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► **To cite this version:**

Philippe Bich, Bernard Cornet. Fixed-point-like theorems on subspaces. Fixed Point Theory and Applications, 2004, 2004, pp.602917 - 602917. 10.1155/S1687182004406056 . halshs-01394846

**HAL Id: halshs-01394846**

**<https://shs.hal.science/halshs-01394846>**

Submitted on 10 Nov 2016

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# FIXED-POINT-LIKE THEOREMS ON SUBSPACES

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*Received 8 June 2004*

We prove a fixed-point-like theorem for multivalued mappings defined on the finite Cartesian product of Grassmannian manifolds and convex sets. Our result generalizes two different kinds of theorems: the fixed-point-like theorem by Hirsch et al. (1990) or Husseini et al. (1990) and the fixed-point theorem by Gale and Mas-Colell (1975) (which generalizes Kakutani's theorem (1941)).

## 1. Introduction

In this paper, we prove a fixed-point-like theorem for multivalued mappings defined on the finite Cartesian product of Grassmannian manifolds and convex sets. Let  $k$  be an integer and let  $V$  be a Euclidean space such that  $0 \leq k \leq \dim V$ , then the  $k$ -Grassmannian manifold of  $V$ , denoted  $G^k(V)$ , is the set of all the  $k$ -dimensional subspaces of  $V$ . The set  $G^k(V)$  is a smooth compact manifold but, in general, it does not satisfy properties such as convexity or acyclicity and its Euler characteristic may be null. This prevents the use of classical fixed-point theorems as Brouwer's [2], Kakutani's [14], or Eilenberg-Montgomery's theorem [7].

Our main result generalizes two different kinds of theorems: the fixed-point-like theorem by Hirsch et al. [11] or Husseini et al. [13] and the fixed-point theorem by Gale and Mas-Colell [8] (which generalizes Kakutani's theorem [14]). As in [11, 13], we will mainly use techniques from degree theory. As a consequence of our main result, we first deduce the standard fixed-point theorems when the variable is in a convex domain (such as Brouwer and Kakutani's theorem) and second Borsuk-Ulam's theorem.

The main result of this paper is directly motivated by the existence problem of equilibria in economic models with incomplete markets; in [1], it is used to extend the classical existence result by Duffie and Shafer [6] to the nontransitive case.

The paper is organized as follows. The main result is stated in Section 2 together with some direct consequences of it, namely, the results by Hirsch et al. [11], Gale and Mas-Colell [8] and Borsuk-Ulam's theorem. The proof of the main result is given in Section 3

and the appendix recalls the main properties of the Grassmannian manifold, used in this paper.

## 2. Statement of the results

**2.1. Preliminaries.** A correspondence  $\Phi$  from a set  $X$  to a set  $Y$  is a map from  $X$  to the set of all the subsets of  $Y$ , and the graph of  $\Phi$ , denoted  $G(\Phi)$ , is defined by  $G(\Phi) = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}$ . A mapping  $\varphi : X \rightarrow Y$  is said to be a selection of  $\Phi$  if  $\varphi(x) \in \Phi(x)$  for all  $x \in X$ . If  $A$  is a subset of  $X$ , we let  $\Phi(A) = \bigcup_{x \in A} \Phi(x)$ , and the restriction of  $\Phi$  to  $A$ , denoted  $\Phi|_A$ , is the correspondence from  $A$  to  $Y$  defined by  $\Phi|_A(x) = \Phi(x)$  if  $x \in A$ . If  $X$  and  $Y$  are topological spaces, the correspondence  $\Phi$  is said to be lower semicontinuous (l.s.c.) (resp., upper semicontinuous (u.s.c.)) if for every open set  $U \subset Y$ , the set  $\{x \in X \mid \Phi(x) \cap U \neq \emptyset\}$  is open in  $X$  (resp., the set  $\{x \in X \mid \Phi(x) \subset U\}$  is open in  $X$  and, for every  $x \in X$ ,  $\Phi(x)$  is compact).

If  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  belong to  $\mathbb{R}^n$ , we denote by  $x \cdot y = \sum_{i=1}^n x_i y_i$  the scalar product of  $\mathbb{R}^n$ ,  $\|x\| = \sqrt{x \cdot x}$  the Euclidian norm. If  $x \in \mathbb{R}^n$  and  $r \in \mathbb{R}_+$ , we let  $B(x, r) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$  and  $\bar{B}(x, r) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$ . If  $E$  is a vector subspace of  $\mathbb{R}^n$ , we denote by  $E^\perp = \{u \in \mathbb{R}^n \mid \forall x \in E, x \cdot u = 0\}$  the orthogonal space to  $E$ . If  $u_1, \dots, u_k$  belong to  $E$ , a vector space, we denote by  $\text{span}\{u_1, \dots, u_k\}$  the vector subspace of  $E$  spanned by  $u_1, \dots, u_k$ .

Let  $V$  be a Euclidean space and let  $k$  be an integer such that  $0 \leq k \leq \dim V$ ; we denote by  $G^k(V)$  the set consisting of all the linear subspaces of  $V$  of dimension  $k$ , called the ( $k$ -)Grassmannian manifold of  $V$ . Then it is known that  $G^k(V)$  is a smooth manifold of dimension  $k(\dim V - k)$  and we refer to the appendix for the properties we will use hereafter, together with the precise definition of the manifold structure on  $G^k(V)$ .

**2.2. The main result and some consequences.** The aim of this paper is to prove the following result.

**THEOREM 2.1.** *Let  $I, J$  be two finite disjoint sets. For every  $i \in I$ , let  $k_i$  be an integer and let  $V_i$  be a Euclidean space such that  $0 \leq k_i \leq \dim V_i$ . For every  $j \in J$ , let  $C_j$  be a nonempty, convex, compact subset of a Euclidean space  $V_j$ , and let  $M = \prod_{i \in I} G^{k_i}(V_i) \times \prod_{j \in J} C_j$ .*

*For  $i \in I$  and  $k = 1, \dots, k_i$ , let  $F_i^k$  be a correspondence from  $M$  to  $V_i$  with convex values, for  $j \in J$ , let  $F_j$  be a correspondence from  $M$  to  $C_j$  with convex values, and suppose that, for every  $i \in I$  and  $k = 1, \dots, k_i$  (resp.,  $j \in J$ ), the correspondence  $F_i^k$  (resp.,  $F_j$ ) is either l.s.c or u.s.c.*

*Then, there exists  $\bar{x} = ((\bar{x}_i)_{i \in I}, (\bar{x}_j)_{j \in J}) \in M$  such that*

- (i) *either  $F_i^k(\bar{x}) \cap \bar{x}_i \neq \emptyset$  or  $F_i^k(\bar{x}) = \emptyset$  for every  $i \in I$  and  $k = 1, \dots, k_i$ ;*
- (ii) *either  $F_j(\bar{x}) \cap \{\bar{x}_j\} \neq \emptyset$  or  $F_j(\bar{x}) = \emptyset$  for every  $j \in J$ .*

The proof of [Theorem 2.1](#) is given in [Section 3](#). A first consequence of [Theorem 2.1](#) is the following theorem by Hirsch et al. [[11](#)].

