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In 1882, Richard Dedekind (1831-1916) and Heinrich Weber (1842-1913), published *Theorie der algebraischen Funktionen einer Veränderlichen* ([Dedekind and Weber, 1882]), in which they offer what we identify as an "algebraico-arithmetic" rewriting of Riemann's function theory. In this paper, which took them two years to write and two more years to publish, they offer a new definition of the concept of Riemann surface, together with the reformulation of several basic concepts of Riemannian function theory and new proofs of important theorems.

A singular paper in the Dedekindian corpus by its subject, [Dedekind and Weber, 1882] exhibits nonetheless typical characteristics of Dedekind's approach in mathematics: ideals as a foundation, definition of new arithmetical operations for ideals, uninhibited use of systems, pure existence proofs, lack of constructive definitions and/or proofs... and it is, for the first half, strictly parallel – sometimes similar word for word – with Dedekind's previous work on algebraic numbers ([Dedekind, 1877][1]). As such, it is often considered as another example of the efficiency of Dedekind's methods and algebraic concepts. This certainly doesn't mean that Weber's input was minimal – in fact, the letters he wrote to Dedekind while working on their paper indicate the opposite. However, it suggests that he considered Dedekind's methods, at that point, as being the best for their work. He later changed his position on the matter, and in his *Lehrbuch der Algebra* ([Weber, 1896]), Weber does not use ideal theory (although he does use fields). For these reasons, I will also consider [Dedekind and Weber, 1882] as intimately linked to Dedekind's conceptions and methods, and will make little references to Weber's other works.[2] More specifically, I will argue that [Dedekind and Weber, 1882] is to be read in relation with Dedekind's conception of arithmetic. My aim, then, will be to show

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[1]References to [Dedekind, 1877] are to Stillwell's translation [Stillwell, 1996]. The French original paper was published in five parts in several different issues of the *Bulletin des sciences mathématiques et astronomiques*, between 1876 and 1877, making the references quite hard to follow. It was only partially reproduced in Dedekind's collected works.

[2]A clarification of Weber's role in the writing of [Dedekind and Weber, 1882] and the changes he chose to make to the theory of algebraic functions will be the subject of a future work.

how their paper provides a rewriting of Riemannian function theory built on elementary arithmetical notions such as divisibility – methods which had a prominent place in Dedekind's approach.[3]

It is important to note that arithmetic is not understood, here, as being an equivalent or a strict synonym for the theory of natural numbers. Rather, it is focused on what it means to be *arithmetical*, in the sense that Dedekind develops an *arithmetical* theory of ideals. More generally, it is important to think of arithmetic in the context of 19th century mathematical practice in order to understand Dedekind's position. Besides, both the attempts to rigorize Riemann's theory, and the resort to arithmetic for more rigorous or general approaches to certain domains of mathematics can be found in several mathematicians' works and were not specific to Dedekind (and Weber). Hence, it is essential to be able to relate this mathematical practice to Dedekind's ideas about numbers and arithmetic, and understand how these elements build up Dedekind's approach – that is, how Dedekind elaborated his concepts and methods from elements which were very common in the mathematical landscape in which he was evolving, and were shared by 'pre-modern' mathematicians. In a first section, I will thus consider some elements of context: practices of arithmetic in Dirichlet's works; elements to understand Dedekind's conception of arithmetic and the role it played in the development of his works; and briefly (and partially) Kronecker's arithmetical approach.[4] These elements allow to place the arithmetical rewriting of Riemann's theory in both a broader and a Dedekindian context. From there, I will concentrate on Dedekind and Weber's paper itself, and present it – like it is written – in two parts: fields and ideals of algebraic functions, and the new definition of the Riemann surface. The 1882 paper was mostly written based on an exchange of letters between Dedekind and Weber, in 1879 and 1880. Some of these letters have been preserved and recently published in [Scheel, 2014].[5] I will

[3]The idea of "rewritings" was notably developed by Norbert Schappacher, in [Schappacher, 2010]. Schappacher explains that "rewritings on all scales make up the very fabric of mathematical activity through the centuries" (*ibid.*, 3260). Rewritings, at least in my use of the word, are an extension of the idea that the meaning that a reader gives to a mathematical theorem depends on her historical and sociological background. See [Goldstein, 1995], [Ehrhardt, 2012]. Goldstein's and Ehrhardt's analyses bring to the fore how the understandings and interpretations of mathematical results, concepts and theories are historically, sociologically and epistemologically correlated. This last point, not present in Goldstein's and Ehrhardt's works, is important in my case.

[4]For more elements on the general mathematical context around number theory and/or algebraic functions, one can refer to, for example, [Bottazzini and Gray, 2013], [Geyer, 1981], [Goldstein et al., 2007], [Strobl, 1982].

[5]The letters published by Scheel are presently in the Niedersächsische Staats- und Universitätsbibliothek Göttingen (Cod. Ms. Riemann, and Cod. Ms. R. Dedekind) and in the Archiv der UB Braunschweig (G 98:11-13). Some of the letters between Dedekind and Weber were given by Paul Epstein to Emmy Noether ([Dedekind, 1932] III, 483) and many can be found in the Göttingen University archive. [Kimberling, 1972] and [Kimberling, 1996] explain how he recovered some of Dedekind's correspondence which Emmy Noether took with her in the United States. In these letters, which have since been given to the Archiv der UB Braunschweig, one can find Weber's letters from 1879. It is not clear what happened to the letters from 1880, however, except for two fragments by Dedekind which were published in [Dedekind, 1932] III, 487-488. In addition, it has so far been impossible to locate Weber's *Nachlass* or ascertain what happened to it ([Scheel, 2014], 16 and personal communications). Consequently, many of Dedekind's letters to Weber – and in our case, all the letters he sent in 1879, are impossible to find.

use elements of the correspondence between Dedekind and Weber for my commentary on the definition of the Riemann surface. Indeed, the letters available concentrate on the correspondence between Riemann's function theory and Dedekind's ideal theory, its nature and how to establish it. They do not address the set up of the theory of ideals of functions, which in the final version, runs parallel to [Dedekind, 1877].

Dedekind's mathematics is often described as *conceptual* mathematics.[6] Roughly speaking, the "conceptual approach" refers to some mathematicians' tendency in the second half of the 19th century to try to substitute concepts to explicit forms of representation (such as the use of indeterminate variables or of series) and/or computations in mathematics, to avoid (or more exactly hide) long computations. A central figure of the so-called conceptual approach is Riemann. Dedekind himself, in a letter to Lipschitz in June 1876 explains that his approach relies on an effort to "bas[e] the research not on arbitrary representations or expressions but on simple fundamental concepts" ([Dedekind, 1932] III, 468, transl. slightly altered in [Edwards, 1983], 11).[7] This qualification has been largely used in the secondary literature to qualify not only Dedekind's approach, but also the opposition to the so-called constructive approach represented by the Berlin tradition.

This description of Dedekind's mathematical practice as relying essentially on 'concepts' suggests that he successfully got rid of *Darstellungsformen*. However, it is well-known that the objects of algebraic numbers theory resist to the abolition of *Darstellungsformen* (as explained in [Avigad, 2006], 170-171). The description of the conceptual approach should, thus, be taken with a grain of salt. Dedekind himself explains, in his correspondence with Lipschitz, in reference to the choice of a variable to describe the elements of the finite field, that finitely generated fields can be defined in different ways: as corresponding to an equation, as having a finite basis, as having a finite number of divisors. This last definition is the one that Dedekind considers as the "best" definition, i.e., the definition based on fundamental properties of the field concept. However, this definition is more difficult to use in that it requires to dig too much into the theory of fields. Hence, he choses to use the definition using an equation, which he explains in the following way:

> I disfigure [by the intrusion of forms of representation], in the introduction, the concept of a finite field Ω, by giving it a presentation (*Darstellungsform*), in which all the numbers of the field are included, and which could as well

[6]See, for example, [Avigad, 2006], [Ehrhardt, 2012], [Edwards, 1983], [Ferreirós, 2008], [Schappacher and Petri, 2007], [Sinaceur, 1990], [Stein, 1988]... This list is not exhaustive.

[7] *"Mein Streben in der Zahlentheorie geht dahin, die Forschung nicht auf zufällige Darstellungsformen oder Ausdrücke sondern auf einfache Grundbegriffe zu stützen."*. In [Dedekind, 1877], he notes that "a theory based on calculation would not yet offer, in my opinion, the highest degree of perfection. It is preferable, as in the modern theory of functions, to try to infer the demonstrations immediately from the fundamental characteristic concepts rather than from calculations, and to construct the theory in a way that it is, on the contrary, able to predict the results of the calculation". ([Dedekind, 1877], 102). Dedekind also explains how to go from *Darstellungsformen* to ideals and the advantages of this approach – notably the fact that computations would be too complicated.

> be replaced by an infinite number of other [equivalent] presentations (...) I have made this concession, in order to borrow as little as possible from the general theory of fields, and in order to make the link with generally known things.[8] (10/06/1876, [Lipschitz, 1986], 60)

Here, by using this *Darstellungsform*, Dedekind bends his own rule to find a way to make his theory of algebraic numbers (and later his theory of algebraic functions) easier to develop. In [Dedekind and Weber, 1882], the same issue appears. In fact, the whole arithmetical apparatus elaborated by Dedekind and Weber depends on the independent variable z chosen, insofar as the ring itself depends on it – a point which Dedekind and Weber notice, since they list which notions depend on the variable z. Dedekind and Weber, then, did not actually completely avoid the reliance on *Darstellungsformen.*

My goal is not to judge to what extent Dedekind lived to his own standards, nor to know whether it is at all possible to avoid all and any use of *Darstellungsformen.* I will not be interested in comparing the so-called conceptual approach and the rival approach(es), nor in judging which approach is the most efficient or the most fruitful. Rather, I wish to consider which strategies were developed by Dedekind (and Weber) to set up their new version of Riemann's theory of algebraic functions without (or with as little as possible) reliance on *Darstellungsformen.* By that, I do not mean that I will try to unveil the hidden computations behind Dedekind and Weber's conceptual apparatus, but that I will try to make more explicit the conceptual move proposed by the authors, i.e. how Dedekind's conceptual reorganization is carried out by replacing (long) computations by simpler ones (e.g. elementary divisibility properties of ideals), using what Dedekind calls the "simplest principles of arithmetic". Such a use of arithmetic is recurrent in Dedekind's mathematics, which strongly suggests that the operations of arithmetic are given a very particular and active role in the development of mathematics. The *conceptual* approach, in Dedekind's mathematics, is taken care of by means of arithmetic and its "inexhaustible wealth" of laws,[9] a strategy which finds a justification in Dedekind's conception of arithmetic as an activity of the human mind. Hence, my insisting on the *arithmetical* aspect of his work does not stem from a desire to take Dedekind's mathematics away from a description as it as being conceptual. Rather, it is an attempt to provide a more precise and contextualized characterization of his methods, which gives the possibility to situate Dedekind's works in the rather vague idea of a "conceptual approach" – after all, Dedekind's and Riemann's approach, while both "conceptual"[10] are very distinct from each other.

[8] "*In ähnlicher Weise verunziere ich in der Einleitung den Begriff eines endlichen Körpers Ω dadurch, dass ich eine Darstellungsform angebe, in welcher alle Zahlen des Körpers enthalten sind und welche ebenso gut durch unendlich viele Darstellungsformen ersetzt werden könnte (...) [I]ch habe diese Concession gemacht, um aus der allgemeinen Theorie der Körper möglichst wenig zu entlehnen und um an allgemein bekannte Dinge anzuknüpfen.*" Unless stated otherwise, the translations are mine.

[9] *Zum Zahlbegriff*, in [Dugac, 1976], 315

[10] As was remarked by Edwards ([Edwards, 2010]), Riemann's mathematics did contain a lot of computations, which is clear from his *Nachlass.* This "algorithmic side" of Riemann's mathematics was subsequently concealed by him through a reorganization, in what we can understand as an attempt by Riemann to clarify the theory.

