϕ_i^ν has convex valued since $\tilde{\beta}_i^\nu$ and \hat{P}_i are so. We now check the irreflexivity. If $x_i \in \alpha_i^\nu(z)$, then, from Assumption (H1)'(b), $x_i \notin \hat{P}_i(x)$, so $x_i \notin \phi_i^\nu(x)$ since $\phi_i^\nu(x) \subset \hat{P}_i(x)$. If $x_i \notin \alpha_i^\nu(z)$, then from Remark 4.3, $p \cdot x_i > \tilde{\gamma}_i^\nu(z)$, so $x_i \notin \tilde{\beta}_i^\nu(z) = \phi_i^\nu(z)$.

For the lower semi-continuity, let V be an open set and z such that $\phi_i^\nu(z) \cap V \neq \emptyset$. If $x_i \notin \alpha_i^\nu(z)$, then $p \cdot x_i > \tilde{\gamma}_i^\nu(z)$. Since $\tilde{\gamma}_i^\nu$ is continuous, there exists a neighborhood W of z such that for all $z' \in W$, $p' \cdot x_i' > \tilde{\gamma}_i^\nu(z')$. Since $\tilde{\beta}_i^\nu$ has an open graph, there exists a neighborhood W' of z such that for all $z' \in W'$, $\tilde{\beta}_i^\nu(z') \cap V \neq \emptyset$. So, for all $z' \in W \cap W'$, $\phi_i^\nu(z') \cap V \neq \emptyset$ and consequently, ϕ_i^ν is lower semi-continuous at z . If $x_i \in \alpha_i^\nu(z)$, we first remark that $\tilde{\beta}_i^\nu \cap \hat{P}_i^\nu$ is lower semicontinuous as an intersection of a lower semicontinuous correspondence with an open graph correspondence. So, there exists a neighborhood W of z such that for all $z' \in W$, $\tilde{\beta}_i^\nu(z') \cap \hat{P}_i^\nu(x') \cap V \neq \emptyset$. This implies that $\tilde{\beta}_i^\nu(z') \cap V \neq \emptyset$. Hence, in both cases, $x_i' \in \alpha_i^\nu(z')$ or $x_i' \notin \alpha_i^\nu(z')$, $\phi_i^\nu(z') \cap V \neq \emptyset$ from the definition of ϕ_i^ν . Thus ϕ_i^ν is also lower semi-continuous at z in this case.

From Theorem 4.2, there exists $\bar{z}^\nu = (\bar{x}^\nu, \bar{y}^\nu, \bar{p}^\nu) \in Z^\nu$ such that, for all $i \in N$

$$\phi_i^\nu(\bar{z}^\nu) = \emptyset \quad (2)$$

As already noticed, since for all $i \in I$, $\bar{x}_i \in \tilde{\beta}_i^\nu(\bar{z}^\nu)$ and $\phi_i^\nu(\bar{z}^\nu) = \emptyset$, we conclude from the definition of ϕ_i^ν that

$$\begin{cases} \bar{p}^\nu \cdot \bar{x}_i^\nu \leq \hat{\gamma}_i^\nu(\bar{z}^\nu) \\ \tilde{\beta}_i^\nu(\bar{z}^\nu) \cap \hat{P}_i^\nu(\bar{x}^\nu) = \emptyset \end{cases} \quad (3)$$

Furthermore, from Remark 4.3, one deduces that $\tilde{\gamma}_i^\nu(\bar{z}^\nu) = \hat{\gamma}_i^\nu(\bar{z}^\nu)$.

In addition, for all $j \in J$, since $\phi_j^\nu(\bar{z}^\nu) = \emptyset$, we deduce that:

$$\forall y_j \in Y_j^\nu, \bar{p}^\nu \cdot y_j \leq \bar{p}^\nu \cdot \bar{y}_j^\nu = \pi_j^\nu(\bar{p}^\nu), \quad (4)$$

and since $\phi_0^\nu(\bar{z}^\nu) = \emptyset$,

$$\forall p \in \bar{B}, p \cdot \left(\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu \right) \leq \bar{p}^\nu \cdot \left(\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu \right) \quad (5)$$

We now prove that $(\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) = 0$. Indeed, if not, it follows from (5) that \bar{p}^ν belongs to the boundary of \bar{B} , that is $\|\bar{p}^\nu\| = 1$ and $\bar{p}^\nu \cdot (\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) > 0$. Now, by (3) and (4), for all i , $\bar{p}^\nu \cdot \bar{x}_i^\nu \leq \hat{\gamma}_i^\nu(\bar{z}^\nu) = \gamma_i^\nu(\bar{z}^\nu) = \bar{p}^\nu \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^\nu \cdot \bar{y}_j^\nu$. Summing up over $i \in I$ these inequalities, one gets, $\bar{p}^\nu \cdot (\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) \leq 0$, which yields a contradiction. We thus have proved that $(\bar{x}^\nu, \bar{y}^\nu) \in \mathcal{A}(\mathcal{E}^\nu)$.

