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# Extracting Information or Resource? The Hotelling Rule Revisited under Asymmetric Information\*

DAVID MARTIMORT<sup>†</sup> JÉRÔME POUYET<sup>‡</sup> FRANCESCO RICCI<sup>§</sup>

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**ABSTRACT.** We characterize the optimal extraction path when a concessionaire has private information on the initial stock of resource. Under asymmetric information, a ‘virtual Hotelling rule’ describes how the resource price evolves over time and how extraction costs are compounded with information costs along an optimal extraction path. In sharp contrast with the case of complete information, fields which are heterogeneous in terms of their initial stocks follow different extraction paths. Some resource might be left unexploited in the long-run as a way to foster incentives. The optimal contract may sometimes be implemented through royalties and license fees. With a market of concessionaires, asymmetric information leads to a ‘virtual Herfindahl principle’ and to a new form of heterogeneity across active concessionaires. Under asymmetric information, the market price converges faster to its long-run limit, exhibiting more stability.

**KEYWORDS.** Non-Renewable resource, Delegated Management, Optimal Contract, Asymmetric Information.

**JEL CODES.** Q38, D86.

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## 1. INTRODUCTION

MOTIVATION. Resource extraction heavily relies on the division between ownership and control. The oil and gas sectors are two prime examples. There, various kinds of contracts rule the relationships between public authorities which own the land where production takes place and specialized firms in charge of resource management.<sup>1</sup> Actually, these firms are most often international ventures operating on a large scale. The first consequence of such specialization is that these firms certainly have the capability to mobilize specific know-how, to finance risky exploration, or to build all infrastructures needed for production and transportation. A second but more perverse consequence of such specialization is that those firms enjoy also a comparative advantage vis-à-vis public authorities when it comes to assess the amount of reserves in a given geographical area.<sup>2</sup> In this paper, we thus ask how public authorities and concessionaires should design contracts for the management of resource in contexts plagued with asymmetric information.

At a broad level, the delegated management of resource should share some of the general features found with other forms of procurement. The existing literature has forcefully stressed that the design of optimal procurement contracts under informational constraints results from a trade-off between efficiency and rent extraction.<sup>3</sup> Operating at the efficient scale might also require leaving excessive and socially costly information rents to the concessionaire. Reducing these rents calls instead for a lower output, lower-powered incentives and, more generally, procurement policies that are less sensitive to information.

Yet, and although our analysis below confirms that these insights still have some bite in the context of resource management, resource extraction is inherently a dynamic phenomenon. This dynamic perspective is probably best illustrated by the importance taken by the Hotelling rule in resource economics. The Hotelling rule shows how the resource price should evolve along an efficient path of extraction. Although much has been said on its empirical validity,<sup>4</sup> this rule provides an important benchmark to assess how the trade-off between efficiency and the extraction of information rent must be revisited in a dynamic model of resource extraction. As our analysis shall demonstrate, such dynamics requires a specific analysis, which unveils new and important effects.<sup>5</sup>

VIRTUAL COST OF EXTRACTION AND VIRTUAL HOTELLING RULE. Before unveiling the new insights brought by asymmetric information, it is useful to briefly remind the

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<sup>1</sup>To illustrate, three different sorts of contracts are most commonly observed in the oil sector. ‘Concessions contracts’ are such that the firm owns oil in the field. For ‘production-sharing agreements’, the firm owns only part of the produced oil. Finally, ‘service contracts’ are such that the firm receives a financial compensation for production. Historically, concession contracts prevailed at the inception of the industry in the U.S. since, there, ownership of an estate includes subsurface mineral rights. They became used throughout the world afterwards. Following the wave of nationalizations in the 1970s, the dominant types of contracts are now production-sharing agreements when the host country contracts with a firm for exploration and management of production. Similar patterns are observed in the gas sector.

<sup>2</sup>Osmundsen (2010, p. 431) provides a detailed account of the various ways in which a firm in charge of the assessment of the stock and the management of the resource strategically uses such information.

<sup>3</sup>See Laffont and Tirole (1993) and Armstrong and Sappington (2007) for accounts of the New Regulatory Economics.

<sup>4</sup>See Gaudet (2007) for a critical overview.

<sup>5</sup>These specificities of models of resource extraction from the perspective of Incentives Theory has also been stressed by Gaudet and Lasserre (2015).

mechanism underlying the Hotelling rule under complete information. Suppose thus that the stock of resource is common knowledge and that, because of depletion, the marginal cost of extraction increases over time. The Hotelling rule stipulates that the resource price must reflect the scarcity rent. Efficiency requires that, at any point in time, the marginal value of the last unit extracted covers not only the current marginal cost (function of current reserves) but also the shadow cost of increasing the extraction cost from that date on by depleting reserves. Those reserves decrease up to the point where the marginal cost of extraction is equal to the choke price and extraction ceases at that point. At the same time, the resource price continuously increases over time to reflect the dynamic arbitrage between consuming resource today or letting more resource in the ground so as to facilitate extraction tomorrow. Under complete information, the public authority can simply capture all the concessionaire's intertemporal profit along this optimal path by imposing an entry fee whose value is perfectly known.<sup>6</sup>

The picture is strikingly different when the firm has private information on its initial stock. Contracts must now induce firms with different stocks to select different extraction paths as a means to reveal information. A firm with a large initial stock and thus a low cost of extraction may just adopt the same extraction pattern as one with lower reserves and a higher extraction cost. By doing so, the high-stock firm produces the same quantity as the low-stock one at any point in time. It also pays a lower license fee and pockets the corresponding cost saving over the whole extraction path. Slowing down extraction is thus a way to capture part of the information rent left to firms with large initial stocks. Underestimating stocks is akin to keeping resource in the ground.

An optimal contract must render such strategy less attractive so as to induce information revelation at the inception of the relationship. This is achieved by the public authority through the commitment to reduce extraction from firms which claim lower levels of reserve to start with. By reducing the production of these firms, the public authority makes less valuable for firms endowed with higher initial stocks to strategically behave as having lower reserves. In other words, everything happens as if the marginal cost of extraction is replaced by a greater 'virtual cost of extraction' that accounts for the cost of extracting information.<sup>7</sup> Dynamic inefficiencies in resource extraction arise when information has also to be extracted. This points to an important dilemma in designing concession contracts: The cost of inducing information revelation is to leave some resource in the ground, stopping extraction before what efficiency would command.

We characterize the extraction path under asymmetric information by means of a 'virtual Hotelling rule' that governs how the resource price optimally evolves over time. This virtual Hotelling rule has again a simple interpretation. The last unit extracted from a given field must be such the marginal benefit of consumption covers not only the current cost of extraction and the shadow cost of increasing future extraction as under complete information but also the cost of the information rents captured by firms with supra-marginal reserves.

COMPARATIVE STATICS. Altogether, scarcity and information rents shape the dynamics of resource extraction. Although under complete information all firms, whatever their initial stock, evolve along the same extraction path and extract up to a point where the

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<sup>6</sup>An alternative would be to hold an auction for the franchise among potential concessionaires.

<sup>7</sup>To use the parlance of Myerson in numerous contributions.

marginal cost of extraction equals the choke price, asymmetric information introduces heterogeneity across trajectories. A firm with a smaller (larger) stock extracts less (more) and leaves thus more (less) resource in the ground in the long-run. The limit level of unexploited stock is now obtained when the virtual marginal cost of extraction is equal to the choke price. That limit varies negatively with the initial stock.

Interestingly, we provide closed-form expressions for the optimal paths and show that asymmetric information does not necessarily slow down resource extraction, at least in the case of a single concessionaire. In the case of linear demand and extraction cost, both the level of resource and the quantity extracted converge towards their long-run limits at the same exponential rates as under complete information. Yet, the overall amount extracted is always lower under asymmetric information.

**IMPLEMENTATION.** We then ask whether the optimal contract can be implemented with simple instruments that might echo real-world practices. Still in the case of linear demand and extraction cost, we first demonstrate that the choice of a dynamic extraction path can be reduced to the choice of the quantity extracted at the start. The whole analysis of a dynamic extraction problem then boils down to a static problem. We then show that a simple nonlinear payment links the firm's compensation to that initial quantity extracted. Under a weak technical condition that guarantees the convexity of this schedule, the optimal contract can be implemented by a menu of linear schemes that specify royalties per unit of output and license fees. Firms with greater stock choose more attractive royalties but also pay higher fees.

**MARKET OF CONCESSIONAIRES.** Finally, we consider the case where the whole sector is run with concession contracts. Under complete information, only firms endowed with larger stocks, and therefore characterized by lower marginal cost of extraction, are initially active. As their remaining stocks decrease, their marginal costs and the resource price rise. Firms with less initial reserves and higher initial marginal cost thus enter the market, according to the Herfindahl Principle.<sup>8</sup> At any given point in time, all active firms keep the same remaining stock, have the same marginal cost and produce the same quantity.

Under asymmetric information, all firms active at a given date produce instead at the same virtual marginal cost of extraction. Since virtual costs depend on initial stocks, all those firms produce different amounts and keep different stocks. Although this heterogeneity contrasts with the case of complete information, a 'virtual Herfindahl principle' still applies. At the optimal path, extraction starts with fields at lower virtual marginal costs of extraction, and then moves to those with higher virtual costs.

Because concessionaires with small stocks might never be active, the price remains quite high along the optimal extraction path and convergence towards its long-run limit is faster than under complete information. With asymmetric information, prices are thus more stable around their long-run equilibrium value.

**LITERATURE REVIEW.** Our paper belongs to a burgeoning literature on resource extraction and incentive contracts. Poudou and Thomas (2000) and Hung, Poudou and Thomas (2006) analyze the design of mining concession contracts in a finite multi-period setting and asymmetric information about a persistent shock on cost. Reducing the firm's

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<sup>8</sup>Herfindahl (1967).

information rent requires downward distortions of production early in the relationship. Yet, with a finite number of periods, this also means that more resource can be extracted in a final phase of extraction. Focusing on a two-period model, Gaudet, Lasserre and Van Long (1995) study optimal royalty contracts for a non-renewable resource under asymmetric cost information with costs independently distributed over time. A common finding of these papers is that asymmetric information shifts production to the future. Castonguay and Lasserre (2016) consider a two-period model with exploration followed by exploitation. Asymmetric information induces inefficiencies in both activities.<sup>9</sup>

We differ from these papers along several lines. First, our model has continuous time and an infinite horizon. All effects that follow from the existence of a terminal date of extraction thus disappear. Although output is reduced for rent extraction reasons, the consequence of such distortions with an infinite horizon are captured through both the long-run value of stocks and the intertemporal profile of extraction. Second, existing models suppose that the cost of extraction only depends on the current production and not on the remaining stock; an assumption that has been often questioned in the literature (Heal, 1976; Solow and Wan, 1976; Pindyck, 1978, 1987; Swierzbinski and Mendelsohn, 1989).<sup>10</sup> Modeling a cost of extraction that depends on depletion allows to see how asymmetric information interacts with the dynamics of extraction. Third, we assume that the cost technology is common knowledge but the value of the initial stock is private information to the firm. This assumption is consistent with an earlier literature in resource economics that has stressed the significant uncertainty on the value of stocks at the inception of an exploitation phase.<sup>11</sup> While assuming private information on the cost of extraction is akin to replace this cost by its virtual value, which is greater, this transformation has only an indirect impact on the dynamics of extraction. To illustrate, assuming heterogeneity on the cost of extraction implies that the complete information Hotelling rules are already and de facto different across firms. Instead assuming, as we do below, that private information is on initial stocks deeply links private information to such dynamics. Although trajectories of the economy remain the same under complete information and the price dynamics follows the same Hotelling rule despite starting from different initial stocks, these trajectories now differ according to the value of the firm's virtual cost of extraction under asymmetric information.

Making similar informational assumptions to ours, Osmudsen (1998) shows that the optimal contract distorts both the extent and the pace of depletion in a two-period (and more generally with a finite number of periods) framework. Having an infinite horizon allows us to get a significantly different set of results. To illustrate, we provide an example (with linear inverse demand function and marginal cost of extraction) which admits closed-form solutions for the optimal trajectory. This example shows that the

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<sup>9</sup>In the context of carbon sequestration in agricultural fields, Chiroleu-Assouline and Roussel (2014) also consider optimal contracts for the management of non-renewable stocks under asymmetric information on the initial size of the stock.

<sup>10</sup>Once the pool of hydrocarbons in a field is precisely identified, the unit operating cost is typically decreasing in the size of the reserve. At the field level, this is due to the fact that the firm starts digging wells where oil is relatively abundant, and, then, moves on to areas within the field where oil is more difficult to extract and less abundant. At the regional level, where several exploration licenses can be handed, the same negative correlation between remaining reserves and unit operating cost may hold. In fact, the first applications for exploration licenses concern the most promising blocks, so that larger pools tend to be discovered first.

<sup>11</sup>See Hoel (1978) and Pindyck (1980) among others.



pace of depletion may remain the same as under complete information, at least in the case of a single operator, although the quantities extracted obviously differ. This, again, shows that the lessons of finite-time models should be taken with a word of caution when asymmetric information is introduced. Following on this author's insights, we also study how royalty payments can be used to implement the optimal allocation. At this stage, we take advantage of the specific trajectory to reduce the infinite horizon dynamic implementation problem to a static one. In this static reduced-form problem, the public authority offers a menu of license contracts. The firm chooses within this menu a particular royalty that applies from date 0 on and pays upfront a license fee. This choice defines the whole trajectory of the economy with outputs and revenues following an exponential decay. Our analysis differs from Osmundsen (1998) because we qualify conditions on technologies under which such implementation through linear contracts is actually feasible: It is only so when the optimal nonlinear payment schedule is convex.<sup>12</sup>

Taking a broader perspective, this paper belongs also to a very active literature in dynamic mechanism design.<sup>13</sup> Our assumption that private information (here the initial stock of resource) is persistent throughout the whole relationship is reminiscent of the work of Baron and Besanko (1984) in a discrete-time dynamic regulation model, although both the dynamics of resource extraction and the locus of informational asymmetries sharply differ from the stationary environment analyzed by these authors. The dynamics of resource extraction and the fact that the marginal cost of extraction depends on the remaining stock introduce an important linkage between production technologies at different points in time. A technical insight of our analysis is that implementability conditions, that are known to be quite challenging in general dynamic mechanism design environments,<sup>14</sup> can be reduced to checking a simple monotonicity condition on the aggregate amount extracted. The dynamic linkage across periods is also reminiscent of other models like Gärtner (2010), Auray, Mariotti and Moizeau (2011) and Lewis and Yildirim (2002). These papers present discrete-time models that analyze how learning-by-doing or quality maintenance affect future cost structures.

**ORGANIZATION OF THE PAPER.** Section 2 describes the model. Section 3 analyzes the optimal extraction of resource under complete information. Section 4 is the core of the analysis. We first derive incentive-compatible allocations and then find the second-best trajectory of the economy under informational constraints. We highlight the role of the virtual cost of extraction as a driver of the dynamics. We also bear a particular attention to the shape of the concession contracts that implement this optimal allocation, and argue that simple licensing contracts with fixed-fee and royalty may implement the optimal allocation. Section 5 shows how our modeling can be adapted to address the behavior of a market with heterogeneous firms, where multiple bilateral concession contracts are signed with each of those. Section 6 builds on our analysis to discuss the recent modernization of the Alberta Royalty Framework. All proofs are relegated to an Appendix.

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<sup>12</sup>In a regulatory environment with both adverse selection and moral hazard elements, Laffont and Tirole (1993, Chapter 1) demonstrate that the optimal cost-reimbursement rule is necessarily convex and can be implemented with a simple menu including fixed-price and cost-plus contracts. In our context, extraction costs remain non-observable.

<sup>13</sup>See Bergemann and Pavan (2015) for an authoritative and up-to-date survey.

<sup>14</sup>See Pavan, Segal and Toikka (2014).



## 2. THE MODEL

RESOURCE DYNAMICS.  $S(\theta, t)$  is the stock of resource at a given date  $t$ , where  $\theta$  is the initial stock of resource at date  $t = 0$

$$(2.1) \quad S(\theta, 0) = \theta.$$

The resource stock evolves according to the following state equation

$$(2.2) \quad \frac{\partial S}{\partial t}(\theta, t) = -q(\theta, t),$$

where  $q(\theta, t)$  denotes the non-negative amount of resource extracted at date  $t$  starting from an initial stock  $\theta$ .

PRODUCTION TECHNOLOGY AND PREFERENCES. We assume that extracting resource is all the more difficult that the remaining stock is low.<sup>15</sup> Formally, the cost of extracting  $q$  units is  $C(S)q$ , where  $S$  is the remaining stock. The marginal cost of extraction  $C(S)$  is decreasing and convex in the stock level ( $C'(\cdot) < 0 \leq C''(\cdot)$ ). For technical reasons, we also assume that  $C(0) < +\infty$  and  $C'''(\cdot) \leq 0$ .

The gross surplus from consuming  $q$  units of resource is denoted by  $V(q)$ , with  $V(\cdot)$  increasing and strictly concave ( $V'(\cdot) > 0 > V''(\cdot)$ ). Let  $P(q) = V'(q)$  denote the inverse demand and assume that the choke price  $P(0)$  is finite. Let  $S_{inf}$  be the stock of resource such that the marginal cost of extraction at that level equals the choke price

$$(2.3) \quad P(0) = C(S_{inf}).$$

Extraction should thus stop when the remaining stock reaches that level.

ASYMMETRIC INFORMATION. The initial stock of resource  $\theta$  is private information to the concessionaire. This parameter is drawn from the set  $\Theta = [\underline{\theta}, \bar{\theta}]$  according to the cumulative (atomless) distribution  $F(\cdot)$  with density  $f(\cdot) = F'(\cdot)$ . Following the screening literature, we assume that the monotone hazard rate property holds:<sup>16</sup>  $\frac{d}{d\theta} \left( \frac{1-F(\theta)}{f(\theta)} \right) \leq 0$  for all  $\theta \in \Theta$ . Finally, we assume that  $\underline{\theta} \geq S_{inf}$  so that the concessionaire always starts with enough resource to find it valuable to begin extraction since  $C(\theta) \leq C(S_{inf}) = P(0)$ .