It is important to separate the arithmetical approach adopted by Dedekind (and Weber) from a structuralist reading. Of course, not all structuralist approaches are arithmetical, and reciprocally arithmetical is not synonymous with either conceptual or structural – Kronecker's arithmetical theory of algebraic magnitudes is a good example of that last point. Moreover, as Corry showed in [Corry, 2004], the concepts invented by Dedekind, and the approach he advocated for, present strong differences with our modern structural algebra.[11]Dedekind (nor Weber) does not consider a general, abstract, theory of ideals (or fields), but always the theory of ideals (or fields) of numbers or functions. In other words, Dedekind does not study ideal (or field) theory for the sake of ideal (or field) theory, but as a new foundation and a tool for number theory or function theory. And, unlike what a modern day reader will first look for in [Dedekind and Weber, 1882], structures are not what matters to Dedekind and Weber. For them, the core of their work is the new definition of a Riemann surface, the new approach for Riemann's theory of algebraic functions.

Emmy Noether, in her editor note in Dedekind's *Werke* as well as in [Noether, 1919], underlined the arithmetical nature and foundation of Dedekind and Weber's paper. She presents their theory as "related to" Riemann's (especially insofar as their proofs are existence proofs), and as preserving the "purity of methods" ([Noether, 1919], 184-185). Her acknowledging the arithmeticity of Dedekind and Weber's work is part of her underlining the structural relationships and parallels between the theory of algebraic functions and the theory of algebraic numbers, and retrospectively placing their contribution in the evolution of mathematics. Her reading of Dedekind, in her 1919 report and in many other places have strongly influenced our reception of his works. From there, as is widely known, Dedekind has been granted a key place in the genealogy with which mathematicians represent the shaping of mathematical knowledge. He introduced several of the most important concepts for the development of modern algebra to the point that he is often said to have influenced our practice and knowledge of mathematics more than any other mathematician of the second half of the 19th century. After the reception of his works remained dormant for about twenty years, Dedekind's concepts and methods eventually changed the face of algebraic number theory – especially through their adoption by David Hilbert in [Hilbert, 1897] and, later, Emmy Noether's works and lectures.

For these reasons, Dedekind's mathematics have been the object of many works from historians as well as philosophers.[12] In these works, the 1882 paper with Weber,

[11]From Corry's viewpoint, there is a distinction between the "body of knowledge" ("theories, 'facts', methods, open problems", [Corry, 2004], 3) and the "image of knowledge" ("claims which express knowledge about the discipline qua discipline", *ibid.*, 3). While Dedekind does possess a relatively extensive part of the body of knowledge for structural algebra (groups, ideals, modules, fields), he does not have the structural image of algebra, that is, he does not consider questions and problems about the *structures* themselves, but rather about their elements.

[12]Dedekind's works have been presented as bringing "the light in the darkness, the order in chaos" by Landau in 1917 ([Landau, 1917], quoted and translated in French in [Dugac, 1976], 29). Bourbaki considers [Dedekind, 1871] a "masterpiece" in which the theory of modules and ideals is "entirely created" and "brilliantly exposed" ([Bourbaki, 1984], 130). It is presented by H. Edwards as the "birthplace of the modern set-theoretic approach to the foundations of mathematics" ([Edwards, 1983]) and by Dugac as

while recognized as a major achievement (of Dedekind and Weber, and in the history of mathematics), is often mentioned rather quickly as yet another example of Dedekind's method. On the other hand, [Dedekind and Weber, 1882] has been largely acknowledged as a milestone of algebraic geometry. Yet, few extensive commentaries have been made on [Dedekind and Weber, 1882]. In addition to [Noether, 1919], notable exceptions are [Geyer, 1981], [Strobl, 1982], and the introductory notes in [Stillwell, 2012].[13]

Commentators such as Dieudonné, Geyer and Strobl insist on the structural properties and relate Dedekind and Weber's paper to later evolutions of mathematics. They provide readings which set up and reformulate Dedekind's works in a modern mathematical and conceptual framework, and offer an invaluable light on the mathematical content of Dedekind and Weber's work. These unfoldings of Dedekind and Weber's approach into our modern mathematical langage put forward how the first corner stones for modern algebraic geometry were laid down, and the crucial importance of their works for future developments of mathematics. To the modern day reader, Dedekind's works often have a quality of undeniable clarity, and almost a feeling of familiarity. Our expectations of Dedekind's conceptions are thus distorted by a feeling of proximity resulting from the preponderance in our mathematics of concepts grown from Dedekind's ideas. As a consequence, the treatment of Dedekind's mathematics slipped through the net of many of the recent historiographical criticisms on retrospective history of mathematics. Dedekind is considered as a sort of founding father of modern algebra – even though he is barely cited in Van der Waerden *Moderne Algebra* if not for Emmy Noether's famous "*es steht alles schon bei Dedekind*" ("everything is already in Dedekind"). Yet, there are no reasons not to submit historical readings of Dedekind's works to the same critical non-retrospective analysis. Indeed, before they arrived to us, Dedekind's ideas were reinterpreted by mathematicians such as Hilbert, Steinitz, and E. Noether, and they are not, in themselves, so close to present day knowledge and practice. Restating the text in modern terms, in the case of Dedekind and Weber's algebraic function theory, implies that many elements of language are lost: the arithmetical terminology deliberately chosen by Dedekind and Weber disappears ("divisible" is replaced by "contained in" for example). The choice of terms is not innocent, in Dedekind and Weber's work, as shown by their correspondence, and it serves to underline the essential arithmetic character of their work.

Finally, such readings tend to detach the original work from their author's own system of conceptions and values to be integrated into our own. The author's epistemological values (e.g. generality), philosophical conceptions (e.g. concept of number), and the value attributed to certain practices (e.g. an arithmetical approach) can constitute, for the

"one of the sources of today mathematics" ([Dugac, 1976], 29). Ferreirós, in [Ferreirós, 2008], also argues that Dedekind is one of the fathers of modern set theory. Dedekind also often plays an important role in more global works, for example, [Corry, 2004] [Gray, 2008]. If the study of his works is not a pressing matter for historians anymore, philosophers, on the other hand, have been increasingly interested in Dedekind's works in the last decade. It is not the place, here, to go over all the philosophical studies of Dedekind's works (neither to consider the philosophical implications of his works).

[13]Dedekind and Weber's 1882 paper is also studied as a part of larger inquiries about history of mathematics, as for example in [Dieudonné, 1974], [Houzel, 2002], or [Schappacher, 2010].

historian, elements of great importance to understand mathematical texts.[14] It has been argued in the case of Kronecker's mathematics that his mathematical practice should be understood in relation to his philosophical views (and reciprocally).[15] This way, it becomes possible to get a better understanding of his statements about natural numbers, computations and arithmetic: methods elaborated by Kronecker express and develop the philosophical position, and one cannot understand one without the other. This point is certainly valid for Dedekind as well (and probably many other mathematicians) and an historian reading his works should take into account his philosophical ideas and conception of numbers and arithmetic. To this, is also related the statement made at the beginning of this introduction, namely that [Dedekind and Weber, 1882] should be read in relation to Dedekind's conception of arithmetic.

As a way to conclude this introduction, let me give a short example. On the 30/10/1880, Dedekind sent a letter to Weber to thank him for their collaboration on this work, and wrote:

> It is a truly special feeling, to be joined by the exploration of the truth, what Pascal in his first letter to Fermat so admirably phrased: "Car je voudrais désormais vous ouvrir mon coeur, s'il se pouvait, tant j'ai de joie de voir notre rencontre. Je vois bien que la vérité est la même à Toulouse et à Paris." Often, had I to think at this place, through the progresses of our work, which though after many oscillations still has assumed the character of intrinsic necessity.[16] ([Scheel, 2014], 271)

[Geyer, 1981] suggests that this "inner necessity" is linked to the construction of normal bases (in particular, the normal basis of the integral closure, studied in the §22 of [Dedekind and Weber, 1882]).[17] This part of their work has indeed been reworked and re-elaborated many times over the years since [Hensel and Landsberg, 1902]. In particular, in 1955, Grothendieck's proof of the Birkhoff-Grothendieck theorem on vector bundles uses "overall the same approach" as Dedekind and Weber "with no knowledge of history" ([Geyer, 1981], 126). Dedekind and Weber's interest in normal bases of the integral closure, at that point, mainly comes from the links between discriminant and ramification and how it sheds light on the ramification number of the surface. And while these technical developments are related to Dedekind's transfer back to number theory of results obtained in [Dedekind and Weber, 1882], it seems that Geyer's interpretation would require for Dedekind to have been fully aware of mathematical developments which still lay in the future. On the other hand, the parallelism with number theory

[14] See, for example, [Chemla, 2003], [Chemla, 2009], and [Chemla et al., 2016].

[15] See [Vlădut, 1991] for more details. See also [Boniface, 2004], [Boniface, 2005], [Edwards, 1980] , the introduction of [Kronecker, 1891], and [Smadja, 2010].

[16] *[E]s ist ein ganz besonderes schönes Gefühl, sich so bei der Erforschung der Wahrheit zu begegnen, was Pascal in seinem ersten Brief an Fermat so trefflich ausdrückt: "Car je voudrais désormais vous ouvrir mon coeur, s'il se pouvait, tant j'ai de joie de voir notre rencontre. Je vois bien que la vérité est la même à Toulouse et à Paris." Oft habe ich an diese Stelle denken müssen bei den Fortschritten unserer Arbeit, die nach mancherlei Oscillationen doch immer mehr den Charakter innerer Nothwendigkeit angenommen haben."*

[17] [Geyer, 1981] also provides a reformulation of these results in modern terms.

and the inherently arithmetical character of their work can certainly be an aspect which induced this kind of reflection from Dedekind, who considered arithmetic as intrinsically linked to the nature of the human mind.

1 Contextual elements

1.1 Arithmetic extended? Some elements of context, before Dedekind

C. F. Gauss used to repeat, according to several witnesses, a motto inspired by a sentence which Plutarch attributed to Plato: "God arithmetizes eternally."[18] As Ferreirós underlines ([Ferreirós, 2007]), the theological assertion behind the motto is somewhat "less interesting than the way in which his adaptation of Plato's words reflects changing perceptions of mathematical knowledge" (*ibid.*, 237). From Gauss's *Disquisitiones Arithmeticae* in 1801 to the emphasis put on a "movement of arithmetization" with Klein in 1895, the meaning of the term "arithmetic" slowly moved from the theory of integers to being associated, rather loosely, to ideals of rigor and certainty. Yet, arithmetic, in the sense of number theory, seems to have stayed at the core of many mathematicians' systems of values, or at least to be the roots of many of their interests, albeit with various importance, for various reasons and in many different forms.

Gauss's *Disquisitiones Arithmeticae* has often been presented as marking the birth of number theory as a discipline in its own right in Germany. However, [Goldstein and Schappacher, 2007] show that the situation is more complicated and requires a more finely-tuned analysis, insofar as the developments following Gauss's works tend to challenge the boundaries between disciplines. To account for the works of mathematicians following Gauss's *Disquisitiones Arithmeticae* such as Dirichlet's, Goldstein and Schappacher propose to consider what they call the "Arithmetic Algebraic Analysis" (*ibid.*, 26), a new field of research which embedded the different approaches to and following the *Disquisitiones Arithmeticae*. Yet, it is interesting to notice that actors present their works either as a part of number theory or as stemming from problems about numbers. In the following, I will vary from Goldstein and Schappacher's approach in that I will not be looking at disciplines *per se*, but will be interested in whether an actor like Dirichlet was referring to arithmetic or number theory as leading his investigations. I will also try to identify what Dirichlet or Dedekind would qualify as "arithmetical" – a delicate point, as the thorough investigation in [Goldstein and Schappacher, 2007] suggests.

