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Asymptotic value in frequency-dependent games: A differential approach.

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Abstract

We study the asymptotic value of a frequency-dependent zero-sum game following a differential approach. In such a game the stage payoffs depend on the current action and on the frequency of actions played so far. We associate in a natural way a differential game to the original game and although it presents an irregularity at the origin, we prove existence of the value on the time interval $[0, 1]$. We conclude, using appropriate approximations, that the limit of $\mathbf{V}_n$, as n tends to infinity, exists and that it coincides with the value of the associated continuous time game.

Keywords: stochastic game, frequency dependent payoffs, continuous-time game, Hamilton-Jacobi-Bellman-Isaacs equation.

JEL Classification: C73 **AMS Classification**: 91A15 91A23 91A25

1 Introduction

Frequency-dependent games (FD games) are a class of dynamic games in which stage payoffs depend on the frequency of past actions. They have been introduced by Brenner and Witt [2003]. Such games consist in the repetition at discrete moments, of an one-shot game in which the stage payoff functions depend on the choices of the players at the current stage, as well as on the relative frequencies of actions played at previous stages. Stage payoffs may be frequency-dependent over time because of several reasons. The actions undertaken by the players at each stage may generate externalities, which accumulate as the game unfolds. For instance, payoffs may change due to learning, habit formation, addiction, or satiation. An extensive review of this class of games can be found in Joosten et al. [2003]. The authors analyse in particular non zero-sum FD games and derive several *folk theorem* like results using a notion of equilibrium that is not standard. Some aspects of FD games have been

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studied in Contou-Carrère [2011], where in particular the author proves that no uniform value exists even for an one player game, but where asymptotic value is shown to exist for any initial state, the convergence of which, however, is not uniform with respect to the state.

In this paper we extend the existence of the asymptotic value to two-player zero-sum FD games. For any $n \in \mathbb{N}^*$, we define a n-stage, two-player zero-sum FD game with finite action sets I and J respectively. At each stage the two players choose simultaneously an action and the stage payoff is the sum of two parts: the first depends only on the current actions and the second on the frequency whereby each pair of actions has been chosen so far. This game can be viewed as a stochastic game with countable state space, namely $\mathbb{N}^{I\times J}$ and deterministic transitions. The current state at the n-th stage is the *aggregate past matrix*, i.e., it reflects how many times each action profile has been selected in the previous $n-1$ stages. Player 1 maximizes and Player 2 minimizes the average payoff on the first n stages and the game is played under *perfect-monitoring* meaning that both players know the current state, as well as the entire history, i.e., the state visited and action pair played at each of the preceding stages. It is already known by the study of the one player game that no uniform value exists. In this paper we prove, by associating to the FD game a differential game, that the asymptotic value exists. From (Mertens et al. [2015]), the n-stage value of the game satisfies a recursive equation and the structure of payoff and transition functions allows us to exhibit a formula for the sequence representing the value. However, it seems difficult to study its asymptotic behavior by the obtained form. We then switch to a differential approach in the sense that we associate to the FD game, a differential game played over $[0,1]\times\mathbb{R}^{I\times J}$. Indeed, by a heuristic reasoning it is possible to conjecture as a limit of the recursive equation, a hypothetical partial differential equation (PDE) that governs the evolution of the value. It turns out that this is precisely the (HJBI) equation of some differential game and furthermore that the value of this continuous game is closely related to the value of our repeated game. Some difficulty arises however since the payoff of the continuous game presents an irregularity at the origin and it is precisely at the origin that the value is needed. Everywhere but the origin, regularity conditions are satisfied by the payoff and dynamics functions and since Isaacs condition is satisfied, i.e., lower and upper hamiltonians coincide, from Evans and Souganidis [1984][1] and Souganidis [1999], the differential game admits a value which is characterized as the unique viscosity solution in the space of bounded, continuous functions of the (HJBI) equation with a boundary condition. Despite the irregularity at the origin, we can prove existence of the value in the differential game starting at $(0,0)$. In order to compare the values of the repeated game with that of the differential game, we proceed by discretization. Precisely we consider the uniformly discretized differential game whose values satisfy the dynamical programming principle (DPP) (Friedman [1970]) that is very similar to the recursive equation of the repeated game, but that differ with it on one important point. While in the discretized game, the state evolves in a deterministic linear space, in the original game the state evolves as a random process. We use the structure of our payoff and transition functions to show coincidence between the values of the original and discretized games. Finally using approximation schemes (Souganidis [1999] for finite horizon and in Bardi and Capuzzo-Dolcetta [2008] for infinite horizon differ-

[1]Existence of the value follows from the standard comparison and uniqueness theorems for viscosity solutions presented in Crandall and Lions [1983].

ential games [2]) we prove that the value of the original game, as the number of stages tends to infinity, converges to the value of its associated differential game starting at the origin.

Let us mention that a similar approach has been proposed by Laraki [2002] in order to prove existence of the asymptotic value in n-stage and λ-discounted repeated games with incomplete information on one side. The two approaches differ in the nature of the state space of the continuous game and chiefly in that, due to the irregularity at the origin in our setting, existence of the value in the continuous game is not straightforward. Cardaliaguet et al. [2012] achieve a transposition to discrete time games of the numerical schemes used to approximate the value function of differential games via viscosity solution arguments, presented in Barles and Souganidis [1991]. The authors prove uniform convergence of the value in absorbing, splitting and incomplete information games. In the model we study, it is not possible to adapt their method since in FD games we deal with a countable state space and following their approach would lead us to an infinite dimensional state space in the associated differential game. Consequently, we cannot deal with the used technics.

Structure of the paper. The remainder of the paper is organized as follows: In Section 2, we give the description of a two-player FD zero-sum game and we provide properties of the n-stage value function, which will be useful, in the sequel. In Section 3, starting from the recursive formula satisfied by the value, we heuristically derive a (PDE). Then, we define the associated differential game and its uniformly discretized version. In Section 4, we prove existence of the value in the differential game played over $[0, 1]$ and starting at initial state 0. In Section 5, we conclude by identifying the value of the continuous time game, as the limit value of the n-stage FD game. Section 6 deals with perspectives and future work.

2 The FD game and some preliminary results

2.1 Definitions

Let I, J be finite sets and denote the space of real matrices with $|I|$ rows and $|J|$ columns by $\mathcal{M}^{I\times J}$. Let $A = [a_{ij}]$ and H be two elements of $\mathcal{M}^{I\times J}$ and let $z_0 \in \mathcal{Z} := \mathbb{N}^{I\times J}$. An FD zero-sum N-times repeated game with initial state z_0, denoted $\Gamma_N(z_0)$, is a dynamic game played by steps as follows:

At stage $t = 1, 2, ..., N$, Player 1 and 2 simultaneously and independently choose an action in their own set of actions, $i_t \in I$ and $j_t \in J$ respectively. The stage payoff to Player 1 is equal to:

$$g_t := g(z_{t-1}, i_t, j_t) = a_{i_t j_t} + h(z_{t-1}),$$

where $z_t = z_0 + e_{i_1 j_1} + ... + e_{i_t j_t}$. The notation $(e_{ij})_{ij}$ stands for the canonical basis in $\mathbb{R}^{I\times J}$

[2]The authors prove that under some regularity conditions on the payoff and dynamics functions, the discrete values converge to the values of the continuous time game as the mesh of the discretization tends to 0. These approximations do not converge in general if the value function is discontinuous.

and,

$$h(z) := \begin{cases} \left\langle H, \frac{z}{|z|} \right\rangle, & z \neq 0 \\ 0, & z = 0, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product in $\mathbb{R}^{I \times J}$ and $|\cdot|$ stands for $\|\cdot\|_1$. The payoff of Player 2 is the opposite of that of Player 1. We assume perfect monitoring of past actions by both players.

Note that, due to the nature of the transition in the state space, announcing the selected moves publicly also reveals the state variable to the players. Therefore, we will denote by $\mathbf{H}_t = \mathcal{Z} \times (I \times J)^{t-1}$ the set of histories at stage t and $\mathbf{H} = \cup_{t \geq 0} \mathbf{H}_t$ will denote the set of all histories.