**COROLLARY 2.2.** *Let  $V_1$  be a Euclidean space, let  $k_1$  be an integer such that  $0 \leq k_1 \leq \dim V_1$ , and for every  $k = 1, \dots, k_1$ , let  $f^k : G^{k_1}(V_1) \rightarrow V_1$  be a continuous mapping. Then, there exists  $\bar{x} \in G^{k_1}(V_1)$  such that for every  $k = 1, \dots, k_1$ ,  $f^k(\bar{x}) \in \bar{x}$ .*

*Proof.* Take  $I = \{1\}$ ,  $J = \emptyset$ , and  $F_1^k(x) = \{f^k(x)\}$  for every  $x \in G^{k_1}(V_1)$  and for every  $k = 1, \dots, k_1$ . From [Theorem 2.1](#), there exists  $\bar{x} \in M = G^{k_1}(V_1)$  such that for every  $k = 1, \dots, k_1$ ,  $F_1^k(\bar{x}) \cap \bar{x} \neq \emptyset$ , that is,  $f^k(\bar{x}) \in \bar{x}$ . □

A second consequence of [Theorem 2.1](#) is the following generalization of Gale and Mas-Colell’s theorem [8], which is also a generalization of Kakutani’s theorem. Hereafter, we use the formulation by Gourdel [9] allowing each correspondence to be either u.s.c. or l.s.c.

**COROLLARY 2.3.** *Let  $J$  be a finite set, for  $j \in J$ , let  $C_j$  be a nonempty, convex, compact subset of a Euclidean space, and let  $F_j$  be a correspondence from  $M := \prod_{j \in J} C_j$  to  $C_j$  with convex values, such that the correspondence  $F_j$  is either l.s.c or u.s.c. Then, there exists  $\bar{x} = (\bar{x}_j)_{j \in J} \in M$  such that for every  $j \in J$ , either  $\bar{x}_j \in F_j(\bar{x})$  or  $F_j(\bar{x}) = \emptyset$ .*

*Proof.* Take  $I = \emptyset$  and apply [Theorem 2.1](#). □

*Remark 2.4.* According to our definition, an u.s.c. correspondence has compact values and without this requirement, [Theorem 2.1](#) may not be true, as we can see in the following counterexample. Let  $M := G^1(\mathbb{R}^2)$ . Each element  $D$  of  $G^1(\mathbb{R}^2)$  can be written as  $D_t = \{\lambda(\cos t, \sin t) \mid \lambda \in \mathbb{R}\}$ , for some  $t \in [0, \pi[$ . We define the correspondence  $F$  from  $M$  to  $\mathbb{R}^2$  by  $F(D_0) = \mathbb{R} \times \{1\}$  and  $F(D_t) = D_t \cap (\mathbb{R} \times \{1\}) + \{(1, 0)\}$  if  $t \in ]0, \pi[$ . We let the reader check that for every open set  $U \subset \mathbb{R}^2$ , the set  $\{x \in M \mid F(x) \subset U\}$  is open in  $M$  and that  $F$  has nonempty, convex (and closed) values. Yet, it is straightforward that  $F(x) \cap x = \emptyset$  for every  $x \in G^1(\mathbb{R}^2)$ .

Another consequence of our main result is the following multivalued version of Borsuk and Ulam’s theorem. We denote by  $S^n$  the unit sphere of  $\mathbb{R}^{n+1}$ .

**COROLLARY 2.5.** *For  $k = 1, \dots, n$ , let  $F^k$  be a correspondence from  $S^n$  to  $\mathbb{R}$  with nonempty and convex values such that for every  $k = 1, \dots, n$ ,  $F^k$  is either l.s.c or u.s.c. Then, there exists  $\bar{x} \in S^n$  such that*

$$\forall k \in \{1, \dots, n\}, \quad F^k(\bar{x}) \cap F^k(-\bar{x}) \neq \emptyset. \tag{2.1}$$

*Proof.* For every  $k = 1, \dots, n$ , let  $\hat{F}^k$  be the correspondence from  $S^n$  to  $\mathbb{R}$  defined by

$$\hat{F}^k(x) = \{u - v \mid u \in F^k(x), v \in F^k(-x)\}. \tag{2.2}$$

We let the reader check that for every  $k = 1, \dots, n$ , the correspondence  $\hat{F}^k$  has nonempty, convex values and that it is u.s.c. (resp., l.s.c.) if  $F^k$  is u.s.c. (resp., l.s.c.). So, to prove [Corollary 2.5](#), it suffices to show the existence of  $\bar{x} \in S^n$  such that  $0 \in \hat{F}^k(\bar{x})$  for every  $k = 1, \dots, n$ .

We define, for every  $k = 1, \dots, n$ , the correspondence  $H^k$  from  $G^n(\mathbb{R}^{n+1})$  to  $\mathbb{R}^{n+1}$  as follows: for every  $E \in G^n(\mathbb{R}^{n+1})$ , we let  $H^k(E) = \hat{F}^k(x)x$ , where  $x$  is an arbitrary element of  $E^\perp \cap S^n$ . The correspondence  $H^k$  is well defined since  $E^\perp \cap S^n = \{x, -x\}$  for some element  $x \in S^n$  and since  $\hat{F}^k(x)x = \hat{F}^k(-x)(-x)$ .

Take  $I = \{1\}$ ,  $V_1 = \mathbb{R}^{n+1}$ ,  $k_1 = n$ ,  $J = \emptyset$ , and apply [Theorem 2.1](#) to the correspondences  $H^k$ , which clearly satisfy the assumptions of [Theorem 2.1](#). So there exists  $\bar{E} \in G^n(\mathbb{R}^{n+1})$  such that  $\bar{E} \cap H^k(\bar{E}) \neq \emptyset$  for every  $k = 1, \dots, n$ .