As Goldstein and Schappacher show, although a domain of research in itself, number theory is developed in close relation with other areas of mathematics at that moment such as binary quadratic forms or analysis. It was notably the case for Dirichlet who imported tools from Fourier analysis into researches which "do not contain any element which is not related to integral numbers" because the theorems were "difficult to establish (...) with purely arithmetical considerations, whereas the mixed method [he used] (...) which

[18] The version attributed to Plato is "God geometrizes eternally". Gauss' quote is immortalized in an etching of Gauss and W. Weber by A. Weger appearing in Zöllner's *Wissenschaftliche Abhandlungen* (1878, vol. 2, part I). See [Ferreirós, 2008], 236.

is based partly on the use of quantities varying by insensible degrees, led [him] to them by the most natural way, and so to speak, effortlessly".[19] Commenting on his own work, Dirichlet presented the theory of quadratic forms as "one of the main branches of the science of numbers" following its development in Gauss's *Disquisitiones Arithmeticae*.[20] Quadratic forms with integral coefficients played a crucial role in 19th century number theory. Dedekind's introduction of ideals is made in a work dedicated to binary quadratic forms. From Gauss' works on, quadratic forms were essentially characterized by the concept of discriminant.[21] Interested by the dependency between the determinant and the number of corresponding distinct forms, Dirichlet is led to the distinction between positive and negative determinants:

> We identify that the expression of the number of forms corresponding to any given determinant presents two very different cases depending on whether the determinant is a negative or a positive number. In the first case, the expression of the law in question has a purely arithmetical character, while for a positive determinant, it is of a more composite nature, mixed in a way, since, in addition to the arithmetic elements on which it depends, it contains others which present themselves in certain auxiliary equations present in the theory of binomial equations, pertaining thus to Algebra.[22] ([Lejeune-Dirichlet, 1897] I, 536)

Here, one can see that arithmetic is more a characteristic of the approach than a discipline – a point that will be important for our understanding of Dedekind's "arithmetical" methods. Dirichlet further explains that, following the "natural" desire to expand the results already obtained, one is led to try to "solve by [these] means other analogous questions but of a higher order" (*ibid.*, 536). One can either consider forms of a higher degree, or quadratic forms with complex integers as coefficients, which is what Dirichlet proposes to do. This "generalized number theory" opens "a new field to arithmetical

[19] *"Quoique les théorèmes précédents ne contiennent aucun élément qui ne soit relatif aux nombres entiers, il paraît difficile de les établir par des considérations purement arithmétiques, tandis que la méthode mixte dont nous venons de faire usage, et qui est fondée en partie sur l'emploi de quantités variant par degrés insensibles, nous y a conduit de la manière la plus naturelle et, pour ainsi dire, sans effort."* ("Recherches sur les formes quadratiques à coefficients et à indéterminées complexes", 1842, in [Lejeune-Dirichlet, 1897] I, 618).

[20] Dirichlet comments on "Recherches sur diverses applications de l'Analyse infinitésimale à la Théorie des nombres" (published in *Journal für die reine und angewandte Mathematik*, vol 19, 324-369; vol 21, 1-12 and 134-155, in 1839 and 1840) in the introduction of the 1842 memoir "Recherches sur les formes quadratiques à coefficients et à indéterminées complexes".

[21] The discriminant, or determinant for some authors like Dirichlet, is a positive or negative integer which allows to distinguish classes of forms for which certain arithmetical properties can be given.

[22] *"On reconnaît que l'expression du nombre des formes qui répondent à un déterminant quelconque présente deux cas très distincts suivant que ce déterminant est un nombre négatif ou positif. Dans le premier cas, l'expression de la loi dont il s'agit a un caractère purement arithmétique, tandis que pour un déterminant positif, elle est d'une nature plus composée et en quelque sorte mixte, puisque, outre les éléments arithmétiques dont elle dépend, elle en renferme d'autres qui ont leur origine dans certaines équations auxiliaires qui se présentent dans la théorie des équations binômes et appartiennent par conséquent à l'Algèbre."*

speculations" (*ibid.*, 537). Dirichlet's goal, here, is to "import, in the thus generalized theory of numbers, the previously considered question" (*ibid.*, 537) about quadratic forms. Hence, it appears that for Dirichlet the consideration of a wider concept of integer, here the Gaussian integers, allows to investigate objects of number theory, here quadratic forms, in a more general framework. By this move are opened new possibilities of "arithmetical speculations," as Dirichlet puts it. The idea that new possibilities in number theory are brought forward by widening the framework through a more general notion of number also constitutes the core of Dedekind's approach, showing the influence of Dirichlet.

Number theory thus developed in an intricate relation with theories such as those of quadratic and biquadratic forms, cyclotomic equations, modular equations and the division equations of elliptic functions.[23] This widened considerably, for the actors, the idea of what "arithmetic" consisted of, which questions pertained to arithmetic, and how these "arithmetic speculations" were to be handled. In the same time, arithmetic was still considered as the study of properties related to integral numbers – whose conception was also widened with, for example, complex integers.

1.2 Dedekind's conception of arithmetic

Dedekind's conception of arithmetic shows the influence of Dirichlet and Gauss. For Dedekind, arithmetic is the "science of numbers" and essentially concerned with the investigation of divisibility properties. It can be used to investigate properties of a "higher order" or a "higher level" – which is precisely the strategy and the terminology adopted by Dedekind for his theory of algebraic numbers and transferred to [Dedekind and Weber, 1882]. Arithmetic can and should be extended and generalized as much as possible, in particular insofar as it holds the possibility of further discoveries, be it because new fields of research are opened, or because results obtained can "ease to the highest degree the discovery and proof" in certain frameworks (here, fields of algebraic numbers, see [Dedekind, 1877]). Arithmetic conceived this way, for Dedekind, is deeply involved in the mathematical practice.

The "science of numbers"

Dedekind characterizes arithmetic as "the science of numbers" in many places ([Dedekind, 1872], 771; [Dedekind, 1888], 791-792, 795, 809; the 1894 version of algebraic number theory in [Dedekind, 1932] III, 24; ...). A most striking example is this quote from a manuscript in his *Nachlass*, entitled "*Zum Zahlbegriff*", which can be dated from after 1888:

> Of all the auxiliary means (*Hilfsmitteln*), that the human mind has until now created to ease its life, i.e., the work in which the thinking consists, none is as far-reaching and as inextricably connected to its innermost Nature, as the concept of number. Arithmetic, whose sole subject matter is this concept, is already by now a science of immeasurable extension, and there is no doubt

[23] See [Goldstein and Schappacher, 2007] for more details.

> that no limits are set to its further developments. Equally limitless is the area of its applications, since every thinking man, even when he is not distinctly aware of it, is a number-man, an arithmetician.[24] (*Zum Zahlbegriff*, in [Dugac, 1976], 315)

The emphasis on the idea of arithmetic as the *science* of numbers is important insofar as there is a specific meaning to "science" for Dedekind: science, for him, represents "the course of human knowledge up to" the truth ([Dedekind, 1854], 756). For him, the "chief goal" of science can be regarded as being "the endeavors to fathom the *truth*" (*ibid.*, 755, emphasis in the original): the pursuit of the truth is an activity of the human understanding, but truth itself does not depend on us. The *theory* of numbers is the collection of truths about numbers (properties, theorems, etc., related to numbers), whereas arithmetic is the investigation of these properties. Science, in consequence, is the human scaffolding of reasonings which attempts to understand, grasp or deepen our understanding of the objective truth and is bound to human understanding.[25] This entails that there may be multiple, if not innumerably many, different ways of trying to fathom or ground the truth, and that science is subjected to the finiteness and arbitrariness of the human mind. It also implies that it is up to the scientist (here, the mathematician) to build and/or invent the means for the development of science.[26]

Arithmetic, numbers, and the human mind

Dedekind famously placed at the head of [Dedekind, 1888] a quote inspired by Gauss('s arithmetization of Plutarch): "Man always arithmetizes", which emphasizes the fact that, for him, arithmetic is not only a creation of the human mind, but deeply embedded in our thinking. By no less than replacing God by the man, Dedekind asserts two key elements of his conception of mathematics: that mathematics, as a science, is a human activity, and that arithmetic is intimately and constitutively linked to the nature of human thought.[27] These points about Dedekind's conception of arithmetic are deeply related.

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[24]"*Von allen Hilfsmitteln, welche der menschliche Geist zur Erleichterung seines Lebens, d.h. der Arbeit, in welcher das Denken besteht, bis jetzt erschaffen hat, ist keines so folgenreich und so untrennbar mit seiner innersten Natur verbunden, wie der Begriff der Zahl. Die Arithmetik, deren einziger Gegenstand dieser Begriff ist, ist schon jetzt eine Wissenschaft von unermesslicher Ausdehnung und es ist keinem Zweifel unterworfen, dass ihrer ferneren Entwicklung gar keine Schranken gesetzt sind; ebenso unermesslich ist das Feld ihrer Anwendung, weil jeder denkende Mensch, auch wenn er dies nicht deutlich fühlt, ein Zahlenmensch, ein Arithmetiker ist.*"

[25]It should be noted that the idea of science as an activity of the human mind can be found in some of Lotze's writings, for example, in [Lotze, 1856], II, 152. Dedekind's statements in [Dedekind, 1854] are very close to Lotze's. See also [Sieg and Morris, pear] for some specific arguments on Lotze's influence on Dedekind

[26]In the introduction to the first edition of [Dedekind, 1888], Dedekind famously advocated for the introduction of new concepts as being particularly fruitful: "(...) the greatest and most fruitful advances in mathematics and other sciences have invariably been made by the creation and introduction of new concepts" ([Dedekind, 1888], 792), in which he also refers to his *Habilitationsvortrag.*

[27]In [Sinaceur, 2015], Hourya Benis Sinaceur suggests an intimate link between arithmetic and the structure of thought, which would attribute a prominent role to arithmetic in thinking. In her view, arithmetic is for Dedekind deeply involved in the mathematician's endeavors to extend mathematical

The idea of arithmetic as a science is indeed closely related to Dedekind's ideas about numbers as creations of the mind, which was fairly common among German mathematicians in the 19th century (notably Gauss). Dedekind, thus, saw arithmetic as an ever evolving intellectual activity and tied it with the human understanding. This aspect of his conception of arithmetic becomes particularly explicit and important in his 1888 essay on natural numbers, *Was sind und was sollen die Zahlen?*, in which Dedekind provides a definition of the natural numbers which he presents as direct productions of the laws of thought. The fact that the arsenal built with mappings and sets, the fundamental operations of the thought,[28] is exclusively used for this definition suggests, as it has been remarked by commentators such as H. Benis Sinaceur ([Sinaceur and Dedekind, 2008]), that Dedekind's definition offers a *justification* of his conviction that numbers are creations of the mind and "flowing from the laws of thought".

Arithmetic and divisibility

These ideas, however, give little concrete information regarding arithmetic in Dedekind's mathematical practice and in a process of arithmetization such as that of [Dedekind and Weber, 1882]. While one can start to understand why Dedekind would want to favor arithmetical methods, it still seems difficult, on these bases, to justify or precisely point out what can be considered as "arithmetical" or to identify Dedekind's practice of arithmetic. Fortunately, Dedekind also gives more practical statements regarding arithmetic, suggesting that the essential means of the science of numbers are its operations (addition, subtraction, multiplication, division). Dedekind follows his predecessors in giving divisibility a prominent role.

During the 19th century, divisibility properties, which appeared to be the first properties whose study impulsed new developments in the theory of numbers (notably with Gauss's congruences) were at the core of most of the inquiries. In 1876/77, Dedekind goes as far as saying that arithmetic is founded on divisibility:

> The theory of divisibility of numbers, which serves as a foundation of arithmetic, was already established in its essentials by Euclid. At any rate, the major theorem that each composite integral number can always be represented as a product of only prime numbers is an immediate consequence of the theorem, proved by Euclid, that a product of two numbers is not divisi-

knowledge. She states that "Arithmetic is fundamental not only because numbers are applied everywhere, but because we can, following the arithmetical laws, calculate with things which are not numbers. What matters is not what can be said of numbers in themselves, but as satisfying the four conditions brought to the light [in [Dedekind, 1888]]. And it is why we can say that arithmetic is a formal structure of our experience. The "logic of the mind" is arithmetic taken generally. [...] Hence, Dedekind's claim that arithmetic is a part of logic as meaning that arithmetic affords also a rational (logical) norm of thinking." ([Sinaceur, 2015], 56)

[28]In *Was sind und was sollen die Zahlen?*, Dedekind defines the system of natural numbers by means of "systems" and "mappings", (*Abbildungen*) which are presented as mathematical representations of the fundamental operations of the thought: to collect things with a common property (i.e. to form sets) and to draw correspondences between things (i.e. to define mappings).

> ble by a prime number unless the prime divides at least one of the factors.[29] ([Dedekind, 1877], transl. modified, 53)

Arithmetic as an epistemic tool

As Dedekind wrote in the quote from *Zum Zahlbegriff* given above, that arithmetic is one of the most powerful and fruitful tools at the human mind's disposal. It is a creation of the human mind to "ease" its work, a means to develop reasoning in all rigor.[30] The conception of arithmetic as the *science* of numbers, a non-rigid theory susceptible to be widened is effective in his mathematical practice and, for example, is essential for Dedekind's introduction of ideals in [Dedekind, 1871] as well as for the arithmetical definition of the linear continuum in [Dedekind, 1872]. It is related to the idea that arithmetic can be remodeled so as to ease proofs and allow more developments.