$\Delta(I)$ and $\Delta(J)$ are the sets of mixed moves of Player 1 and Player 2 respectively. A behavioral strategy for Player 1 is a family of maps $\sigma = (\sigma_t)_{t \geq 1}$, such that $\sigma_t : \mathbf{H}_t \to \Delta(I)$. Similarly, a behavioral strategy for Player 2 is a family of maps $\tau = (\tau_t)_{t \geq 1}$, where $\tau_t : \mathbf{H}_t \to \Delta(J)$. Σ and T denote the sets of behavioral strategies of Player 1 and Player 2, respectively. Given $z_0 \in \mathcal{Z}$, each strategy profile (σ, τ) induces a unique probability distribution $\mathbb{P}^{z_0}_{\sigma,\tau}$ on the set $\mathcal{Z} \times (I \times J)^{\infty}$ of plays (endowed with the σ-field generated by the cylinders). $\mathbb{E}^{z_0}_{\sigma,\tau}$ stands for the corresponding expectation.

2.2 The value of $\Gamma_N(z_0)$

Given $N \in \mathbb{N}^*$, and $z_0 \in \mathcal{Z}$, we will be interested in the finite N-stage game of initial state z_0. In this game, the expected average payoff of Player 1 is $\gamma_N(z_0, \sigma, \tau) = \mathbb{E}^{z_0}_{\sigma,\tau}\left(\frac{1}{N} \sum_{t=1}^{N} g_t\right)$ and the lower value is defined as:

$$\mathbf{V}^-_N(z_0) = \sup_{\sigma \in \Sigma} \inf_{\tau \in T} \gamma_N(z_0, \sigma, \tau)$$

Symmetrically, we define the upper value, i.e.,

$$\mathbf{V}^+_N(z_0) = \inf_{\tau \in T} \sup_{\sigma \in \Sigma} \gamma_N(z_0, \sigma, \tau)$$

Given $z_0 \in \mathcal{Z}$, if $\mathbf{V}^-_N(z_0) = \mathbf{V}^+_N(z_0)$, then the game $\Gamma_N(z_0)$ admit a *value* denoted by $\mathbf{V}_N(z_0)$.

Proposition 2.1. *Given $n \in \mathbb{N}^*$ and a state $z \in \mathcal{Z}$, the game $\Gamma_n(z)$ has a value, $\boldsymbol{V}_n(z)$. Moreover, the value of the game satisfies the following recursive formula:*

$$(n+1)\, \boldsymbol{V}_{n+1}(z) = h(z) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \big(a_{ij} + n\, \boldsymbol{V}_n(z + e_{ij})\big) \right) \tag{2.1}$$

Proof. Existence of the value in $\Gamma_n(z)$ in mixed strategies follows from the minmax theorem of von Neumann [1928] and since the game is played under perfect-recall by Kuhn's theorem the value can be achieved by using behavioral strategies. By Mertens et al. [2015] we have:

$$\mathbf{V}_{n+1}(z) = \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \mathbb{E}_{u\otimes v}\left(\frac{1}{n+1}g(i^*,j^*,z) + \frac{n}{n+1}\mathbf{V}_n(z+e_{i^*j^*})\right)$$

where $\mathbb{E}_{u\otimes v}$ is the expectation operator of the probability space $(I\times J, u\otimes v)$.
Since the payoff function h does not depend on the selected mixed moves (u,v) one gets

$$(n+1)\mathbf{V}_{n+1}(z) = h(z) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \mathbb{E}_{u\otimes v}\big(a_{i^*j^*} + n\mathbf{V}_n(z+e_{i^*j^*})\big),$$

which concludes the proof. □

In the remainder of this section, we provide a formula for the value. This formula is too complex to allow the study of the limit, nevertheless it sheds a light on the asymptotic behavior of the value.

Notation. We use the following notations:

- Given $t\in\mathbb{N}$, let Π_t denote the subset of the state space $\mathcal{Z}$ defined as follows:

$$\Pi_t = \{z\in\mathcal{Z} : |z| = t\}.$$

- We denote the max min operator by **val**.
- For any $(a,p)\in\mathbb{R}_+^*\times\mathbb{N}^*$, we put :

$$\Lambda_p(a) := \frac{1}{a} + \frac{1}{a+1}\ldots + \frac{1}{a+p-1} = \sum_{k=0}^{p-1}\frac{1}{a+k}.$$

Proposition 2.2. *For all $n\in\mathbb{N}^*$, for all $t\in\mathbb{N}^*$, there exist $K_{n,t}\in\mathcal{M}^{I\times J}$ and $C_{n,t}\in\mathbb{R}$, such that for all $z\in\Pi_t$*

$$n\,\boldsymbol{V}_n(z) = \langle K_{n,t}, z\rangle + C_{n,t},$$

where $K_{n,t} = \Lambda_n(t)H$ and $C_{n,t} = \sum_{k=1}^{n-1}\mathbf{val}\big(A + \Lambda_{n-k}(t+k)H\big)$.

Proof. We proceed by induction on the variable n:
For $n=1$, for any $t\in\mathbb{N}^*$ and any $z\in\Pi_t$:

$$\mathbf{V}_1(z) = \left\langle H, \frac{z}{|z|}\right\rangle + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)}\left(\sum_{ij} u_i v_j a_{ij}\right).$$

Then, $\mathbf{V}_1(z) = \langle K_{1,t}, z\rangle + C_{1,t}$, where $K_{1,t} = \frac{H}{t}$ and $C_{1,t} = \mathbf{val}(A)$.

The recursive formula (2.1) and the induction hypothesis for $n = m$ implies, for all $z \in \Pi_t$:

$$(m+1)\mathbf{V}_{m+1}(z) = \left\langle \frac{H}{t} + K_{m,t+1}, z \right\rangle + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{ij} u_i v_j \Big(a_{ij} + \langle K_{m,t+1}, e_{ij}\rangle + C_{m,t+1} \Big) \right)$$
$$= \left\langle \frac{H}{t} + K_{m,t+1}, z \right\rangle + \mathbf{val}\Big(A + K_{m,t+1} \Big) + C_{m,t+1},$$

the middle equality folows from the inner product properties and the fact that $\sum_{ij} u_i v_j = 1$, and the last one from the **val** operator properties. Hence,

$$(m+1)V_{m+1}(z) = \langle K_{m+1,t}, z\rangle + C_{m+1,t},$$

where $K_{m+1,t} = \frac{H}{t} + K_{m,t+1}$ and $C_{m+1,t} = \mathbf{val}\Big(A + K_{m,t+1} \Big) + C_{m,t+1}$, This concludes the proof of the assumption. The rest is routine algebra.

□

Corollary 2.3. *Let $\rho \in (0,1)$. If $N \to +\infty$ and $\frac{n}{N} \to \rho$, then* $\lim K_{n,N-n} = -H \ln(1-\rho)$.

Proof. Writing $\Lambda_n(N-n) = \sum_{k=1}^{N-1} \frac{1}{k} - \sum_{k=1}^{N-n-1} \frac{1}{k}$, the limit follows readily from the fact that the sequence $\sum_{k=1}^{n} \frac{1}{k} - \ln n$ converges to the Euler constant γ when n goes to infinity. □

3 A differential approach

Guided by the corollary, we switch to the study of the asymptotic value by means of a differential approach. To begin with, moving from the recursive formula obtained in Proposition 2.1, we heuristically derive a Partial Differential Equation (PDE) (Section 3.1). It turns out that the latter is precisely the (HJBI) equation of some differential game that we shall define.

For any $N \in \mathbb{N}^*$, we define the quotient state space and the uniform partition of $[0,1]$:

$$\mathcal{Q}_N := \left\{ q | q = \frac{z}{N}, z \in \mathcal{Z} \right\}, \qquad \mathcal{I}_N := \left\{ 0, \frac{1}{N}, ..., 1 \right\}.$$

3.1 The heuristic PDE and the differential game $\mathcal{G}(t,q)$

We define the function $\Psi_N : \mathcal{I}_N \times \mathcal{Q}_N \to \mathbb{R}$, such that

$$\Psi_N(t,q) := (1-t)\mathbf{V}_n(z), \tag{3.1}$$

where $z = Nq$ and $n = N(1-t)$. Ψ_N satisfies:

$$\begin{cases} \Psi_N(t,q) = \frac{h(q)}{N} + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \Psi_N\left(t+\frac{1}{N}, q+\frac{e_{ij}}{N}\right)\right)\right), & t \in \mathcal{I}_N \setminus \{1\} \\ \Psi_N(1,q) = 0. \end{cases} \tag{3.2}$$

The first formula of (3.2), for any $t \in \mathcal{I}_N \setminus \{1\}$ can be written equivalently:

$$0 = h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(a_{ij} + N\Big(\Psi_N\left(t+\frac{1}{N}, q+\frac{e_{ij}}{N}\right) - \Psi_N(t,q)\Big)\right)\right). \tag{3.3}$$