That the concessionaire has better information on the stock of resource is in line with casual evidence that firms have more precise signals on the value of reserves than outsiders (public authorities, financiers, local communities).<sup>17</sup> Although in practice, operators may learn more about the exact reserves as they further exploit, we take the shortcut that the learning process is lumpy.

RUNNING EXAMPLE. We sometimes provide closed-form solutions for some of the results below by assuming linear inverse demand and marginal cost, i.e.,  $P(q) = P(0) + qP'(0)$ ,  $C(S) = C(S_{inf}) + (S - S_{inf})C'(S_{inf})$ .<sup>18</sup>

<sup>15</sup>See Heal (1976), Solow and Wan (1976), Pindyck (1978, 1987) and Swierzbinski and Mendelsohn (1989).

<sup>16</sup>See Bagnoli and Bergstrom (2005).

<sup>17</sup>Much scrutiny is given in practice on the requirement that firms disclose operating conditions.

<sup>18</sup>We also assume that  $\bar{\theta} \leq S_{inf} - \frac{C(S_{inf})}{C'(S_{inf})}$  to ensure that the marginal cost of extraction remains

### 3. THE DYNAMICS OF RESOURCE EXTRACTION: COMPLETE INFORMATION

We now focus on a benchmark where the initial stock of resource is common knowledge. Although the dynamics of extraction in this setting is well-known, it is useful to remind its main features to anchor comparisons with the case of asymmetric information.

#### 3.1. Pareto-Optimal Trajectory

Let  $r$  be the discount factor, common to the public authority and the concessionaire.<sup>19</sup> The optimal path of resource extraction maximizes the intertemporal discounted surplus subject to resource dynamics

$$(\mathcal{P}^*(\theta)) : \max_{q, S} \int_0^{+\infty} (V(q(\theta, t)) - C(S(\theta, t))q(\theta, t)) e^{-rt} dt$$

subject to (2.1) and (2.2).

Denote by  $q^*(\theta, t)$  and  $S^*(\theta, t)$  the quantity extracted and the remaining stock of resource at time  $t$  that solve problem  $(\mathcal{P}^*(\theta))$ . The next proposition presents properties of the optimal dynamics of extraction.

**PROPOSITION 1.** *Under complete information, the optimal path of resource extraction converges towards no extraction and a finite stock of resource in infinite time*

$$(3.1) \quad \lim_{t \rightarrow +\infty} q^*(\theta, t) = 0 \text{ and } \lim_{t \rightarrow +\infty} S^*(\theta, t) = S_{inf}.$$

The price of resource  $p^*(\theta, t) = V'(q^*(\theta, t))$  evolves according to the Hotelling rule

$$(3.2) \quad \frac{\partial p^*}{\partial t}(\theta, t) = r(p^*(\theta, t) - C(S^*(\theta, t))).$$

**IDENTICAL EXTRACTION PATH FOR HETEROGENEOUS FIELDS.** All fields evolve along the same path despite some prior heterogeneity in initial stocks. Indeed, after some extraction phase decreasing the initial stock  $\theta$  to a lower value  $\theta'$ , the continuation path is the same had the stock started from this lower value  $\theta'$ . Since firms have the same extraction technology and differ only in their initial stocks, this feature just follows from Bellman Principle. In the long-run, the initial stock of resource has an impact neither on the rate of extraction nor on the final level of the stock. Whatever the starting point, the resource is extracted following the same path and up to the point where the marginal cost of extraction is equal to the choke price, which occurs in infinite time in our framework.

**HOTELLING RULE.** The optimal extraction path obeys the standard Hotelling rule. At the optimum, the planner must be indifferent at date  $t$  between consuming some extra amount  $dq$  today or delaying extraction for an extra length of time so as to facilitate extraction later on. Producing and consuming  $dq$  more units beyond  $q^*(\theta, t)$  for

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positive in the relevant range.

<sup>19</sup>In this model where consumers have quasi-linear utility over consumption and money, the discount factor is also the interest rate of the economy; a fact used below to compute the intertemporal profit of the concessionaire.

a length of time  $dt$  around date  $t$  increases overall surplus at date  $t$  by approximately  $(V'(q^*(\theta, t)) - C(S^*(\theta, t))) dqdt$ . However, extracting these extra units  $dq$  reduces the stock of resource from date  $t + dt$  onwards to approximately  $S^*(\theta, \tau) - dqdt$ . Future costs are thus increased from date  $t + dt$  onwards by an amount worth approximately  $-\int_t^{+\infty} C'(S^*(\theta, \tau))q^*(\theta, \tau)e^{-r(\tau-t)}d\tau dqdt$ . Therefore, at the optimal path we must have

$$V'(q^*(\theta, t)) - C(S^*(\theta, t)) = - \int_t^{+\infty} C'(S^*(\theta, \tau))q^*(\theta, \tau)e^{-r(\tau-t)}d\tau.$$

Differentiating with respect to  $t$  immediately yields the Hotelling rule (3.2).

IMPLEMENTATION. The planner could force the firm to extract the socially optimal quantity  $q^*(\theta, t)$  at date  $t$  and compensate the firm for the cost  $C(S^*(\theta, t))q^*(\theta, t)$  of producing this quantity. Under complete information, there is no real obstacle to implementing this policy, even though such forcing contracts are rarely seen in practice.

The optimal allocation can also be implemented if the firm adopts a competitive behavior and takes the price for resource  $p^*(\theta, t)$  as given. As the resource rarefies, its price increases towards the choke price  $P(0)$  and the amount extracted decreases over time. The overall profit  $\int_0^{+\infty} (p^*(\theta, t) - C(S^*(\theta, t))q^*(\theta, t))e^{-rt}dt$  can be fully extracted through a lump-sum tax on profit paid upfront if the firm has enough cash or can be distributed over time to extract profit at any point in time.

An alternative implementation, much in the spirit of Loeb and Magat (1979), would be to organize an *ex ante* competitive tender for the right of managing the field among several identical concessionaires (each of them already knowing the realization of  $\theta$ ). Through this process, the public authority selects arbitrarily a winner, which has bid the whole intertemporal profit of running operations. If it behaves *ex post* like a monopolist practicing first-degree discrimination, this firm may extract all consumer surplus to cover his prior bid. The process clearly replicates the trajectory highlighted in Proposition 1.

RUNNING EXAMPLE. Closed-form solutions for the optimal trajectory are obtained as

$$(3.3) \quad S^*(\theta, t) - S_{inf} = (\theta - S_{inf})e^{-\frac{t}{z^*}}$$

$$(3.4) \quad q^*(\theta, t) = \frac{\theta - S_{inf}}{z^*}e^{-\frac{t}{z^*}}$$

where  $\frac{1}{z^*} = \frac{2\kappa^*}{1 + \sqrt{1 + \frac{4\kappa^*}{r}}}$  and  $\kappa^* = \frac{C'(S_{inf})}{P'(0)} > 0$ . The stock of resource and the quantity extracted decrease towards their long-run limits ( $S_{inf}$  and 0 respectively) at the same exponential rate with relaxation time  $z^*$ . Intuitively, convergence is faster as the marginal cost of extraction is more sensitive to resource depletion.

#### 4. THE DYNAMICS OF RESOURCE EXTRACTION: ASYMMETRIC INFORMATION

As pointed out by the regulatory economics literature,<sup>20</sup> the true impediment to optimal contracting is asymmetric information. We now assume that the initial stock of resource is privately known by the firm.

<sup>20</sup>Baron and Myerson (1982), Laffont and Tirole (1993) and Armstrong and Sappington (2007).

#### 4.1. Incentive Compatibility

A long-term contract governs the relationship between the public authority and the firm. The contract stipulates how much quantity should be extracted and possibly a payment at any point in time. From the Revelation Principle,<sup>21</sup> the contract is viewed without loss of generality as a direct and truthful revelation mechanism that stipulates the output  $q(\hat{\theta}, t)$  and the payment  $\omega(\hat{\theta}, t)$  profiles over time as a function of the firm's announcement  $\hat{\theta}$  on its initial stock. Allowing such communication in our theoretical model certainly echoes real world practices. Indeed, most contracts allow experts working for the host government to access data or use equipment from the international operating companies in order to check the likelihood of these declared reserves. Because the public authority commits to the mechanism, this announcement takes place once for all at date  $0^-$ .<sup>22</sup>

Henceforth, let  $U(\theta)$  denote the firm's informational rent (or intertemporal payoff) when adopting the truthful strategy (i.e.,  $\hat{\theta} = \theta$ )

$$(4.1) \quad U(\theta) = \int_0^{+\infty} (\omega(\theta, t) - C(S(\theta, t))q(\theta, t))e^{-rt} dt$$

where the trajectory  $S(\theta, t)$  is defined through (2.1) and (2.2).

Incentive compatibility requires that this payoff be greater than what the firm with type  $\theta$  would obtain by adopting the extraction pattern of a type  $\hat{\theta}$  but starting from the initial stock  $\theta$ . Denote by  $\hat{S}(\theta, \hat{\theta}, t)$  the corresponding trajectory. From (2.1) and (2.2), this trajectory can be defined in integral form as  $\hat{S}(\theta, \hat{\theta}, t) = \theta - Q(\hat{\theta}, t)$  where  $Q(\hat{\theta}, t) = \int_0^t q(\hat{\theta}, \tau) d\tau$  is the cumulative amount extracted up to date  $t$  by a firm endowed with a stock of resource  $\hat{\theta}$ . Of course,  $\hat{S}(\theta, \theta, t) = S(\theta, t)$  and the trajectory following the truthful strategy satisfies  $S(\theta, t) = \theta - Q(\theta, t)$ . Incentive compatibility can then be written in a more compact form as

$$(4.2) \quad U(\theta) = \max_{\hat{\theta} \in \Theta} \int_0^{+\infty} (\omega(\hat{\theta}, t) - C(\hat{S}(\theta, \hat{\theta}, t))q(\hat{\theta}, t))e^{-rt} dt.$$

A contract  $\{(\omega(\hat{\theta}, t), q(\hat{\theta}, t))_{t \geq 0}\}_{\hat{\theta} \in \Theta}$  induces an allocation  $\{(U(\theta), q(\theta, t))_{t \geq 0}\}_{\theta \in \Theta}$ . The next lemma provides a useful characterization of the allocations that are incentive compatible, i.e., that can be implemented by a particular contract. This dual approach proves useful to characterize the optimal trajectories under asymmetric information.

LEMMA 1. *A necessary condition for incentive compatibility of an allocation  $\{(U(\theta), q(\theta, t))_{t \geq 0}\}_{\theta \in \Theta}$  is that  $U(\cdot)$  is absolutely continuous with*

$$(4.3) \quad \dot{U}(\theta) = - \int_0^{+\infty} C'(S(\theta, t))q(\theta, t)e^{-rt} dt.$$

*A sufficient condition for incentive compatibility is that (4.3) holds with*

$$(4.4) \quad Q(\theta, t) \text{ non-decreasing in } \theta.$$

<sup>21</sup>Myerson (1982).

<sup>22</sup>Observe also that there is no loss of generality in making the convention of an infinite planning horizon for the firm because the contract can just stipulate zero output and zero payment from a given date on if extraction was chosen over a finite period of time.

Condition (4.3) is a standard envelope condition that describes how incentive compatibility shapes the profile of the firm's information rent as its type varies. To understand this condition, it is useful to think of a firm knowing that it starts extraction from a stock of size  $\theta$ . That  $\theta$ -type firm can mimic the behavior of a firm with a type  $\theta - d\theta$ , that is, of a firm endowed with  $d\theta$  less units of resource and facing a greater cost of extraction. So doing means that, at any point in time, the  $\theta$ -type firm receives the same compensation destined to the less well-endowed type  $\theta - d\theta$  and extracts the same amount  $q(\theta - d\theta, t)$ , but it does so at a lower marginal cost of extraction  $C(\theta - Q(\theta - d\theta, t))$ . This mimicking strategy is thus akin to moving ahead along the extraction path. The intertemporal gains of adopting such mimicking strategy are thus approximately equal to  $-d\theta \int_0^{+\infty} C'(\theta - Q(\theta, t))q(\theta, t)e^{-rt}dt$ . To be compensated for the opportunity cost of not adopting this mimicking strategy, a firm with type  $\theta$  must get an incremental rent above that of type  $\theta - d\theta$ , namely  $U(\theta) - U(\theta - d\theta) \approx \dot{U}(\theta)d\theta$ , which is just worth the cost saving above.

It is standard in the screening literature to summarize implementability conditions by an envelope condition together with a monotonicity requirement on output.<sup>23</sup> The familiar approach consists in studying a so-called relaxed optimization problem where the monotonicity requirement is omitted, and, then, checking ex post that the solution of the relaxed problem is indeed monotonic. Things are rather easy in the context of a single-dimensional screening variable, even in the more involved settings where this monotonicity condition may be binding.<sup>24</sup> Here, our dynamic model allows the principal to control output at all dates, making the sufficient conditions for implementability harder to summarize into a single condition. Fortunately, this reduction is possible in our structured model and the sufficient condition (4.4) states that, for any incentive compatible allocation, the total amount of resource extracted up to any date  $t$  must be non-decreasing in the initial stock. If the optimal path is such that the greater this stock is, the greater is total extraction up to any date, then the solution to the relaxed problem is incentive-compatible. This is thus a rather simple condition that needs to be checked ex post. In our context, it turns out to be always satisfied.<sup>25</sup>

Finally, the envelope condition (4.3) also shows that the firm's information rent is non-decreasing in its initial stock. In other words, any type of firm accepts the contract if the firm with the lowest reserve  $\underline{\theta}$  already does so. On top of the incentive compatibility conditions (4.3) and (4.4), we thus impose the following participation constraint

$$(4.5) \quad U(\underline{\theta}) \geq 0.$$

#### 4.2. Virtual Cost of Extraction and Virtual Hotelling Rule

The public authority's objective is to maximize the expected intertemporal payoff<sup>26</sup>

$$\mathbb{E}_{\theta} \left( \int_0^{+\infty} (V(q(\theta, t)) - \omega(\theta, t))e^{-rt}dt \right)$$

<sup>23</sup>See Laffont and Martimort (2002, Chapter 3) for instance.

<sup>24</sup>See Guesnerie and Laffont (1984).

<sup>25</sup>See Proposition 4 below.

<sup>26</sup>We choose to give no weight to the firm's profit in the public authority's objective to simplify notations. We already know from Baron and Myerson (1982) that giving a weight less than one to the firm generates the same kind of rent-efficiency trade-off.

subject to first, equations (2.2) and (2.3) that describe how the stock of resource evolves on path and second, to the firm's incentive and participation constraints (4.3), (4.4) and (4.5). As suggested above, we first focus on a relaxed optimization problem where the sufficient condition (4.4) is omitted and check ex post that this condition holds.

Expressing the firm's intertemporal payment in terms of its information rent, we rewrite the public authority's objective function as follows

$$\mathbb{E}_\theta \left( \int_0^{+\infty} (V(q(\theta, t)) - C(S(\theta, t)q(\theta, t))e^{-rt} dt - U(\theta) \right).$$

This expression highlights the rent-efficiency trade-off faced by the public authority in this dynamic environment. On the one hand, the public authority wants to choose a path of resource extraction which maximizes the overall surplus. On the other hand, doing so means leaving excessive information rents to the firm and the corresponding payments are costly for the public authority.<sup>27</sup>

To get a more compact expression of the optimization problem, we observe that the participation constraint (4.5) is necessarily binding at the optimum. Otherwise, all payments could be reduced by a small amount, still ensuring participation while keeping the same dynamics of resource extraction, and such modification of the allocation would improve the public authority's expected payoff. Henceforth,  $U(\theta) = 0$  and a simple integration by parts together with (4.3) then yields:

$$\mathbb{E}_\theta (U(\theta)) = -\mathbb{E}_\theta \left( \int_0^{+\infty} \frac{1 - F(\theta)}{f(\theta)} C'(S(\theta, t)) q(\theta, t) e^{-rt} dt \right).$$

This expression highlights that the firm's expected information rent is an extra cost incurred by the public authority. Inserting this cost into the public authority's objective, we obtain a more compact expression of the maximand as

$$\mathbb{E}_\theta \left( \int_0^{+\infty} \left( V(q(\theta, t)) - \left( C(S(\theta, t)) - \frac{1 - F(\theta)}{f(\theta)} C'(S(\theta, t)) \right) q(\theta, t) \right) e^{-rt} dt \right).$$

Maximizing this expression amounts to solving a collection of maximization problems for each possible realization of  $\theta$ , each problem writing as follows

$$(\mathcal{P}^{sb}(\theta)) : \quad \max_{q, S} \int_0^{+\infty} (V(q(\theta, t)) - \tilde{C}(\theta, S(\theta, t))q(\theta, t))e^{-rt} dt$$

subject to (2.1) and (2.2).

The maximand shows that the marginal cost of extraction  $C(S)$  is now replaced by the 'virtual marginal cost of extraction'  $\tilde{C}(\theta, S)$  defined by

$$(4.6) \quad \tilde{C}(\theta, S) = C(S) - \frac{1 - F(\theta)}{f(\theta)} C'(S) \quad \forall (S, \theta).$$

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<sup>27</sup>Observe also in passing that the public authority can always choose to extract nothing, so that the value of this problem is always non-negative, and it is so for any possible realization of the initial stock of resource. In other words, the strategy consisting in 'shutting down' the least productive units to reduce the rent of the most productive ones is already considered in the above formulation.