It is in particular essential for Dedekind's approach in number theory, in which he developed new methods in which arithmetical notions are applied to objects which are not, properly speaking, numbers. In particular, in many of his works, Dedekind defines new divisibility relationships between objects such as groups, modules, ideals as a way to make the development of the theory easier:

> If α is contained in [the ideal] $\mathfrak{a}$, we will say that α is *divisible by* $\mathfrak{a}$, and that $\mathfrak{a}$ *divides* α, since by this manner of expression we gain in facility.[31] ([Dedekind, 1871], 452)

He then proceeds to study their divisibility laws and to import notions and methods from arithmetic, such as GCDs and LCMs. This allows to reformulate theories and problems such as the study of binary quadratic forms (in [Dedekind, 1871] and [Dedekind, 1877]) or Riemannian function theory in terms emulating rational arithmetic. Arithmetic is used by Dedekind as a tool for producing new knowledge in mathematics.

The resort to arithmetic, even if it requires to define new operations for new objets or long preliminaries (as it is the case, for both, in [Dedekind and Weber, 1882]), is a way to bypass the difficulties such as the long tedious computations that Dedekind dislikes so much or the set up of an analytic framework for the study of Riemann surfaces. It is a way to simplify the inferences by reducing the complicated computations to very simple ones – manipulations of arithmetical operations similar to elementary number theory. As Dedekind tells us, "if the theory has not been shortened it has at least been simplified a little" ([Dedekind, 1877], 119). In the same way, in 1882, Dedekind and Weber's use

[29] *"La théorie de la divisibilité des nombres, qui sert de fondement à l'arithmologie, a déjà été établie par Euclide dans ce qu'elle a d'essentiel ; du moins, le théorème capital que tout nombre entier composé peut toujours se mettre, et cela d'une seule manière, sous la forme d'un produit de nombres tous premiers, est une conséquence immédiate de ce théorème démontré par Euclide, qu'un produit de deux nombres ne peut être divisible par un nombre premier que si celui-ci divise au moins l'un des facteurs."*

[30] The association between arithmetic and rigor was a rather widely spread conception among 19th century mathematicians (albeit not all of them). On the topic of Dedekind and rigor, one can refer to [Detlefsen, 2012] and [Haffner, 2014a].

[31] "*Ist* α *in* $\mathfrak{a}$ *enthalten, so sagen wir,* α *sei theilbar durch* $\mathfrak{a}$, $\mathfrak{a}$ *gehe in* α *auf, weil die Ausdrucksweise hierdurch an Leichtigkeit gewinnt.*"

of arithmetical notions openly aimed at a more rigorous and clearer theory, which they obtained adopting a strongly arithmetical approach with several layers of arithmetical notions – an approach which was not perpetuated by Weber.

1.3 A brief note on Kronecker's theory of algebraic magnitudes

The field of research described in [Goldstein and Schappacher, 2007] was very active until the 1860s. Both Dedekind and Kronecker, who were students and close collaborators of Dirichlet and deeply influenced by Gauss, had been in contact with it during their formative years and the beginning of their mathematical career. By the time Dedekind started to publish important works,[32] what Goldstein and Schappacher coined the Arithmetic Algebraic Analysis had mostly died off. From interactions between "higher arithmetic" and other domains such as quadratic forms, the focus shifted towards the desire to find the 'right' methods: for example, for algebraic numbers with Dedekind and Kronecker, or for algebraic functions with Weierstrass, Riemann, Clebsch, M. Noether, Brill, and later again Kronecker, Dedekind and Weber.

Algebraic function theory was a widely spread subject among mathematicians, at the end of the 19th century – and not limited to the reception of Riemann's works. For Clebsch, Noether and Brill, the most appropriate approach for algebraic functions was to be found in algebraic methods, working directly with specific equations,[33] whereas for Dedekind and Kronecker, the methods to be favored were arithmetical methods (with strikingly different conceptions of what "arithmetical" meant). From their common inheritance, Kronecker and Dedekind took diverging roads. I will give, here, a partial presentation of Kronecker's theory of algebraic functions.

Kronecker's *Grundzüge*

Kronecker's theory was announced as a forthcoming work by Kummer in 1857 but – as the well-known story goes – Kronecker considered that Weierstrass' results made his own works unnecessary and renounced to publish them. It is only after reading [Dedekind and Weber, 1882] and seeing the similarity of his ideas with Dedekind and Weber's, that Kronecker came back to algebraic magnitudes (numbers and functions). He then published his theory in 1881 and 1882 ([Kronecker, 1881], [Kronecker, 1882]) after having delayed the publication of [Dedekind and Weber, 1882] in Crelle's Journal. Dedekind and Weber briefly mention Kronecker's work in a footnote, but had no knowledge of it while writing their own theory, since they started working on their "theory of ideals of functions" (*Idealtheorie für algebraische Funktionen*, letter from Weber, 02/02/1879, [Scheel, 2014], 220) in 1879.

Kronecker's *Grundzüge einer arithmetischen Theorie der algebraischen Größen*[34]

[32]Dedekind's first paper in number theory was published in 1857 ([Dedekind, 1857]) and what is considered as his first major works was his theory of algebraic integers, first published in 1871 ([Dedekind, 1871]).

[33]See [Bottazzini and Gray, 2013], especially pp. 311-343.

[34]Kronecker's *Grundzüge* are a notoriously difficult (and long) work. [Edwards, 1990] gives a thorough mathematical commentary on Kronecker's theory.

([Kronecker, 1881]) are not just a theory of algebraic functions (nor are they just a theory of algebraic numbers) but a theory of algebraic *magnitudes* (*Größen*)[35] which encompass both algebraic numbers and algebraic functions. For Kronecker, "magnitudes" have only an arithmetico-algebraic meaning ([Kronecker, 1882], 249-250). This implies an extension of the notion of magnitude to *any* algebraic expression, i.e. rational function with integral coefficients. The standpoint taken by Kronecker in [Kronecker, 1882] is more general than Dedekind and Weber's in the sense that it considers a larger scope of objects (i.e. not only algebraic functions of one complex variable, but any algebraic expression). Kronecker's ideas about magnitudes are directly linked to the importance of computations in his mathematics, often emphasized by commentators. Kronecker, who walks in Kummer's footprints, has a position diametrically opposed to Dedekind's, and considers that concepts are (must be) the result of computations.[36]

Algebraic magnitudes can be added, multiplied, subtracted and divided. The divisibility of algebraic magnitudes is the core of Kronecker's investigations, and leads to the notion of divisor (*Divisor*).[37] The aim of Kronecker's theory of divisors is not, as it is often claimed, the theorem of unique factorization in primes. Rather as [Edwards, 1990] tells us, it is to define greatest common divisors. Working with greatest common divisors is preferable to primality because they are independent of the field in which one is working (see [Edwards, 1990], v). It is an essential difference with Dedekind's (and Weber's) approach: primality and the unique factorization in primes are at the core of their theory. The advantage seen by Kronecker in working with GCD is not one that interests Dedekind. To avoid the problem of primality being linked to the domain, Dedekind transfers primality to ideals (for algebraic numbers in [Dedekind, 1871] and its later rewritings, then for algebraic functions in [Dedekind and Weber, 1882]). As we will see, primality and divisibility properties are the focus of Dedekind and Weber's works, and transferred to many different levels and objects in function theory, leading to the divisibility of complexes of points of a Riemann surface.

Kronecker's theory can be understood, in modern terms, as aiming at decomposing the manifold "generated by an ideal into irreducible components" ([Vlădut, 1991], 11). However, Kronecker does not refer to Riemann's works nor to Riemann surfaces.[38] Unlike Dedekind and Weber, Kronecker shows no interest in studying algebraic functions with Riemann surfaces. This point is a second important difference between their

[35]In [Edwards et al., 1982], Dedekind puts "(?)" after uses of the word "*Größe*" which he considers, at least since [Dedekind, 1872], as imprecise and avoidable.

[36]See [Vlădut, 1991], [Boniface, 2004], and [Smadja, 2010].

[37]For brevity's sake, the modernized definition of a divisor given by Edwards is the following: "Let K be an algebraic extension of [a ring] r. A polynomial f, in any number of indeterminates and with coefficients in K, is said to divide another such polynomial g if $g = fq$, where q is a polynomial whose coefficients are elements of K integral over r. We will say that a polynomial f (in any number of indeterminates and with coefficients in K) represents a divisor, and will denote the divisor it represents by $[f]$" ([Edwards, 1990], 19). Edwards adds that if one wants to know what a divisor *is*, one should understand it by what it *does*: it "*divides* things. Specifically, it divides (or does not divide) polynomials with coefficients in K".

[38]As far as I am aware – but my knowledge of Kronecker can certainly be challenged – direct references to Riemann's works by Kronecker are, to say the least, very rare.

works. For Kronecker, algebraic functions must be part of a "general arithmetic" and studied only as such using the theory of divisors. His 1881 and 1882 works expose this general arithmetization of algebraic magnitudes, and the "general arithmetic" is to be a theory in itself. Neither Dedekind nor Weber were interested in a general theory of algebraic magnitudes. Their theories (of algebraic numbers and of algebraic functions) are tightly framed and considered independent of each other, and they did not express a desire to unify both theories. It shows how both arithmetizations stem from very different conceptions. For Dedekind and Weber, algebraic functions are an independent theory for which Riemann introduced a very fruitful concept – the Riemann surface – but which suffers from a crucial lack of rigor. They, thus, propose to rewrite Riemann's theory using Dedekind's ideal theory. There is no desire to subsume algebraic functions to arithmetic in Dedekind's and Weber's works. Rather, arithmetical strategies appear as a tool to produce a 'better' (Riemannian) theory of algebraic functions of one variable.

As is well known, there are many interesting similarities between Dedekind and Kronecker: both were deeply influenced by Gauss and Dirichlet, as well as by Kummer whose works they both wanted to pursue and generalize. They often worked on the same subjects and both insisted on the importance of developing them with the 'right' arithmetical approach... but with very different conceptions of arithmetic, to the point that the 'arithmeticity' of the other's work would be a central point of discord. Both of them acknowledged that the core of their dissensions can be found in two aspects of their practices: their conceptions of arithmetic, and their opinions on constructivity and actual infinity. When Kronecker qualifies his own works as being "arithmetical", Dedekind denies him this quality. In Dedekind's view, Kronecker's extensive use of indeterminate variables and polynomials implies that his work is too formal to be arithmetical and rather belongs to algebra. In comments about Kronecker's *Grundzüge*, Dedekind wrote:

> Neither from this introduction nor from the essay itself is it clear why exactly this theory should be called an *arithmetical* one. Under this name, one should expect that the consideration of the realm of numbers (the absolute constants) would form the main foundation, but it is by no means the case. I would very much like to rather call this theory *formal*, because it predominantly is based on the "auxiliary methods of indefinite coefficients" (p. 47, 48, 69) and the "association of forms (formed by these coefficients or auxiliary variables u, u', u'') (§15 and §22, p. 93-96).[39] ([Edwards et al., 1982], 54)

[39] "*Weder aus dieser Einleitung noch aus der Abhandlung selbst wird es deutlich, weshalb diese Theorie gerade eine* arithmetische *genannt wird. Nach diesem Namen sollte man vermuthen, daß die Betrachtung des Reiches der Zahlen (der absoluten Constanten) die hauptsächliche Grundlage bilden würde, was keineswegs der Fall ist. Viel eher möchte ich diese Theorie eine* formale *nennen, weil sie vorwiegend auf dem 'methodischen Hülfsmittel der unbestimmten Coefficienten' (S. 47, 48, 69) und der 'Association der (aus diesen Coefficienten oder Hülfsvariabelen u, u′, u″ ... gebildeten) Formen' (§15 und §22, S. 93-96) beruht.*"
Dedekind's "Bunte Bemerkungen" are comments in which he wrote about Kronecker's *Grundzüge*, and which were never published by Dedekind – although according to H. Edwards, he seemed to intend to do so. Edwards, Neumann and Purkert edited and published the "Bunte Bemerkungen" in 1982

I will not develop further the comparison between Kronecker's Dedekind's (and Weber's) arithmetizations. While such a comparison would certainly give us a better understanding of these works, and more generally of the importance of arithmetic in the 19th century, it is not the point of this paper... and it would require more space than is possible in one paper.