When $N \to +\infty$, we heuristically assume that there exists a sufficiently differentiable function $\Psi : [0,1] \times \mathbb{R}_+^{I\times J} \setminus \{0\} \to \mathbb{R}$ as the limit of Ψ_N, which will therefore satisfy the following (PDE) with boundary condition of (3.2):

$$\begin{cases} \frac{\partial \Psi}{\partial t}(t,q) + h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \sum_{i,j} u_i v_j \left(a_{ij} + \frac{\partial \Psi}{\partial q_{ij}}(t,q)\right) = 0, & (t,q) \in [0,1) \times \mathbb{R}_+^{I\times J} \setminus \{0\}, \\ \Psi(1,q) = 0, \qquad q \in \mathbb{R}_+^{I\times J} \setminus \{0\}. \end{cases} \tag{3.4}$$

Given $(t,q) \in [0,1] \times \mathbb{R}_+^{I\times J}$, we define a differential zero-sum game, denoted by $\mathcal{G}(t,q)$ starting at time t with initial state q. It consists of:

- The state space $\mathcal{Q} = \mathbb{R}_+^{I\times J}$.
- The time interval of the game $T = [t,1]$.
- Player 1 uses a measurable control $\tilde{u} : [t,1] \to \Delta(I)$ and his control space is $\mathcal{U}_t$.
- Player 2 uses a measurable control $\tilde{v} : [t,1] \to \Delta(J)$ and his control space is $\mathcal{V}_t$.
- If Player 1 uses $\tilde{u}$ and Player 2 uses $\tilde{v}$, then the dynamics in the state space is defined as follows:

$$\begin{cases} \frac{dq}{dt}(s) = \tilde{u}(s) \otimes \tilde{v}(s), & s \in (t,1), \\ q(t) = q. \end{cases} \tag{3.5}$$

  Clearly, the dynamics is driven by a bounded, continuous function, which is Lipschitz in q and thus (3.5) admits a unique solution.

- The running-payoff at time $s \in [t,1]$ that Player 1 receives from Player 2 is given by $g : \mathcal{Q} \times \Delta(I) \times \Delta(J) \to \mathbb{R}$ and defined as:

$$g(q,u,v) = h(q) + \langle u \otimes v, A \rangle \tag{3.6}$$

where,

$$h(q) := \begin{cases} \left\langle H, \frac{q}{|q|} \right\rangle, & q \neq 0 \\ 0, & q = 0. \end{cases}$$

It is easy to see that g is bounded by $\|H\|_\infty + \|A\|_\infty$ and since $q : [t, 1] \to \mathcal{Q}$ is a differentiable function of time (see (3.5)), g is differentiable on $\mathcal{Q} \setminus \{0\}$.

- The payoff associated to the pair of controls $(\tilde{u}, \tilde{v}) \in \mathcal{U}_t \times \mathcal{V}_t$ that Player 2 pays to Player 1 at time 1 is given by:

$$G(t, q, \tilde{u}, \tilde{v}) = \int_t^1 g(q_s, \tilde{u}_s, \tilde{v}_s) ds. \tag{3.7}$$

Following Varaiya [1967], Roxin [1969] and Elliott and Kalton [1972], we allow the players to update their controls using non-anticipative strategies. A non-anticipative strategy for Player 1 is a map $\alpha : \mathcal{V}_t \to \mathcal{U}_t$ such that for any time $\tilde{t} > t$,

$$\tilde{v}_1(s) = \tilde{v}_2(s) \qquad \forall s \in [t, \tilde{t}] \quad \Rightarrow \quad \alpha[\tilde{v}_1(s)] = \alpha[\tilde{v}_2(s)] \quad \forall s \in [t, \tilde{t}].$$

The definition of non-anticipative strategies for Player 2 is analogous. Denote by $\mathcal{A}_t$ and $\mathcal{B}_t$ the sets of non-anticipative strategies of the players respectively. With respect to this notion of strategies, the lower and upper values are defined as follows:

$$W^-(t, q) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tilde{v} \in \mathcal{V}_t} G(t, q, \alpha[\tilde{v}], \tilde{v})$$
$$W^+(t, q) := \inf_{\beta \in \mathcal{B}_t} \sup_{\tilde{u} \in \mathcal{U}_t} G(t, q, \tilde{u}, \beta[\tilde{u}]).$$

When both functions coincide, we say that the game $\mathcal{G}(t, q)$ has a *value*.

Following Cardaliaguet [2000] and Bardi and Capuzzo-Dolcetta [2008], the lower and upper hamiltonian functions of the game $\mathcal{G}(t, q)$, $\mathcal{H}^\pm : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ are given by:

$$\begin{aligned} \mathcal{H}^-(\xi, q) &= h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \langle u \otimes v, A + \xi \rangle \\ \mathcal{H}^+(\xi, q) &= h(q) + \min_{v \in \Delta(J)} \max_{u \in \Delta(I)} \langle u \otimes v, A + \xi \rangle. \end{aligned} \tag{3.8}$$

Notation. In the sequel, $\mathcal{Q}^*$ stands for $\mathcal{Q} \setminus \{0\}$.

Given $(t, q) \in [0, 1] \times \mathcal{Q}^*$, since: (i) $\Delta(I)$ and $\Delta(J)$ are compact sets; (ii) the dynamics in the state space (3.5) and the running payoff (3.6) are bounded, continuous in all their variables and Lipschitz in the state variable q functions, by Elliott and Kalton [1974], Evans and Souganidis [1984] and Souganidis [1999] the lower and upper values of minorant and majorant games respectively are characterized by means of the (DPP). Namely, for all $(t,q) \in [0,1) \times \mathcal{Q}^*$ and all $\delta \in (0,1-t]$, we have:

$$W^-(t,q) = \left\langle H, \int_t^{t+\delta} \frac{q(s)}{|q|+s} ds \right\rangle + \sup_{\alpha \in \mathcal{A}_t} \inf_{\tilde{v} \in \mathcal{V}_t} \left\{ \left\langle \int_t^{t+\delta} \alpha[\tilde{v}(s)] \otimes \tilde{v}(s) ds, A \right\rangle + W^* \right\}, \tag{3.9}$$

where $W^* := W^-\big(t+\delta, q(t+\delta)\big)$ with $q(t+\delta) = q + \int\limits_t^{t+\delta} \alpha[\tilde{v}(s)] \otimes \tilde{v}(s) ds$.
In a symmetric way, a characterization is obtained for the upper value W^+ as follows:

$$W^+(t,q) = \left\langle H, \int_t^{t+\delta} \frac{q(s)}{|q|+s} ds \right\rangle + \inf_{\beta \in \mathcal{B}_t} \sup_{\tilde{u} \in \mathcal{U}_t} \left\{ \left\langle \int_t^{t+\delta} \tilde{u}(s) \otimes \beta[\tilde{u}(s)] ds, A \right\rangle + W^* \right\},$$

where $W^* := W^+\big(t+\delta, q(t+\delta)\big)$ with $q(t+\delta) = q + \int\limits_t^{t+\delta} \tilde{u}(s) \otimes \beta[\tilde{u}(s)] ds$.

Furthermore under the preceding assumptions, by Evans and Souganidis [1984] and Souganidis [1999], $W^-(t,q)$ is the unique solution in the space of real-valued, bounded, continuous functions defined over $[0,1] \times \mathcal{Q}^*$ of the following (HJBI) equation:

$$\begin{cases} \frac{\partial W^-}{\partial t}(t,q) + \mathcal{H}^-\big(\nabla_q W^-(t,q), q\big) = 0, & (t,q) \in [0,1) \times \mathcal{Q}^*, \\ W^-(1,q) = 0, \quad q \in \mathcal{Q}^*. \end{cases} \tag{3.10}$$

where, $\mathcal{H}^-$ is the lower hamiltonian defined previously.
Likewise, the upper value W^+ is the unique solution in the space of real-valued, bounded, continuous functions defined over $[0,1] \times \mathcal{Q}^*$ of the symmetric equation:

$$\begin{cases} \frac{\partial W^+}{\partial t}(t,q) + \mathcal{H}^+\big(\nabla_q W^+(t,q), q\big) = 0, & (t,q) \in [0,1) \times \mathcal{Q}^*, \\ W^+(1,q) = 0, \quad q \in \mathcal{Q}^*, \end{cases} \tag{3.11}$$

where $\mathcal{H}^+$ is the upper hamiltonian presented earlier.