4.2 The limit argument

We first show that we can apply Assumption (H3)' to the sequence $((\bar{x}_i^\nu))$ built in the previous sub-section. We have already proved that \bar{x}^ν is attainable in the truncated economy \mathcal{E}^ν , so it is also attainable in the economy \mathcal{E}' . It remains to show that $\underline{x}_i \in \hat{P}_i(\bar{x}^\nu)^c$ for all i .

There are two cases. First, if $\bar{p}^\nu \cdot \underline{x}_i < \hat{\gamma}_i^\nu(\bar{z}^\nu)$, which means that $\underline{x}_i \in \tilde{\beta}_i^\nu(\bar{z}^\nu)$, then, from (3), $\underline{x}_i \notin \hat{P}_i^\nu(\bar{x}^\nu) = \hat{P}_i(\bar{x}^\nu) \cap B^\nu$. Since $\underline{x}_i \in B^\nu$ as ν has been chosen large enough, one deduces that $\underline{x}_i \notin \hat{P}_i(\bar{x}^\nu)$ and therefore $\underline{x}_i \in \hat{P}_i(\bar{x}^\nu)^c$.

If $\bar{p}^\nu \cdot \underline{x}_i \geq \hat{\gamma}_i^\nu(\bar{z}^\nu)$, as $\underline{x}_i \in \tilde{\beta}_i^\nu(\bar{z}^\nu)$, we actually have the equality $\bar{p}^\nu \cdot \underline{x}_i = \hat{\gamma}_i^\nu(\bar{z}^\nu)$. We remark that $\hat{\gamma}_i^\nu(\bar{z}^\nu) = \gamma_i^\nu(\bar{z}^\nu) + \frac{1 - \|\bar{p}^\nu\|}{\sharp I} = \bar{p}^\nu \cdot \underline{x}_i = \bar{p}^\nu \cdot (\omega_i + \sum_{j \in J} \theta_{i,j} \underline{y}_{i,j}) \leq \gamma_i^\nu(\bar{z}^\nu)$. So, $\|\bar{p}^\nu\| = 1$. By contradiction, we prove that $\underline{x}_i \in \hat{P}_i(\bar{x}^\nu)^c$. Indeed, if not, $\underline{x}_i \in \text{int } \hat{P}_i(\bar{x}^\nu)$ and there exists $\rho > 0$ such that $B(\underline{x}_i, \rho) \subset \hat{P}_i(\bar{x}^\nu)$ and $B(\underline{x}_i, \rho) \subset B^\nu$. Since $\bar{p}^\nu \neq 0$, there exists $\xi_i^\nu \in B(\underline{x}_i, \rho)$ such that $\bar{p}^\nu \cdot \xi_i^\nu < \bar{p}^\nu \cdot \underline{x}_i = \hat{\gamma}_i^\nu(\bar{z}^\nu)$ and this contradicts (3) since $\xi_i^\nu \in B(\underline{x}_i, \rho) \subset \hat{P}_i^\nu(\bar{x}^\nu)$.

Applying Assumption (H3)', there exists a subsequence $((\bar{x}_i^{\varphi(\nu)}))$ of \hat{X} and $((\bar{x}_i)) \in \hat{X}$ such that, for all $i \in I$, for all $\xi_i \in \hat{P}_i(\bar{x})$, there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i with $\xi_i^{\varphi(\nu)} \in \hat{P}_i(\bar{x}^{\varphi(\nu)})$, $\forall \nu \geq \nu_1$. From the definition of \hat{X} , there exists $(\bar{y}_j) \in \prod_{j \in J} Y'_j$ such that $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} \omega_i + \sum_{j \in J} \bar{y}_j$.

Since \bar{B} is compact, we can assume without any loss of generality that the sequence $(\bar{p}^{\varphi(\nu)})$ converges to $\bar{p} \in \bar{B}$.

Now let $(y_j) \in \prod_{j \in J} Y'_j$, $i \in I$, $\xi_i \in \hat{P}_i(\bar{x})$ and $\lambda \in [0, 1]$. Such ξ_i exists from Assumption (H1)'(e). Furthermore, from the definition of the extended preferences, note that $\xi_i^\lambda = \lambda \bar{x}_i + (1 - \lambda) \xi_i \in \hat{P}_i(\bar{x})$.