2 Rewriting Riemann

Dedekind and Weber's correspondance started with their collaboration on the edition of Riemann's *Gesammelte Werke*, in 1874. It was the beginning of a long friendship between Dedekind and Weber, as attested by their rich correspondence. One of the most significant side-effects of their collaboration and correspondance, and on a longer term, is the co-writing of [Dedekind and Weber, 1882], in which Dedekind and Weber transfer to algebraic functions of one complex variable, the methods elaborated by Dedekind for the theory of algebraic integers. The goal of their paper was to redefine the basic notions of Riemann's function theory and to set up a solid ground for the investigation of complex functions and Riemann surfaces. Their paper is presented as a criticism of previous treatments of Riemann's ideas (such as the algebraization of his ideas by Clebsch, Brill and Noether) and an attempt to go back to (what they took to be) Riemann's 'true' methodological and epistemological principles. Dedekind and Weber do not explicitly refer to any of their contemporaries's works, and one can only try to guess which works they might be alluding to. Considering the specificity of their reproaches, it is likely that the criticisms were meant for approaches relying on algebraic equations (Clebsch, Noether, Brill, Gordan) or even on explicit series (Weierstrass).

Dedekind and Weber propose a new definition for the notions introduced by Riemann (the surface, the ramification, the genus...) attempting to follow these guidelines, which were *for them* a most essential advantage of Riemann's work. On the other hand, Clebsch, Noether, Brill, and Gordan, attempted an algebraization of the theory, relying on explicit equations and studying restricted cases, because they considered the generality of Riemann's works to be "bewildering" ([Clebsch and Gordan, 1866], v). And Roch and Prym considered that the Dirichlet's principle was at the core of Riemann's theory – they saw it as Riemann's true way to "create" functions.

2.1 Dedekind and Weber and Riemann

Dedekind and Riemann

Dedekind greatly admired Riemann's works and followed his lectures while they were both *Privatdozenten* in Göttingen. While the main focus of Dedekind's mathematical works, throughout his career, was number theory, he was deeply influenced by Riemann's mathematics and claimed to pursue, in his own number-theoretical works, the epistemological and methodological principles adopted by Riemann, which he considered to be "to seek to infer the proofs immediately from the fundamental characteristics of

([Edwards et al., 1982]). References are to their paper.

concepts, rather than from calculation, and indeed to construct the theory in such a way that it is able to predict the results of calculation" ([Dedekind, 1877], transl. slightly altered, 102). It was for him the best way to avoid founding a theory on computations, which would be problematic for two reasons: firstly it made the difficulties much greater when one would try to adopt a more general viewpoint (e.g. not only for cyclotomic integers, but for any algebraic number) and even made mistakes more likely, secondly it "would still not be of the highest degree of perfection" ([Dedekind, 1877], 102).

For Dedekind, the avoidance of these computations or *Darstellungsformen* is related to the setting up of a "higher level", the resort to collections of numbers or functions to simplify and to avoid, or hide away, the computations – and, to some extent, it was already the case for Riemann.[40] This point is explicitly stated by Dedekind when he introduces ideals in [Dedekind, 1871] or explains the genesis of his theory in [Dedekind, 1877]. In particular, ideals allowed to hide away long arduous computations (on algebraic numbers or functions) by replacing them with the simple(r) ones between ideals – as we will see in the following paragraphs.

A (very cursory) note on Weber

Weber was recognized as a specialist of function theory and knew Riemann's works well, as testified by the fact that he was brought in by Dedekind to take over on the edition of Riemann's *Werke*. In 1870, Weber was one of the very few mathematicians to take interest in (and the defense of) Riemann's *Inauguraldissertation*. He tried to prove the Dirichlet principle in [Weber, 1870], an article which went unnoticed by his contemporaries. Weber was able to point out, before Weierstrass's publication of his own criticisms, that the core of the problem with the Dirichlet principle was the assumption of arbitrary functions. Weber explains that he does not want to try to prove the Dirichlet principle on the basis of "essentially new foundations" but rather wishes to "complete, at some vulnerable points, the proof that was more suggested than carried out by Riemann" ([Weber, 1870], 29). The important point, for us, is the fact that Weber, in his 1870 paper, is tackling a prevailing problem, and does so with the desire to provide a rigorous ground to Riemann's theory without departing from Riemann's initial proposal.

Another important point regarding Weber's practice in Riemannian function theory is the observation that he wants to adopt an approach valid for *any* algebraic function, in keeping with [Riemann, 1857]. It is, for example, noticeable in his 1873 paper "*Zur Theorie der Transformation algebraischer Functionen*". It is not the place, here, to give a detailed account of Weber's works. However, it seems important to underline that in many of his papers, he adopted an approach which did not present the kind of strong methodological and/or mathematical statements against Riemann of some his contemporaries, such as Clebsch or Weierstrass.

The edition of Riemann's *Gesammelte Werke*

The edition of Riemann's *Gesammelte Werke* started under the responsibility of Dedekind

[40] See, for example, [Ferreirós, 2008].

and Alfred Clebsch. In 1874, two years after Clebsch's death, Dedekind contacted Weber to ask him to take over the editorial work. From 1874 to 1876,[41] Weber did a large part of the editorial work on Riemann's *Werke*, which contains in addition to Riemann's published works a selection of twenty-two texts from his *Nachlass*, during which he regularly sought Dedekind's help who is credited as second editor of Riemann's *Werke.* Their editorial work was very careful, and most of it was done through letters.[42]

From 1 November 1874 to the end of 1876, Dedekind and Weber exchanged about 70 letters. These letters show us most clearly that Dedekind and Weber read very thoughtfully and thoroughly all the texts which they edited – and a few which were ultimately not published. It was very important for them to publish works which would be good for Riemann's standing, which would be interesting for the readers, relevant to his published works, true to his thought (and obviously correct). They proofread Riemann's published works, and embarked in the tedious process of deciphering, clearing up and reconstructing Riemann's (sometimes messy and often elusive) manuscripts.

For them, the editorial process was a real, deep, important mathematical work: their reading Riemann's papers implied to redo and verify the proofs, the computations, and reconstruct Riemann's reasoning in some of the less developed manuscripts.[43] Dedekind and Weber, the letters tell us, also sent each other Riemann's manuscripts, their notes on the manuscripts, and as time passes, they start sending each other notes from some of their personal works which they saw as related to what they were studying for Riemann's *Werke.* It was, thus, truly a part of their mathematical activity, which likely had an important impact on their future works – as suggested by the recurrence of their discussions around the subjects studied by Riemann such as Abelian functions, or of course the 1882 paper which is studied in this paper.

This first collaboration around Riemann's works was the foundation of their collaboration and their common reading of Riemann. While there is no textual evidence that their desire to rewrite Riemann's theory of algebraic functions was actually born while they were editing his *Werke*, Dedekind wrote to Weber early in their exchanges that he thought that himself would only truly understand Riemann's works when he would be able to "overcome in [Dedekind's] way, with the rigor that is customary in number theory, a whole series of obscurities" in them (11/11/1874, [Scheel, 2014], 50). However, of course, at the time of the edition of Riemann's *Werke*, the version of Dedekind's theory of ideals transferred to algebraic functions was not written. The fact that this new theory of ideals of algebraic functions is first mentioned in early 1878 suggests that using ideals for algebraic functions was only considered after the publication of [Dedekind, 1877].

[41]Considering that Dedekind and Clebsch worked together on the edition before Clebsch's death and that Dedekind tried to continue on his own, the overall editorial work probably lasted five to six years.

[42]See [Scheel, 2014].

[43]It was notably the case for "*Fragmente über die Grenzfälle der elliptischen Modulfunctionen*", [Riemann, 1876], 427-447. I study Dedekind and Weber's correspondence on the edition of Riemann's *Werke* in a forthcoming paper published by the FMSH, and in a forthcoming paper to be published in the proceedings of the conference in honor of the 100th anniversary of Dedekind's death as a special issue of the *Mathematische Semesterberichte.*

2.2 On writing the 1882 paper

A couple of years after the edition of Riemann's *Werke* and the in depth study of Riemann's works that accompanied the process, Dedekind and Weber started exchanging letters regarding a "ideal theory of algebraic functions" ([Scheel, 2014], 220). Throughout 1879 (and probably 1880), Dedekind and Weber's correspondence is essentially concerned with the investigation of this new theory. The letters written in 1879 are partially reproduced in [Scheel, 2014] and give us insights of the slow process of setting up this new theory.[44]

Weber's first letter referencing this theory indicates that the initial idea of an "ideal theory for algebraic functions" came from Dedekind. Dedekind sent his work to Weber, who responded with undeniable enthusiasm on 18/01/1879:

> Although I am not yet completely finished with studying your last message, I wanted already to thank you for it today, and inform you that I am extremely interested in it. It also promises great benefits for the theory of Abelian functions. Maybe we will manage, this way, to successfully obtain a useful normal form for Abelian functions.
>
> I can't write more about the details, today, because I need first to study the matter more precisely. In any case, I also hope at this occasion to penetrate for the first time deeply into your ideal theory.[45] ([Scheel, 2014], 220)

His following letter goes through Dedekind's ideas point by point, answering difficulties and proposing alternative approaches (for example for the basis of the field). After this first exchange on what Weber called the "ideal theory of algebraic functions", there is in [Scheel, 2014] more than 50 pages of letters from Weber on this theory. Weber sent a letter every two weeks, and most of the elaboration of their paper is done through these letters, with only a few visits.

Weber's letters in 1879 explore the relationships between function theory and ideal theory, linking, for example, the discriminant of the field to the ramification points or attempting to provide a satisfying proof of the Riemann-Roch theorem ([Scheel, 2014], 239-251). His letters show that he tries to translate Riemann's theory into the language of Dedekind's ideal theory:

> There is something, in the concept of an integral algebraic function which seemed strange to me from the beginning, and which in fact makes it a difficulty, namely the different behavior in the different sheets at infinity.

[44] As mentioned in the introduction of this paper, Dedekind's letters from 1879 and most of the letters from 1880 seem to have been lost.

[45] *"Obgleich ich noch nicht vollständig mit dem Studium Deiner letzten Mittheilung durch bin, will ich Dir doch schon heute dafür danken und Dir mittheilen, daß mich dieselbe über die Maßen interessirt. Ich verspreche mir großen Nutzen davon auch für die Theorie der Abelschen Functionen. Vielleicht gelingt es auf diesem Wege zur eher brauchbaren Normalform für die Abelschen Functionen zu gelangen.*
Ueber Einzelnes kann ich Dir heute noch nicht schreiben, da ich erst die Sache genauer durchstudieren muß. Jedenfalls hoffe ich bei dieser Gelegenheit auch einmal gründlich in Deine Ideal-Theorie hineinzukommen."

> I have therefore wondered already, whether one should not also here introduce something similar to *order*, by considering the collection (*Inbegriff*) of those integral functions which stay finite in determinate sheets at infinity.[46] (05/03/1879, *ibid.*, 229)

This exploration was difficult and Weber was both enthusiastic and cautious. In the same letter, after considering how to prove, for ideals, some results which are easy to prove in the Riemannian setting, Weber wrote to Dedekind:

> We must wait and see if the whole thing is going to lead to something new. What we have so far is not fundamentally new. Anyway, it is a very elegant and neat presentation for known propositions, and in this respect, it satisfies an esthetic requirement. What I expect from this in the first place is in fact a rigorous, or at least more general foundation for Riemann's theory.[47] (05/03/1879, *ibid.*, 230)

In their published paper, Dedekind and Weber, indeed, do not provide any new results, but give new definitions (e.g. of the genus) and new proofs (e.g. of the Riemann-Roch theorem). But despite their difficulties and the disappointment of not being able to provide new results, Weber is insistent that their work is worth pursuing and writes in December 1879:

> What we have done so far is however too nice to abandon it completely (...). It would however be good now, if soon we could come forward with something, when precisely so many [works] appear in this field which are not so good as ours, for example a thick book by Briot on Abelian functions.[48] (26/12/1879, *ibid.*, 265)

Dedekind and Weber did not reach Weber's goal, and Abelian functions do not appear in their paper. In the letters from 1879, Weber uses notions and elements of language which disappear from the 1882 presentation, such as the reference to the sheets of the surface.

Insofar as more than half of the letters are missing, a tentative genesis of their theory would be a dangerous enterprise and will not be attempted in this paper.