Moreover, from the minmax theorem in von Neumann [1928], it clearly follows that the Isaacs condition, i.e., $\mathcal{H}^- = \mathcal{H}^+$ holds true (see (3.8)). Hence, by Evans and Souganidis [1984] and Souganidis [1999], the differential game $\mathcal{G}(t,q)$, starting at time $t \in [0,1)$ with initial state $q \in \mathcal{Q}^*$ admits a value, i.e., $W(t,q) := W^-(t,q) = W^+(t,q)$. Consequently, one can identify the (PDE) obtained in (3.4) with the (HJBI) equations of (3.10) and (3.11).

3.2 The discretized game $\mathcal{G}_{\mathcal{P}}(t_0, q_0)$

In this section we prove a coincidence result between the function Ψ_N and the value of the associated differential game in which players are allowed to choose their actions only on the nodes of the uniform partition of the time interval. To that purpose, we next consider finite subdivisions of $[0,1]$ and we define a family of discretized games played on them.

Notations. Let us consider the following subdivisions of the time interval $[0,1]$:

- For all $t_0 \in [0,1)$, $\mathcal{P}$ stands for any finite subdivision of $[t_0, 1]$ and $h_{\mathcal{P}}$ denotes the number of intervals of the subdivision $\mathcal{P}$.
- Given $N \in \mathbb{N}^*$, $\mathcal{P}_N = \left(t_k^N\right)_{0\le k\le N}$, where $t_k^N := \frac{k}{N}$ stands for the uniform subdivision of $[0,1]$ in N intervals. We will also use the notation $\mathcal{P}_N = \left(t_n^N\right)_{0\le n\le N}$ for $n = N-k$.

Fix $\mathcal{P}$ and let $\pi_k := t_{k+1} - t_k$ for $k \in \{0, ..., h_{\mathcal{P}} - 1\}$ be the k-th increament of it. We denote by $|\mathcal{P}|$ the mesh of the subdivision $\mathcal{P}$, i.e., $|\mathcal{P}| = \sup_k |\pi_k|$.

Given $\mathcal{P}$, for all $(t_0, q_0) \in [0,1) \times \mathcal{Q}$ we associate to $\mathcal{G}(t_0, q_0)$ a discrete time game adapted to the subdivision $\mathcal{P}$ denoted by $\mathcal{G}_{\mathcal{P}}(t_0, q_0)$. Such discrete time game starts at time t_0, has initial state $q_0 \in \mathcal{Q}$ and is repeated $h_{\mathcal{P}}$ times. At time t_k with $k = 0, .., h_{\mathcal{P}} - 1$, both players observe the current state q_k and choose simultaneously and independently actions u_{k+1} and v_{k+1} in $\Delta(I)$ and $\Delta(J)$ respectively. The control sets are denoted by $\Delta(I)^{h_{\mathcal{P}}}$ and $\Delta(J)^{h_{\mathcal{P}}}$, indicating that players now choose piecewise constant functions defined over the $h_{\mathcal{P}}$-times cartesian product of their corresponding mixed strategy sets. We will use the notation $\hat{u} = (u_k)_{k=1}^{h_{\mathcal{P}}}$ and $\hat{v} = (v_k)_{k=1}^{h_{\mathcal{P}}}$. The state evolves according to:

$$(\tilde{\mathcal{S}}) \qquad \begin{cases} q_{k+1} = q_k + \pi_k u_{k+1} \otimes v_{k+1}, & k \in \{0, ..., h_{\mathcal{P}} - 1\}, \\ q_0 = q. \end{cases}$$

At stage k, the expected payoff that Player 1 receives from Player 2 is given by:

$$g(q_{k-1}, u_k, v_k) = h(q_{k-1}) + u_k A v_k \tag{3.12}$$

and given $(\hat{u}, \hat{v}) \in \Delta(I)^{h_{\mathcal{P}}} \times \Delta(J)^{h_{\mathcal{P}}}$, the total payoff of the game is

$$G_{\mathcal{P}}(q_0, \hat{u}, \hat{v}) = \sum_{k=1}^{h_{\mathcal{P}}} \pi_{k-1} g(q_{k-1}, u_k, v_k). \tag{3.13}$$

The *minorant game* $\mathcal{G}_{\mathcal{P}}^-(t_0, q_0)$ is similar to $\mathcal{G}_{\mathcal{P}}(t_0, q_0)$, except that Player 1 announces his move to Player 2 at each stage before Player 2 chooses his move. In the *majorant game* $\mathcal{G}_{\mathcal{P}}^+(t_0, q_0)$, the advantage in the information pattern is reversed and goes to Player 1 because Player 2 must commit himself first to each move. The notation $W_{\mathcal{P}}^-(t_0, q_0)$ and $W_{\mathcal{P}}^+(t_0, q_0)$ stands for the lower and upper values of $\mathcal{G}_{\mathcal{P}}(t_0, q_0)$ respectively.

Following Friedman [1970], we characterize $W_{\mathcal{P}}^-$ and $W_{\mathcal{P}}^+$ by means of discrete versions of the lower and upper (HJBI) equations (3.10), (3.11):

- $W_{\mathcal{P}}^+(t_k, q_k) = \pi_k h(q) + \min_{v\in\Delta(J)} \max_{u\in\Delta(I)} \left\{\langle \pi_k u \otimes v, A\rangle + W_{\mathcal{P}}^+\left(t_{k+1}, q_k + \pi_k u \otimes v\right)\right\}$,
- $W_{\mathcal{P}}^-(t_k, q_k) = \pi_k h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left\{\langle \pi_k u \otimes v, A\rangle + W_{\mathcal{P}}^-\left(t_{k+1}, q_k + \pi_k u \otimes v\right)\right\}$,
- $W_{\mathcal{P}}^{\pm}(1, q) = 0$.

We will refer to the first two equations above as the *discrete* Dynamic Programing Principle that will be abbreviated to (discrete DPP).

Recall that the map $\Psi_N : \mathcal{I}_N \times \mathcal{Q}_N \to \mathbb{R}$ has been defined by (3.1) and is characterized by the recursive formula (3.2). It is then clear that an extension of Ψ_N to a map $\Psi_N : \mathcal{P}_N \times \mathcal{Q} \to \mathbb{R}$ is obtained if we define it by the same recursive formula and terminal condition, namely:

$$\begin{cases} \Psi_N\left(t_k^N, q\right) = \frac{h(q)}{N} + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \Psi_N\left(t_{k+1}^N, q + \frac{e_{ij}}{N}\right)\right)\right), \ 0 \le k \le N-1, q \in \mathcal{Q}, \\ \Psi_N(1,q) = 0, \quad k = N, \ q \in \mathcal{Q}. \end{cases} \tag{3.14}$$

In what follows we are going to compare the similar recursive formulas of $W_{\mathcal{P}}^{\pm}$ and Ψ_N. The difference between them is essentially that in the discretized game $\mathcal{G}_{\mathcal{P}}(t_0, q_0)$ the play generated by pure strategies is deterministic and lives in $\mathbb{R}^{I\times J}$, whereas the play generated by the original game $\Gamma_N(z_0)$ in behavior strategies is random and takes its values in a discrete subset of $\mathbb{R}^{I\times J}$.

Proposition 3.1. *Let $N \in \mathbb{N}^*$. There exists a collection $(k_{n,t}, c_{n,t}) \in \mathcal{M}^{I\times J} \times \mathbb{R}$ where $n \in \{0, ..., N\}, t \in \mathbb{R}_+$ such that for all $q \in \mathcal{Q}$, and $n \in \{0, ..., N\}$:*

$$\Psi_N(t_n^N, q) = \left\langle k_{n,|q|}, q \right\rangle + c_{n,|q|}. \tag{3.15}$$