This expression accounts for the extra cost of the information rent that has to be given up to the  $\theta$ -type firm to induce revelation of value of the stock. As a result, the virtual marginal cost of extraction is always above the true marginal cost of extraction, that is

$$\tilde{C}(\theta, S) \geq C(S) \quad \forall (S, \theta)$$

with an equality only for the highest possible type  $\bar{\theta}$ . This increase in the cost of extraction induced by asymmetric information is akin to introducing a delay in extraction. This points at the core dilemma between extracting information rent and extracting resource.

The virtual marginal cost of extraction is increasing and convex in  $S$  under our assumptions on  $C(\cdot)$ . Importantly, the virtual marginal cost of extraction can also be ranked with respect to the initial stock  $\theta$  since

$$\frac{\partial \tilde{C}}{\partial \theta}(\theta, S) \leq 0 \quad \forall (S, \theta).$$

A firm starting with a higher initial stock has thus a lower virtual marginal cost of extraction and is then more willing to extract at any stock level along the trajectory.

Importantly for the dynamics soon described, the cost of extraction at any given date now depends not only on the current reserves but also on the initial stock  $\theta$ . In other words, different types of firms become heterogeneous in terms of their costs of extraction whereas, under symmetric information, all types had the same technology and just differed by their initial stocks. As we stress below, the consequence of this heterogeneity induced by asymmetric information is that extraction paths now differ across firms with different initial stocks of resource.

To stress those dynamics, observe first that firms with different initial reserves become now also heterogeneous in terms of the minimal level of resource beyond which extraction ceases to be valuable. Let define  $\tilde{S}_{inf}(\theta)$  as the level of resource such that the marginal cost of extraction is equal to the choke price for a firm with type  $\theta$

$$(4.7) \quad P(0) = \tilde{C}(\theta, \tilde{S}_{inf}(\theta)).$$

Thanks to the monotone hazard rate property, different types are unambiguously ranked according to their limiting stocks  $\tilde{S}_{inf}(\theta)$  as shown in next lemma.

LEMMA 2.  $\tilde{S}_{inf}(\theta)$  is a decreasing function of  $\theta$  with  $\tilde{S}_{inf}(\bar{\theta}) = S_{inf}$ .

We are now ready to characterize optimal trajectories under asymmetric information. Because virtual marginal costs of extraction play the same role as true marginal costs of extraction under complete information, the next proposition bears some strong resemblance with Proposition 1.

PROPOSITION 2. *Under asymmetric information and when  $\theta > \tilde{S}_{inf}(\theta)$ , the optimal path of resource extraction solving problem  $(\mathcal{P}^{sb}(\theta))$  converges towards no extraction and a finite stock of resource in infinite time, but this limiting level now depends on the initial stock*

$$(4.8) \quad \lim_{t \rightarrow +\infty} q^{sb}(\theta, t) = 0 \text{ and } \lim_{t \rightarrow +\infty} S^{sb}(\theta, t) = \tilde{S}_{inf}(\theta).$$



The price of resource  $p^{sb}(\theta, t) = V'(q^{sb}(\theta, t))$  now evolves according to the virtual Hotelling rule

$$(4.9) \quad \frac{\partial p^{sb}}{\partial t}(\theta, t) = r(p^{sb}(\theta, t) - \tilde{C}(\theta, S^{sb}(\theta, t))).$$

When  $\theta \leq \tilde{S}_{inf}(\theta)$ , there is no exploitation of the resource at any point in time.

For a given initial stock, the dynamics looks quite similar to the case of complete information with the proviso that the long-run limit of the stock remains at a higher level. To understand this effect, it is important to come back on the incentive constraint (4.3), which shows that a type- $\theta$  firm enjoys some information rent by extracting the same amount as a firm with a lower initial stock and does so at a lower marginal cost. That rent is reduced when the public authority calls for lower extraction levels. Asking for less extraction by a given type  $\theta$  indeed reduces the information rent of all supra-marginal types  $\theta' \geq \theta$ . In other words, doing as if more of the resource had already been exploited by a given type and pushing forward the extraction path for that type helps to extract rents from types endowed with higher initial stocks. The second-best policy results from a trade-off between implementing extraction paths close to efficiency, providing the firm with excessive information rent, and extracting more that rent by extracting less resource.

The outcome under asymmetric information highlights a strong inefficiency that takes the form of insufficient long-run extraction. The long-run limit  $\tilde{S}_{inf}(\theta)$  remains indeed above  $S_{inf}$ . In other words, the public authority chooses to leave resource, and therefore gains from trade, in the ground. While it would be optimal to keep on extracting under complete information, this limited extraction now acts as a device to induce information revelation by making less attractive the extraction path for a concession that claims a low initial stock and a high cost of extraction to start with.

Perhaps the most spectacular expression of this inefficient extraction is for those firms with initial stocks (if they exist) such that

$$\theta \leq \tilde{S}_{inf}(\theta).$$

Because  $\tilde{S}_{inf}(\cdot)$  is decreasing in the initial stock  $\theta$ , this corresponds to a (maybe empty) lower tail of the types distribution. Those types are such that

$$\tilde{C}(\theta, \theta) \geq \tilde{C}(\theta, \tilde{S}_{inf}(\theta)) = P(0) = C(S_{inf}) > C(\theta).$$

In other words, for those types, producing even the first unit is inefficient under asymmetric information although it was efficient under complete information. Extraction does not even start for those fields. Everything happens as if the public authority is putting aside the whole field if it is too small. Doing so extracts more rents from larger fields.

**VIRTUAL HOTELLING RULE.** To better understand the virtual Hotelling rule, it is again useful to see the main trade-offs involved when considering a slight modification of the production plan. These computations must now take into account that any such modification bears on a subset of types and has consequences on supra-marginal types' information rents. Consider producing  $dq$  more units beyond  $q^{sb}(\theta, t)$  for a length of time  $dt$  after date  $t$  and for a subset of types of mass  $f(\theta)d\theta$  around  $\theta$ . Such a change increases

overall surplus at date  $t$  roughly by

$$(4.10) \quad (V'(q^{sb}(\theta, t)) - C(S^{sb}(\theta, t))) f(\theta) d\theta dq dt.$$

where  $S^{sb}(\theta, t)$  is the current stock of resource at a date  $t$  for a type  $\theta$ . However, consuming those extra units  $dq$  reduces the stock of resource from date  $t + dt$  onwards for all firms around  $\theta$  to approximately  $S^{sb}(\theta, \tau) - dq dt$ . Because of discounting, the increase in welfare from date  $t + dt$  onwards is worth approximately

$$(4.11) \quad \left( \int_t^{+\infty} C'(S^{sb}(\theta, \tau)) q^{sb}(\theta, \tau) e^{-r(\tau-t)} d\tau \right) f(\theta) d\theta dq dt.$$

Under asymmetric information, such a change in the production plan for all types around type  $\theta$  also impacts the information rents of all supra-marginal types  $\theta' \geq \theta$  whose mass is  $1 - F(\theta)$ . Increasing by  $dq$  the production of all types around  $\theta$  increases the slope of the information rent around that point. The overall incremental rent from date  $t$  for the mass of supra-marginal types is approximately equal to

$$(4.12) \quad (1 - F(\theta)) \left( - \int_t^{+\infty} C'(S^{sb}(\theta, \tau)) e^{-r(\tau-t)} d\tau \right) dq dt d\theta.$$

At the second-best optimum, the public authority should not find such modification of the production plan attractive, which requires that (4.10) just compensates for the shadow cost of increased extraction costs (4.11) and increased information rents (4.12), or

$$\begin{aligned} V'(q^{sb}(\theta, t)) - C(S^{sb}(\theta, t)) = & \underbrace{- \left( \int_t^{+\infty} C'(S^{sb}(\theta, \tau)) q^{sb}(\theta, \tau) e^{-r(\tau-t)} d\tau \right)}_{\text{Shadow cost associated to increased future extraction costs}} \\ & - \underbrace{\frac{1 - F(\theta)}{f(\theta)} \int_t^{+\infty} C'(S^{sb}(\theta, \tau)) e^{-r(\tau-t)} d\tau}_{\text{Shadow cost of increased information rents}}. \end{aligned}$$

Differentiating with respect to  $t$  immediately gives us the virtual Hotelling rule (4.9).

RUNNING EXAMPLE (CONTINUED). Observe first that from (4.6) and (4.7)

$$(4.13) \quad \tilde{S}_{inf}(\theta) = S_{inf} + \frac{1 - F(\theta)}{f(\theta)}.$$

Closed-form solutions for the optimal trajectories under asymmetric information are now readily obtained as

$$(4.14) \quad S^{sb}(\theta, t) - \tilde{S}_{inf}(\theta) = (\theta - \tilde{S}_{inf}(\theta)) e^{-\frac{t}{z^*}},$$

$$(4.15) \quad q^{sb}(\theta, t) = \frac{\theta - \tilde{S}_{inf}(\theta)}{z^*} e^{-\frac{t}{z^*}}.$$

Both the stock of resource and the quantity extracted decrease towards their long-run limit at the same rate than under complete information. Asymmetric information has an impact on total extraction but not on the rate of extraction with the above specification.

To ensure that output remains positive under all circumstances, we shall assume that

$$\underline{\theta} - S_{inf} > \frac{1}{f(\underline{\theta})}.$$

These formulas are useful to immediately check the sufficient condition for implementability (4.4) since the total quantity extracted

$$(4.16) \quad Q^{sb}(\theta, t) = (\theta - \tilde{S}_{inf}(\theta)) \left(1 - e^{-\frac{t}{z^*}}\right)$$

is increasing in  $\theta$ , since  $\tilde{S}_{inf}(\theta)$  (defined in (4.13)) is decreasing under the monotone hazard rate property.

REMARK. Our analysis of the set of incentive compatible allocations has implicitly supposed that a firm, whatever the level of its initial stock  $\theta$ , can always adopt the same extracting pattern as if that stock was actually  $\hat{\theta}$ . This means that the optimal mechanism must be such that the maximal amount extracted when claiming reserves  $\hat{\theta}$  remains less than the stock  $\theta$ . The following condition must thus be verified

$$\theta \geq \lim_{t \rightarrow +\infty} Q(\hat{\theta}, t), \quad \forall (\theta, \hat{\theta}) \in \Theta^2.$$

For the optimal contract, we indeed have  $\lim_{t \rightarrow +\infty} Q^{sb}(\hat{\theta}, t) = \hat{\theta} - \tilde{S}_{inf}(\hat{\theta})$  where  $\tilde{S}_{inf}(\theta)$  is a non-increasing function of  $\theta$ . In other words, the maximal amount that can be extracted is  $\max_{\hat{\theta} \in \Theta} \lim_{t \rightarrow +\infty} Q^{sb}(\hat{\theta}, t) = \bar{\theta} - \tilde{S}_{inf}(\bar{\theta})$ . To make it always possible, for any type, to extract the quantity targeted for another type, the following feasibility condition must thus hold

$$(4.17) \quad \underline{\theta} \geq \bar{\theta} - \tilde{S}_{inf}(\bar{\theta}).$$

This condition is left implicit in the above analysis. Whenever (4.17) fails, some low-reserve types may not be able to mimic the large extraction pattern of firms endowed with better stocks. The mechanism design problem falls into the framework of type-dependent message space that was studied by Green and Laffont (1986). We leave the analysis of such problems for future research.

### 4.3. Comparative Statics

Our results so far have uncovered a first type of inefficiency: There is less long-run extraction under asymmetric information. The second and novel feature, which is highlighted by the RUNNING EXAMPLE, is that the quantity extracted is also lower at any point in time under asymmetric information.<sup>28</sup> Interestingly, these results hold over a broader ranges of functional forms.

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<sup>28</sup>The path of resource extraction under asymmetric information may appear at a first sight similar to the path that would be obtained with an unregulated monopoly. Under an unregulated monopoly, resource extraction is limited and the stock remains also above its optimal value. Although similar, those patterns also exhibit some differences. An unregulated monopoly drives the price above the marginal cost of extraction at any point in time because it cares about the marginal revenue and not the marginal surplus. That wedge keeps extraction low but depends only on the inverse demand function. Instead, the distortions found under asymmetric information are only driven by cost considerations. The virtual marginal cost of extraction being greater than the true value, the price is again above the marginal cost of extraction; a first difference with complete information. As a consequence, the long-run amount of resource remains above its complete information value while the unregulated monopolist is able to reach the same limit as a competitive market; a second significant difference. Proofs are available upon request.

PROPOSITION 3. *Under asymmetric information, the remaining stock of resource is always larger and the quantity extracted is always smaller than under complete information, that is, for all  $\theta \in \Theta$  and for all  $t \geq 0$*

$$(4.18) \quad S^{sb}(\theta, t) \geq S^*(\theta, t),$$

$$(4.19) \quad q^{sb}(\theta, t) \leq q^*(\theta, t),$$

with equality only for  $\theta = \bar{\theta}$  when  $t > 0$ .

The next proposition compares extraction paths across firms with different initial stocks. It shows in passing that the sufficient condition (4.4) is satisfied by the solution to the relaxed problem ( $\mathcal{P}^{sb}(\theta)$ ). This validates our approach of focusing on the relaxed problem in the first place.

PROPOSITION 4. *The quantity extracted  $Q^{sb}(\theta, t) = \theta - S^{sb}(\theta, t)$  under asymmetric information is increasing in  $\theta$*

$$(4.20) \quad \frac{\partial Q^{sb}}{\partial \theta}(\theta, t) > 0.$$

#### 4.4. Implementation

This section addresses what sort of instruments may implement the second-best outcome in Proposition 3. To begin with, it is worth stressing that these instruments have to be consistent with the information structure. For instance, profit-sharing arrangements or cost-reimbursement rules presuppose that profits or costs are observable, an assumption which would imply, given our choice of functional forms, that the initial resource stock would be perfectly observed and verifiable by the public authority. Hence, the only instruments that can be used in our setting can be contingent on production. Within the relevant class, we bear a particular attention to royalty payments.<sup>29</sup>

For our RUNNING EXAMPLE, the formulas (4.14) and (4.15) show the optimal extraction and resource levels are separable in time and type. Accordingly, denote by  $q_0^{sb}(\theta) = \frac{\theta - \tilde{S}_{inf}(\theta)}{z^*}$  the optimal output at date  $t = 0$ . Observe that this output contains all information on the firm's initial stock  $\theta$  and, that  $q_0^{sb}(\theta)$  is strictly increasing with  $\theta$  because  $\tilde{S}_{inf}(\theta)$  is itself decreasing. Let  $\vartheta_0(q)$  denote the corresponding inverse function. Starting from this initial value of output, output declines over time at an exponential rate,  $q^{sb}(\theta, t) = q_0^{sb}(\theta)e^{-\frac{t}{z^*}}$ , and the total amount extracted also increases at such rate since  $Q^{sb}(\theta, t) = z^* q_0^{sb}(\theta)(1 - e^{-\frac{t}{z^*}})$ .

By picking a particular item within the menu of output and payment profiles available  $\{q(\hat{\theta}, t); \omega(\hat{\theta}, t)\}_{\hat{\theta} \in \Theta}$ , the firm determines the whole trajectory of the economy once and for all. The separability stressed above suggests that this choice is akin to picking only

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<sup>29</sup>The resource owner or public authority typically keeps control on output in production-sharing agreements, but does not do so in concession contracts. In that case, however, the public authority validates the pace of fields development, influencing the intertemporal production scheme. In production sharing-agreements, the sharing rule concerns in part cost reimbursement. To the extent that part of the cost of extraction remains non-observable and is related to the remaining stock of resource, our analysis is still relevant.

an initial output level. In other words, the dynamic contracting model can be somehow reduced to a static problem. To illustrate, suppose that the firm is offered a nonlinear payment schedule  $T(q_0)$  that first, depends on the starting level of extraction  $q_0$  and second, stipulates the firm's upfront payment for its services. Any departure from the trajectory  $q_0 e^{-\frac{t}{z^*}}$  could be punished with severe penalties for non-provision of the right production at any point in time. With such punishments, the firm is deterred from deviating from this extraction path but, still, it has the choice of which extraction path to choose within the proposed family. Formally,  $q_0^{sb}(\theta) = \frac{\theta - \tilde{S}_{inf}(\theta)}{z^*}$  should thus solve

$$(4.21) \quad q_0^{sb}(\theta) = \arg \max_{q_0 \geq 0} T(q_0) - \mathcal{C}(\theta, q_0)$$

where

$$\mathcal{C}(\theta, q_0) = \int_0^{+\infty} C \left( \theta - z^* q_0 \left( 1 - e^{-\frac{t}{z^*}} \right) \right) q_0 e^{-\frac{t}{z^*}} e^{-rt} dt$$

denotes the intertemporal cost associated to an extraction path with starting value  $q_0$  and an initial stock  $\theta$ .

Notice that the public authority's problem ( $\mathcal{P}^{sb}(\theta)$ ) is also akin to choosing that initial value  $q_0^{sb}(\theta)$  so as to instead maximize the virtual surplus of the allocation, taking again into account the exponential decay of output and stock that are specific to our functional forms. By construction of the optimal path under asymmetric information,  $q_0^{sb}(\theta) = \frac{\theta - \tilde{S}_{inf}(\theta)}{z^*}$  must also solve

$$(4.22) \quad q_0^{sb}(\theta) = \arg \max_{q_0 \geq 0} \mathcal{V}(q_0) - \mathcal{C}(\theta, q_0) + \frac{1 - F(\theta)}{f(\theta)} \frac{\partial \mathcal{C}}{\partial \theta}(\theta, q_0)$$

where the intertemporal consumer surplus associated to a path of extraction with starting value  $q_0$  is defined as

$$\mathcal{V}(q_0) = \int_0^{+\infty} V \left( q_0 e^{-\frac{t}{z^*}} \right) e^{-rt} dt.$$

The nonlinear payment schedule  $T(q_0)$  implements the optimal choice  $q_0^{sb}(\theta)$  when Problems (4.21) and (4.22) have the same solution. The next proposition unveils the properties of such a schedule.