[46] "*Es ist mir in dem Begriff der ganzen algebraischen Function etwas von vorn herein fremdartig vorgekommen, was auch in der That Schwierigkeiten macht, nämlich das verschiedene Verhalten in den verschiedenen Blättern im Unendlichen. Ich habe deshalb schon daran gedacht, ob man nicht auch hier etwas den* Ordnungen *Analoges einführen müßte, indem man den Inbegriff derjenigen ganzen Functionen betrachtet, die in bestimmten Blättern im Unendlichen endlich bleiben.*"

[47] "*Ob die ganze Sache zu Etwas Neuem führen wird, müssen wir abwarten. Was wir bis jetzt haben ist im Grunde nicht neu. Immerhin ist es eine sehr elegante und hübsche Ausdruckweise für bekannte Sätze und genügt insofern einem ästhetischen Bedürfnis. Was ich zunächst davon hoffe ist übrigens eine strenge oder wenigstens allgemeinere Begründung der Riemannschen Theorie*".

[48] "*Was wir darin gemacht haben, ist doch zu hübsch, um es ganz liegen zu lassen (...). Es wäre doch schön wenn wir jetzt bald einmal mit etwas hervortreten könnten, wo gerade so vielerlei auf diesem Gebiete erscheint, was entschieden nicht so gut ist als unseres, z. B. aber ein dickes Buch von Briot über Abel'sche Functionen.*"

2.3 Motivations for the 1882 paper

For Dedekind and Weber, since Riemann's works,[49] mathematicians had failed to follow some of the key precepts of his elaboration of the concept of Riemann surface, namely, to treat entire classes of functions without distinguishing between special cases, without relying on individual or explicit expressions, and without taking computations as a ground of the theory.[50] Dedekind and Weber explain, in the introduction of their paper, that by proposing approaches to the theory attached to definite equations or to explicit representations in series, the previous authors had to impose initial restrictions on the functions treated (e.g., on their singularities). This is something which Dedekind and Weber want to avoid. They wish to offer a "general" theory, that is to treat, in one move, entire classes of functions, to avoid restrictive initial hypotheses, distinctions between cases and the "so-called exceptional cases" which, according to Dedekind and Weber, were often "mentioned casually as limit cases, or even left aside entirely" in previous works ([Dedekind and Weber, 1882], 238, transl. [Stillwell, 2012], 41).

They also wish to avoid any use of geometric intuition, especially if it is taken as a reason to admit the truth of certain theorems about functions, such as their developability. Facing what they perceived as conspicuous lack of rigor and recurring restrictive assumptions that can be found in the treatments of Riemannian function theory they opposed, Dedekind and Weber wish to reformulate the fundamental concepts of the theory, so as to solve these issues. In addition, from Dedekind and Weber's viewpoint, the treatments of function theory before their work had a tendency to unduly accept the validity of certain unproven theorems at the basis of their researches. The authors explicitly state so, in the introduction of the paper:

> The purpose of the investigations communicated in what follows is to found the theory of algebraic functions of one variable, one of the most important results of Riemann's creation, from a standpoint which would be simple, and at the same time rigorous and completely general.[51] ([Dedekind and Weber, 1882], 238, transl. modified [Stillwell, 2012], 41)

To do so, they import in algebraic function theory the concepts and methods introduced by Dedekind in algebraic number theory, namely the concepts of field, module and ideal,

[49]Riemann's core ideas were first presented in 1851 in his doctoral dissertation, "*Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*" ([Riemann, 1851], which was never published (e.g. in Crelle's journal) but only printed and distributed to a number of German universities, as was customary, and later reproduced in [Riemann, 1876]). Riemann published his theory six years later in a reworked and extended form, with an application to Abelian functions in "*Theorie der Abel'schen Functionen*" ([Riemann, 1857]).

[50]The amount of problems to investigate and of possible directions that could be taken after Riemann's function theory is very wide, and is well illustrated by the variety of authors studied in works such as [Houzel, 2002], [Chorlay, 2007], or [Bottazzini and Gray, 2013], to which the reader can refer for a more comprehensive overview.

[51]"*Die im nachstehenden mitgeteilten Untersuchungen verfolgen den Zweck, die Theorie der algebraischen Funktionen einer Veränderlichen, welche eines der Hauptergebnisse der Riemannschen Schöpfung ist, von einem einfachen und zugleich strengen und völlig allgemeinen Gesichtspunkt aus zu begründen.*"

and methods of proofs developed for them. In particular, the core of their approach is the definition of arithmetical operations for ideals in such a way that it becomes possible to "calculate" with ideals:

> With a proper definition of multiplication, it is possible to calculate with these ideals using exactly the same rules as for rational functions.[52] ([Dedekind and Weber, 1882], 240, transl. slightly altered [Stillwell, 2012], 42)

Since the notion of field of functions, as Dedekind and Weber write in the introduction, "coincides completely with that of Riemann's class of algebraic functions", they first study in depth the properties of the field. On this basis, in the second part of the paper, they exhibit the said "coincidence" by establishing a one-to-one correspondence between the field and the surface. The arithmetical methods used for ideal theory are transferred to the Riemann surface as well, through the new definition itself.

The Riemann surface remains the core of the theory, despite the lengthy technical preliminaries. Dedekind and Weber explain, in fact, that for the time being, providing a rigorous foundation to the theory is too difficult to be done directly:

> Admittedly, all these results can be obtained from Riemann's theory with a much more limited investment of means, and as special cases of a much more extensive general presentation. However, it is known that a rigorous foundation of this theory still presents certain difficulties, and until one has managed to completely overcome these difficulties, it may very well be that the path we have taken, or at least one related to it, is in fact the only one leading to this goal for algebraic function theory with satisfactory rigor and generality.[53] ([Dedekind and Weber, 1882], 240, transl. modified [Stillwell, 2012], 42)

While Dedekind and Weber keep the central ideas of Riemann surfaces, the theory is stripped of any geometrical component. Dedekind and Weber also give up the infamous Dirichlet principle, without even discussing it. Any kind of infinitesimal characterization or the idea of a mobile point to join the sheets of the surface completely disappear from their work.

3 Fields of algebraic functions of one complex variable

In the first part of their paper, which will be studied in this section, Dedekind and Weber, even if they are not telling their readers so, set up the framework and the toolbox which

[52] *"Mit diesen Idealen läßt sich nach gehöriger Erklärung der Multiplikation ganz nach denselben Regeln rechnen, wie mit rationalen Funktionen."*

[53] *"Freilich ergeben sich alle diese Resultate durch einen weit geringeren Aufwand von Mitteln und als Spezialfälle einer vielumfassenden Allgemeinheit aus Riemanns Theorie; allein es ist bekannt, daß diese Theorie bezüglich einer strengen Begründung noch gewisse Schwierigkeiten bietet, und bis es gelungen ist, diese Schwierigkeiten vollständig zu überwinden, dürfte der von uns betretene Weg oder wenigstens ein verwandter, wohl der einzige sein, der für die Theorie der algebraischen Funktionen mit befriedigender Strenge und Allgemeinheit zum Ziele führt."*

serves to establish what they hope to be a more general and more rigorous treatment of Riemann's function theory. Most of the considerations about the arithmetical structure of the field Ω and the definition of special ideals will have a counterpart in the definition of the surface.

In order to develop the "theory of ideals of algebraic functions", Dedekind and Weber take as a basis

> a generalization of the theory of rational functions of one variable, in particular of the theorem according to which every polynomial[54] of one variable admits a decomposition into linear factors.[55] ([Dedekind and Weber, 1882], 238, transl. modified [Stillwell, 2012], 41)

In the case of surfaces of genus 0, such as the sphere, the approach has already been studied and is "simple and well known". Dedekind and Weber's aim is to treat the general case, that is, to provide a treatment valid for any genus p. This general case is related to the surfaces of genus 0 "in the same way that general algebraic numbers are related to rational numbers" (*ibid.*).[56] To treat the general case, they will thus transfer and adapt to functions of one complex variable the number theoretical methods "derived from Kummer's creation of ideal numbers", which proved to be "most successful" in Dedekind's own works in number theory.

This comparison is the core of the so-called analogy between fields of numbers and fields of functions, for Dedekind and Weber: the similarity of two relations presented as a proportion ($A : B :: C : D$) – a rather classical view of analogy. This comparison guides the "transfer" (*Übertragung*) of methods from number theory to function theory. It does not indicate, thus, a structural conception of the notions of field and ideal. For Geyer, the "fundamental idea and fundamental result" (*[d]ie Grundidee und das Grundresultat*, [Geyer, 1981], 113) of the first part of their paper is "the exposition" (*Aufzeigen*) of the analogy between fields of numbers and fields of functions. It is important, however, to underline that Dedekind and Weber are not investigating 'structural similarities' between fields of functions and fields of numbers. Not only is it not their aim, but Dedekind and Weber don't show any interest in thinking about fields and ideals in and for themselves – rather, fields and ideals appear as new fundamental notions and tools for specific theories (of numbers, of functions). Fields, ideals or modules are not thought of as abstract concepts, in the sense of (our) modern algebra. Although fields, ideals and modules have

[54]Dedekind and Weber, as it was common at the time, use "integral rational function" (*ganze rationale Funktion*) to designate polynomials. To avoid confusion with the integral functions defined later in the text as the integers of the field, I will use the term "polynomial".

[55]"*Eine sichere Basis für die Grundvorstellungen sowie für eine allgemeine und ausnahmslose Behandlung der Theorie läßt sich gewinnen, wenn man von einer Verallgemeinerung der Theorie der rationalen Funktionen einer Veränderlichen, insbesondere des Satzes, daß jede ganze rationale Funktion einer Veränderlichen sich in lineare Faktoren zerlegen läßt, ausgeht.*"

[56]Dedekind uses "rational numbers" and "rational integers", since algebraic numbers (and integers) are defined as an extension of rational numbers (and integers). He does not consider "natural numbers", here. I will use "rational integers" in the same way. In particular, since it is, for Dedekind a way to contrast the properties of algebraic integers as an extension of the concept of integer, I will be talking about "rational number theory" to designate number theory done with rational integers.

an ‘abstract’ definition (which Weber calls “pure formalism” ([Weber, 1893], 521) and Dedekind calls “invariant” ([Edwards et al., 1982], 63)), that is, a definition that does not refer to the individual nature of their elements, never is a general theory of these concepts developed by Dedekind himself.[57] Moreover, Dedekind’s concept of field is linked to equations (see [Dedekind, 1873], 409) rather than structures.

3.1 The transfer

The main idea behind the transfer of algebraic number methods to the theory of algebraic functions is to study divisibility in a field of algebraic functions. While they do mention Dedekind’s algebraic integer theory and its relation to the 1882 paper, they also emphasize that they do not expect their reader to be acquainted with that theory, and consequently do not want to presuppose any of the definitions and theorems to be already known. All definitions and proofs are given *in extenso*: every result, although already proved in 1877, is proved again. The fact that module and ideal theories are entirely developed for the theory of functions without relying on their development and validity in number theory emphasizes the idea that Dedekind and Weber are proceeding to a “transfer” or “translation” of the methods, implying no presupposition of the validity of the methods in a different (or general) framework.

Dedekind and Weber take full advantage of the analogy mentioned in the introduction and the well-known similarities between the arithmetical behaviors of numbers and of functions, and define a notion of field of algebraic functions:

> In an analogy with number theory, a *field of algebraic functions* is understood to be a system of such functions with the property that application of the four fundamental operations of arithmetic (*Spezies*) to the functions of this system always leads to functions of the same system.[58] ([Dedekind and Weber, 1882], 239, transl. slightly altered [Stillwell, 2012], 42)

For the transfer of methods from number theory, they follow the analogy and define, for functions, counterparts of the notions used in number theory, in order to be able to unfold the ideal theory in function fields:

rational integers	$\to$	polynomials
rational numbers	$\to$	rational functions
algebraic numbers	$\to$	algebraic functions
algebraic integers	$\to$	integral algebraic functions

Thus, one has to identify the integers of the field. A notion of *integral* algebraic function can thus be defined in very similar way as the algebraic integers were defined:[59]

[57] It will be developed in the works of Steinitz, E. Noether, etc. On these matters, see [Corry, 2004].

[58] “*Versteht man, analog der Zahlentheorie, unter einem* Körper algebraischer Funktionen *ein System solcher Funktionen von der Beschaffenheit, daß die Anwendung der vier Spezies auf Funktionen des Systems immer zu Funktionen desselben Systems führt.*”

[59] See [Geyer, 1981], 116, for a modern presentation of Dedekind and Weber’s concept of integral algebraic function.

A number θ is an *algebraic integer* if it satisfies an equation

$$\theta^n + a_1\theta^{n-1} + ... + a_{n-1}\theta + a_n = 0$$

of finite degree n and whose coefficients a_1, a_2, ..., a_n are rational integers.