The general terms of the sequence $(k_{n,t})$ are given for $t \in \mathbb{R}_+^*$ by:

$$k_{n,t} = \Lambda_{N-n}(Nt)H$$

Proof. For $t = 0$ we take by convention $k_{n,0} = 0$ for all $n \in \{0, ..., N\}$. Let $q \in \mathcal{Q}^*$, we proceed by backward induction on the variable n:
For $n = N$, $\Psi_N(1, q) = 0$ for any $q \in \mathcal{Q}^*$ and thus, one can take $k_{N,t} = 0$ and $c_{N,t} = 0$ for all $t > 0$.
Assume the result is true for $n = m$, i.e., for all $q \in \mathcal{Q}^*$, there exist $k_{m,t} \in \mathcal{M}^{I\times J}$ and $c_{m,t} \in \mathbb{R}$, such that (3.15) is satisfied. For $n = m$, for all $q \in \mathcal{Q}^*$, we get from (3.14):

$$\begin{aligned} \Psi\left(t_m^N, q\right) &= \left\langle \frac{H}{N}, \frac{q}{|q|} \right\rangle + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \left\langle k_{m+1,|q|+\frac{1}{N}}, q + \frac{e_{ij}}{N} \right\rangle + c_{m+1,|q|+\frac{1}{N}} \right)\right) \\ &= \left\langle \frac{H}{N|q|} + k_{m+1,|q|+\frac{1}{N}}, q \right\rangle + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \left\langle k_{m+1,|q|+\frac{1}{N}}, \frac{e_{ij}}{N} \right\rangle\right)\right) + c_{m+1,|q|+\frac{1}{N}}. \end{aligned}$$

and thus (3.15) is satisfied if we put:

$$k_{m,t} = \frac{H}{Nt} + k_{m+1,t+\frac{1}{N}}$$
$$c_{m,t} = \frac{1}{N}\mathbf{val}\left(A + k_{m+1,t+\frac{1}{N}}\right) + c_{m+1,t+\frac{1}{N}}.$$

This ends the induction. ∎

Notation. Given $N \in \mathbb{N}^*$ and $q_0 \in \mathcal{Q}$, for all $t \in \mathcal{P}_N$, we define the subset of $\mathcal{Q}$:

$$\mathcal{Q}_N(t, q_0) = \big\{q \in \mathcal{Q} : |q| = |q_0| + t\big\}.$$

Proposition 3.2. *Given $N \in \mathbb{N}^*$ and $q_0 \in \mathcal{Q}$, for all $t \in \mathcal{P}_N$ and all $q \in \mathcal{Q}_N(t, q_0)$,*

$$\Psi_N(t, q) = W^-_{\mathcal{P}_N}(t, q)\,.$$

Proof. Both functions share the same terminal condition, i.e $\Psi_N(1, q) = W^-_{\mathcal{P}_N}(1, q) = 0$, for all $q \in \mathcal{Q}$, $\big($see (3.1), and characterization of $W^-_{\mathcal{P}_N}$ in terms of (discrete DPP)$\big)$. Thus, it suffices to prove that Ψ_N and $W^-_{\mathcal{P}_N}$ satisfy the same recursive formula. To that purpose, fix $q_0 \in \mathcal{Q}$ and time $t = \frac{k}{N}$, where $k \in \big\{0, ..., N-1\big\}$. From the (discrete (DPP)), it follows that for all $q \in \mathcal{Q}_N\left(\frac{k}{N}, q_0\right)$,

$$W^-_{\mathcal{P}_N}\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u \in U}\min_{v \in V}\left(\left\langle \frac{u \otimes v}{N}, A\right\rangle + W_{\mathcal{P}_N}\left(\frac{k+1}{N}, q + \frac{u \otimes v}{N}\right)\right) \tag{3.16}$$

By (3.14), for any $k \in \{0, ..., N-1\}$,

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u \in \Delta(I)}\min_{v \in \Delta(J)}\left(\sum_{ij} u_i v_j \left(\frac{a_{ij}}{N} + \Psi_N\left(\frac{k+1}{N}, q + \frac{e_{ij}}{N}\right)\right)\right),$$

where $q \in \mathcal{Q}_N\left(\frac{k}{N}, q_0\right)$. Equivalently:

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u \in \Delta(I)}\min_{v \in \Delta(J)}\left(\sum_{ij} u_i v_j \frac{a_{ij}}{N} + \sum_{ij} u_i v_j \Psi_n\left(\frac{k+1}{N}, q + \frac{e_{ij}}{N}\right)\right).$$

By Lemma 3.1, it follows that

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u \in \Delta(I)}\min_{v \in \Delta(J)}\left(\sum_{ij} u_i v_j \frac{a_{ij}}{N} + \Psi_n\left(\frac{k+1}{N}, \sum_{ij} u_i v_j \left(q + \frac{e_{ij}}{N}\right)\right)\right).$$

Hence,

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u \in \Delta(I)}\min_{v \in \Delta(J)}\left(\sum_{ij} u_i v_j \frac{a_{ij}}{N} + \Psi_n\left(\frac{k+1}{N}, q + \frac{\sum_{ij} u_i v_j e_{ij}}{N}\right)\right),$$

which implies,

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u \in \Delta(I)}\min_{v \in \Delta(J)}\left(\left\langle \frac{u \otimes v}{N}, A\right\rangle + \Psi_N\left(\frac{k+1}{N}, q + \frac{u \otimes v}{N}\right)\right).$$

This in view of (3.16), proves that $\Psi_N = W^-_{\mathcal{P}_N}$. $\square$

In a similar way on can prove $\Psi_N = W^+_{\mathcal{P}_N}$. This result will allow us to use approximation schemes for differential games in the subsequent parts of the proof. Since the value of the original game is equal to that of the discretized approximated game, proving convergence of the value of the latter will prove the convergence of the value of the former. One difficulty arises however in applying approximation schemes, namely that the differential game presents an irregularity at the origin.

4 Existence of the value in $\mathcal{G}(0,0)$

In this section, we prove existence of the value in the differential game played over $[0,1]\times\mathcal{Q}$. Next lemma will be useful in the sequel.

Lemma 4.1. *Let $t\in[0,1]$, $(q,\tilde{q})\in\mathcal{Q}^*\times\mathcal{Q}$ and $(\tilde{u},\tilde{v})\in\mathcal{U}_t\times\mathcal{V}_t$. Denote by $q(\cdot)$ and $\tilde{q}(\cdot)$ the trajectories with initial conditions $q(t)=q$ and $\tilde{q}(t)=\tilde{q}$ obtained from (3.5). Then,*

1. *For all $s\in[t,1]$ such that $\tilde{q}(s)\neq 0$, one has: $\left|\frac{q(s)}{|q(s)|}-\frac{\tilde{q}(s)}{|\tilde{q}(s)|}\right|\leq 2\frac{|q-\tilde{q}|}{|q(s)|}$,*
2. *For all $s\in[t,1]$, one has $|h(q(s))-h(\tilde{q}(s))|\leq 2\|H\|_\infty\frac{|q-\tilde{q}|}{|q(s)|}$.*

Proof. For any $s\in[t,1]$, such that $\tilde{q}(s)\neq 0$, since $q(s)\neq 0$ one has:

$$\begin{aligned}\left|\frac{q(s)|\tilde{q}(s)|-\tilde{q}(s)|q(s)|}{|q(s)||\tilde{q}(s)|}\right| &= \left|\frac{|\tilde{q}(s)|(q(s)-\tilde{q}(s))}{|q(s)||\tilde{q}(s)|}+\frac{\tilde{q}(s)(|\tilde{q}(s)|-|q(s)|)}{|q(s)||\tilde{q}(s)|}\right| \\ &= \left|\frac{q(s)-\tilde{q}(s)}{|q(s)|}+\frac{\tilde{q}(s)(|\tilde{q}(s)|-|q(s)|)}{|q(s)||\tilde{q}(s)|}\right|\end{aligned}$$

Hence,

$$\begin{aligned}\left|\frac{q(s)}{|q(s)|}-\frac{\tilde{q}(s)}{|\tilde{q}(s)|}\right| &\leq \frac{|q(s)-\tilde{q}(s)|}{|q(s)|}+\frac{|\tilde{q}(s)|\,\big||\tilde{q}(s)|-|q(s)|\big|}{|q(s)|\,|\tilde{q}(s)|} \\ &\leq \frac{|q(s)-\tilde{q}(s)|}{|q(s)|}+\frac{|\tilde{q}(s)-q(s)|}{|q(s)|}.\end{aligned}$$

Since the controls of the players depend only on the time variable, by the ordinary differential equation (3.5), for all $s\in[t,1]$,

$$|q(s)-\tilde{q}(s)| = |q-\tilde{q}|\,. \tag{4.1}$$

Then, it follows that,

$$\left|\frac{q(s)}{|q(s)|}-\frac{\tilde{q}(s)}{|\tilde{q}(s)|}\right|\leq 2\frac{|q-\tilde{q}|}{|q(s)|}.$$

For the second statement, if $\tilde{q}(s)\neq 0$ the result follows from statement 1 and the definition of h, and if $\tilde{q}(s)=0$ from the fact that $h(\tilde{q}(s))=0$ and $|h(q(s)|\leq\|H\|_\infty$. ∎

Since for all $t, \tilde{t} \in [0,1]$ and all $q \in \mathcal{Q}^*$, the value in the game $\mathcal{G}(t,q)$ exists, i.e., $W(t,q) := W^-(t,q) = W^+(t,q)$, we establish next result on the value W; we obtain an upper bound of the value function with respect to the variation of the state variable.