By (H3)', there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i^λ with $\xi_i^{\varphi(\nu)} \in \hat{P}_i(\bar{x}^{\varphi(\nu)})$, $\forall \nu \geq \nu_1$. Since the sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ is convergent, it is bounded and for ν large enough $\xi_i^{\varphi(\nu)} \in B^\nu$, so $\xi_i^{\varphi(\nu)} \in \hat{P}_i^\nu(\bar{x}^{\varphi(\nu)})$. We deduce from (3) that $\xi_i^{\varphi(\nu)} \notin \tilde{\beta}_i^\nu(\bar{z}^{\varphi(\nu)})$. Recalling that $\tilde{\gamma}_i^\nu(\bar{z}^\nu) = \hat{\gamma}_i^\nu(\bar{z}^\nu)$, this means that

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \hat{\gamma}_i^\nu(\bar{z}^\nu) = \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot \bar{y}_j + \frac{1 - \|\bar{p}^{\varphi(\nu)}\|}{\sharp I}$$

For ν large enough, for all $j \in J$, $y_j \in \bar{B}^\nu$, so $(y_j) \in \prod_{j \in J} Y_j^\nu$, and from (4),

one gets

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot y_j + \frac{1 - \|\bar{p}^{\varphi(\nu)}\|}{\#I} \quad (6)$$

Using the same argument for $(\bar{y}_j) \in \prod_{j \in J} Y'_j$, one gets

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot \bar{y}_j + \frac{1 - \|\bar{p}^{\varphi(\nu)}\|}{\#I} \quad (7)$$

Passing to the limit in (6) and (7), we obtain:

$$\bar{p} \cdot \xi_i^\lambda \geq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot y_j + \frac{1 - \|\bar{p}\|}{\#I}. \quad (8)$$

and

$$\bar{p} \cdot \xi_i^\lambda \geq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j + \frac{1 - \|\bar{p}\|}{\#I}. \quad (9)$$

The two above inequalities hold true for any $i \in I$, $\xi_i \in \hat{P}_i(\bar{x})$, $\lambda \in [0, 1[$ and $(y_j) \in \prod_{j \in J} Y'_j$. Knowing that (\bar{x}, \bar{y}) is an attainable allocation, we will show that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of the economy \mathcal{E}' , which completes the proof.

When λ goes to 1 in (8) and (9), one gets

$$\bar{p} \cdot \bar{x}_i \geq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot y_j + \frac{1 - \|\bar{p}\|}{\#I} \quad (10)$$

and

$$\bar{p} \cdot \bar{x}_i \geq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j + \frac{1 - \|\bar{p}\|}{\#I} \quad (11)$$

Summing over i in (11) and using the fact that $(\bar{x}, \bar{y}) \in \mathcal{A}(\mathcal{E}')$, then, one gets

$$\begin{aligned} 1 - \|\bar{p}\| &\leq \sum_{i \in I} \bar{p} \cdot (\bar{x}_i - \omega_i - \sum_{j \in J} \theta_{i,j} \bar{y}_j) \\ &= \bar{p} \cdot \left(\sum_{i \in I} (\bar{x}_i - \omega_i) - \sum_{j \in J} \bar{y}_j \right) \\ &= 0 \end{aligned}$$

which implies $\|\bar{p}\| = 1$ and $\bar{p} \cdot \bar{x}_i = \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j$ for all $i \in I$. So the budget constraint is satisfied by each consumer.

Taken $\lambda = 0$ in (9), we obtain for all $i \in I$, for all $\xi_i \in \hat{P}_i(\bar{x})$,

$$\bar{p} \cdot \xi_i \geq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j \quad (12)$$

so the quasi-demand condition (b) of Definition 2.2 is satisfied.

Finally, from (10) and (11), for all $(y_j) \in \prod_{j \in J} Y'_j$, one gets

$$\bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot y_j \leq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j \quad (13)$$

Summing over i , we get

$$\sum_{j \in J} \bar{p} \cdot y_j \leq \sum_{j \in J} \bar{p} \cdot \bar{y}_j$$

For any $j \in J$, applying this inequality to $y' \in \prod_{j \in J} Y'_j$ defined by $y'_{j'} = \begin{cases} y_j & \text{if } j' = j \\ \bar{y}'_j & \text{if } j' \neq j \end{cases}$, it readily follows that

$$\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{y}_j \quad (14)$$

which means that the profit maximization condition (a) of Definition 2.2 is also satisfied. \square

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