Now, if  $\bar{x}$  is an arbitrary point of  $\bar{E}^\perp \cap S^n$ , then we have  $\bar{E} \cap \hat{F}^k(\bar{x})\bar{x} \neq \emptyset$ ; from  $\bar{x} \in \bar{E}^\perp$  and  $\bar{x} \neq 0$ , we finally obtain  $0 \in \hat{F}^k(\bar{x})$  for every  $k = 1, \dots, n$ , which ends the proof of [Corollary 2.5](#).  $\square$

### 3. Proof of [Theorem 2.1](#)

The proof is given in three steps, corresponding to the following three subsections. The first step gives the proof under the additional assumptions that  $J = \emptyset$  and the correspondences  $F_i^k$  are single-valued. The second step only assumes in addition that  $J = \emptyset$ . Finally, the third step gives the proof under the assumptions of [Theorem 2.1](#).

**3.1. Proof when  $J = \emptyset$  and  $F_i^k$  are single-valued.** We first prove [Theorem 2.1](#) under the additional assumptions that  $J = \emptyset$  and the  $F_i^k$  are single-valued. This is exactly the statement below.

**THEOREM 3.1.** *Let  $I$  be a finite set and for  $i \in I$ , let  $k_i$  be an integer and let  $V_i$  be a Euclidean space such that  $0 \leq k_i \leq \dim V_i$ . Let  $M := \prod_{i \in I} G^{k_i}(V_i)$  and for  $i \in I$ , let  $f_i : M \rightarrow (V_i)^{k_i}$  be a continuous mapping. Then, there exists  $\bar{x} = (\bar{x}_i)_{i \in I} \in M$  such that*

$$\forall i \in I, \quad f_i(\bar{x}) \in (\bar{x}_i)^{k_i}. \tag{3.1}$$

The proof of [Theorem 3.1](#) is given in two steps. In the first step, we additionally assume that the mappings are smooth, and the second step gives the proof in the general case.

**3.1.1. Proof of [Theorem 3.1](#) when the  $f_i$  are smooth.** Let  $M := \prod_{i \in I} G^{k_i}(V_i)$  and define  $f : M \rightarrow \prod_{i \in I} V_i^{k_i}$  by

$$f(x) = (\text{proj}_{(x_i)^{k_i}{}^\perp} f_i(x))_{i \in I} \quad \text{for } x = (x_i)_{i \in I} \in M, \tag{3.2}$$

and the subsets  $Z$ ,  $Z_1$ , and  $Z_2$  of  $M \times \prod_{i \in I} V_i^{k_i}$  by

$$\begin{aligned} Z &= \left\{ (x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid \forall i \in I, y_i \in (x_i^{k_i})^\perp \right\}, \\ Z_1 &= \left\{ (x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = f(x) \right\}, \\ Z_2 &= \left\{ (x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = 0 \right\}. \end{aligned} \tag{3.3}$$

Proving [Theorem 3.1](#) amounts to showing the existence of  $\bar{x} \in M$  such that  $f(\bar{x}) = 0$  or, equivalently, such that  $Z_1 \cap Z_2 \neq \emptyset$ . For this, we will use the following [Intersection Theorem 3.2](#), which is a direct consequence of mod 2 intersection theory (see, e.g., [10, page 79] and [5, page 127]).

**INTERSECTION THEOREM 3.2.** *Let  $Z$  be a smooth boundaryless manifold of dimension  $2m$  and let  $Z_1, Z_2$  be two compact boundaryless submanifolds of  $Z$  of dimension  $m$ . If  $\bar{Z}_1$  is a compact boundaryless  $m$ -submanifold of  $Z$  homotopic to  $Z_1$  and if the manifolds  $\bar{Z}_1$  and  $Z_2$  intersect transversally in a unique point  $\bar{z}$  (which means that  $T_{\bar{z}}\bar{Z}_1 + T_{\bar{z}}Z_2 = T_{\bar{z}}Z$ ), then  $Z_1 \cap Z_2 \neq \emptyset$ .*

The proof of [Theorem 3.1](#) consists of checking that the above-defined sets  $Z, Z_1,$  and  $Z_2$  (together with the set  $\bar{Z}_1$  defined below) satisfy the assumptions of [Intersection Theorem 3.2](#).

The sets  $Z, Z_1,$  and  $Z_2$  satisfy the assumptions of [Intersection Theorem 3.2](#). We recall that for every  $i \in I, G^{k_i}(V_i)$  is a smooth, boundaryless, compact manifold of dimension  $k_i(\dim V_i - k_i)$  (see [Lemma A.1](#) in the appendix). Thus  $M := \prod_{i \in I} G^{k_i}(V_i)$  is a boundaryless, smooth, compact manifold of dimension  $m = \sum_{i \in I} k_i(\dim V_i - k_i)$ . Clearly  $Z$  is a fiber bundle whose base space is  $M$  and whose fiber at  $x = (x_i)_{i \in I} \in M$  is the vector space  $\prod_{i \in I} (x_i^{k_i})^\perp$  which has the dimension of  $M$ . Hence,  $Z$  is a smooth manifold of dimension  $2m$ .

The mapping  $f : M \rightarrow \prod_{i \in I} V_i^{k_i}$  is a smooth mapping from Parts (c), (d), and (e) of [Lemma A.1](#) in the appendix. Consequently,  $Z_1$  is a smooth compact boundaryless submanifold of  $Z$  of dimension  $m$ . Finally,  $Z_2$  is clearly a smooth boundaryless compact submanifold of  $Z$  of dimension  $m$ .

The manifold  $Z_1$  is homotopic to the manifold  $\bar{Z}_1$  that we now define. For every  $i \in I,$  let  $\bar{x}_i \in G^{k_i}(V_i)$  and let  $\{\bar{e}_i^1, \dots, \bar{e}_i^{k_i}\}$  be an orthonormal basis of  $\bar{x}_i$ . For every  $i \in I,$  let  $g_i : G^{k_i}(V_i) \rightarrow V_i^{k_i}$  and  $g : M \rightarrow \prod_{i \in I} V_i^{k_i}$  be the mappings defined as follows:

$$\begin{aligned} \forall x_i \in G^{k_i}(V_i), \quad g_i(x_i) &= (\text{proj}_{x_i^\perp}(\bar{e}_i^1), \dots, \text{proj}_{x_i^\perp}(\bar{e}_i^{k_i})) \in (x_i^\perp)^{k_i} = (x_i^{k_i})^\perp, \\ \forall x &= (x_i)_{i \in I} \in M, \quad g(x) = (g_i(x_i))_{i \in I}. \end{aligned} \tag{3.4}$$

We let

$$\bar{Z}_1 := \left\{ (x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = g(x) \right\}. \tag{3.5}$$

To show that the manifold  $Z_1$  is homotopic to  $\bar{Z}_1,$  we let  $H : [0, 1] \times Z_1 \rightarrow Z$  be the continuous mapping defined by  $H(t, (x, f(x))) = (x, (1 - t)f(x) + tg(x))$ . Then  $H(0, \cdot)$  is the canonical inclusion from  $Z_1$  to  $Z,$  and  $H(1, \cdot)(Z_1) = \bar{Z}_1$ .