**PROPOSITION 5.** *Assume that functional forms are given as in our RUNNING EXAMPLE. The optimal choice  $q_0^{sb}(\theta)$  is implemented by the following nonlinear schedule  $T(q_0)$ :*

$$(4.23) \quad T(q_0) = \mathcal{V}(q_0) - \mathcal{V}(q_0^{sb}(\theta)) + \mathcal{C}(\theta, q_0^{sb}(\theta)) + \frac{C'(S_{inf})}{r + \frac{1}{z^*}} \int_{q_0^{sb}(\theta)}^{q_0} \frac{1 - F(\vartheta_0(\tilde{q}))}{f(\vartheta_0(\tilde{q}))} d\tilde{q}.$$

The marginal payment  $T'(q_0)$  is non-negative,<sup>30</sup> always lower than the marginal social value  $\mathcal{V}'(q_0)$  so as to induce the firm to choose lower extraction paths as a means to reduce information rents. The difference

$$\mathcal{V}'(q_0) - T'(q_0) = -\frac{C'(S_{inf})}{r + \frac{1}{z^*}} \frac{1 - F(\vartheta_0(q_0))}{f(\vartheta_0(q_0))} \geq 0$$

represents the corresponding discount. This discount is decreasing thanks to the mono-

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<sup>30</sup>See the Appendix.

tone hazard rate property, non-negative and worth zero only at  $q_0(\bar{\theta})$  so that the firm has efficient incentives to produce at that point.

As the firm chooses a more ambitious extraction path, and reveals by doing so that it has a larger stock, its marginal incentives to produce are more aligned with the socially optimal ones. The extraction path for such a firm comes closer to efficiency. The counterpart is that such a firm under high-powered incentives also gets a large and costly information rent. A firm that instead chooses a modest extraction path, and reveals by doing so that its initial stock is somewhat limited, evolves under low-powered incentives, produces below the efficient level and gets a low information rent.

**RUNNING EXAMPLE (CONTINUED).** To get sharper results, we now specialize these findings under the additional assumption that  $\theta$  is uniformly distributed on  $[\bar{\theta} - 1, \bar{\theta}]$  with  $\bar{\theta} > S_{inf} + 2$  to ensure that the second-best extraction level is positive for all  $\theta$ . We immediately derive that  $\tilde{S}_{inf}(\theta) = S_{inf} + \bar{\theta} - \theta$ ,  $q_0^{sb}(\theta) = \frac{1}{z^*}(2\theta - \bar{\theta} - S_{inf})$  and  $\vartheta_0(q_0) = \frac{1}{2}(\bar{\theta} + S_{inf} + z^*q_0)$ . Inserting into the above expression gives

$$(4.24) \quad T'(q_0) = \frac{P(0) + \frac{1}{2}C'(S_{inf})(\bar{\theta} - S_{inf})}{r + \frac{1}{z^*}} + \frac{P'(0)}{2(2 + rz^*)} \left( z^* - \frac{2}{r} \right) q_0,$$

$$(4.25) \quad T''(q_0) = \frac{P'(0)}{2(2 + rz^*)} \left( z^* - \frac{2}{r} \right).$$

Observe that the nonlinear payment schedule  $T(q_0)$  is convex only when  $z^* < \frac{2}{r}$  or  $C'(S_{inf}) < \frac{3}{4}rP'(0)$ , i.e., when the marginal cost of extraction decreases sufficiently quickly with the level of remaining resource in comparison with the inverse demand. In practice, we might expect this condition to hold given that it is easier to satisfy when  $r$  is small. When  $T(\cdot)$  is convex, we may replace this nonlinear schedule with the menu of its tangents, which are expressed as follows

$$(4.26) \quad \tilde{T}(q, q_0) = T(q_0) + T'(q_0)(q - q_0).$$

Indeed, convexity of  $T(\cdot)$  ensures that the maximization problem of a firm facing such a menu of linear schemes, namely  $\max_{q, q_0} \tilde{T}(q, q_0) - \mathcal{C}(\theta, q)$ , is globally concave.<sup>31</sup> The contract (4.26) can then be readily interpreted as a menu of license fees and royalties. The firm chooses an output target  $q_0$  at date 0, pays a license fee  $q_0T'(q_0) - T(q_0)$  and a royalty leaving to the firm  $T'(q_0)$  for each unit of output produced at date 0.

Instead of receiving all the payment at date 0, the firm may as well receive a fraction of the per-period revenue associated to resource extraction. The revenue at date  $t$  is given by  $P(q(t))q(t)$  where  $q(t) = q_0e^{-\frac{t}{z^*}}$ . If the firm is given a share  $\alpha(t, q_0)$  of that per-period revenue, its incentives are unchanged provided that its intertemporal payoff remains equal to  $T(q_0)$ . Assuming that the share of revenue is constant across time, this

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<sup>31</sup>This argument is familiar from the incentive regulation literature (Laffont and Tirole, 1993, Chapter 1, for instance) but it was developed there in a static framework. Osmudsen (1998) in his two-period setting does not check for the global concavity of the firm's maximization problem when the optimal nonlinear schedule is replaced with the menu of its tangents. That global concavity condition requires the convexity of  $T(\cdot)$ , a condition that is not always satisfied as shown in our RUNNING EXAMPLE.

requires that

$$\int_0^{+\infty} \alpha(q_0) P\left(q_0 e^{-\frac{t}{z^*}}\right) q_0 e^{-\frac{t}{z^*}} e^{-rt} dt = T(q_0) \Leftrightarrow \alpha(q_0) = \frac{T(q_0)}{\frac{P(0)q_0}{r+\frac{1}{z^*}} + \frac{P'(0)q_0^2}{r+\frac{2}{z^*}}}.$$

It then comes immediately that  $\alpha(\cdot)$  is an increasing function of  $q_0$ .<sup>32</sup> a firm endowed with a larger initial reserve receives a larger fraction of the revenue associated to resource extraction. This implementation thus echoes real-world instruments observed in practice.

## 5. FROM A SINGLE FIRM TO A MARKET OF CONCESSIONAIRES

Our analysis has so far supposed that the public authority contracts with a single firm whose initial stock of resource is private information. One implication is that the resource price follows from matching the demand with the supply of that firm according to its virtual cost of extraction. As a result, there was not a single price for the resource but a collection of such prices, one for each possible realization of the initial stock. However, in practice, we may expect that the public authority is involved into many simultaneous relationships with different concessionaires producing on different fields so that only their aggregate supply meets the demand.<sup>33</sup> We thus need to adapt our previous findings to the case of a market where a continuum of firms operate.<sup>34</sup> The paths of resource extraction and the allocation of production among concessionaires are somewhat different when one moves from the single-firm scenario to a market context, although much of our previous insights carry over.

### 5.1. Complete Information

With a continuum of firms, the public authority collects individual production from each of them according to their respective cost of extraction. Formally, let  $Q(t) = \int_0^{\bar{\theta}} q(\theta, t) f(\theta) d\theta$  be the overall production at date  $t$  and let  $p(t)$  be the market price. Optimality of the public authority's plan of production requires that aggregate supply meets demand at the market price so that the consumers' marginal benefit of an extra unit of resource is equal to the shadow cost of the aggregate feasibility constraint above,

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<sup>32</sup>This comes from the fact that  $T'(q_0) \geq 0$  (see (4.21)) and the fact that the public authority implements, at any date, a quantity larger than the monopoly quantity, so that the revenue from resource extraction is decreasing over the relevant range of extraction levels. Proof available upon request.

<sup>33</sup>This is the case of natural resource characterized by high transportation costs sold on a regional market in large countries, such as gas in China for instance. In other cases, national public authorities may have little influence on the resource market, for instance, on the international oil market. The case of exogenous price dynamics is studied in our companion paper Martimort, Pouyet and Ricci (2016). Most cases are in between these two extremes: the public authority can affect a large share of production on a market that is partially segmented due to transportation costs, as for instance coal in the U.S. where 42% of production came from federal and Indian lands in 2014. To the extent that the public authority takes into account its influence on the resource price, the lessons from our analysis carry over qualitatively.

<sup>34</sup>Screening models do not make much differences between these two scenarios. To illustrate, the optimal nonlinear pricing of a monopolist would be the same whether the monopolist faces a single consumer with private information or a continuum of consumers each with his own preference parameter as long as the distribution of types remains the same in both scenarios and the monopolist offers an anonymous nonlinear price.



i.e., the price of the resource<sup>35</sup>

$$(5.1) \quad p^*(t) = P(Q^*(t)) \quad \forall t \geq 0.$$

Following Herfindahl's (1967) principle, production units with the lowest costs are first mobilized. Given that the marginal cost of extraction increases with depletion, the model generates an interesting dynamics for the last unit involved. For future reference, we define  $\sigma(t)$  as the lowest level of the resource stock across firms active at date  $t$ . Since all firms will follow the same extraction path under complete information,  $\sigma(t)$  also characterizes the stock of the 'marginal' firm, i.e., the firm with the lowest level of stock that starts being active at any date  $t$ . When all firms are already active, a case that arises when the price has sufficiently raised to justify that even the firm endowed with the minimal stock  $\underline{\theta}$  has started extraction, we may use the convention that  $\sigma(t) = S(\underline{\theta}, t)$ .

Equipped with these notations, we can define the optimal path of the economy under complete information as follows.

**PROPOSITION 6.** *In a market context with complete information, all firms follow the same path of resource extraction. This path converges in infinite time towards no extraction and the same final stock of resource  $S_{inf}$  for all firms:*

$$\lim_{t \rightarrow +\infty} q^*(\theta, t) = 0 \text{ and } \lim_{t \rightarrow +\infty} S^*(\theta, t) = S_{inf}.$$

*All firms active at a given date  $t$  produce the same amount*

$$(5.2) \quad q^*(\theta, t) = -\dot{\sigma}^*(t) \quad \forall \theta \geq \sigma^*(t)$$

*where  $\sigma^*(t)$  is the remaining stock of resource across active firms at date  $t$  and satisfies  $\sigma^*(0) = \bar{\theta}$ ,  $\lim_{t \rightarrow +\infty} \sigma^*(t) = S_{inf}$ .*

*The price of resource is*

$$(5.3) \quad p^*(t) = P(-\dot{\sigma}^*(t)(1 - F(\sigma^*(t))).$$

*which evolves according to the following Hotelling rule*

$$(5.4) \quad \dot{p}^*(t) = r(p^*(t) - C(\sigma^*(t))),$$

The price  $p^*(t)$  is determined by applying the Hotelling rule for a cost function which is that of the latest production unit to have started producing. In a market context, all active firms at a given date face the same cost of extraction, and thus also have the same amount of remaining stock. They extract the same quantity at all future dates and follow the same extraction path. In particular, the remaining stock of these firms converges towards  $S_{inf}$ , just as in the case of a single production unit. This finding reiterates, in a market context, our previous observation that under complete information firms follow the same trajectory. The difference with the single-firm scenario is that the set of active firms corresponds to an upper tail of the firms' distribution, that is, to those firms such

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<sup>35</sup>Slightly abusing notation, we denote by  $q^*(\theta, t)$ ,  $S^*(\theta, t)$  and  $p^*(t)$  the optimal values of extraction, stock and price under complete information, though these values do not necessarily coincide with those determined in the analysis of Section 3. Similarly, in the current section we adopt the notation 'sb' to denote the constrained-optimal solution under asymmetric information.

that  $\theta \geq \sigma^*(t)$  according to the Herfindahl principle. This explains the term  $1 - F(\sigma^*(t))$  in the market-clearing condition (5.3).

Importantly in view of the results to come in Section 5.2 below, all firms end up being active at some point in time. Again, this result captures the idea that an efficient path of resource extraction should not leave any valuable resource in the ground.

### 5.2. Asymmetric Information

We now turn to the scenario where firms have private information on their stocks. The public authority offers a similar contract to all these firms and thus incentive compatibility and participation constraints are expressed as above. As in Section 5.1, the fact that there is now a continuum of firms has nevertheless consequences on how supply equals demand at the market clearing price.

To see how, we may again define  $\sigma(t)$  as the level of resource stock of the latest firm becoming active at date  $t$ . Under asymmetric information, the allocation of aggregate production  $\mathcal{Q}(t)$  across firms does not depend on marginal costs of extraction but, instead, on virtual marginal costs. Constrained efficiency now requires that virtual marginal costs be identical across all active firms, even though these firms may have started with different initial stock of resource. On top, this value of virtual marginal cost is determined by the latest firm to become active, which is endowed with a stock  $\sigma^{sb}(t)$ . In other words, the resource stock  $S^{sb}(\theta, t)$  of the firm that starts from an initial level worth  $\theta$  and is active at date  $t$  must be such that<sup>36</sup>

$$(5.5) \quad \tilde{C}(\theta, S^{sb}(\theta, t)) = \tilde{C}(\sigma^{sb}(t), \sigma^{sb}(t)) \Leftrightarrow S^{sb}(\theta, t) = \Phi(\theta, \sigma^{sb}(t))$$

where  $\Phi(\theta, \sigma)$  is defined as

$$\tilde{C}(\theta, \Phi(\theta, \sigma)) = \tilde{C}(\sigma, \sigma).$$

Importantly, from its definition (4.6), the virtual marginal extraction cost is equalized across firms with different initial stocks only if marginal extraction costs differ. It follows that, at a given point in time, all firms active keep different stocks and extract different amounts. The fact that heterogeneous firms follow various extraction paths is thus compatible with optimal supply in a market context plagued with asymmetric information.

**PROPOSITION 7.** *In a market context with asymmetric information, firms with different initial stocks of resource follow different paths of resource extraction. Those paths converge in infinite time towards no extraction and a finite stock of resource  $\tilde{S}_{inf}(\theta)$  for a firm with initial stock  $\theta$ :*

$$\lim_{t \rightarrow +\infty} q^{sb}(\theta, t) = 0 \text{ and } \lim_{t \rightarrow +\infty} S^{sb}(\theta, t) = \tilde{S}_{inf}(\theta).$$

*The resource stock  $\sigma^{sb}(t)$  of the latest firm to become active at a given date  $t$  satisfies  $\sigma^{sb}(0) = \bar{\theta}$ ,  $\lim_{t \rightarrow +\infty} \sigma^{sb}(t) = \underline{s} > S_{inf}$  such that*

$$\tilde{C}(\underline{s}, \underline{s}) = P(0).$$

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<sup>36</sup>In this sub-section we consider the case where there is always a new entrant, i.e.,  $\sigma^{sb}(t) \geq \underline{s} \forall t$ . In the proof of Proposition 7 we consider the possibility that the endowment and the remaining stock of the latest firm becoming active differ.

The set of active firms at date  $t$  is  $[\sigma^{sb}(t), \bar{\theta}]$  and firms active at date  $t$  extract different amounts characterized by

$$(5.6) \quad q^{sb}(\theta, t) = -\dot{\sigma}^{sb}(t) \frac{\partial \Phi}{\partial \sigma}(\theta, \sigma^{sb}(t)) > 0$$

The price of the resource  $p^{sb}(\theta, t)$  is

$$(5.7) \quad p^{sb}(t) = P \left( -\dot{\sigma}^{sb}(t) \int_{\sigma^{sb}(t)}^{\bar{\theta}} \frac{\partial \Phi}{\partial \sigma}(\theta, \sigma^{sb}(t)) f(\theta) d\theta \right)$$

and evolves according to the following virtual Hotelling rule

$$(5.8) \quad \dot{p}^{sb}(t) = r(p^{sb}(t) - \tilde{C}(\sigma^{sb}(t), \sigma^{sb}(t))).$$

At the aggregate level, the dynamics of resource extraction and price resemble those found in Section 4 for the case of a single firm. The dynamics are also somewhat similar to that found in the complete information scenario of Section 5.1; it is again fully characterized by the stock of the latest firm entering the market. Contrary to the complete information scenario, that minimal surplus now converges over time towards a firm-specific limit  $\tilde{S}_{inf}(\theta)$ , and this limit remains above  $S_{inf}$ , meaning that inefficient extraction remains at both the individual and the aggregate level.

As under complete information, the set of active firms at any given date remains an upper tail of the types distribution, but all those firms produce different amounts at any point in time, in sharp contrast with complete information contexts. Indeed, because active firms produce according to their virtual marginal costs of production, as the price increases, a firm already active keeps on producing from a smaller remaining stock and extracts more than firms with lower stocks that start to become active from that date on.<sup>37</sup> Although, under asymmetric information, some inactive firms may have smaller technical marginal cost than some active firms, a ‘virtual Herfindahl principle’ applies. The optimal sequence of exploitation of resource fields starts using fields characterized by lower virtual marginal costs of extraction, and then move to more expensive ones, as the virtual marginal cost in currently active fields increases with resource depletion.

**RUNNING EXAMPLE (CONTINUED).** The comparison of the price trajectory under complete and asymmetric information is significantly more complex than in the case of a single firm because it entails a detailed description of aggregate supply at any point in time, and because this aggregate supply depends itself on the price level. For tractability, we again consider our **RUNNING EXAMPLE** but also assume that  $\theta$  is uniformly distributed on  $[S_{inf}, S_{inf} + 1]$ . While  $\sigma^*(t)$  converges towards  $S_{inf}$  as  $t$  goes to  $+\infty$ ,  $\sigma^{sb}(t)$  converges towards the greater value  $S_{inf} + \frac{1}{2}$ . Half of the resource that could have been extracted under complete information thus remains in the ground under asymmetric information. Yet, both  $p^*(t)$  and  $p^{sb}(t)$  converge towards the choke price  $P(0)$  with less and less extraction over time under both scenarios.

The next proposition ranks those prices in terms of their asymptotic properties under the assumptions of our **RUNNING EXAMPLE**.

<sup>37</sup>Indeed, as shown in the Appendix,  $\partial^2 \Phi / \partial \sigma \partial \theta \geq 0$  and  $\partial \Phi / \partial \theta < 0$ . Together with (5.6), this implies that  $\partial q^{sb}(\theta, t) / \partial \theta > 0$ .