Lemma 4.2. *For all $t \in [0,1]$ and all $(q,\tilde{q}) \in \mathcal{Q}^* \times \mathcal{Q}^*$ such that $|q|, |\tilde{q}| < \frac{1}{4}$:*

1. *For all $(\tilde{u},\tilde{v}) \in \mathcal{U}_t \times \mathcal{V}_t$:*

$$|G(t,q,\tilde{u},\tilde{v}) - G(t,\tilde{q},\tilde{u},\tilde{v}| \leq 4\,\|H\|_\infty \min\left(\big|\ln(|q|)\big|, \big|\ln(|\tilde{q}|)\big|\right)|q-\tilde{q}| \tag{4.2}$$

2. *Consequently:*

$$\big|W(t,q) - W(t,\tilde{q})\big| \leq 4\,\|H\|_\infty \min\left(\big|\ln(|q|)\big|, \big|\ln(|\tilde{q}|)\big|\right)|q-\tilde{q}|, \tag{4.3}$$

Proof. Let us put $G(q) := G(t,q,\tilde{u},\tilde{v})$ and $G(\tilde{q}) := G(t,\tilde{q},\tilde{u},\tilde{v})$. Then,

$$|G(q) - G(\tilde{q})| = \left|\int_t^1 \left(h\left(q(s)\right) - h\left(\tilde{q}(s)\right)\right)ds\right|.$$

For all $s \in [t,1]$, it holds true that $|q(s)| = |q| + s - t$. By the statement 2 of Lemma 4.1,

$$\begin{aligned}
|G(q) - G(\tilde{q})| &\leq 2\,\|H\|_\infty \left|\int_t^1 \frac{|q-\tilde{q}|}{|q|+s-t}ds\right| = 2\,\|H\|_\infty\, |q-\tilde{q}|\left|\int_t^1 \frac{ds}{|q|+s-t}\right| \\
&= 2\,\|H\|_\infty \left|\Big(\ln\left(1+|q|-t\right) - \ln\left(|q|\right)\Big)\right||q-\tilde{q}| \\
&= 2\,\|H\|_\infty \left|\ln\left(\frac{1+|q|-t}{|q|}\right)\right||q-\tilde{q}|.
\end{aligned}$$

Now since $t \in [0,1]$, $|q|, |\tilde{q}| \leq \frac{1}{4}$, we claim that $\big|\ln(1+|q|-t)\big| \leq |\ln(|q|)|$. Indeed if $|q|-t \leq 0$: $\ln(|q|) \leq \ln(|q|+1-t) \leq 0$ and if $|q|-t \geq 0$, we have $1 \leq |q|+1-t \leq |q|+1 \leq \frac{5}{4} \leq 4$, so that $0 \leq \ln(|q|+1-t) \leq \ln 4 \leq -\ln(|q|)$. It follows:

$$|G(q) - G(\tilde{q})| \leq 4\,\|H\|_\infty \big|\ln(|q|)\big||q-\tilde{q}|.$$

A symmetric inequality follows by exchanging the role of q and $\tilde{q}$. Thus (4.2) is proved. Now, in value terms, we have:

$$\begin{aligned}
W(t,q) &= \sup_{\alpha\in\mathcal{A}_t}\inf_{\tilde{v}\in\mathcal{V}_t} G\big(t,q,\alpha[\tilde{v}],\tilde{v}\big) \\
&\leq \sup_{\alpha\in\mathcal{A}_t}\inf_{\tilde{v}\in\mathcal{V}_t} G\big(t,\tilde{q},\alpha[\tilde{v}],v\big) + 4\,\|H\|_\infty \big|\ln(|q|)\big||q-\tilde{q}| \\
&\leq W(t,\tilde{q}) + 4\,\|H\|_\infty \big|\ln(|q|)\big||q-\tilde{q}|.
\end{aligned}$$

Exchanging the role of q and $\tilde{q}$ on the left hand side, we have:

$$|W(t,\tilde{q}) - W(t,q)| \leq 4\,\|H\|_\infty \big|\ln(|q|)\big||q-\tilde{q}|.$$

Now exhanging the role of q and $\tilde{q}$ on the right hand side, inequality (4.3) is thus proved. ∎

Next, we extend the results on the differential game $\mathcal{G}(t,q)$ over the set $[0,1]\times\mathcal{Q}$. Namely, we show that $W(0,q)$ has a limit as q tends to 0 and we further establish that such limit is the value of the game starting at $(0,0)$, which therefore exists.

Lemma 4.3. *For any $\varepsilon>0$, there exists $\eta>0$, such that for all $t\in[0,1]$ and $(q,\tilde{q})\in\mathcal{Q}^*\times\mathcal{Q}^*$, if $|q|,|\tilde{q}|<\eta$, we have:*

$$\big|W(t,q)-W(t,\tilde{q})\big|<\varepsilon.$$

Proof. For any $\varepsilon>0$, we choose $\eta\in\big(0,\frac{1}{4}\big)$, such that for all ρ with $0<\rho<\eta$, one has $\rho\ln(\rho)<\frac{\varepsilon}{8\|H\|_\infty+1}$. Then let $t\in[0,\frac{1}{4})$, $q,\tilde{q}\in\mathcal{Q}^*$ such that $|q|,|\tilde{q}|<\eta$. If $|q|\le|\tilde{q}|$ it follows that $|\ln(|\tilde{q}|)|\cdot|\tilde{q}-q|\le|\ln(|\tilde{q}|)|\cdot(|\tilde{q}|+|q|)\le 2(\ln|\tilde{q}|)|\tilde{q}|$. Similarly if $|q|>|\tilde{q}|$, we have $|\ln(|q|)|\cdot|\tilde{q}-q|\le 2(\ln|q|)|q|$. It follows that $\min\{\big|\ln(|q|)\big|,|\ln(|\tilde{q}|)|\}|q-\tilde{q}|\le 2\max\{|\ln(|q|)|)|q|,(|\ln(|\tilde{q}|)|)|\tilde{q}|\}$. Thus, using Lemma 4.2, one has $\big|W(t,q)-W(t,\tilde{q})\big|<2\times 4\|H\|_\infty\frac{\varepsilon}{8\|H\|_\infty+1}<\varepsilon$. $\square$

Proposition 4.4. $\lim\limits_{\substack{q\to 0\\ q\in\mathcal{Q}\setminus\{0\}}} W(0,q)$ *exists.*

Proof. The result follows from Lemma 4.3 by simple application of the Cauchy principle. $\square$

Notation. In the sequel, we use the notations:

- We put $\lim\limits_{\substack{q\to\mathbf{0}\\ q\in\mathcal{Q}\setminus\{0\}}} W(0,q)=\ell$.
- $\mathcal{G}(0,0)$, the differential game starting at time $t=0$ and initial state $q=0$.
- $G_{[t,t']}(\cdot)$ the payoff that Player 1 receives over the time interval $[t,t']$.
- We denote the evaluation at time $s>t$ of the solution in (3.5) by $\mathfrak{q}[t,q,\tilde{u},\tilde{v}](s)$ in order to indicate the dependence on the initial conditions (t,q) and on the control processes $(\tilde{u}_s)$ and $(\tilde{v}_s)$.

Next theorem proves existence of the value in $\mathcal{G}(0,0)$ and we further show that $W(0,0)=\ell$. In doing so, we prove that lower and upper values are continuous functions at $(0,0)$ and their images at $(0,0)$ are both equal to ℓ. The idea of the proof lies in the consideration of $\frac{\varepsilon}{4}$-optimal strategies in the game $\mathcal{G}(0,q)$, which will be ε-optimal in the game $\mathcal{G}(0,0)$.