The manifolds  $\bar{Z}_1$  and  $Z_2$  intersect transversally in a unique point. First, notice that  $\bar{Z}_1 \cap Z_2 = \{(x, 0) \in M \times \prod_{i \in I} V_i^{k_i} \mid g(x) = 0\}$  is the singleton  $(\bar{x}, 0) = ((\bar{x}_i)_{i \in I}, 0)$ . But that  $\bar{Z}_1$  and  $Z_2$  intersect each other transversally in  $Z$  means that  $T_{(\bar{x}, 0)}\bar{Z}_1 + T_{(\bar{x}, 0)}Z_2 = T_{(\bar{x}, 0)}Z$ . Recalling that  $\dim T_{(\bar{x}, 0)}\bar{Z}_1 + \dim T_{(\bar{x}, 0)}Z_2 = \dim T_{(\bar{x}, 0)}Z = 2m,$  we only have to show that  $T_{(\bar{x}, 0)}\bar{Z}_1 \cap T_{(\bar{x}, 0)}Z_2 = \{0\}$ . Finally, noticing that  $T_{(\bar{x}, 0)}\bar{Z}_1 = \{(u, Dg(\bar{x})(u)) \mid u \in T_{\bar{x}}M\}$  and  $T_{(\bar{x}, 0)}Z_2 = \{(u, 0) \mid u \in T_{\bar{x}}M\},$  we only have to prove that  $Dg(\bar{x})$  is injective, which is proved in the following lemma.

**LEMMA 3.3.**  *$Dg(\bar{x})$  is injective.*

*Proof.* Recalling that for every  $x = (x_i)_{i \in I} \in M, g(x) = (g_i(x_i))_{i \in I},$  the mapping  $Dg(\bar{x})$  is injective if and only if for every  $i \in I, Dg_i(\bar{x}_i)$  is injective.  $\square$

So, let  $i \in I$ , let  $(\varphi, U)$  be a local chart of  $G^{k_i}(V_i)$  at  $\bar{x}_i$ , and let  $\psi : (\bar{x}_i^\perp)^{k_i} \rightarrow G^{k_i}(V_i)$  be the inverse mapping of  $\varphi : U \rightarrow (\bar{x}_i^\perp)^{k_i}$ . From the appendix, if  $\{\bar{e}_1^1, \dots, \bar{e}_1^{k_1}\}$  is a given orthonormal basis of  $\bar{x}_i$ ,  $\psi$  can be defined by

$$\psi(u^1, \dots, u^{k_i}) = \text{span} \{\bar{e}_i^1 + u^1, \dots, \bar{e}_i^{k_i} + u^{k_i}\} \quad \text{for every } (u^1, \dots, u^{k_i}) \in (\bar{x}_i^\perp)^{k_i}. \quad (3.6)$$

Since the mapping  $g_i \circ \psi$  is the local representation  $g_i$  in the chart  $(\varphi, U)$ , proving that  $Dg_i(\bar{x}_i)$  is injective amounts to proving that  $D(g_i \circ \psi)(0)$  is injective. This is a consequence of the following claim.

**CLAIM 3.4.** For all  $(h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$ ,  $D(g_i \circ \psi)(0)(h_1, \dots, h_{k_i}) = -(h_1, \dots, h_{k_i})$ .

*Proof of Claim 3.4.* Let  $p : V_i \times (\bar{x}_i^\perp)^{k_i} \rightarrow V_i$  be defined by

$$p(y, u) := \text{proj}_{\psi(u)} y. \quad (3.7)$$

If we prove that for every  $y \in V_i$ , the derivative of the mapping  $p_y : u \rightarrow p(y, u)$  is the linear mapping  $Dp_y(0) : (\bar{x}_i^\perp)^{k_i} \rightarrow V_i$  defined by

$$Dp_y(0)(h) = \sum_{k=1}^{k_i} (y \cdot \bar{e}_i^k) h_k, \quad \forall h = (h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}, \quad (3.8)$$

then **Claim 3.4** will be proved. Indeed, taking  $y = \bar{e}_i^k$  for every  $k = 1, \dots, k_i$ , we would obtain  $D_{\bar{e}_i^k} p(0)(h_1, \dots, h_{k_i}) = h_k$ . Thus, since  $g_i \circ \psi(u) = (\bar{e}_i^1, \dots, \bar{e}_i^{k_i}) - (p_{\bar{e}_i^1}(u), \dots, p_{\bar{e}_i^{k_i}}(u))$ , it would entail **Claim 3.4**.

Now, for every  $u = (u_1, \dots, u_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$ , there exists  $\lambda(y, u) = (\lambda_k(y, u))_{k=1, \dots, k_i} \in \mathbb{R}^{k_i}$  such that

$$p(y, u) = \text{proj}_{\psi(u)} y = \sum_{k=1}^{k_i} \lambda_k(y, u) (\bar{e}_i^k + u^k), \quad (3.9)$$

with  $(\lambda_k(y, u))$  satisfying

$$\left( -y + \sum_{k=1}^{k_i} \lambda_k(y, u) (\bar{e}_i^k + u^k) \right) \cdot (\bar{e}_i^j + u^j) = 0 \quad \text{for every } j = 1, \dots, k_i. \quad (3.10)$$

This can be equivalently rewritten as follows:

$$(I_{k_i} + G(u))\lambda(y, u) = (y \cdot (\bar{e}_i^1 + u^1), \dots, y \cdot (\bar{e}_i^{k_i} + u^{k_i})), \quad (3.11)$$

where  $I_{k_i}$  is the  $k_i \times k_i$  identity matrix and  $G(u)$  is the  $k_i \times k_i$  matrix  $G(u) = (u^j \cdot u^k)_{j, k=1, \dots, k_i}$ . Besides, for  $u$  in a neighborhood  $\mathcal{N}$  of 0 small enough, the matrix  $(I_{k_i} + G(u))$  is invertible. Consequently, the mapping  $\lambda(\cdot, \cdot)$  is smooth on  $V \times \mathcal{N}$ , which implies

that the mapping  $p(\cdot, \cdot)$  is smooth on  $V \times \mathcal{N}$ . Differentiating, with respect to  $u$ , the above equality at  $u = 0$ , we obtain, for every  $h = (h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$ ,

$$DG(0)(h)\lambda(y, 0) + D_u\lambda(y, 0)(h) = 0. \tag{3.12}$$

But it is clear that  $DG(0) = 0$ . Consequently,  $D_u\lambda(y, 0) = 0$ .