PROPOSITION 8. *The price  $p^{sb}(t)$  under asymmetric information converges faster to  $P(0)$  than the price  $p^*(t)$  under complete information:*

$$p^{sb}(t) - P(0) \underset{t \rightarrow +\infty}{\sim} \frac{P'(0)}{z^{sb}} e^{-\frac{t}{z^{sb}} + t\eta^{sb}(t)}$$

while

$$p^*(t) - P(0) \underset{t \rightarrow +\infty}{\sim} \frac{P'(0)}{z^*} e^{-\frac{t}{z^*} + t\eta^*(t)}$$

where  $\frac{1}{z^{sb}} = \frac{4\kappa^*}{1 + \sqrt{1 + \frac{8\kappa^*}{r}}} > \frac{1}{z^*}$  and  $\lim_{t \rightarrow +\infty} \eta^{sb}(t) = \lim_{t \rightarrow +\infty} \eta^*(t) = 0$ .

Thus, the market price converges faster under asymmetric information. What is remarkable here is that, when a single firm is contracted on, the price dynamics under asymmetric information and under complete information exhibit similar patterns. Bringing back our previous findings, the price dynamics under both scenarios are indeed given by

$$p^{sb}(\theta, t) = P(0) + P'(0) \max \left\{ 2 \frac{\theta - (S_{inf} + \frac{1}{2})}{z^*}; 0 \right\} e^{-\frac{t}{z^*}}$$

while

$$p^*(\theta, t) = P(0) + P'(0) \frac{\theta - S_{inf}}{z^*} e^{-\frac{t}{z^*}}.$$

Yet, the subtlety is that these dynamics are similar provided that  $\theta \geq S_{inf} + \frac{1}{2}$  so that extraction starts at all. Under asymmetric information, however, smaller fields such that  $\theta \leq S_{inf} + \frac{1}{2}$  do not even start extraction. At the aggregate level, this contraction of supply induced by asymmetric information raises the price quickly towards its long-run limit. Only firms with initial stocks  $\theta \geq S_{inf} + \frac{1}{2}$  start extraction and, even though they do extract over the whole time horizon, the price has in the meantime converged more quickly towards its long-run limit.

This important finding may somewhat reconcile the Hotelling rule with empirical evidence that has repeatedly shown that prices tend to be more stable than what the rule predicts.<sup>38</sup> Stability follows here from the faster convergence of prices towards their long-run limit  $P(0)$ . From an empirical viewpoint, regressing  $\log(p^{sb}(t) - P(0))$  on time should, according to the model, give a coefficient of greater magnitude than its complete information counterpart together with a residual that converges towards zero slowly over time. We leave to further research the empirical validation of such prediction.

## 6. APPLICATION

Our analysis may be used to shed light on the recent modernization of the royalty framework undertaken in the Province of Alberta, Canada. After an extensive investigation by the Royalty Review Advisory Panel,<sup>39</sup> the Alberta government decided to implement the panel's recommendations, leading to the so-called Alberta's Modernized Royalty Framework, which will be effective as of January 1, 2017.<sup>40</sup>

<sup>38</sup>See Gaudet (2007) for a nice account of the criticisms of this rule but also of their own limits.

<sup>39</sup>See "Alberta at a Crossroads," Royalty Review Advisory Panel Report, available at <http://www.energy.alberta.ca/Org/pdfs/RoyaltyReportJan2016.pdf>.

<sup>40</sup>See [http://www.energy.alberta.ca/About\\_Us/Royalty.asp](http://www.energy.alberta.ca/About_Us/Royalty.asp).

The recommendations of the advisory panel reflect the fact that several major changes have affected the competitiveness of Alberta's energy industry. First, while Alberta has been for a long time a major supplier of natural gas to North America, the shale gas revolution that started in the U.S. around 2006 (mostly due to advances in the technology of hydraulic fracturing) has led the U.S. to flood North America with natural gas and unconventional oil. The 2008 collapse of natural gas prices and the higher costs of Alberta's production both implied a fall in Alberta's competitive position in energy markets and the erosion of capital investment. Second, the previous royalty framework was complex and lacked transparency.<sup>41</sup>

Figure 1 illustrates the main features of the modernized royalty framework, which applies at the well level.<sup>42</sup> There are three distinct phases. First, a proxy for well cost  $C^*$  is computed on the basis of average industry drilling and completion costs.<sup>43</sup> In this pre-payout phase, a low 5% royalty is applied until the revenue generated covers that fixed cost. Observe that exploitation costs are not included, perhaps because they are hardly verifiable and can be manipulated, but certainly also to promote incentives for innovation and cost-reduction. Second, in the post-payout phase, the royalty is set at a higher value. Changes in the royalty do not depend on the well's production, but only on fluctuations of the international price. Third, once production falls below the maturity threshold, the royalty is reduced downwards in order to avoid early shut-in due to higher extraction costs.

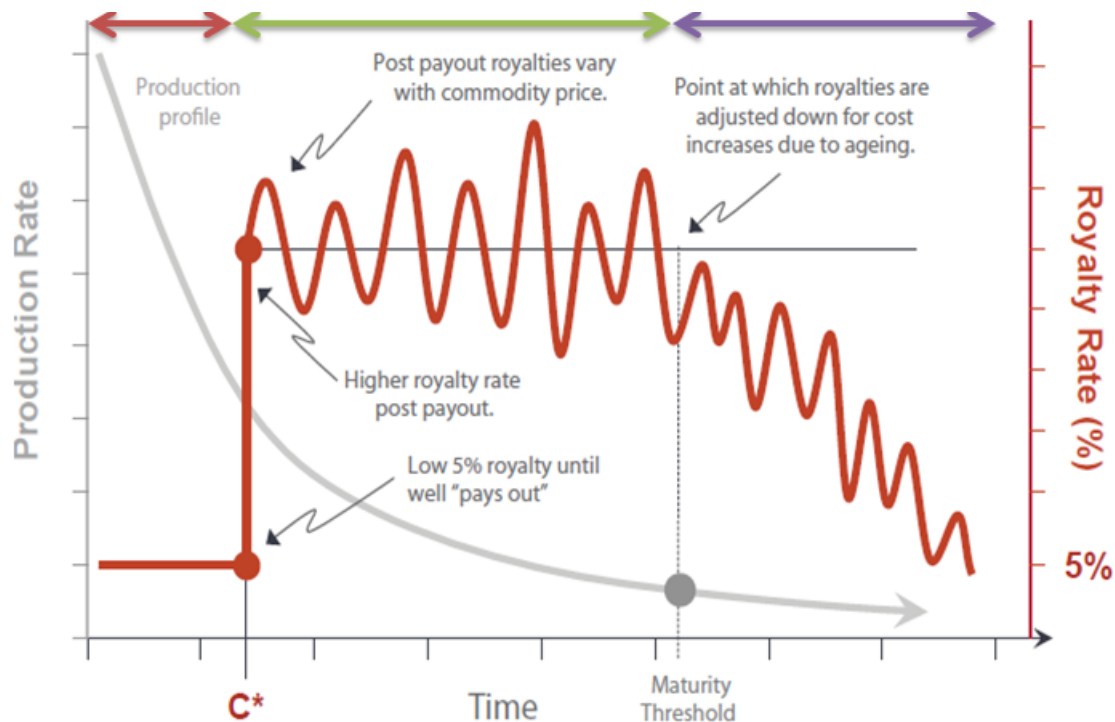


Figure 1: Overview of Alberta Modernized Royalty Framework.

<sup>41</sup>See <http://www.energy.alberta.ca/Org/pdfs/EnergyEconomics.pdf>.

<sup>42</sup><http://www.energy.alberta.ca/Org/pdfs/MRFFactsheet.pdf>

<sup>43</sup>The calculation does not depend on the type of hydrocarbon the well produces and depends on three key parameters: the vertical and lateral lengths, and the amount of proppant (typically sand) used to stimulate a well during its completion and to hold fractures that have been opened.

Our analysis suggests several remarks on the post-payout and the maturity phases.

First, royalties need to provide incentives for an efficient extraction path. This requires to set output-based royalties and to differentiate these royalties according to the extraction path chosen by the concessionaire. A well endowed with a high initial stock of resource should pay a lower royalty than one with a low initial stock.

Our implementation requires that lower royalties also come with higher fixed payments paid for the right of exploitation. On top of royalties, the Province of Alberta receives fixed payments through bids revenue from the successful auctions of mineral leases and rentals associated with these leases. This suggests two possible improvements of the current regime. Either the payment of the auction winner and the royalty should be determined endogenously through the competitive tendering, in the spirit of Laffont and Tirole (1987)'s auction of incentive contracts. Or, the concessionaire should pay a higher rental fee when it chooses a lower royalty level. In a nutshell, our analysis suggests to link explicitly the royalty and the fixed payment paid by a concessionaire.

Finally, our analysis also suggests that the maturity threshold should be dependent on the total amount of resource extracted. Wells which produce less overall should not be allowed, and let alone encouraged, to pursue extraction until the resource is exhausted.

## 7. CONCLUSION

We have analyzed optimal contracts for resource extraction in a context of asymmetric information. This analysis has stressed the fundamental dilemma between extracting information and extracting resource. Reducing the information rent of concessionaires who hold private information on the initial stock of their fields requires diminishing extraction and leaving some resource in the ground. We have derived a virtual Hotelling rule that characterizes the dynamics of resource extraction in such scenarios. Asymmetric information requires to replace the marginal cost of extraction by a virtual marginal cost of extraction that is greater, which explains the inefficient amount of extraction. Although fields of different initial values would follow the same extraction paths under complete information, these paths significantly differ under asymmetric information with fields of smaller magnitude being less extracted.

From an implementation viewpoint, we showed that the optimal contract for a concessionaire may be (but not always) implemented with a menu of license contracts. Contractors operating on fields with larger initial stocks receive a higher royalty per unit of output but must also pay upfront a greater license fee to have the right to extract the resource.

Finally, we have generalized our model to the case of a market of concessionaires. Under asymmetric information, constrained efficiency requires that all active fields at a given point in time operate at the same virtual marginal cost of extraction. Since those costs differ across firms with different initial stocks, all active firms are not producing the same amount even though their technologies for extraction are identical. We provide the virtual Hotelling rule in that environment and demonstrate that the market price converges faster to its long-run limit under asymmetric information. This gives us an important empirical check to assess the validity of our approach in future research.

Beyond this empirical aspect, our model also offers a building block that could certainly be extended into various directions that might impact on the magnitude of the rent-efficiency trade-off highlighted in our analysis.

ON LIMITED COMMITMENT, RENEGOTIATION AND EXPROPRIATION. The optimal contract found above requires credibility on the side of the public authority; a standard blessing in the theory of dynamic contracts.<sup>44</sup> Once information is revealed, which arises immediately when the firm chooses among the various options offered by the incentive menu, the extraction path  $S^{sb}(\theta, t)$  is certainly no longer optimal and the public authority would like to switch to the complete information path  $S^*(\theta, t)$ . Anticipating such move, the firm may be reluctant to report the initial stock of resource in the first place. We leave the analysis of those complex dynamics under limited commitment for future research.<sup>45</sup>

ENFORCEMENT AND WEAK INSTITUTIONAL ENVIRONMENTS. Renegotiation and limited enforcement of contractual provisions may be particularly important in the context of developing countries which delegate to foreign companies the management of their resource. In weak institutional environments with insecure property rights, the threat of expropriation and hold-up might induce concessionaires to extract resource early on.<sup>46</sup> This effect thus introduces countervailing incentives<sup>47</sup> that go counter the incentives to downplay the size of the stock. These countervailing incentives may thus limit the inefficient extraction stressed in our full commitment environment.

COMMON POOL. As pointed out by Gaudet (2007), countervailing incentives are also likely in the context of common pool extraction. When various concessionaires extract from the same field, they certainly tend to over-harvest.<sup>48</sup> This effect is likely to be reinforced when firms operating on the same fields have learned private information on the value of the pool. A phenomenon akin to the standard ‘winner’s curse’ in auction theory emerges in this context. This effect would tend to push extraction forward.

EXPLORATION AND LEARNING. The discussion above suggests that a relevant information structure in practice is that concessionaires may only have better signals than public authorities on the value of the field. When it operates alone on a field, a concessionaire may learn about the stock as it is explored. The process of information gathering is dynamic and linked to the production process. Gathering information pushes to extract early and introduces again some kind of countervailing incentives.

We plan to investigate some of these issues in future research.

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<sup>44</sup>See Dewatripont (1989) and Laffont and Tirole (1993, Chapter 9) among others.

<sup>45</sup>Gaudet, Lasserre and Van Long (1995) analyze a two-period model without commitment and private information on the extraction costs but the assumptions that those costs are drawn independently over time rules out the difficult issues of imperfect information revelation that would arise in our context where private information is persistent.

<sup>46</sup>On this issue, see Aghion and Quesada (2010) and Engel and Fischer (2010). These authors adopt a reduced-form approach and take both the resource price and the probability of expropriation as given.

<sup>47</sup>Lewis and Sappington (1989) and Maggi and Rodriguez-Clare (1995).

<sup>48</sup>See Libecap (1994) and Gaudet, Moreaux and Salant (2002) for analyses along this line.



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## APPENDIX

PROOF OF PROPOSITION 1. In order to determine in a rigorous way that convergence takes place in infinite time, we modify the statement of problem  $(\mathcal{P}^*(\theta))$  by introducing a free-end point  $T$ . Formally, the so modified problem  $(\tilde{\mathcal{P}}^*(\theta))$  can be written as a maximization problem with free-end point to which we apply Pontryagin Principle:<sup>49</sup>

$$(\tilde{\mathcal{P}}^*(\theta)) : \max_{T, q, S} \int_0^T (V(q(\theta, t)) - C(S(\theta, t))q(\theta, t))e^{-rt} dt$$

subject to (2.1) and (2.2).

The Hamiltonian associated to problem  $(\tilde{\mathcal{P}}^*(\theta))$  writes as follows:

$$\mathcal{H}(S, q, t, \lambda) = (V(q) - C(S)q)e^{-rt} - \lambda q$$

where  $\lambda$  is the costate variable for (2.2). Let  $(T^*(\theta), q^*(\theta), S^*(\theta))$  denote a maximizer of  $(\tilde{\mathcal{P}}^*(\theta))$ . The Hamiltonian  $\mathcal{H}(S, q, t, \lambda)$  is strictly concave in  $(S, q)$  and thus this maximizer is unique. Observe also that the regularity conditions of Theorem 16, p.244, in Seierstad and Sydsaeter (1987) all trivially hold. In particular, both the “growth conditions” (210) and (211) and the condition  $\int_0^{+\infty} e^{-rt} |V(q^*(\theta, t)) - C(S^*(\theta, t))q^*(\theta, t)| dt < +\infty$  are satisfied. The first of these conditions is satisfied because any admissible pair  $(q(t), S(t))$  is bounded above; the second one because the optimizer  $(q^*(\theta, t), S^*(\theta, t))$  is also bounded and because of exponential discounting. Moreover, the condition  $\int_0^{+\infty} q^*(\theta, t) dt = \theta - \lim_{t \rightarrow +\infty} S^*(\theta, t) < \theta < +\infty$  in this Theorem also holds thanks to the fact that extraction cannot exceed initial reserves. The necessary conditions for optimality that are satisfied by such maximizer (whether  $T^*(\theta)$  is finite or not) write thus as follows:

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<sup>49</sup>For a similar analysis, although mostly developed in the case of discrete time, we refer to Salant, Eswaran and Lewis (1983).

- *Costate variable.*

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial S}(S^*(\theta, t), q^*(\theta, t), t, \lambda(t))$$

which amounts to

$$(A.1) \quad \dot{\lambda}(t) = e^{-rt} q^*(\theta, t) C'(S^*(\theta, t)).$$

- *Control variable.*

$$q^*(\theta, t) \in \arg \max_{q \geq 0} \mathcal{H}(S^*(\theta, t), q, t, \lambda(t))$$

which amounts to

$$(A.2) \quad e^{-rt}(V'(q^*(\theta, t)) - C(S^*(\theta, t))) - \lambda(t) \leq 0 \text{ with } = 0 \text{ if } q^*(\theta, t) > 0.$$

- *Free-end point and transversality conditions.* Following Note 21 in Seierstad and Sydsaeter (1987, p.245), those conditions are expressed as:

$$\lim_{t \rightarrow T^*(\theta)} \lambda(t) = 0 \text{ and } \lim_{t \rightarrow T^*(\theta)} e^{-rt}(V(q^*(\theta, t)) - C(S^*(\theta, t))q^*(\theta, t)) = 0.$$

In more details, these conditions become:

1. If  $T^*(\theta) < +\infty$ , then we should have:

$$(A.3) \quad \lambda(T^*(\theta)) = 0 \text{ and } V(q^*(\theta, T^*(\theta))) - C(S^*(\theta, T^*(\theta)))q^*(\theta, T^*(\theta)) = 0.$$

2. If  $T^*(\theta) = +\infty$ , then we should have:

$$(A.4) \quad \lim_{t \rightarrow +\infty} \lambda(t) = 0 \text{ and } \lim_{t \rightarrow +\infty} e^{-rt}(V(q^*(\theta, t)) - C(S^*(\theta, t))q^*(\theta, t)) = 0.$$

Because the Hamiltonian  $\mathcal{H}(S, q, t, \lambda)$  is strictly concave in  $(S, q)$  and because the condition  $\lim_{t \rightarrow T^*(\theta)} \lambda(t)(S(t) - S^*(\theta, t)) \geq 0$  trivially holds for all admissible paths  $S(t)$  when the necessary conditions above hold (and especially (A.4)), those necessary conditions are also sufficient conditions of the Mangasarian type (Seierstad and Sydsaeter (1987, Theorem 13, p. 234).