A function ω is an *integral algebraic function* of z if in the equation of lowest degree satisfied by ω

$$\omega^e + b_1\omega^{e-1} + ... + b_{e-1}\omega + b_e = 0$$

the coefficients b_1, b_2, ..., b_e are all polynomials of z.

The development of ideal theory is very much similar to what was done in 1877, it is sometimes the same word for word, with "function" in place of "number". However, it is *not strictly* identical. In particular, some results are simpler to prove for functions – although, as Dieudonné mentions, Dedekind and Weber do not seem to clearly know why it is simpler and, in any case, do not attempt to explain it. Dedekind and Weber mention, in footnotes, the differences when they appear, but their aim is not to try to expose what the theory of algebraic numbers and the theory of algebraic functions have in common. They do not explain these differences and do not try to extract a common core for the two theories.[60] Following their lead, I will make few references to [Dedekind, 1877], in this paper. This allows to stress the fact that the arithmetical aspect of the 1882 paper should not be taken as a mere result of the *application* of number theoretical methods.

The strategy adopted by Dedekind and Weber allows them to transfer all the questions, theorems and proofs regarding divisibility of algebraic functions to divisibility of ideals. Indeed, as they explain themselves, with an appropriate definition of arithmetical operations between ideals, it becomes possible to "calculate with ideals using the same rules as for rational functions", to define notions such as "prime ideal" and to prove the validity of divisibility theorems such as the existence of a unique factorization in primes for ideals of algebraic functions. The arithmetic properties appear, then, to be of a striking simplicity. Such a strategy, the study of divisibility properties for systems of elements, is the basis for their overall approach in the paper and used for the definition of the Riemann surface as well.

3.2 Fields of functions

An algebraic function θ of an independent variable z is defined as the root of a polynomial equation of degree n in θ whose coefficients are polynomials (without common divisors) in z ([Stillwell, 2012], 45). Dedekind and Weber establish that θ does not satisfy any equation of lower degree in θ as well as in z, and that the "system $\Phi(\theta, z)$ of all rational functions of θ and z" is "a *field* Ω *of algebraic functions of degree* n" (*ibid.*, 46, emphasis in the original). Dedekind and Weber's first and essential move is thus the introduction of the notion of a field of functions. While mathematicians were well aware that rational and polynomial functions reproduce themselves (*sich reproduzieren*) by the four fundamental operations of arithmetic $(+, -, \times, \div)$, the choice to consider this very property as the ground of the theory is an important departure from the usual approaches to the theory.

[60] For such a comparison, see [Noether, 1919] and [Strobl, 1982].

Having defined integral algebraic functions, Dedekind and Weber work in the integral closure $\mathfrak{o}$. The system $\mathfrak{o}$ is proved to be closed under addition, subtraction and multiplication (in §3, 5.). Arithmetical operations (including divisibility of functions[61]) are defined for the integral functions, following the known rules of manipulation of rational functions. Dedekind and Weber mainly investigate arithmetical properties of algebraic functions as elements of $\mathfrak{o}$ such as the transitivity of divisibility. The complete study of the divisibility laws governing fields of algebraic functions is done further down the paper by resorting to ideal theory.

Dedekind and Weber *do not* introduce a notion of ring, or a notion similar to a ring, as are Dedekind's *Ordnungen* in number theory. As it was pointed out to me by Norbert Schappacher, if Dedekind and Weber were to work in Dedekind's theory of *Ordnungen* rather than simply in the integral closure, they would not succeed in avoiding the singularities as they do here – a point of which they are very proud. It is not clear whether they purposely avoided to work in *Ordnungen*, or whether they just did not see the point in doing so. Indeed, the importance of *Ordnungen*, only became clear later for Dedekind: they are especially important in the 1894 version of his algebraic number theory (yet still very different from our rings). Dedekind, in [Dedekind, 1877], uses *Ordnungen* as a change of setting to bypass a difficulty in ideal theory (namely the proof that the set-theoretic definition of the divisibility of ideals is equivalent to the arithmetical definition of divisibility, see page 31 of this paper). They do not have the status and role that rings have for modern algebra. In algebraic function theory, this result is not difficult to prove. Hence, the need for *Ordnungen* does not arise. Following Dedekind and Weber, I will avoid using the term "ring", which will lead me to write about ideals in a field. This slight misuse highlights that Dedekind and Weber's work is far from our contemporary conceptual setting.

While I do not intend to consider this in any detail here, it seems important to underline that before the introduction of modules and ideals, Dedekind and Weber develop for the field Ω notions such as the norm of a function, the determinant, the trace[62] and the discriminant. The "analogy" with number theory likely played an important role in the set up of these notions, which are most useful in the proofs.[63] Indeed, in number theory, since the norm of an algebraic number is a rational number, using norms allows to reduce the problems to the study of rational numbers. This strategy was used by Gauss and Kummer (e.g. [Kummer, 1851]) and was well-known. In the case of algebraic functions, the norm is a rational function, which continues further the correspondence

[61] [Dedekind and Weber, 1882], 249

[62] "Trace", in German *Spur*, was not a very widespread term (although the concept was known) and seems to have been introduced by Dedekind in algebraic number theory. It was usually solely referred to as a "coefficient of the characteristic equation" ([Hawkins, 2008], 500). As such, it was well-known, and in 1895, Dedekind referred to "norms, discriminants and traces" as "common" ([Dedekind, 1895], in [Dedekind, 1932] II, 59). According to Hawkins, it was not such a common expression, and Frobenius, "following the lead of Dedekind", was the one to begin to popularize it in his works on matrices. Weber also uses it in [Weber, 1896].

[63] See [Stillwell, 2012], 47-55. See also [Geyer, 1981], 116-117, and [Strobl, 1982], 235-246, for modern expositions of some of their results.

stated earlier. Determinants and discriminants were usual tools for the theory of forms, and can be found in works from Gauss, Sylvester, Gordan, Kronecker... The notion of discriminant is also particularly important because of its relations with the ramification of the field, which we will see later. Dedekind's and Weber's innovations thus stem from prevalent methods in number theory.[64]

3.3 Ideals of functions

Dedekind and Weber define and study modules before considering ideals – a step that is considered "tedious" by Dedekind (in the letter from 30/10/1880, in [Scheel, 2014], 271) but necessary, if only to simplify ideal theory. A module is defined by the following closure conditions:

> A system of functions (in Ω) is called a module if its functions reproduce themselves by addition, subtraction and multiplication by a polynomial function of z.[65] ([Dedekind and Weber, 1882], 251-252, transl. modified [Stillwell, 2012], 55)

An important aspect of their approach is the definition of (new) arithmetical operations. The idea used here is the one that constitutes the core of Dedekind's algebraic number theory and allowed him to prove the general validity of the propositions and theorems of arithmetic in fields of algebraic numbers. A new notion of divisibility is defined for modules:

> A module $\mathfrak{a}$ is said to be divisible by a module $\mathfrak{b}$, or $\mathfrak{b}$ is said to be a divisor (*Teiler* (*Divisor*)) of $\mathfrak{a}$, or $\mathfrak{a}$ a multiple (*Vielfaches* (*Multiplum*)) of $\mathfrak{b}$ ($\mathfrak{b}$ taken up in $\mathfrak{a}$ [$\mathfrak{b}$ *geht in* $\mathfrak{a}$ *auf*]), if every function in $\mathfrak{a}$ also belongs to $\mathfrak{b}$. We call $\mathfrak{b}$ a proper divisor of $\mathfrak{a}$ if $\mathfrak{a}$ is divisible by $\mathfrak{b}$ but not identical to $\mathfrak{b}$.[66] ([Dedekind and Weber, 1882], 251, transl. [Stillwell, 2012], 57)

This definition, which will be given again for ideals, gives Dedekind and Weber the means to develop an arithmetic of ideals emulating rational arithmetic – arithmetic at the "higher level" of ideals – a strategy which will be consistently used in their paper, including for other objects such as complexes of points.

Modules are said to be an "auxiliary theory", to which Dedekind referred in earlier works as an "independent" general theory whose interest "lies mainly in [its] applications" ([Dedekind, 1877], 62). Dedekind and Weber explain that since "every ideal is at the same time a module, all the terms and notations defined for modules can be applied to

[64] [Geyer, 1981] and [Strobl, 1982] offer modern expositions and interpretations of this part of Dedekind and Weber's works.

[65] *"Ein Funktionensystem (in Ω) heißt ein Modul, wenn sich die Funktionen desselben durch Addition, Subtraktion und durch Multiplikation mit ganzen rationalen Funktionen von z reproduzieren."*

[66] *"Ein Modul* $\mathfrak{a}$ *heißt durch einen Modul I*$\mathfrak{b}$ *teilbar, oder* $\mathfrak{b}$ *ein Teiler (Divisor) von* $\mathfrak{a}$, $\mathfrak{a}$ *ein Vielfaches (Multiplum) von* $\mathfrak{b}$ *(*$\mathfrak{b}$ *geht in* $\mathfrak{a}$ *auf), wenn jede Funktion in* $\mathfrak{a}$ *zugleich in* $\mathfrak{b}$ *enthalten ist.* $\mathfrak{b}$ *soll ein echter Teiler von* $\mathfrak{a}$ *heißen, wenn* $\mathfrak{a}$ *durch* $\mathfrak{b}$ *teilbar, aber nicht mit* $\mathfrak{b}$ *identisch ist."*

ideals”[67] ([Dedekind and Weber, 1882], 264, transl. [Stillwell, 2012], 66). In particular, the notions of divisibility and product are carried over to ideals. For this reason, and because they play a much more crucial role in the theory, I will focus on ideals.[68]

In the field Ω, an ideal is a system $\mathfrak{a}$ of integral algebraic functions (i.e., contained in $\mathfrak{o}$) defined by the following necessary and sufficient conditions:

> I. The sum and difference of every pair of functions of $\mathfrak{a}$ are again functions of $\mathfrak{a}$.
>
> II. The product of any function in $\mathfrak{a}$ by any function in $\mathfrak{o}$ is again a function of $\mathfrak{a}$.[69] ([Dedekind and Weber, 1882], 264, transl. slightly altered [Stillwell, 2012], 66)

An ideal $\mathfrak{a}$ is said to be divisible by an ideal $\mathfrak{b}$ if every function of $\mathfrak{a}$ is also contained in $\mathfrak{b}$. Since every ideal is contained in $\mathfrak{o}$ (which is of course itself an ideal), each ideal is divisible by $\mathfrak{o}$ which plays the role of the unit for the arithmetic of ideals.

The definition of divisibility for ideals here is a way of expressing the relation of inclusion: to divide is to contain. Thus, the divisor “contains a greater quantity of functions” than the ideal it divides (*ibid.*, 57), which simply corresponds to the idea that the multiples of $\mathfrak{a}$ are included in the multiples of $\mathfrak{b}$. Once accepted this definition of divisibility, the study of divisibility properties will be developed following the lines of rational integer divisibility. Dedekind and Weber rely on the reformulation of inclusion of ideals as a divisibility relation to put into play arithmetical notions and methods of proof in their study of the properties of the field.

Having defined a “principal ideal” $\mathfrak{o}\mu$ as the system of all the integral functions divisible by μ (with μ any non-zero function in $\mathfrak{o}$), Dedekind and Weber consider the divisibility between functions and ideals:

> A principal ideal $\mathfrak{o}\mu$ is divisible by a principal ideal $\mathfrak{o}\nu$ if and only if the integral function μ is divisible by the integral function ν. (...)
>
> A function α in $\mathfrak{o}$ is said to be *divisible* by the ideal $\mathfrak{a}$ if the principal ideal $\mathfrak{o}\alpha$ is divisible by $\mathfrak{a}$ or, which means the same thing, if α is a function in $\mathfrak{a}$.[70] ([Dedekind and Weber, 1882], 265, transl. slightly altered [Stillwell, 2012], 68)

[67] *“Jedes Ideal ist also zugleich ein Modul und alle für die Moduln erklärten Begriffe und Bezeichnungen können auf die Ideale angewandt werden.”*

[68] The reader who wishes to refer to Dedekind and Weber’s original text will find that observations made here about divisibility of ideals were made by Dedekind and Weber regarding modules. They only refer to the definitions and propositions given for modules, for the basis of the divisibility of ideals.