Finally, differentiating the equality  $p(y, u) = \sum_{k=1}^{k_i} \lambda_k(y, u)(\bar{e}_i^k + u^k)$  at  $(y, 0)$ , one obtains, for every  $h = (h_k)_{k=1}^{k_i} \in (\bar{x}_i^\perp)^{k_i}$ ,

$$D_u p(y, 0)(h) = \sum_{k=1}^{k_i} \lambda_k(y, 0)h_k = \sum_{k=1}^{k_i} (y \cdot \bar{e}_i^k)h_k, \tag{3.13}$$

which ends the proof of [Claim 3.4](#). □

*3.1.2. Proof of [Theorem 3.1](#) in the general case.* Since  $M$  is a compact manifold and  $V_i^{k_i}$  is a Euclidean space, for every  $i \in I$ , each continuous mapping  $f_i : M \rightarrow V_i^{k_i}$  can be approximated by a sequence of smooth mappings  $f_i^n : M \rightarrow V_i^{k_i}$  converging to  $f_i$ , in the sense that  $\lim_{n \rightarrow \infty} \|f_i^n - f_i\|_\infty = 0$  (see, e.g., Hirsch [[12](#)]). Applying the first step to the smooth mappings  $f_i^n$ , we deduce the existence of  $(x_i^n)_{i \in I} \in M$  such that

$$\forall i \in I, \quad f_i^n(x_i^n) \in (x_i^n)^{k_i} \tag{3.14}$$

or, equivalently,

$$\text{proj}_{(x_i^n)^{k_i}} f_i(x_i^n) = 0. \tag{3.15}$$

From the compactness of  $M$ , without any loss of generality, one can suppose that the sequence  $(x_i^n)_{i \in I}$  converges to some element  $(\bar{x}_i)_{i \in I} \in M$ . We have

$$\begin{aligned} & \left\| \text{proj}_{(\bar{x}_i^\perp)^{k_i}} f_i(\bar{x}_i) - \text{proj}_{(\bar{x}_i^\perp)^{k_i}} f_i^n(x_i^n) \right\| \\ & \leq \left\| \text{proj}_{(\bar{x}_i^\perp)^{k_i}} f_i(\bar{x}_i) - \text{proj}_{(\bar{x}_i^\perp)^{k_i}} f_i(x_i^n) \right\| + \|f_i^n - f_i\|_\infty. \end{aligned} \tag{3.16}$$

Consequently, from the convergence of  $f_i^n$  to  $f_i$  and the continuity of the mapping  $(u, v) \rightarrow \text{proj}_{(u^\perp)^{k_i}} v$  (see [Lemma A.1](#) in the appendix), we obtain

$$\text{proj}_{(\bar{x}_i^\perp)^{k_i}} f_i(\bar{x}_i) = 0 \tag{3.17}$$

or, equivalently,

$$\forall i \in I, \quad f_i(\bar{x}_i) \in \left( (\bar{x}_i^\perp)^{k_i} \right)^\perp = (\bar{x}_i)^{k_i}, \tag{3.18}$$

which ends the proof of [Theorem 3.1](#).



**3.2. Proof of Theorem 2.1 when  $J = \emptyset$ .** We now prove Theorem 2.1 when  $J = \emptyset$ . The proof rests on the following claim.

CLAIM 3.5. *For every  $i \in I$  and every  $k \in \{1, \dots, k_i\}$ , there exists an u.s.c. correspondence  $\hat{F}_i^k$  from  $M$  to  $V_i$ , with nonempty convex values, such that*

$$\forall x \in M, [F_i^k(x) \neq \emptyset] \implies [\forall y \in \hat{F}_i^k(x), \exists \lambda \in \mathbb{R}, \lambda y \in F_i^k(x)]. \quad (3.19)$$

*Proof of Claim 3.5.* Let  $i \in I$  and  $k \in \{1, \dots, k_i\}$ . We distinguish two cases.

Assume first that  $F_i^k$  is l.s.c. Let  $U_i^k = \{x \in M \mid F_i^k(x) \neq \emptyset\}$ . Then  $U_i^k$  is an open subset of  $M$  and  $F_i^k|_{U_i^k}$  is a l.s.c. correspondence with nonempty convex values. By Michael [15], there exists a continuous selection  $f_i^k$  of  $F_i^k|_{U_i^k}$ , that is,  $f_i^k : U_i^k \rightarrow V_i$  is a continuous mapping such that  $f_i^k(x) \in F_i^k(x)$  for every  $x \in U_i^k$ . Let  $B_i$  be the closed unit ball of  $V_i$ , and we define the correspondence  $\hat{F}_i^k$  from  $M$  to  $B_i$  by  $\hat{F}_i^k(x) = \{f_i^k(x)/\|f_i^k(x)\|\}$  if  $x \in U_i^k$  and  $f_i^k(x) \neq 0$  and  $\hat{F}_i^k(x) = B_i$  otherwise. We let the reader check that the correspondence  $\hat{F}_i^k$  satisfies the conclusion of Claim 3.5.

We now consider the case where  $F_i^k$  is u.s.c. Let  $U_i^k = \{x \in M \mid F_i^k(x) \neq \emptyset\}$ . Then  $U_i^k$  is a closed subset of  $M$ . By Cellina [4], one can extend  $F_i^k|_{U_i^k}$  as follows: there exists a correspondence  $\hat{F}_i^k$  from  $M$  to  $V_i$  which is u.s.c., with nonempty, convex, and compact values, such that for every  $x \in U_i^k$ ,  $F_i^k(x) = \hat{F}_i^k(x)$ .  $\square$

We now come back to the proof of Theorem 2.1 when  $J = \emptyset$ . For every  $i \in I$  and  $k = 1, \dots, k_i$ , let  $\hat{F}_i^k$  be the u.s.c. correspondence from  $M$  to  $V_i$  with nonempty convex (compact) values defined in Claim 3.5. By Cellina [3], for every integer  $n$ , there exists a continuous mapping  $f_i^{k,n} : M \rightarrow V_i$  such that

$$G(f_i^{k,n}) \subset G(\hat{F}_i^k) + B\left(0, \frac{1}{n}\right). \quad (3.20)$$

Now, for  $i \in I$ , let  $f_i^n : M \rightarrow (V_i)^{k_i}$  be defined as follows:

$$\forall x \in M, f_i^n(x) = (f_i^{1,n}(x), \dots, f_i^{k_i,n}(x)). \quad (3.21)$$

Applying Theorem 3.1 to the mappings  $f_i^n$ , we deduce the existence of  $(\bar{x}_i^n)_{i \in I} \in M$  such that for every  $i \in I$ ,  $f_i^n(\bar{x}^n) \in (\bar{x}_i^n)^{k_i}$ , hence

$$y_i^{k,n} := f_i^{k,n}(\bar{x}^n) \in \bar{x}_i^n. \quad (3.22)$$

Since the correspondence  $\hat{F}_i^k$  is bounded ( $M$  is compact and  $\hat{F}_i^k$  is u.s.c.), the sequence  $(y_i^{k,n})$  is bounded. Thus, without any loss of generality, one can suppose that the sequence  $(y_i^{k,n})$  converges to some  $y_i^k \in V_i$  when  $n$  tends to  $+\infty$ .