Theorem 4.5. *The game $\mathcal{G}(0,0)$ has a value, $W(0,0)=\ell$.*

Proof. For any $\varepsilon > 0$, we choose $\eta \in (0, \frac{1}{8})$, such that $\eta|\ln(\eta)| < \frac{\varepsilon}{128\|H\|_\infty+1}$ and such that (in view of Proposition 4.4) for all $q \in \mathcal{Q}^*$ with $|q| < \eta$ one has $|W(0,q) - \ell| < \frac{\varepsilon}{4}$.
Choose $q \in \mathcal{Q}^*$ such that $|q| < \frac{\eta}{2}$ and let $\alpha_q \in \mathcal{A}_0$ be an $\frac{\varepsilon}{4}$-optimal strategy for Player 1 in $\mathcal{G}(0,q)$. Then, for all $\tilde{v} \in \mathcal{V}_0$, Player 1 guarantees,

$$G(0, q, \alpha_q, \tilde{v}) \geq W(0,q) - \frac{\varepsilon}{4}.$$

We put $q(s) = \mathfrak{q}[0, q, \alpha_q, \tilde{v}](s)$ and $q_0(s) = \mathfrak{q}[0, 0, \alpha_q, \tilde{v}](s)$. We further define $\tilde{q} := q(\eta)$ and $\hat{q} := q_0(\eta)$. It follows that $|\tilde{q}| = |q| + \eta$, $|\hat{q}| = \eta$ and $|\tilde{q} - \hat{q}| = |q|$.

$$\begin{aligned}
\left|G_{[0,\eta]}(0, q, \alpha_q, \tilde{v}) - G_{[0,\eta]}(0, 0, \alpha_q, \tilde{v})\right| &\leq \left|\left\langle H, \int_0^\eta \left(\frac{q(s)}{|q(s)|} - \frac{q_0(s)}{|q_0(s)|}\right) ds \right\rangle\right| \\
&\leq \|H\|_\infty \int_0^\eta \left(\left|\frac{q(s)}{|q(s)|}\right| + \left|\frac{q_0(s)}{|q_0(s)|}\right|\right) ds \\
&\leq 2\eta \|H\|_\infty \\
&< \frac{\varepsilon}{8}.
\end{aligned}$$

In view of Lemma 4.2, inequality (4.2) implies:

$$\begin{aligned}
\left|G_{[\eta,1]}(\eta, \tilde{q}, \alpha_q, \tilde{v}) - G_{[\eta,1]}(\eta, \hat{q}, \alpha_q, \tilde{v})\right| &\leq 4\|H\|_\infty \min\left(\left|\ln(|\hat{q}|)\right|, |\ln(|\tilde{q}|)|\right)|\hat{q} - \tilde{q}| \\
&\leq 8\|H\|_\infty \max\{|\ln(|\hat{q}|)|)|\hat{q}|, (|\ln(|\tilde{q}|)|)|\tilde{q}|\} \\
&\leq 8\|H\|_\infty (2\eta)\left|\ln\left(2\eta\right)\right| \\
&\leq 8\|H\|_\infty (4\eta|\ln(\eta)|) \\
&< \frac{\varepsilon}{4}
\end{aligned}$$

where the second inequality is obtained like in the proof of Lemma 4.3 since $|\tilde{q}|, |\hat{q}| < 2\eta < \frac{1}{4}$. Hence, we have:

$$\left|G(0, q, \alpha_q, \tilde{v}) - G(0, 0, \alpha_q, \tilde{v})\right| < \frac{\varepsilon}{2},$$

and so, in the game $\mathcal{G}(0,0)$ α_q guarantees for Player 1:

$$G(0, 0, \alpha_q, \tilde{v}) \geq G(0, q, \alpha_q, \tilde{v}) - \frac{\varepsilon}{2} \geq W(0,q) - \frac{\varepsilon}{4} - \frac{\varepsilon}{2}.$$

Therefore:

$$\begin{aligned}
G(0, 0, \alpha_q, \tilde{v}) - \ell &\geq W(0,q) - \ell - \frac{3\varepsilon}{4} \\
&> -\frac{\varepsilon}{4} - \frac{3\varepsilon}{4} \\
&> -\varepsilon.
\end{aligned}$$

Following similar arguments, we can prove that for all $q \in \mathcal{Q}^*$ Player 2 chooses an $\frac{\varepsilon}{4}$-optimal strategy in $\mathcal{G}(0,q)$, denoted by $\beta_q \in \mathcal{B}_0$, such that in $\mathcal{G}(0,0)$, for all $\tilde{u} \in \mathcal{U}_0$,

$$G(0,0,\tilde{u},\beta_q) < \ell + \varepsilon.$$

that concludes the proof. $\square$

5 Existence of the limit value in $\Gamma_N(z)$

In this section, we provide the main result of the paper. We prove that $\lim_{N\to+\infty} \mathbf{V}_N(z)$ exists and it is independent of z. We first prove some useful lemmas on the value of the original game and the associated function Ψ_N.

Lemma 5.1. *If for some $z_0 \in \mathcal{Z}$, there exists $\ell \in \mathbb{R}$, such that $\lim\limits_{N\to+\infty} \boldsymbol{V}_N(z_0) = \ell$, then for all $z \in \mathcal{Z}$, $\lim\limits_{N\to+\infty} \boldsymbol{V}_N(z) = \ell$.*

Proof. Given $N \in \mathbb{N}^*$, fix $z \in \mathcal{Z}^*$ and let us consider the games $\Gamma_N(z)$ and $\Gamma_N(0)$. For a pair of behavioral strategies (σ,τ) we denote by $\mathbb{P}_{\sigma,\tau}$ the probability induced on $(I \times J)^N$. Let $((i_t,j_t), t \geq 1)$ be the process of actions and $\gamma_N(z) := \gamma_N(z, i_1, j_1, \cdots, i_N, j_N)$ the payoff generated by the actions and initial state z. Let $((z_t), t \geq 1)$ be the process in $\mathcal{Z}$, induced by the initial condition $z_0 = 0$. It follows:

$$\begin{aligned}
\big|\gamma_N(0) - \gamma_N(z)\big| &= \frac{1}{N}\Big|\sum_{t=1}^{N} h(z_t) - h(z+z_t)\Big| = \frac{1}{N}\Big|\Big\langle H, \frac{z}{|z|}\Big\rangle + \sum_{t=2}^{N}\Big\langle H, \frac{z_t}{t} - \frac{z+z_t}{|z|+t}\Big\rangle\Big| \\
&= \frac{1}{N}\Big|\Big\langle H, \frac{z}{|z|}\Big\rangle + \sum_{t=2}^{N}\Big\langle H, \frac{z_t(|z|+t) - t(z+z_t)}{t(|z|+t)}\Big\rangle\Big| \\
&= \frac{1}{N}\Big|\Big\langle H, \frac{z}{|z|}\Big\rangle + \sum_{t=2}^{N}\Big\langle H, \frac{z_t\,|z| - tz}{t(|z|+t)}\Big\rangle\Big| \\
&\leq \frac{\|H\|_\infty}{N}\Big(1 + \sum_{t=2}^{N} \frac{|z_t|\,|z| + t\,|z|}{t(|z|+t)}\Big) \\
&\leq \frac{\|H\|_\infty}{N}\Big(1 + 2|z|\sum_{t=2}^{N} \frac{1}{|z|+t}\Big) \\
&\leq \frac{\|H\|_\infty}{N}\Big(1 + 2\,|z|\,\big(\ln(|z|+N) + \gamma + \varepsilon_N\Big).
\end{aligned}$$

where $\varepsilon_N \to 0$ as $N \to \infty$ and γ is the Euler constant. Then, passing to expectations with respect to the probability $\mathbb{P}_{\sigma,\tau}$, it follows that:

$$\big|\gamma_N(0,\sigma,\tau) - \gamma_N(z,\sigma,\tau)\big| \leq \frac{\|H\|_\infty}{N}\Big(1 + 2\,|z|\,\big(\ln(|z|+N) + \gamma + \varepsilon_N\Big).$$

Since the right hand term is independent of (σ, τ), by an easy argument:

$$\left|\mathbf{V}_N(z) - \mathbf{V}_N(0)\right| \leq \frac{\|H\|_\infty}{N}\Big(1 + 2\,|z|\,\big(\ln(|z| + N\big) + \gamma + \varepsilon_N\Big).$$

The conclusion follows by remarking that the right hand side goes to zero when N goes to infinity. ☐

Lemma 5.2. *For any $\varepsilon > 0$ and any $(q, \tilde{q}) \in \mathcal{Q}^* \times \mathcal{Q}$, there exists $\tilde{N} \in \mathbb{N}^*$, such that for all $N \geq \tilde{N}$,*

$$\left|\Psi_N(0, q) - \Psi_N(0, \tilde{q})\right| \leq 2\,\|H\|_\infty \ln\left(\frac{|q| + 1}{|q|}\right)|q - \tilde{q}| + \varepsilon$$