Several facts follow from those optimality conditions.

1. First, observe that differentiating (A.2) with respect to  $t$  when  $q^*(\theta, t) > 0$  yields:

$$(A.5) \quad \dot{\lambda}(t) + r\lambda(t) = e^{-rt} \left( V''(q^*(\theta, t)) \frac{\partial q^*}{\partial t}(\theta, t) + C'(S^*(\theta, t))q^*(\theta, t) \right).$$

Using (A.1) and (A.2) when  $q^*(\theta, t) > 0$  and simplifying yields:

$$(A.6) \quad V''(q^*(\theta, t)) \frac{\partial q^*}{\partial t}(\theta, t) = r(V'(q^*(\theta, t)) - C(S^*(\theta, t)))$$

which can be written as (3.2).

2. If  $T^*(\theta) < +\infty$  and  $q^*(\theta, T^*(\theta)) > 0$ , then (A.2) and (A.3) imply:

$$V(q^*(\theta, T^*(\theta))) = q^*(\theta, T^*(\theta))V'(q^*(\theta, T^*(\theta)))$$

which requires  $q^*(\theta, T^*(\theta)) = 0$ , a contradiction. Thus, if  $T^*(\theta) < +\infty$ , we must have  $q^*(\theta, T^*(\theta)) = 0$ . Then, (A.2) together with  $\lambda(T^*(\theta)) = 0$  from (A.3) also imply  $V'(0) - C(S^*(\theta, T^*(\theta))) \leq 0$  and thus  $S^*(\theta, T^*(\theta)) \leq S_{inf}$ . But  $S^*(\theta, 0) = \theta \geq S_{inf}$ , by assumption. Because  $S^*(\theta, t)$  is non-increasing, there exists  $t_0 \leq T^*(\theta)$  such that  $S^*(\theta, t_0) = S_{inf}$ . Hence, from (A.2) and the fact that  $\lambda(t)$  (being non-increasing and zero at  $T^*(\theta)$ ) remains non-negative, we deduce that  $q^*(\theta, T^*(\theta)) = 0$  over  $[t_0, T^*(\theta)]$ . There is thus no loss of generality in considering that  $T^*(\theta)$  is the lowest time at which  $q^*(\theta, t) = 0$ . Taking left-side limit of (A.2) at  $T^*(\theta)$  immediately gives that  $S^*(\theta, T^*(\theta)) = S_{inf}$ .

3. For the optimal path of resource extraction, we rewrite (2.2) as follows:

$$(A.7) \quad \frac{\partial S^*}{\partial t}(\theta, t) = -q^*(\theta, t).$$

We also rewrite (A.6) as:

$$(A.8) \quad \frac{\partial q^*}{\partial t}(\theta, t) = \frac{r}{V''(q^*(\theta, t))}(V'(q^*(\theta, t)) - C(S^*(\theta, t))).$$

This defines  $q^*(\theta, t)$  for  $q^*(\theta, t) > 0$ , otherwise  $C(S^*(\theta, t)) = V'(q^*(\theta, t)) = P(0)$ .

Observe that the system made of (A.7) and (A.8) admits a unique stationary point  $(S_{inf}, 0)$ . We now prove that this system together with the initial condition  $(S^*(\theta, 0), q^*(\theta, 0)) = (\theta, q^*(\theta, 0))$  converges towards this stationary point in infinite time.

If  $T^*(\theta) < +\infty$ , Item 2 above shows that the stationary point is achieved at  $T^*(\theta)$ . Suppose now that, for  $t < T^*(\theta)$  but close enough,  $q^*(\theta, t) > 0$  and thus  $S^*(\theta, t) > S_{inf}$ . Then using Item 1 above shows that (A.6) holds, and thus so does (A.8). But by the Cauchy-Lipschitz Theorem, there is a unique solution to the system made of (A.7) and (A.8) with the initial condition  $(S^*(\theta, T^*(\theta)), q^*(\theta, T^*(\theta))) = (S_{inf}, 0)$  and this is  $(S^*(\theta, t), q^*(\theta, t)) = (S_{inf}, 0)$  itself. But this contradicts the boundary initial condition  $S^*(\theta, 0) = \theta > S_{inf}$ . Hence, the system cannot converge in finite time towards  $(S_{inf}, 0)$  and  $q^*(\theta, t)$  remains strictly positive.

We deduce from this last fact that  $S^*(\theta, t)$  is decreasing over time. Since it is bounded below, it converges. We must have from (A.1) that  $\lambda(t)$  is decreasing, and, from (A.4), that it remains non-negative. Since  $q^*(\theta, t)$  remains positive, we also deduce from (A.2) that  $V'(q^*(\theta, t)) - C(S^*(\theta, t)) \geq 0$ . From (A.8),  $q^*(\theta, t)$  is thus also decreasing, and since it is positive, it converges. Hence, the system made of (A.7) and (A.8) admits a limit, which is necessarily the unique stationary point  $(S_{inf}, 0)$ .

Finally, gathering everything, the system (A.7)-(A.8) converges towards  $(S_{inf}, 0)$  in infinite time. Hence, (3.1).

4. For future reference, we propose an alternative description of the solution to the system (A.7)-(A.8). The first step is to notice that (A.6) can be rewritten as:

$$(A.9) \quad V''(q^*(\theta, t)) \frac{\partial q^*}{\partial t}(\theta, t) - rV'(q^*(\theta, t)) = e^{rt} \frac{d}{dt} (V'(q^*(\theta, t))e^{-rt}) = -rC(S^*(\theta, t)).$$

Integrating yields:

$$(A.10) \quad V'(q^*(\theta, t)) = re^{rt} \int_t^{+\infty} C(S^*(\theta, \tau)) e^{-r\tau} d\tau.$$

Using now (A.7), we find that  $S^*(\theta, t)$  must solve the following differential equation together with the initial condition  $S^*(\theta, 0) = \theta$  and the limit behavior (3.1):

$$(A.11) \quad \frac{\partial S^*}{\partial t}(\theta, t) = -(V')^{-1} \left( re^{rt} \int_t^{+\infty} C(S^*(\theta, \tau)) e^{-r\tau} d\tau \right).$$

5. For future reference, we observe that the system (A.7)-(A.8) is autonomous in time:  $q$  can be expressed in terms of  $S$  along the optimal trajectory through a simple differential equation. Denote this trajectory by the function  $q = \Omega^*(S)$ . Taking the ratio between (A.7) and (A.8) (and slightly abusing notations), we obtain that  $\Omega^*(\cdot)$  solves the first-order differential equation:

$$(A.12) \quad \Omega^*(S) V''(\Omega^*(S)) \frac{d\Omega^*}{dS}(S) + r V'(\Omega^*(S)) = r C(S)$$

where  $S \in [S_{inf}, \theta]$ .

Defining the elasticity of the inverse demand function as  $\varepsilon(q) = -\frac{qP'(q)}{P(q)} = -\frac{qV''(q)}{V'(q)}$ , and by  $\varphi(\cdot)$  the inverse function of  $\int_0^q \varepsilon(\tilde{q}) d\tilde{q}$ , the solution to (A.12) is of the form:

$$(A.13) \quad \Omega^*(S) = \varphi(r(S - k^*(S)))$$

where  $k^*(S)$  solves

$$(A.14) \quad \frac{dk^*}{dS}(S) = \frac{C(S)}{V'(\varphi(r(S - k^*(S))))}$$

with the boundary condition  $k^*(S_{inf}) = S_{inf}$  since  $\Omega^*(S_{inf}) = 0$ . Observe that  $\frac{dk^*}{dS}(S_{inf}) = 1$  so that the Cauchy-Lipschitz Theorem applies and such solution is unique.

Finally, using the definition of  $k^*(S)$  and (2.2) gives that  $S^*(\theta, t)$  solves:

$$(A.15) \quad \frac{\partial S^*}{\partial t}(\theta, t) = -\varphi(r(S^*(\theta, t) - k^*(S^*(\theta, t))))$$

with the initial condition  $S^*(\theta, 0) = \theta$ .

**RUNNING EXAMPLE.** Observe that we may rewrite (A.12) as a differential equation for the inverse function  $\mathcal{S}^*(\Omega)$  for  $\Omega^*(S)$ , namely:

$$(A.16) \quad \frac{\partial \mathcal{S}^*}{\partial \Omega}(\Omega) = -\frac{\Omega V''(\Omega)}{r(V'(\Omega) - C(\mathcal{S}^*(\Omega)))}.$$

For the specific functional forms of the RUNNING EXAMPLE, (A.16) becomes:

$$(A.17) \quad \frac{\partial \mathcal{S}^*}{\partial \Omega}(\Omega) = -\frac{1}{r \left(1 - \kappa^* \frac{\mathcal{S}^*(\Omega) - S_{inf}}{q}\right)}$$

where  $\kappa^* = \frac{C'(S_{inf})}{P'(0)} > 0$ . This is a first-order differential equation of the Clairaut kind. It admits a singular solution, and this is the only one satisfying the terminal condition, such that:

$$(A.18) \quad \mathcal{S}^*(\Omega) - S_{inf} = z^* \Omega \quad \text{or} \quad \frac{q^*}{\mathcal{S}^*(\Omega) - S_{inf}} = \frac{1}{z^*}$$

for some constant  $z^* > 0$ . Inserting into (A.17),  $z^*$  must satisfy:

$$(A.19) \quad z^* = -\frac{1}{r(1 - \kappa^* z^*)} \quad \text{or} \quad \frac{1}{z^*} = r(\kappa^* z^* - 1),$$

and, selecting the relevant solution to this second-degree equation, we obtain:

$$(A.20) \quad z^* = \frac{1}{2\kappa^*} \left(1 + \sqrt{1 + \frac{4\kappa^*}{r}}\right).$$

Using (A.18) and inserting into (A.7) and noticing that  $\mathcal{S}^*(q^*(\theta, t)) = S^*(\theta, t)$ , we obtain

$$(A.21) \quad \frac{\partial S^*}{\partial t}(\theta, t) = -\frac{S^*(\theta, t) - S_{inf}}{z^*}$$

which can be integrated to obtain (3.3). Inserting then (3.3) into (A.18) again, we find (3.4).  $\square$

**PROOF OF LEMMA 1.** Absolute continuity follows from an argument in Milgrom and Segal (2002). The envelope condition for the maximization problem (4.2) then gives us (4.3).

We now turn to sufficiency. Absolute continuity allows us to use the following integral representation of  $U(\theta)$ :

$$(A.22) \quad U(\theta) - U(\hat{\theta}) = -\int_{\hat{\theta}}^{\theta} \left( \int_0^{+\infty} C'(S(\tilde{\theta}, t)) q(\tilde{\theta}, t) e^{-rt} dt \right) d\tilde{\theta}.$$

Incentive compatibility requiring that for all pairs  $(\theta, \hat{\theta})$ , the following string of inequalities holds:

$$\begin{aligned} U(\theta) &= \int_0^{+\infty} (\omega(\theta, t) - C(S(\theta, t)) q(\theta, t)) e^{-rt} dt \\ &\geq \int_0^{+\infty} (\omega(\hat{\theta}, t) - C(\hat{S}(\theta, \hat{\theta}, t)) q(\hat{\theta}, t)) e^{-rt} dt \\ &= U(\hat{\theta}) + \int_0^{+\infty} \left( C(\hat{\theta} - Q(\hat{\theta}, t)) - C(\theta - Q(\hat{\theta}, t)) \right) q(\hat{\theta}, t) e^{-rt} dt. \end{aligned}$$



Using (A.22), the latter condition holds when:

$$\begin{aligned} \int_0^{+\infty} \left( C(\theta - Q(\hat{\theta}, t)) - C(\hat{\theta} - Q(\hat{\theta}, t)) \right) q(\hat{\theta}, t) e^{-rt} dt \\ \geq \int_{\hat{\theta}}^{\theta} \left( \int_0^{+\infty} C'(S(\tilde{\theta}, t)) q(\tilde{\theta}, t) e^{-rt} dt \right) d\tilde{\theta}. \end{aligned}$$

This last condition can be rewritten as:

$$\begin{aligned} \text{(A.23)} \quad & - \int_{\theta}^{\hat{\theta}} \left( \int_0^{+\infty} C'(\tilde{\theta} - Q(\hat{\theta}, t)) q(\hat{\theta}, t) e^{-rt} dt \right) d\tilde{\theta} \\ & \geq - \int_{\theta}^{\hat{\theta}} \left( \int_0^{+\infty} C'(S(\tilde{\theta}, t)) q(\tilde{\theta}, t) e^{-rt} dt \right) d\tilde{\theta} \end{aligned}$$

or

$$- \int_{\theta}^{\hat{\theta}} \left( \int_0^{+\infty} \left( C'(\tilde{\theta} - Q(\hat{\theta}, t)) q(\hat{\theta}, t) - C'(\tilde{\theta} - Q(\tilde{\theta}, t)) q(\tilde{\theta}, t) \right) e^{-rt} dt \right) d\tilde{\theta} \geq 0.$$

We integrate by parts the left-hand side to obtain:

$$r \int_{\theta}^{\hat{\theta}} \left( \int_0^{+\infty} \left( C(\tilde{\theta} - Q(\hat{\theta}, t)) - C(\tilde{\theta} - Q(\tilde{\theta}, t)) \right) e^{-rt} dt \right) d\tilde{\theta}.$$

Hence, we may rewrite (A.23) as:

$$\int_{\theta}^{\hat{\theta}} \left( \int_0^{+\infty} \left( \int_{Q(\hat{\theta}, t)}^{Q(\tilde{\theta}, t)} C'(\tilde{\theta} - \tilde{Q}) d\tilde{Q} \right) e^{-rt} dt \right) d\tilde{\theta} \geq 0.$$

Using Fubini Theorem, (A.23) holds for all pairs  $(\theta, \hat{\theta})$  if and only if

$$\text{(A.24)} \quad \int_0^{+\infty} \left( \int_{\theta}^{\hat{\theta}} \left( \int_{Q(\hat{\theta}, t)}^{Q(\tilde{\theta}, t)} C'(\tilde{\theta} - \tilde{Q}) d\tilde{Q} \right) d\tilde{\theta} \right) e^{-rt} dt \geq 0 \quad \forall (\theta, \hat{\theta}) \in \Theta^2.$$

Because  $C(\cdot)$  is convex, the latter condition holds provided that the monotonicity condition (4.4) is satisfied.  $\square$

PROOF OF LEMMA 2. Differentiating (4.7) with respect to  $\theta$  yields:

$$\frac{d\tilde{S}}{d\theta}(\theta) = \frac{\frac{d}{d\theta} \left( \frac{1-F(\theta)}{f(\theta)} \right) C'(\tilde{S}(\theta))}{C'(\tilde{S}(\theta)) - \frac{1-F(\theta)}{f(\theta)} C''(\tilde{S}(\theta))}.$$

Since  $\frac{d}{d\theta} \left( \frac{1-F(\theta)}{f(\theta)} \right) \leq 0$  and  $C''(\cdot) \geq 0 > C'(\cdot)$ , we conclude that  $\frac{d\tilde{S}}{d\theta}(\theta) < 0$ .  $\square$

PROOF OF PROPOSITION 2. The proof is almost identical to the one of Proposition 1 provided that  $C(S)$  and  $S_{inf}$  are replaced by  $\tilde{C}(\theta, S)$  and  $\tilde{S}_{inf}(\theta)$  respectively. Details are thus omitted. The only minor difference comes from Item 3, which has to be rewritten with some care. Indeed, for the optimal path of resource extraction, we may again rewrite

(2.2) as follows:

$$(A.25) \quad \frac{\partial S^{sb}}{\partial t}(\theta, t) = -q^{sb}(\theta, t).$$

Replacing with  $\tilde{C}(\theta, S)$ , (A.8) becomes:

$$(A.26) \quad \frac{\partial q^{sb}}{\partial t}(\theta, t) = \frac{r}{V''(q^{sb}(\theta, t))} \left( V'(q^{sb}(\theta, t)) - \tilde{C}(\theta, S^{sb}(\theta, t)) \right).$$

The system made of (A.25) and (A.26) admits a limit, which is necessarily the unique stationary point  $(\tilde{S}_{inf}(\theta), 0)$ . If convergence towards that point occurs in finite time, then  $T^{sb}(\theta) < +\infty$ , and we now prove that output is necessarily null along the whole trajectory. Indeed, by Cauchy-Lipschitz Theorem, there is a unique solution to the system made of (A.25) and (A.26) with the initial condition  $(S^{sb}(\theta, T^{sb}(\theta)), q^{sb}(\theta, T^{sb}(\theta))) = (\tilde{S}_{inf}(\theta), 0)$  and this is  $(S^{sb}(\theta, t), q^{sb}(\theta, t)) = (\tilde{S}_{inf}(\theta), 0)$  itself. But this contradicts the initial condition  $S^{sb}(\theta, 0) = \theta > \underline{\theta}$  if  $\theta > \tilde{S}_{inf}(\theta)$ . Hence, when  $\theta > \tilde{S}_{inf}(\theta)$ , the system cannot converge in finite time towards  $(\tilde{S}_{inf}(\theta), 0)$  and  $q^{sb}(\theta, t)$  remains positive. Therefore, the system (A.25)-(A.26) converges towards  $(\tilde{S}_{inf}(\theta), 0)$  in infinite time when  $\theta > \tilde{S}_{inf}(\theta)$ .