Besides, from the compactness of  $M$ , without any loss of generality, one can suppose that  $(\bar{x}_i^n)_{i \in I}$  converges to  $\bar{x} = (\bar{x}_i)_{i \in I} \in M$  when  $n$  tends to  $+\infty$ .

Moreover, from Lemma A.1(d) in the appendix and from  $y_i^{k,n} \in \bar{x}_i^{k,n}$ , at the limit we have that

$$\forall i \in I, \forall k = 1, \dots, k_i, y_i^k \in \bar{x}_i. \quad (3.23)$$

Since the graph of  $\hat{F}_i^k$  is closed (it is u.s.c. with compact values) and from  $G(f_i^{k,n}) \subset G(\hat{F}_i^k) + B(0, 1/n)$ , one obtains

$$y_i^k \in \hat{F}_i^k(\bar{x}). \tag{3.24}$$

To end the proof, we assume that  $F_i^k(\bar{x}) \neq \emptyset$ . Since  $y_i^k \in \hat{F}_i^k(\bar{x})$ , by Claim 3.5, there exists  $\lambda \in \mathbb{R}$  such that  $\lambda y_i^k \in F_i^k(\bar{x})$ . Hence  $\lambda y_i^k \in F_i^k(\bar{x}) \cap \bar{x}_i \neq \emptyset$  (since  $y_i^k \in \bar{x}_i$ ). This ends the proof of Theorem 2.1.

**3.3. Proof of Theorem 2.1 in the general case.** We first prove the following lemma.

LEMMA 3.6. *Let  $C$  be a nonempty, convex, compact subset of a Euclidean space  $V$ . Then there exists a continuous mapping  $\rho : G^1(V \times \mathbb{R}) \rightarrow C$  such that*

$$\forall x \in G^1(V \times \mathbb{R}), \quad x \cap [C \times \{1\}] \neq \emptyset \implies x \cap [C \times \{1\}] = \{(\rho(x), 1)\}. \tag{3.25}$$

*Proof.* Since  $C$  is compact, it is included in a closed ball  $\bar{B}(0, k)$  of  $V$ . We let  $r : V \rightarrow \bar{B}(0, k+1)$  be defined by  $r(u) = \alpha(\|u\|)u$ , where  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is defined by

$$\begin{aligned} \alpha(t) &= 1 && \text{if } t \in [0, k], \\ \alpha(t) &= k+1-t && \text{if } t \in [k, k+1], \\ \alpha(t) &= 0 && \text{if } t \geq k+1. \end{aligned} \tag{3.26}$$

Let  $\pi_1 : V \times \mathbb{R} \rightarrow V$  and  $\rho : G^1(V \times \mathbb{R}) \rightarrow C$  be defined by  $\pi_1(x, t) = x$  and

$$\rho(x) = \begin{cases} \text{proj}_C \circ r \circ \pi_1(x \cap [V \times \{1\}]) & \text{if } x \cap [V \times \{1\}] \neq \emptyset, \\ \text{proj}_C(0) & \text{if } x \cap [V \times \{1\}] = \emptyset, \end{cases} \tag{3.27}$$

where  $\text{proj}_C : V \rightarrow C$  denotes the projection from  $V$  to  $C$ . Then, one easily sees that  $\rho$  satisfies the conclusion of Lemma 3.6. □

*Proof of Theorem 2.1.* Using Lemma 3.6, we first modify the correspondences  $F_j$  for every  $j \in J$  and replace each nonempty compact convex set  $C_j \subset V_j$  by the Grassmannian manifold  $G^1(V_j \times \mathbb{R})$ . For every  $j \in J$ , let  $\rho_j : G^1(V_j \times \mathbb{R}) \rightarrow C_j$  be the mapping associated to  $C_j \subset V_j$  by Lemma 3.6. Let

$$\rho : \tilde{M} := \prod_{i \in I} G^{k_i}(V_i) \times \prod_{j \in J} G^1(V_j \times \mathbb{R}) \longrightarrow M := \prod_{i \in I} G^{k_i}(V_i) \times \prod_{j \in J} C_j \tag{3.28}$$

be defined by

$$\rho(x) = \left( (x_i)_{i \in I}, \rho_j(x_j)_{j \in J} \right), \quad \text{for } x = \left( (x_i)_{i \in I}, (x_j)_{j \in J} \right). \tag{3.29}$$

For  $i \in I$  and  $k = 1, \dots, k_i$ , let  $\tilde{F}_i^k$  be the correspondence from  $\tilde{M}$  to  $V_i$  defined by

$$\tilde{F}_i^k(x) = F_i^k(\rho(x)). \tag{3.30}$$

For  $j \in J$ , let  $\tilde{F}_j$  be the correspondence from  $\tilde{M}$  to  $V_j \times \mathbb{R}$  defined by

$$\tilde{F}_j(x) = F_j(\rho(x)) \times \{1\}. \tag{3.31}$$

Now, applying the result proved in Section 3.2 (i.e., Theorem 2.1 when  $J = \emptyset$ ) to the correspondences  $\tilde{F}_i^k$  and  $\tilde{F}_j$ , there exists  $x = ((x_i)_{i \in I}, (x_j)_{j \in J}) \in \tilde{M}$  such that

- (i) either  $\tilde{F}_i^k(x) \cap x_i \neq \emptyset$  or  $\tilde{F}_i^k(x) = \emptyset$  for every  $i \in I$  and  $i = 1, \dots, k_i$ ,
- (ii) either  $\tilde{F}_j(x) \cap x_j \neq \emptyset$  or  $\tilde{F}_j(x) = \emptyset$  for every  $j \in J$ .