Proof. Given $N \in \mathbb{N}$ and $q, \tilde{q} \in \mathcal{Q}^*$, we prove by induction that for all $n \leq N$:

$$\left|\Psi_N\left(1 - \frac{n}{N}, q\right) - \Psi_N\left(1 - \frac{n}{N}, \tilde{q}\right)\right| \leq 2\,\|H\|_\infty\,|q - \tilde{q}| \sum_{k=0}^{n-1} \frac{1}{N|q| + k}. \tag{5.1}$$

- For $n = 0$, the result is trivial since $\Psi(1, q) = 0$ for all $q \in \mathcal{Q}$.
- Assume the result is true for all $n \leq N$, i.e.,

$$\left|\Psi\left(1 - \frac{n}{N}, q\right) - \Psi\left(1 - \frac{n}{N}, \tilde{q}\right)\right| \leq 2\,\|H\|_\infty\,|q - \tilde{q}| \sum_{k=0}^{n-1} \frac{1}{N|q| + k}.$$

- We prove the result for $n + 1$, for all $n \leq N - 1$.

$$\begin{aligned}\left|\Psi\left(1 - \frac{n+1}{N}, q\right) - \Psi\left(1 - \frac{n+1}{N}, \tilde{q}\right)\right| &\leq \frac{1}{N}\left|\left\langle H, \frac{q}{|q|} - \frac{\tilde{q}}{\tilde{q}}\right\rangle\right| \\ &\quad + \left|\mathbf{val}\Big(\Psi_N\left(1 - \frac{n}{N}, q + \frac{e_{ij}}{N}\right)\Big) - \mathbf{val}\Big(\Psi_N\left(1 - \frac{n}{N}, \tilde{q} + \frac{e_{ij}}{N}\right)\Big)\right|.\end{aligned}$$

By the Lipschitz property of the **val** operator, we get:

$$\begin{aligned}\left|\Psi\left(1 - \frac{n+1}{N}, q\right) - \Psi\left(1 - \frac{n+1}{N}, \tilde{q}\right)\right| &\leq \frac{1}{N}\left|\left\langle H, \frac{q}{|q|} - \frac{\tilde{q}}{\tilde{q}}\right\rangle\right| \\ &\quad + \sup_{i,j}\left|\Big(\Psi_N\left(1 - \frac{n}{N}, q + \frac{e_{ij}}{N}\right)\Big) - \Big(\Psi_N\left(1 - \frac{n}{N}, \tilde{q} + \frac{e_{ij}}{N}\right)\Big)\right|.\end{aligned}$$

By Lemma 4.1 and the induction hypothesis, it follows that:

$$\left|\Psi\left(1 - \frac{n+1}{N}, q\right) - \Psi\left(1 - \frac{n+1}{N}, \tilde{q}\right)\right| \leq \frac{2\,\|H\|_\infty}{N|q|}|q - \tilde{q}| + 2\,\|H\|_\infty\,|q - \tilde{q}| \sum_{k=0}^{n-1} \frac{1}{N|q + \frac{1}{N}| + k}.$$

Therefore,

$$\begin{aligned}\left|\Psi\left(1-\frac{n+1}{N},q\right)-\Psi\left(1-\frac{n+1}{N},\tilde{q}\right)\right| &\leq \frac{2\left\|H\right\|_\infty}{N|q|}|q-\tilde{q}|+2\left\|H\right\|_\infty|q-\tilde{q}|\sum_{k=1}^{n}\frac{1}{N|q|+k}\\ &\leq 2\left\|H\right\|_\infty|q-\tilde{q}|\left(\frac{1}{N|q|}+\sum_{k=1}^{n}\frac{1}{N|q|+k}\right)\\ &\leq 2\left\|H\right\|_\infty|q-\tilde{q}|\sum_{k=0}^{n}\frac{1}{N|q|+k},\end{aligned}$$

which concludes the proof of the induction; it is a simple exercise to show: $\sum_{k=0}^{n-1}\frac{1}{N|q|+k}\to \ln\left(\frac{1+|q|}{|q|}\right)$ when $N\to\infty$.

□

Theorem 5.3. *For any $z\in\mathcal{Z}$, $\lim_{N\to+\infty}\boldsymbol{V}_N(z)=W(0,0)$.*

Proof. For any $\varepsilon>0$, we choose $\eta\in(0,\frac{1}{4})$ such that $\eta|\ln(\eta)|<\frac{\varepsilon}{36\|H\|_\infty+1}$ and in view of Proposition 4.4 and Theorem 4.5, we also require η to be such that for all $q\in\mathcal{Q}^*$ with $|q|\leq\eta$,

$$\big|W(0,q)-W(0,0)\big|<\frac{\varepsilon}{3}.$$

Choose some q_0 such that $|q_0|=\eta$. Assumptions on the strategy sets, the dynamics and running payoff functions of Theorem 4.4 in Souganidis [1999] are established in $\mathcal{G}\big(0,q_0\big)$. Accordingly, there exists $\delta>0$, such that for all $|\mathcal{P}|<\delta$, the value $W^-_{\mathcal{P}}$ converges uniformly on every compact set of $\mathcal{Q}$ to W, as the mesh of the discretization $|\mathcal{P}|$ tends to 0. Fix $N_0=\lfloor\frac{1}{\delta}\rfloor+1$ and associate to $\mathcal{G}\big(0,q_0\big)$, for all $N\geq N_0$, a discrete time game adapted to the subdivision $\mathcal{P}_N$, denoted by $\mathcal{G}_{\mathcal{P}_N}\big(0,q_0\big)$. Then,

$$\left|W^-_{\mathcal{P}_N}(0,q_0)-W(0,q_0)\right|<\frac{\varepsilon}{3}.$$

From Proposition 3.2, $W_{\mathcal{P}_N}(0,q_0)=\Psi_N(0,q_0)$ and by Lemma 5.2, for any $\varepsilon>0$ there exists $N_1\in\mathbb{N}^*$, such that for every integer $N\geq N_1$,

$$\big|\Psi_N(0,q_0)-\Psi_N(0,0)\big|\leq 2\left\|H\right\|_\infty\eta\ln\left(\frac{\eta+1}{\eta}\right)+\frac{2\varepsilon}{9}.$$

Since $\eta<\frac{1}{4}$, we get $\big|\ln(\eta+1)\big|<\big|\ln(\eta)\big|$ and thus,

$$\begin{aligned}\big|\Psi_N(0,q_0)-\Psi_N(0,0)\big| &\leq 4\left\|H\right\|_\infty\eta\big|\ln(\eta)\big|+\frac{2\varepsilon}{9}\\ &<\frac{\varepsilon}{9}+\frac{2\varepsilon}{9}\\ &<\frac{\varepsilon}{3}.\end{aligned}$$

Therefore, there exists $\tilde{N} := \max\{N_0, N_1\}$, such that for every integer $N \geq \tilde{N}$,

$$\begin{aligned}\left|\Psi_N(0,0) - W(0,0)\right| &\leq \left|\Psi_N(0,0) - \Psi_N(0,q_0)\right| + \left|\Psi_N(0,q_0) - W(0,q_0)\right| + \left|W(0,q_0) - W(0,0)\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &< \varepsilon.\end{aligned}$$

From (3.1) $\Psi_N(0,0) = \mathbf{V}_N(0)$. It follows that $V_N(0) \to 0$ when $N \to \infty$. In view of Lemma 5.1 we conclude that for any $z \in \mathcal{Z}$, $\mathbf{V}_N(z)$ converges to $W(0,0)$ as N tends to infinity. $\square$

6 Conclusion and perspectives

In this paper, we study two-player zero-sum FD games and establish convergence of $\mathbf{V}_n$, as n tends to infinity, to the value of the associated differential game starting at the origin, $W(0,0)$. A first extension of this result concerns the λ-discounted game and precisely the convergence of $\mathbf{V}_\lambda$, as λ tends to 0. Ziliotto [2015] ensures equivalence between uniform convergence with respect to the state variable of $\mathbf{V}_n$, as n tends to infinity and uniform convergence of $\mathbf{V}_\lambda$, as λ goes to 0. However we know from Contou-Carrère [2011] that even for a one-player FD game (a particular case of our model) convergence of $\mathbf{V}_n$ is not uniform in the state, and therefore we cannot use the above result.

A further interesting generalization of the existence result is to consider a stage payoff function $g(q,i,j)$ where q stands for the average of past actions and (i,j) for current actions and that is assumed to be linear in q. In such a model the impact of the past and that of present actions are not additive, but they combine together in some way.

Lastly, let us mention that since existence of the asymptotic value in the zero-sum case is established, a study of limits of Nash equilibria payoffs in non zero-sum FD games that leads to some Folk-like theorem now seems to be possible.

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