Suppose instead that  $\theta \leq \tilde{S}_{inf}(\theta)$ , or equivalently (because  $\tilde{C}(\theta, S)$  is decreasing in  $S$ ) that  $\tilde{C}(\theta, \theta) \geq \tilde{C}(\theta, \tilde{S}_{inf}(\theta)) = P(0)$ . Extraction is then not valuable even for the first unit and no extraction occurs for these lower values of  $\theta$ .  $\square$

PROOF OF PROPOSITION 3. Remind the following definition for  $k^*(S)$  from (A.14) and  $k^{sb}(\theta, S)$  (obtained mutatis mudandis after noticing that the function  $\varphi(\cdot)$  is relevant in both cases):

$$(A.27) \quad \frac{dk^*}{dS}(S) = \frac{C(S)}{V'(\varphi(r(S - k^*(S))))} \quad \text{with } k^*(S_{inf}) = S_{inf},$$

$$(A.28) \quad \frac{\partial k^{sb}}{\partial S}(\theta, S) = \frac{\tilde{C}(\theta, S)}{V'(\varphi(r(S - k^{sb}(\theta, S))))} \quad \text{with } k^{sb}(\theta, \tilde{S}_{inf}(\theta)) = \tilde{S}_{inf}(\theta).^{50}$$

Observe that  $k^*(S)$  is defined over the domain  $S \in [S_{inf}; \bar{\theta}]$  whereas  $k^{sb}(\theta, S)$  is defined over the domain  $S \in [\tilde{S}_{inf}(\theta); \theta]$ . From this, we construct  $S^*(\theta, t)$  and  $S^{sb}(\theta, t)$  as solutions to:

$$(A.29) \quad \frac{\partial S^*}{\partial t}(\theta, t) = -\varphi(r(S^*(\theta, t) - k^*(S^*(\theta, t)))),$$

$$(A.30) \quad \frac{\partial S^{sb}}{\partial t}(\theta, t) = -\varphi(r(S^{sb}(\theta, t) - k^{sb}(\theta, S^{sb}(\theta, t)))).$$

with the same initial condition  $S^*(\theta, 0) = S^{sb}(\theta, 0) = \theta$ .

LEMMA A.1.

$$(A.31) \quad k^{sb}(\theta, S) \geq k^*(S) \quad \forall S \in [\tilde{S}_{inf}(\theta), \theta]$$

with equality only for  $\theta = \bar{\theta}$ , in which case  $k^{sb}(\bar{\theta}, S) = k^*(S)$  for all  $S \in [S_{inf}, \bar{\theta}]$ .

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<sup>50</sup>This boundary condition being now function of  $\theta$ , we make the dependency of  $k^{sb}$  on  $\theta$  explicit.

PROOF OF LEMMA A.1. Because  $\tilde{S}_{inf}(\theta) \geq S_{inf}$  with equality only at  $\theta = \bar{\theta}$ , (A.31) certainly holds as an equality at  $\theta = \bar{\theta}$  and for all  $S$ .

Consider now the case  $\theta < \bar{\theta}$ . Observe that  $\tilde{S}_{inf}(\theta) > S_{inf}$  implies that  $k^{sb}(\theta, \tilde{S}_{inf}(\theta)) > k^*(\tilde{S}_{inf}(\theta)) > k^*(S_{inf}) = S_{inf}$  since  $k^*(\cdot)$  is increasing. Moreover, by continuity,  $k^{sb}(\theta, S) > k^*(S)$  in a right-neighborhood of  $\tilde{S}_{inf}(\theta)$ . Suppose also that there exists  $S_0 \in [\tilde{S}_{inf}(\theta), \theta]$  such that  $k^{sb}(\theta, S_0) = k^*(S_0)$ , and, if there are more than one such values, then just denote by  $S_0$  the lowest one. By construction,  $S_0 > \tilde{S}_{inf}(\theta) > S_{inf}$ . At  $S_0$ , we have:

$$\frac{dk^*}{dS}(S_0) = \frac{C(S_0)}{V'(\varphi(r(S_0 - k^*(S_0))))} < \frac{\tilde{C}(\theta, S_0)}{V'(\varphi(r(S_0 - k^*(S_0))))}.$$

Therefore,  $k^{sb}(\theta, S) < k^*(S)$  for  $S$  in a left-neighborhood of  $S_0$ : a contradiction with the definition of  $S_0$ .  $\square$

From Lemma A.1, (A.29) and (A.30), it follows that:

$$(A.32) \quad \frac{\partial S^*}{\partial t}(\theta, t) = -q^*(\theta, t) < \frac{\partial S^{sb}}{\partial t}(\theta, t) = -q^{sb}(\theta, t) < 0.$$

Hence, (4.19). Taking into account that  $S^*(\theta, 0) = S^{sb}(\theta, 0) = \theta$  yields (4.18).  $\square$

PROOF OF PROPOSITION 4. As a preliminary, we prove the following Lemma.

LEMMA A.2.  $k^{sb}$  satisfies the following properties:

1.  $k^{sb}(\theta, S) < S$ ,  $\forall S \in (\tilde{S}_{inf}(\theta), \theta]$ ;
2.  $\frac{\partial k^{sb}}{\partial S}(\theta, S) \leq 1$ ,  $\forall S \in [\tilde{S}_{inf}(\theta), \theta]$ , and with equality at  $\tilde{S}_{inf}(\theta)$  only;
3. For all  $\theta_1 > \theta_2$ ,  $k^{sb}(\theta_1, S) < k^{sb}(\theta_2, S)$  for all  $S \in [\tilde{S}_{inf}(\theta_2), \theta_2]$ .

PROOF OF LEMMA A.2.

1. The first item immediately follows from the fact that  $q^{sb}(\theta, t) > 0$  along any extraction path and from the fact that (adapting (A.12) to the case of asymmetric information):

$$(A.33) \quad q^{sb}(\theta, t) = \Omega^{sb}(\theta, S^{sb}(\theta, t)) = \varphi(r(S^{sb}(\theta, t) - k^{sb}(\theta, S^{sb}(\theta, t))))$$

where  $\Omega^{sb}(\theta, S)$  solves:

$$(A.34) \quad \Omega^{sb}(\theta, S)V''(\Omega^{sb}(\theta, S))\frac{\partial \Omega^{sb}}{\partial S}(\theta, S) + rV'(\Omega^{sb}(\theta, S)) = r\tilde{C}(\theta, S).$$

2. Under asymmetric information, (A.1), (A.2) and (A.4) are respectively replaced by:

$$(A.35) \quad \dot{\lambda}(t) = e^{-rt}q^{sb}(\theta, t)\frac{\partial \tilde{C}}{\partial S}(\theta, S^{sb}(\theta, t))$$

and

$$(A.36) \quad e^{-rt}(V'(q^{sb}(\theta, t)) - \tilde{C}(\theta, S^{sb}(\theta, t))) - \lambda(t) \leq 0 \text{ with } = 0 \text{ if } q^{sb}(\theta, t) > 0,$$

and

$$(A.37) \quad \lim_{t \rightarrow +\infty} \lambda(t) = 0 \text{ and } \lim_{t \rightarrow +\infty} e^{-rt}(V(q^{sb}(\theta, t)) - \tilde{C}(\theta, S^{sb}(\theta, t)))q^{sb}(\theta, t) = 0.$$

From (A.1) and (A.37), we deduce that  $\lambda(t) \geq 0$  for all  $t$  and inserting into (A.36) gives:

$$V'(q^{sb}(\theta, t)) \geq \tilde{C}(\theta, S^{sb}(\theta, t)).$$

Using (A.33) finally yields that for any  $S \in [\tilde{S}_{inf}(\theta), \theta]$  (which is the range of  $S^{sb}(\theta, \cdot)$ )

$$V'(\varphi(r(S - k^{sb}(\theta, S)))) \geq \tilde{C}(\theta, S).$$

Inserting into (A.28) shows that  $\frac{\partial k^{sb}}{\partial S}(\theta, S) \leq 1$  for all such  $S \in [\tilde{S}_{inf}(\theta), \theta]$ .

3. Consider  $\theta_1 > \theta_2$  and the corresponding solutions to (A.28), say  $k^{sb}(\theta_1, S)$  and  $k^{sb}(\theta_2, S)$ , which have respective domains  $[\tilde{S}_{inf}(\theta_1), \theta_1]$  and  $[\tilde{S}_{inf}(\theta_2), \theta_2]$ . Because  $\tilde{S}_{inf}(\cdot)$  is decreasing,  $\tilde{S}_{inf}(\theta_1) < \tilde{S}_{inf}(\theta_2)$  and  $[\tilde{S}_{inf}(\theta_2), \theta_2] \subset [\tilde{S}_{inf}(\theta_1), \theta_1]$ . In particular, Item 1 implies that  $k^{sb}(\theta_1, \tilde{S}_{inf}(\theta_2)) < \tilde{S}_{inf}(\theta_2) = k^{sb}(\theta_2, \tilde{S}_{inf}(\theta_2))$ . So the property  $k^{sb}(\theta_1, S) < k^{sb}(\theta_2, S)$  holds for  $S = \tilde{S}_{inf}(\theta_2)$ . Suppose thus that there exists  $S_0 \in (\tilde{S}_{inf}(\theta_2), \theta_2]$  such that  $k^{sb}(\theta_1, S_0) = k^{sb}(\theta_2, S_0)$  and consider the smallest such value. At that point, we would have:

$$\frac{\partial k^{sb}}{\partial S}(\theta_1, S_0) = \frac{\tilde{C}(\theta_1, S_0)}{V'(\varphi(r(S_0 - k^{sb}(\theta_1, S_0))))} < \frac{\tilde{C}(\theta_2, S_0)}{V'(\varphi(r(S_0 - k^{sb}(\theta_2, S_0))))} = \frac{\partial k^{sb}}{\partial S}(\theta_2, S_0)$$

where the strict inequality follows from the fact that  $\tilde{C}(\theta, S)$  is decreasing in  $\theta$ . We deduce from the above inequality that  $k^{sb}(\theta_1, S) > k^{sb}(\theta_2, S)$  in a right-neighborhood of  $S_0$ ; a contradiction with the definition of  $S_0$ . This proves the last item. □

Observe now that:

$$(A.38) \quad Q^{sb}(\theta, 0) = 0,$$

$$(A.39) \quad \frac{\partial Q^{sb}}{\partial t}(\theta, t) = -\frac{\partial S^{sb}}{\partial t}(\theta, t) = \varphi(r(\theta - Q^{sb}(\theta, t) - k^{sb}(\theta, \theta - Q^{sb}(\theta, t)))).$$

Hence, we get:

$$(A.40) \quad \frac{\partial Q^{sb}}{\partial t}(\theta, 0) = \varphi(r(\theta - k^{sb}(\theta, \theta))).$$

Take now  $\theta_1 > \theta_2$  and observe that:

$$\theta_1 - k^{sb}(\theta_1, \theta_1) > \theta_2 - k^{sb}(\theta_2, \theta_2)$$

amounts to:

$$\theta_1 - \theta_2 > k^{sb}(\theta_1, \theta_1) - k^{sb}(\theta_1, \theta_2) + (k^{sb}(\theta_1, \theta_2) - k^{sb}(\theta_2, \theta_2)).$$

From Item 2 in Lemma A.2, the last bracketed term on the right-hand side is negative.

Hence, it suffices to prove that:

$$(A.41) \quad \theta_1 - \theta_2 > k^{sb}(\theta_1, \theta_1) - k^{sb}(\theta_1, \theta_2).$$

But this inequality immediately follows from Item 1 in Lemma A.2. Inserting into (A.40), we deduce that:

$$(A.42) \quad \frac{\partial^2 Q^{sb}}{\partial \theta \partial t}(\theta, 0) > 0.$$

From (A.38), we deduce that, locally around  $t = 0$ ,

$$(A.43) \quad \frac{\partial Q^{sb}}{\partial \theta}(\theta, t) > 0.$$

Consider now two values  $\theta_1 < \theta_2$ . From (A.38) and (A.43), locally around  $t = 0$ ,  $Q^{sb}(\theta_1, t) \leq Q^{sb}(\theta_2, t)$  with equality only at  $t = 0$ . Suppose that there exists  $t_0 > 0$  such that  $Q^{sb}(\theta_1, t_0) = Q^{sb}(\theta_2, t_0)$  and, as previously, take the lowest such value if there are several. Using (A.39), we obtain:

$$(A.44) \quad \frac{\partial Q^{sb}}{\partial t}(\theta_1, t_0) = \varphi(r(\theta_1 - Q^{sb}(\theta_1, t_0) - k^{sb}(\theta_1, \theta_1 - Q^{sb}(\theta_1, t_0)))).$$

Using arguments similar to those used to prove (A.41), we can prove that:

$$(A.45) \quad \frac{\partial}{\partial \theta}(\theta - Q - k^{sb}(\theta, \theta - Q)) > 0$$

for any  $Q$ . Using (A.45) and (A.44) for  $Q = Q^{sb}(\theta_1, t_0) = Q^{sb}(\theta_2, t_0)$  leads to:

$$\frac{\partial Q^{sb}}{\partial t}(\theta_1, t_0) < \frac{\partial Q^{sb}}{\partial t}(\theta_2, t_0).$$

Thus, for  $t < t_0$  but close enough, we would have  $Q^{sb}(\theta_1, t) > Q^{sb}(\theta_2, t)$ : a contradiction. This proves that (4.20) holds.  $\square$

**PROOF OF PROPOSITION 5.** The nonlinear schedule  $T(q_0)$  implements the optimal choice  $q_0^{sb}(\theta)$  when Problems (4.21) and (4.22) have the same optimality condition. The first-order condition for (4.21) writes as follows:

$$T'(q_0^{sb}(\theta)) = \frac{\partial \mathcal{C}}{\partial q_0}(\theta, q_0^{sb}(\theta)),$$

which may be rewritten using the functional forms of our RUNNING EXAMPLE as follows:

$$T'(q_0^{sb}(\theta)) = \frac{C(S_{inf}) + C'(S_{inf})(\theta - S^*)}{r + \frac{1}{z^*}} - \frac{2C'(S_{inf})q_0^{sb}(\theta)}{(r + \frac{1}{z^*})(r + \frac{2}{z^*})}.$$

Observe that  $T'(q_0^{sb}(\theta)) > 0$  since the two terms above are positive.

The first-order condition for (4.22) is:

$$\mathcal{V}'(q_0^{sb}(\theta)) = \frac{\partial \mathcal{C}}{\partial q}(\theta, q_0^{sb}(\theta)) + \frac{1 - F(\theta)}{f(\theta)} \frac{\partial^2 \mathcal{C}}{\partial \theta \partial q}(\theta, q_0^{sb}(\theta))$$

or, using again the linearity of the marginal cost function,

$$\mathcal{V}'(q_0^{sb}(\theta)) = \frac{C(S_{inf}) + C'(S_{inf})(\theta - S_{inf})}{r + \frac{1}{z^*}} - \frac{2C'(S_{inf})q_0^{sb}(\theta)}{\left(r + \frac{1}{z^*}\right)\left(r + \frac{2}{z^*}\right)} - \frac{1 - F(\theta)}{f(\theta)} \frac{C'(S_{inf})}{r + \frac{1}{z^*}}.$$

Identifying these two first-order conditions requires that, for all  $\theta \in \Theta$ , the following identity holds:

$$T'(q_0^{sb}(\theta)) = \mathcal{V}'(q_0^{sb}(\theta)) + \frac{1 - F(\theta)}{f(\theta)} \frac{C'(S_{inf})}{r + \frac{1}{z^*}} = \frac{\partial \mathcal{C}}{\partial q}(\theta, q_0^{sb}(\theta)).$$

With the functional forms of the RUNNING EXAMPLE, we have:

$$\mathcal{V}'(q_0) = \int_0^{+\infty} P\left(q_0 e^{-\frac{t}{z^*}}\right) e^{-\frac{t}{z^*}} e^{-rt} dt = \frac{P(0)}{r + \frac{1}{z^*}} + \frac{P'(0)q_0}{r + \frac{2}{z^*}}.$$

Eliminating  $\theta$  from the above expression immediately gives the following expression of the marginal reward:

$$(A.46) \quad T'(q_0) = \mathcal{V}'(q_0) + \frac{1 - F(\vartheta_0(q_0))}{f(\vartheta_0(q_0))} \frac{C'(S_{inf})}{r + \frac{1}{z^*}}.$$

Finally, the value of  $T(q_0)$  is anchored by the fact that the firm with initial stock  $\underline{\theta}$  makes no profit, that is:

$$(A.47) \quad T(q_0^{sb}(\underline{\theta})) = \mathcal{C}(\underline{\theta}, q_0^{sb}(\underline{\theta})).$$

Integrating (A.46) and taking into account (A.47) yields (4.23).  $\square$

**PROOF OF PROPOSITION 6.** The public authority's problem can now be written as:

$$(A.48) \quad \begin{aligned} (\mathcal{P}^*) : \quad & \max_{T, q \geq 0, S, \mathcal{Q}} \int_0^{+\infty} \left( V(\mathcal{Q}(t)) - \int_{\underline{\theta}}^{\bar{\theta}} C(S(\theta, t)) q(\theta, t) f(\theta) d\theta \right) e^{-rt} dt \\ & \text{subject to (2.1), (2.2) and} \\ & \int_{\underline{\theta}}^{\bar{\theta}} q(\theta, t) f(\theta) d\theta = \mathcal{Q}(t). \end{aligned}$$

Let denote by  $p^*(t)$  the multiplier of the feasibility condition (A.48). Optimality with respect to  $\mathcal{Q}(t)$  immediately gives (5.1). Denoting by  $\lambda(\theta, t)$  the costate variable for (2.2), the necessary conditions for optimality (which, again, turn out to be sufficient) can be written as follows.

- *Costate variable.*

$$(A.49) \quad \frac{\partial \lambda}{\partial t}(\theta, t) = e^{-rt} q^*(\theta, t) C'(S^*(\theta, t)).$$

- *Control variable.*

$$(A.50) \quad e^{-rt} (p^*(t) - C(S^*(\theta, t))) - \lambda(\theta, t) \leq 0 \text{ with } = 0 \text{ if } q^*(\theta, t) > 0.$$

- *Free-end point and transversality condition.* Let denote by  $\bar{T}^*(\theta) = \sup\{t \geq 0 \text{ such that } q^*(\theta, t) > 0\}$  and  $\underline{T}^*(\theta) = \inf\{t \geq 0 \text{ such that } q^*(\theta, t) > 0\}$ .