Let  $\bar{x} = \rho(x) \in M$ ; we end the proof by showing that it satisfies the conclusion of Theorem 2.1. From the above, it is clearly the case for  $i \in I$  and  $k = 1, \dots, k_i$ , that is, we have that

- (i) either  $F_i^k(x) \cap x_i \neq \emptyset$  or  $F_i^k(x) = \emptyset$  for every  $i \in I$  and  $i = 1, \dots, k_i$ .

Now, let  $j \in J$ . We first notice that  $\tilde{F}_j(x) = \emptyset$  if and only if  $F_j(x) = \emptyset$ . Assume now that  $\tilde{F}_j(x) \cap x_j \neq \emptyset$  and recall that  $x_j \cap \tilde{F}_j(x) = x_j \cap (F_j(\bar{x}) \times \{1\})$  and  $F_j(\bar{x}) \subset C_j$ . Consequently,  $x_j \cap (C_j \times \{1\}) \neq \emptyset$  and from Lemma 3.6 we get

$$\emptyset \neq x_j \cap (F_j(\bar{x}) \times \{1\}) \subset x_j \cap (C_j \times \{1\}) = \{(\rho_j(x_j), 1)\}. \tag{3.32}$$

Hence, the equality holds and  $\bar{x}_j = \rho_j(x_j) \in F_j(\bar{x})$ . This ends the proof of Theorem 2.1. □

## Appendix

### The Grassmannian manifold $G^k(V)$

Let  $V$  be a Euclidean space and let  $k$  be an integer such that  $0 \leq k \leq \dim V$ . In this section, we recall the properties of  $G^k(V)$  which are used in this paper.

First, we recall that  $G^k(V)$  is a smooth boundaryless manifold of dimension  $k(\dim V - k)$  (see, e.g., Hirsch [12] and Lemma A.1). The local charts can be defined as follows. Let  $\bar{E} \in G^k(V)$  and let  $\{\bar{e}_1, \dots, \bar{e}_k\}$  be some given orthonormal basis of  $\bar{E}$ ; we define the mapping  $\psi_{\bar{E}} : (\bar{E}^\perp)^k \rightarrow G^k(V)$  by

$$\psi_{\bar{E}}(u) = \text{span} \{\bar{e}_1 + u_1, \dots, \bar{e}_k + u_k\}, \quad \text{for } u = (u_1, \dots, u_k) \in (\bar{E}^\perp)^k. \tag{A.1}$$

Then it is easy to check that the mapping  $\psi_{\bar{E}}$  is injective (see Claim A.2); so  $\psi_{\bar{E}}$  is a bijection from  $(\bar{E}^\perp)^k$  onto  $U_{\bar{E}} = \psi_{\bar{E}}((\bar{E}^\perp)^k)$ . We can now consider the inverse mapping  $\varphi_{\bar{E}} : U_{\bar{E}} \rightarrow (\bar{E}^\perp)^k$  defined by  $\varphi_{\bar{E}}(E) = \psi_{\bar{E}}^{-1}(E)$ , which is clearly a bijection.

LEMMA A.1. (a)  $G^k(V)$  is a smooth boundaryless (i.e.,  $C^\infty$ ) manifold of dimension  $k(\dim V - k)$  without boundary and  $(U_{\bar{E}}, \varphi_{\bar{E}})_{\bar{E} \in G^k(V)}$  defines a  $C^\infty$  atlas of  $G^k(V)$ .

(b) The set  $G^k(V)$  is compact.

(c) The mapping  $E \rightarrow E^\perp$  from  $G^k(V)$  to  $G^\ell(V)$  ( $\ell = \dim V - k$ ) is a smooth diffeomorphism.

(d) The mapping  $p : V \times G^k(V) \rightarrow V$  defined by  $p(x, E) = \text{proj}_E(x)$  is smooth. Hence, the set  $\{(x, E) \in V \times G^k(V) \mid x \in E\}$  is a closed subset of  $V \times G^k(V)$ .

(e) The mapping  $x \rightarrow x^p$  from  $G^k(V)$  to  $(G^k(V))^p$  is smooth.

We prepare the proof of the lemma with a claim.

CLAIM A.2. Let  $\bar{E} \in G^k(V)$  and let  $\{\bar{e}_1, \dots, \bar{e}_k\}$  be an orthonormal basis of  $\bar{E}$ .

(a) The mapping  $\psi_{\bar{E}}$  is injective.

(b) For every  $u \in (\bar{E}^\perp)^k$ ,  $\psi(0) \cap \psi(u)^\perp = \psi(0)^\perp \cap \psi(u) = \{0\}$ .

*Proof of Claim A.2.* Part (a). Let  $u = (u_1, \dots, u_k)$  and  $v = (v_1, \dots, v_k)$  in  $(\bar{E}^\perp)^k$  such that

$$\psi_{\bar{E}}(u) = \text{span}\{\bar{e}_1 + u_1, \dots, \bar{e}_k + u_k\} = \psi_{\bar{E}}(v) = \text{span}\{\bar{e}_1 + v_1, \dots, \bar{e}_k + v_k\}. \tag{A.2}$$

Then, there exist some real numbers  $\lambda_{ij}$  ( $i = 1, \dots, k, j = 1, \dots, k$ ) such that

$$\bar{e}_i + u_i = \sum_{j=1}^k \lambda_{ij}(\bar{e}_j + v_j), \quad \text{for every } i = 1, \dots, k. \tag{A.3}$$

Taking, for each inequality, the scalar product with  $\bar{e}_l$ , where  $l = 1, \dots, k$ , we obtain  $\lambda_{il} = 0$  if  $i \neq l$  and  $\lambda_{ii} = 1$ . Hence  $u_l = v_l$  for every  $l = 1, \dots, k$ , and finally  $u = v$ .