1. If  $\bar{T}^*(\theta) < +\infty$ , then we have:

$$(A.51) \quad \lambda(\theta, \bar{T}^*(\theta)) = 0 \text{ and } p^*(\bar{T}^*(\theta)) - C(S^*(\theta, \bar{T}^*(\theta)))q^*(\theta, \bar{T}^*(\theta)) = 0.$$

2. If  $\bar{T}^*(\theta) = +\infty$ , then we have:

$$(A.52) \quad \lim_{t \rightarrow +\infty} \lambda(\theta, t) = 0 \text{ and } \lim_{t \rightarrow +\infty} e^{-rt}(p^*(t) - C(S^*(\theta, t)))q^*(\theta, t) = 0.$$

3. If  $\underline{T}^*(\theta) > 0$ , then we have:

$$(A.53) \quad e^{-r\underline{T}^*(\theta)}(p^*(\underline{T}^*(\theta)) - C(S^*(\theta, \underline{T}^*(\theta)))) = \lambda(\theta, \underline{T}^*(\theta)).$$

Several facts follow from these optimality conditions.

1. Differentiating (A.50) with respect to  $t$  when  $q^*(\theta, t) > 0$  yields:

$$\dot{\lambda}(t) + r\lambda(t) = e^{-rt}(\dot{p}^*(t) + C'(S^*(\theta, t))q^*(\theta, t)).$$

Using (A.49) and (A.50) and simplifying implies that at any point where  $q^*(\theta, t) > 0$ :

$$(A.54) \quad \dot{p}^*(t) = r(p^*(t) - C(S^*(\theta, t))).$$

2. From (A.54), it follows that all firms which are active at date  $t$  and extract a positive amount ( $q^*(\theta, t) > 0$ ) have the same remaining stock at such date  $t$ :

$$(A.55) \quad S^*(\theta, t) = \sigma^*(t),$$

where  $\sigma^*(t)$  is defined as the lowest level of resource currently extracted at date  $t$ . Formally,  $\sigma^*(t) = \min\{\theta \in \Theta \text{ such that } S^*(\theta, t) = \sigma^*(t)\}$  when  $\sigma^*(t) \geq \underline{\theta}$ , and,  $\sigma^*(t) = S^*(\underline{\theta}, t)$  otherwise. From (A.54), we obtain that  $p^*(t)$  satisfies:

$$(A.56) \quad \dot{p}^*(t) = r(p^*(t) - C(\sigma^*(t))).$$

3. Differentiating (A.55) with respect to time, it follows from (2.2) that:

$$(A.57) \quad q^*(\theta, t) = -\dot{\sigma}^*(t) \quad \forall \theta \in [\sigma^*(t), \bar{\theta}].$$

Together with (A.48), we deduce (5.3) or, equivalently:

$$(A.58) \quad \dot{\sigma}^*(t)(1 - F(\sigma^*(t))) = -D(p^*(t))$$

where  $D(\cdot) = P^{-1}(\cdot)$ .

4. Consider the system made of (A.56) and (A.58) together with the initial condition  $\sigma^*(0) = \bar{\theta}$ . By an argument that replicates the one given in the proof of Proposition 1, this system  $(\sigma^*(t), p^*(t))$  converges in infinite time towards  $(S_{inf}, P(0))$ . In particular,  $\dot{\sigma}^*(t) < 0$  for all  $t$  and thus, from (A.57),  $q^*(\theta, t) > 0$  for  $t$  large enough (indeed,  $t \geq \underline{T}^*(\theta)$  and for all  $\theta$ ).



5.  $\underline{T}^*(\theta)$  is determined by the simple condition:

$$(A.59) \quad \sigma^*(\underline{T}^*(\theta)) = \theta.$$

□

PROOF OF PROPOSITION 7. The proof is to a large extent similar to the proof of Proposition 6 modulo the fact that the virtual cost of extraction must now be taken into account. The public authority's problem can be written as:

$$(A.60) \quad \begin{aligned} (\mathcal{P}^{sb}) : \quad & \max_{T, q \geq 0, S, \mathcal{Q}} \int_0^{+\infty} \left( V(\mathcal{Q}(t)) - \int_{\underline{\theta}}^{\bar{\theta}} \tilde{C}(\theta, S(\theta, t)) q(\theta, t) f(\theta) d\theta \right) e^{-rt} dt \\ & \text{subject to (2.1), (2.2) and} \\ & \int_{\underline{\theta}}^{\bar{\theta}} q(\theta, t) f(\theta) d\theta = \mathcal{Q}(t). \end{aligned}$$

Denote by  $p^{sb}(t)$  the multiplier of the feasibility condition (A.60).

1. We proceed as in the proof of Proposition 6 and obtain that, for  $q^{sb}(\theta, t) > 0$ :

$$(A.61) \quad \dot{p}^{sb}(t) = r(p^{sb}(t) - \tilde{C}(\theta, S^{sb}(\theta, t))).$$

Define now  $\sigma^{sb}(t)$  as the resource stock at date  $t$  of the latest firm that has become active. Formally,  $\sigma^{sb}(t) = \min \{ \theta \in \Theta \text{ such that } q^{sb}(\theta, t) > 0 \}$  when  $\sigma^{sb}(t) \geq \underline{\theta}$ , and,  $\sigma^{sb}(t) = S^{sb}(\underline{\theta}, t)$  otherwise. Additionally, define  $s^{sb}(t) = \max \{ \sigma^{sb}(t), \underline{\theta} \}$  as the initial stock endowment of the latest firm that has become active.

From (A.61) and the equalization of virtual marginal costs across active firms, we obtain that  $p^{sb}(t)$  satisfies:

$$(A.62) \quad \dot{p}^{sb}(t) = r(p^{sb}(t) - \tilde{C}(s^{sb}(t), \sigma^{sb}(t))).$$

2. It also follows from (A.61) that all firms which are active at date  $t$  and extract a positive amount ( $q^{sb}(\theta, t) > 0$ ) are such that:

$$(A.63) \quad S^{sb}(\theta, t) = \Phi(\theta, \sigma^{sb}(t)),$$

where  $\Phi(\cdot, \cdot)$  is implicitly defined as the solution to:

$$\tilde{C}(\theta, \Phi(\theta, \sigma)) = \tilde{C}(s, \sigma) \quad \text{where } s = \max \{ \sigma, \underline{\theta} \}.$$

Notice now that  $C'(\cdot) < 0 \leq C''(\cdot)$  implies:

$$\frac{\partial}{\partial \sigma} \tilde{C}(s, \sigma) = C'(\sigma) - \frac{1 - F(s)}{f(s)} C''(\sigma) < 0.$$

Using the monotone hazard rate property, this implies:

$$\frac{d}{d\sigma} \tilde{C}(\sigma, \sigma) = C'(\sigma) - \frac{1 - F(\sigma)}{f(\sigma)} C''(\sigma) - \frac{d}{d\sigma} \left( \frac{1 - F(\sigma)}{f(\sigma)} \right) C'(\sigma) < 0.$$

Hence, as  $\sigma^{sb}(t)$  falls, the virtual marginal cost increases.

3. Straightforward manipulations show that:

$$\begin{aligned}\frac{\partial \Phi}{\partial \sigma}(\theta, \sigma) &= \frac{\frac{d\tilde{C}}{d\sigma}(\sigma, \sigma)}{\frac{\partial \tilde{C}}{\partial S}(\theta, S)} > 0, \\ \frac{\partial^2 \Phi}{\partial \sigma \partial \theta}(\theta, \sigma) &= \frac{\frac{\partial^2 \tilde{C}}{\partial S \partial \theta}(\theta, S)}{\frac{\partial^2 \tilde{C}}{\partial S^2}(\theta, S)} \geq 0, \\ \frac{\partial \Phi}{\partial \theta}(\theta, \sigma) &= -\frac{\frac{\partial \tilde{C}}{\partial \theta}}{\frac{\partial \tilde{C}}{\partial S}} < 0.\end{aligned}$$

4. Differentiating (A.63) with respect to time and using (2.2), it follows that:

$$(A.64) \quad q^{sb}(\theta, t) = -\dot{\sigma}^{sb}(t) \frac{\partial \Phi}{\partial \sigma}(\theta, \sigma^{sb}(t)).$$

From Item 3 above, extraction is increasing in  $\theta$ . Hence, at a given date, active firms with larger initial endowments produce relatively more.

Together with (A.60), we deduce:

$$(A.65) \quad \dot{\sigma}^{sb}(t) \int_{\sigma^{sb}(t)}^{\bar{\theta}} \frac{\partial \Phi}{\partial \sigma}(\theta, \sigma^{sb}(t)) f(\theta) d\theta = -D(p^{sb}(t)).$$

5. Consider the system made of (A.62) and (A.65) together with the initial condition  $\sigma^{sb}(0) = \bar{\theta}$ . By an argument that replicates the one developed in the proof of Proposition 1, the system  $(\sigma^{sb}(t), p^{sb}(t))$  converges in infinite time towards  $(\tilde{S}_{inf}(\underline{s}), P(0))$  where  $\underline{s} \equiv \lim_{t \rightarrow \infty} s^{sb}(t) \geq \underline{\theta}$  is the initial stock of the last marginal firm to become active. We have that  $\tilde{S}_{inf}(\underline{s}) = \underline{s} > S_{inf}$  if  $\tilde{C}(\underline{s}, \underline{s}) = P(0)$  for  $\underline{s} > \underline{\theta}$ , or  $\underline{s} = \underline{\theta}$  in which case  $\tilde{S}_{inf}(\underline{s}) = \tilde{S}_{inf}(\underline{\theta}) > S_{inf}$  defined by  $\tilde{C}(\underline{\theta}, \tilde{S}_{inf}(\underline{\theta})) = P(0)$ .

In particular,  $\dot{\sigma}^{sb}(t) < 0$  for all  $t$  and thus, from (A.64),  $q^{sb}(\theta, t) > 0$  for  $t \geq \underline{T}^{sb}(\theta)$  and for all  $\theta$ . From this, it follows that  $S^{sb}(\theta, t)$  defined in (A.63) converges in infinite time also, but, now, towards  $\tilde{S}_{inf}(\theta)$ . Thus,  $\bar{T}^{sb}(\theta) = +\infty$ .

6. For  $\theta \in [\underline{\theta}, \bar{\theta}]$ ,  $\underline{T}^{sb}(\theta)$  is determined by the simple condition:

$$\tilde{C}(\sigma^{sb}(\underline{T}^{sb}(\theta)), \sigma^{sb}(\underline{T}^{sb}(\theta))) = \tilde{C}(\theta, \theta).$$

Given that  $d\tilde{C}(\sigma, \sigma)/d\sigma < 0$ , we simplify this condition as:

$$(A.66) \quad \sigma^{sb}(\underline{T}^{sb}(\theta)) = \theta.$$

Since  $\sigma^{sb}(t)$  is decreasing, this last condition implies that the subset of active types at any date  $t$  is  $[\sigma^{sb}(t), \bar{\theta}]$  as long as  $\sigma^{sb}(t) \geq \underline{\theta}$ .

□

PROOF OF PROPOSITION 8. The system (5.4)-(5.3) can be rewritten with the new variables  $\tilde{p}^*(t) = \frac{p^*(t)-P(0)}{P'(0)}$  and  $\tilde{\sigma}^*(t) = \sigma^*(t) - S^*$  as follows:

$$(A.67) \quad \begin{aligned} \dot{\tilde{p}}^*(t) &= r(\tilde{p}^*(t) - \kappa^* \tilde{\sigma}^*(t)), \\ \dot{\tilde{\sigma}}^*(t) &= -\frac{\tilde{p}^*(t)}{1 - \tilde{\sigma}^*(t)} \text{ with } \tilde{\sigma}^*(0) = 1. \end{aligned}$$

Define  $x^*(t) = \tilde{\sigma}^*(t) - \frac{1}{2}(\tilde{\sigma}^*(t))^2 \Leftrightarrow \tilde{\sigma}^*(t) = 1 - \sqrt{1 - 2x^*(t)}$  with  $x^*(0) = \frac{1}{2}$ . We have:

$$(A.68) \quad \dot{x}^*(t) = -\tilde{p}^*(t).$$

Taking the ratio between (A.67) and (A.68), we can express  $\tilde{p}^*(\cdot)$  in terms of  $x^*(\cdot)$  and observe that the corresponding function  $P^*(x)$  solves the following ordinary differential equation:

$$(A.69) \quad \frac{dP^*}{dx}(x) = -r \left( 1 - \frac{\kappa^*}{P^*(x)}(1 - \sqrt{1 - 2x}) \right) \text{ with } P^*(0) = 0.$$

Importing this finding above,  $x^*(t)$  now solves:

$$(A.70) \quad \dot{x}^*(t) = -P^*(x^*(t)).$$

It is then immediate from (A.70) that  $x^*(t)$  is decreasing and, since it remains non-negative, converges as  $t$  goes to  $+\infty$  to its limit  $\lim_{t \rightarrow +\infty} x^*(t) = 0$ .

Using L'Hospital rule in (A.69) to determine the derivative  $\dot{P}^*(0)$ , we find  $\dot{P}^*(0) = \frac{1}{z^*}$ . Thus, when  $t$  goes to  $+\infty$ ,  $P^*(x) \sim_{x \rightarrow 0} \frac{x}{z^*}$ . Thus, fix an arbitrary  $\epsilon > 0$ , because  $x^*(t)$  converges towards zero, there exists  $t_0$  such that for  $t \geq t_0$ :

$$\frac{1 - \epsilon}{z^*} x^*(t) \leq P^*(x^*(t)) = -\dot{x}^*(t) \leq \frac{1 + \epsilon}{z^*} x^*(t).$$

By Grönwall Lemma, we deduce that, for  $t \geq t_0$ , the following inequalities hold:

$$Be^{-\frac{1+\epsilon}{z^*}t} \leq x^*(t) \leq Ae^{-\frac{1-\epsilon}{z^*}t}.$$

for some constants  $A$  and  $B$ . From this, it follows that for  $t$  large enough, we can write:

$$-\frac{\epsilon}{z^*} \leq \frac{\log(x^*(t))}{t} + \frac{1}{z^*} \leq \frac{\epsilon}{z^*}$$

for any  $\epsilon > 0$ , or:

$$\lim_{t \rightarrow +\infty} \frac{\log(x^*(t))}{t} + \frac{1}{z^*} = 0.$$

Thus, there exists a function  $\eta^*(t)$  such that  $\lim_{t \rightarrow +\infty} \eta^*(t) = 0$  and one can write:

$$x^*(t) = e^{-\frac{t}{z^*} + t\eta^*(t)}.$$

Therefore, from (A.68) and (A.70) we get, as claimed:

$$\tilde{p}^*(t) = P^*(x^*(t)) \sim_{t \rightarrow +\infty} \frac{1}{z^*} e^{-\frac{t}{z^*} + t\eta^*(t)}.$$

Consider now the asymmetric information scenario. We now define  $\tilde{p}^{sb} = \frac{p^{sb}(t) - P(0)}{P'(0)}$  and  $\tilde{\sigma}^{sb}(t) = \sigma^{sb}(t) - \hat{S}$  with  $\hat{S} = S_{inf} + \frac{1}{2}$ . Notice that  $\Phi(\theta, \sigma) = 2\sigma - \theta$ . These new variables solve the system:

$$(A.71) \quad \begin{aligned} \dot{\tilde{p}}^{sb}(t) &= r(\tilde{p}^{sb}(t) - 2\kappa^* \tilde{\sigma}^{sb}(t)), \\ \dot{\tilde{\sigma}}^{sb}(t) &= -\frac{\tilde{p}^{sb}(t)}{1 - 2\tilde{\sigma}^{sb}(t)} \text{ with } \tilde{\sigma}^{sb}(0) = 1. \end{aligned}$$

Define now  $x^{sb}(t) = \tilde{\sigma}^{sb}(t) - (\tilde{\sigma}^{sb}(t))^2 \Leftrightarrow \tilde{\sigma}^{sb}(t) = 1/2(1 - \sqrt{1 - 4x^{sb}(t)})$  with  $x^{sb}(0) = 0$ . We now have:

$$(A.72) \quad \dot{x}^{sb}(t) = -P^{sb}(x^{sb}(t)).$$

Taking now the ratio between (A.71) and (A.72), we can express  $\tilde{p}^{sb}(\cdot)$  in terms of  $x^{sb}(\cdot)$ , namely  $P^{sb}(x)$ , and, observe that this function solves the following differential equation:

$$(A.73) \quad \frac{dP^{sb}}{dx} = -r \left( 1 - \frac{\kappa^*}{P^{sb}(x)} (1 - \sqrt{1 - 4x}) \right) \text{ with } P^{sb}(0) = 0.$$

Importing this finding above,  $x^{sb}(t)$  now solves:

$$\dot{x}^{sb}(t) = -P^{sb}(x^{sb}(t)).$$

It is immediate from the last equation that  $x^{sb}(t)$  is decreasing and, since it remains non-negative, converges as  $t$  goes to  $+\infty$  to its limit  $\lim_{t \rightarrow +\infty} x^{sb}(t) = 0$ .

Using L'Hospital rule in (A.73) to determine the derivative  $\dot{P}^{sb}(0)$ , we find  $\dot{P}^{sb}(0) = \frac{1}{z^{sb}}$  with  $z^{sb}$  being defined in the text. Thus, locally for  $x$  small enough, we have:  $P^{sb}(x) \sim_{x \rightarrow 0} \frac{x}{z^{sb}}$ . Proceeding as in the complete information scenario, we demonstrate, as claimed, that:

$$\tilde{p}^{sb}(t) = P^{sb}(x^{sb}(t)) \sim_{t \rightarrow +\infty} \frac{1}{z^{sb}} e^{-\frac{t}{z^{sb}} + t\eta^{sb}(t)}.$$

□