Part (b). Let  $x \in \bar{E} \cap \psi(u)^\perp$ . Then there exists some real numbers  $\lambda_i$  ( $i = 1, \dots, k$ ) such that  $x = \sum_{i=1}^k \lambda_i \bar{e}_i$ . Taking the scalar product with  $\bar{e}_j + u_j$  ( $j = 1, \dots, k$ ), we obtain  $\lambda_j = 0$  for every  $j = 1, \dots, k$ , which proves  $\psi(0) \cap \psi(u)^\perp = \{0\}$ . Similarly, let  $x \in \bar{E}^\perp \cap \psi(u)$ . Then there exists some real numbers  $\lambda_i$  ( $i = 1, \dots, k$ ) such that  $x = \sum_{i=1}^k \lambda_i(\bar{e}_i + u_i)$ . Taking the scalar product with  $\bar{e}_j$  ( $j = 1, \dots, k$ ), we obtain  $\lambda_j = 0$  for every  $j = 1, \dots, k$ , which proves  $\psi(0) \cap \psi(u)^\perp = \{0\}$  and ends the proof of the claim.  $\square$

*Proof of Lemma A.1.* Part (a). We prove that  $(U_E, \varphi_E)_{E \in G^k(V)}$  is a smooth (i.e.,  $C^\infty$ ) atlas of  $G^k(V)$  and it is then clear that  $\dim M = \dim(E^\perp)^k = k(\dim V - k)$ . Let  $(U_E, \varphi_E)$  and  $(U_F, \varphi_F)$  be two local charts at  $E$  and  $F$ , respectively, such that  $U_E \cap U_F \neq \emptyset$ . We will prove that  $\varphi_F \circ \varphi_E^{-1}$  is smooth (i.e.,  $C^\infty$ ). We let  $\{e_1, \dots, e_k\}$  and  $\{f_1, \dots, f_k\}$  be two orthonormal bases of  $E$  and  $F$ , respectively. Let  $(v_1, \dots, v_k) = \varphi_F \circ \varphi_E^{-1}(u_1, \dots, u_k)$  and  $u = (u_1, \dots, u_k)$ ; then there exist real numbers  $\lambda_{ij}(u)$  ( $i, j = 1, \dots, k$ ) such that

$$f_i + v_i = \sum_{j=1}^k \lambda_{ij}(u)(e_j + u_j) \quad (i = 1, \dots, k). \tag{A.4}$$

The proof will be complete by showing that the real-valued functions  $\lambda_{ij}(u)$  are differentiable with respect to  $u$ . Taking the scalar product with  $f_l$  ( $l = 1, \dots, k$ ), we obtain

$$f_l \cdot f_l = \sum_{j=1}^k \lambda_{lj}(u)(e_j + u_j) \cdot f_l \quad (l = 1, \dots, k). \tag{A.5}$$

Thus, for every  $i = 1, \dots, k$ , the vector  $\lambda_i(u) = (\lambda_{ij}(u))_{j=1}^k$  is the solution of a linear system whose matrix  $G(u) = ((e_j + u_j) \cdot f_l)_{j,l=1, \dots, k}$  is now shown to be invertible (which clearly implies the differentiability of  $\lambda_i(u)$ ). Indeed, if  $G(u)\lambda = 0$  for some  $\lambda \in \mathbb{R}^k$ , then  $\sum_{j=1}^k \lambda_j(e_j + u_j) \cdot f_l = 0$  (for  $l = 1, \dots, k$ ), thus  $\sum_{j=1}^k \lambda_j(e_j + u_j) \in F^\perp$ . Besides, since  $\sum_{j=1}^k \lambda_j(e_j + u_j) \in \psi_E(u_1, \dots, u_j) = \psi_F(v_1, \dots, v_j)$ , we finally obtain  $\sum_{j=1}^k \lambda_j(e_j + u_j) \in F^\perp \cap \psi_F(v_1, \dots, v_k) = \{0\}$  (from Claim A.2). Now, since  $(e_j + u_j)_{j=1, \dots, k}$  is a basis, we obtain  $\lambda_j = 0$  for every  $j = 1, \dots, k$ .

Part (b). Let  $(E^\nu)$  be a sequence in  $G^k(V)$  and for every  $\nu$ , let  $\{e_1^\nu, \dots, e_k^\nu\}$  be an orthonormal basis of  $E^\nu$ . Without any loss of generality, we can assume that for every  $i = 1, \dots, k$ , the sequence  $(e_i^\nu)$  converges to some element  $e_i$  in  $V$ . Clearly,  $\{e_1, \dots, e_k\}$  is an orthonormal family in  $V$ , and we let  $E = \text{span}\{e_1, \dots, e_k\}$ . We will now prove that the sequence  $(E^\nu)$  converges to  $E$ . Indeed, for  $\nu$  large enough, there exists  $u^\nu = (u_1^\nu, \dots, u_k^\nu) \in (E^\perp)^\nu$  such that  $E^\nu = \psi_E(u_1^\nu, \dots, u_k^\nu)$ . It can be written as  $e_i + u_i^\nu = \sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu$  ( $i = 1, \dots, k$ ). Multiplying these equalities by  $e_l$  ( $l = 1, \dots, k$ ), we obtain  $\sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu \cdot e_l = 0$  if  $i \neq l$  and  $\sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu \cdot e_i = 1$ . This can be written (for every  $i = 1, \dots, k$ ) as  $(e_j^\nu \cdot e_l)_{j,l=1,\dots,k} (\lambda_{ij}^\nu)_{j=1}^k = (e_i \cdot e_l)_{l=1,\dots,k}$ . If  $\nu$  is large enough, then  $(e_j^\nu \cdot e_l)_{j,l=1,\dots,k}$  is invertible and converges to  $\text{Id}$ , which proves that the sequence  $(\lambda_{ij}^\nu)_{j=1}^k$  converges to  $(e_i \cdot e_l)_{l=1,\dots,k}$  for every  $i = 1, \dots, k$ , that is,  $(\lambda_{ii}^\nu)$  converges to 1 and  $(\lambda_{ij}^\nu)$  converges to 0 for  $i \neq j$ . We finally obtain that  $(u_i^\nu)$  converges to 0, which proves that  $E^\nu$  converges to  $E$ .

Part (c). Let  $\bar{E} \in G^k(V)$  and let  $(\bar{e}_1, \dots, \bar{e}_k)$  and  $(\bar{f}_1, \dots, \bar{f}_\ell)$  be orthonormal bases of  $\bar{E}$  and  $\bar{E}^\perp$ , respectively. Let  $(u_1, \dots, u_k) \in (\bar{E}^\perp)^\nu$  and let  $E = \psi(u)$ . Then it is easy to see that  $E^\perp = \text{span}\{f_1 + v_1, \dots, f_\ell + v_\ell\}$ , where  $v_i = -\sum_{j=1}^k (f_i \cdot u_j) \bar{e}_j$ . So each  $v_i$  is a smooth mapping with respect to the  $u_i$  and, conversely, from  $(E^\perp)^\perp = E$ , each  $u_i$  is a smooth mapping with respect to the  $v_i$ . This ends the proof of part (c).

Part (d). The differentiability of the mapping  $p$  is left to the reader. Then notice that  $\{(x, E) \in V \times G^k(V) \mid x \in E\} = \{(x, E) \in V \times G^k(V) \mid x = \text{proj}_E(x)\}$ , which is clearly closed since the mapping  $p : (x, E) \rightarrow \text{proj}_E(x)$  is continuous.  $\square$

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