[69] “*Ein System $\mathfrak{a}$ von ganzen Funktionen von z im Körper Ω heißt ein Ideal, wenn es die beiden folgenden Bedingungen erfüllt: I. Summe und Differenz je zweier Funktionen in $\mathfrak{a}$ ergeben wieder eine Funktion in $\mathfrak{a}$. II. Das Produkt einer jeden Funktion in $\mathfrak{a}$ mit einer jeden Funktion in $\mathfrak{o}$ ist wieder eine Funktion in $\mathfrak{a}$.*”

[70] “*Ein Hauptideal $\mathfrak{o}\mu$ ist dann und nur dann teilbar durch ein Hauptideal $\mathfrak{o}\nu$, wenn die ganze Funktion μ teilbar ist durch die ganze Funktion ν. (...) Eine Funktion α in $\mathfrak{o}$ soll durch das Ideal l $\mathfrak{a}$ teilbar heißen, wenn das Hauptideal $\mathfrak{o}\alpha$ durch $\mathfrak{a}$ teilbar, oder, was dasselbe sagt, wenn α eine Funktion in $\mathfrak{a}$ ist.*”

From there, it can easily be seen that the divisibility for principal ideals is equivalent to the divisibility for the integral functions generating them. Then the study of the divisibility of the functions of the field is equivalent to studying the laws of divisibility governing the ideals. Dedekind and Weber show that the laws of divisibility for ideals are the same as the laws governing the polynomials. Having shown that, Dedekind and Weber have thus shown that the laws of divisibility governing the (integral) algebraic functions are the same as the laws governing polynomials (thus, the same as those governing the rational integers). Hence, Dedekind and Weber set up an arithmetical framework for the study of ideals, so that they can transfer the study of the divisibility laws of the field Ω from integral algebraic functions to modules and ideals and handle ideal theory with methods of proof that are similar to those in elementary number theory. The reformulation of inclusion as a divisibility relation is not an *ad hoc* introduction of some familiar denomination, the inquiry takes an overall arithmetical form, and the theorems and their proofs are given under an arithmetical form.

Additional definitions and properties for the divisibility of ideals are given, presenting a striking similarity with rational arithmetic. Dedekind and Weber develop the ideal counterparts of familiar arithmetical notions. For example, the least common multiple and greatest common divisor of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are defined:

– The collection $\mathfrak{m}$ of all the functions belonging to both ideals $\mathfrak{a}$ and $\mathfrak{b}$, unless it consists only of the "zero" function, is an ideal called the *least common multiple of* $\mathfrak{a}$ *and* $\mathfrak{b}$.

– If α is an arbitrary function in $\mathfrak{a}$, β an arbitrary function in $\mathfrak{b}$, then the collection of all the functions of the form $\alpha + \beta$ forms an ideal $\mathfrak{d}$ called the *greatest common divisor of* $\mathfrak{a}$ *and* $\mathfrak{b}$.[71]

– Two ideals are said to be relatively prime when their GCD is $\mathfrak{o}$.

A prime ideal is defined as "an ideal $\mathfrak{p}$ different from $\mathfrak{o}$" which is such that "no other ideal than $\mathfrak{p}$ and $\mathfrak{o}$ divides $\mathfrak{p}$" ([Dedekind and Weber, 1882], 266, transl. slightly altered [Stillwell, 2012], 68).[72] The propositions and theorems given about the divisibility of ideals appear familiar to the reader, as they are parallel to well-known theorems about the divisibility properties of rational integers, such as:

[71] Dedekind and Weber *justify* the choice of an arithmetical terminology by explaining that any ideal which is a multiple of both $\mathfrak{a}$ and $\mathfrak{b}$ (i.e., included in both $\mathfrak{a}$ and $\mathfrak{b}$) is also a multiple of $\mathfrak{m}$ (i.e., contained in their intersection). The justification for the GCD goes along the same lines.

[72] Dedekind and Weber prove, later in their paper, that an ideal is prime if and only if it is an ideal of first degree (i.e., its norm is a linear polynomial), which is, for them, the most essential difference between the theory of algebraic numbers and the theory of algebraic functions. The difference lies in the notion of norm of an ideal. In algebraic number theory, the degree of an ideal is defined in the following way: the norm of an ideal $\mathfrak{a}$ is a rational integer, the (always finite) number of incongruence classes in $\mathfrak{o}$ modulo $\mathfrak{a}$ (i.e., the number of integers in $\mathfrak{o}$ which are not congruent modulo $\mathfrak{a}$). In the case of a prime ideal $\mathfrak{p}$, its norm is a rational integer divisible by $\mathfrak{p}$ and which put together with all the rational numbers divisible by $\mathfrak{p}$ forms a module $[p]$ in which p is the smallest rational number greater than 0 divisible by $\mathfrak{p}$. This p is necessarily indecomposable (or it wouldn't be the smallest number divisible by $\mathfrak{p}$) and it cannot be 1, because in that case $\mathfrak{p} = \mathfrak{o}$. Since $\mathfrak{o}p$ is divisible by $\mathfrak{p}$, $N(\mathfrak{o}p) = p^n$ is divisible by the norm of $\mathfrak{p}$ which is thus of the form p^f and not necessarily of the first degree. See also [Strobl, 1982], 238-239.

> Each ideal $\mathfrak{a}$ different from $\mathfrak{o}$ is divisible by a prime ideal $\mathfrak{p}$. (...)
>
> If $\mathfrak{a}$ is relatively prime to $\mathfrak{c}$ and $\mathfrak{ab}$ is divisible by $\mathfrak{c}$, then $\mathfrak{b}$ is divisible by $\mathfrak{c}$.[73] ([Dedekind and Weber, 1882], 267-268, transl. [Stillwell, 2012]., 69-70)

In order to completely unfold and explain the divisibility laws governing ideals, and bring to the fore the fact that these laws are the same as the ones governing polynomials, Dedekind and Weber give additional properties, labelled "Laws of divisibility of ideals". In particular, two core results are the existence of a unique decomposition in primes for every ideal, and the equivalence between the notion of divisibility defined as an inclusion relationship and the arithmetical notion of divisibility, that is:

> If $\mathfrak{c}$ is an ideal divisible by an ideal $\mathfrak{a}$, then there exists *one and only one* ideal $\mathfrak{b}$ that satisfies the condition $\mathfrak{ab} = \mathfrak{c}$. It is called the *quotient of* $\mathfrak{c}$ *by* $\mathfrak{a}$.[74] ([Dedekind and Weber, 1882], 271, transl. slightly altered [Stillwell, 2012], 73)

There is a clear distinction made between the quotient and the notion of divisibility. With the proof of this theorem, it becomes clear that the notion of divisibility given for ideals can indeed be considered as an extension of the arithmetical notion. This equivalence between the 'set-theoretic' notion of divisibility and the arithmetical one was, in algebraic number theory, a difficult property to prove – Dedekind even presents it as the main difficulty of the theory. For algebraic functions, the property is easier to prove. In fact, the paragraph on the laws of divisibility of ideals in algebraic function theory "proceed[s] significantly more simply than" algebraic number theory (*ibid.*, 70) according to Dedekind and Weber.[75] This simplification of the (proofs for the) study of the divisibility laws of ideals is a second important difference (with the one mention in note 72) between the study of divisibility in fields of algebraic and in fields of algebraic functions which Dedekind and Weber notice. Dedekind and Weber do not explain this simplification.[76]

Ideal theory is developed with no other means than these arithmetical notions[77] and following a path very similar to rational number theory. Dedekind and Weber are

[73]"*Jedes von* $\mathfrak{o}$ *verschiedene Ideal* $\mathfrak{a}$ *is durch ein Primideal* $\mathfrak{p}$ *teilbar. (...)*
Ist $\mathfrak{a}$ *relativ prim zu* $\mathfrak{c}$ *und* $\mathfrak{ab}$ *durch* $\mathfrak{c}$ *teilbar, so ist* $\mathfrak{b}$ *durch* $\mathfrak{c}$ *teilbar.*"

[74] *"Ist ein Ideal* $\mathfrak{c}$ *teilbar durch ein Ideal* $\mathfrak{a}$*, so gibt es ein and nur ein Ideal* $\mathfrak{b}$*, welches der Bedingung* $\mathfrak{ab} = \mathfrak{c}$ *genügt, welches der Quotient von* $\mathfrak{c}$ *durch* $\mathfrak{a}$ *heißt."*

[75]It results from the fact that it is possible to prove early in the theory that for an ideal $\mathfrak{a}$ and an arbitrary polynomial k, one can choose a function α in $\mathfrak{a}$ such that there is no common divisor between k and the norm of $\mathfrak{o}\alpha$ relative to $\mathfrak{a}$ (defined – in modern terms for brevity's sake – as the determinant up to units of a change of basis from $\mathfrak{a}$ to $\mathfrak{o}\alpha$). See [Stillwell, 2012], 61-65. [Geyer, 1981], 116-117, provides a modern reconstruction and proof of the theorem. In Geyer's reconstruction of Dedekind and Weber's result, the arithmetical terminology naturally disappears and the result is no longer a divisibility result as it was in [Dedekind and Weber, 1882] – recall that it is the first in the paragraph "§9 Laws of divisibility of ideals" and holds the key to the other divisibility results.

[76]It is, in fact, related to peculiar properties of polynomials, in particular the possibility to use linear methods ([Geyer, 1981], 116-117 and [Stillwell, 2012], 70).

[77]In keeping with their strategy of transferring number-theoretical notions to the level of ideals, Dedekind and Weber also define a notion of norm for ideals. See [Stillwell, 2012], 66-67. [Strobl, 1982], 239-242 gives a modern presentation and a comparison with norms of ideals in algebraic number theory.

able not only to give analogous theorems, but also demonstrations mirroring the proofs given in rational number theory. They can prove non-trivial results by means of methods emulating elementary arithmetic. Indeed, everything is done by means of divisibility of ideals, the statements of propositions and theorems *and* their proofs. The proofs of the theorems, such as the unique factorization theorem, are modeled on the same tactics as in algebraic number theory in which Dedekind, very carefully, proves the theorems by using only operations of elementary arithmetic between ideals, giving demonstrations strikingly resembling the usual ones in rational number theory.[78] By moving the study of the properties of algebraic functions up to the level of *systems* of such functions and setting up a new arithmetical framework for these systems considered as objects, Dedekind and Weber are replacing the long tedious computations by very simple ones.

The set up of the theory of ideals and their divisibility leads to more algebraic investigations: the first part of the 1882 paper continues for four more sections, in which Dedekind and Weber pursue their analysis of the properties of the field.[79] An essential new concept for the definition of the Riemann surface is what they call the "ramification ideal" (*Verzweigungsideal*). This notion, as its name indicates, will later in the paper be used to define the ramification on the Riemann surface (in modern terms, they describe the ramification of the field). It is defined as an ideal $\mathfrak{z}$ such that

$$\mathfrak{z} = \prod \mathfrak{p}^{e-1}$$

in which the product is "taken over all the prime ideals $\mathfrak{p}$ for which a power higher than the first, namely the eth, divides their norm" (*ibid.*, 83). One of the first and most remarkable properties of the ramification ideal is that its norm is the discriminant of the field: $N(\mathfrak{z}) = D$.[80] The ramification ideal is introduced without mention of its relation with the ramification of the (to be defined) Riemann surface.

4 The new definition of the Riemann surface

Dedekind and Weber qualify the first part of their paper as "formal":

> The previous considerations on the functions in the field Ω were of purely formal nature. (...) The numerical values of these functions did not come under consideration at any moment. (...) But now that we have carried

[78] See, for example, [Stillwell, 2012], 73-74.

[79] In particular, Dedekind and Weber study linear properties of the field, related to the basis of the field. See [Geyer, 1981] and [Strobl, 1982] for more on that matter. They also prove the birational invariance of their results ([Stillwell, 2012], 91).

[80] See [Strobl, 1982], 242-244, for a modern presentation as part of a comment on Dedekind and Weber's concept of point on which I will come back in the next section. [Geyer, 1981], 118-119, gives a modern presentation of the ideal of ramification, with links to number theory. I will not develop this aspect. See also [Stillwell, 2012], 75-85. Dedekind, in [Dedekind, 1882], transferred the ramification ideal (renamed *Grundideal*, fundamental ideal) back to number theory. The back transfer is allowed by the generality and the "formal" character of the investigations conducted here, as it was the case for the transfer from number theory to function theory.

> the formal part of the investigation this far, the pressing question is *to what extent is it possible to assign particular numerical values to the functions in Ω so that all rational relations (identities) existing between these functions become correct numerical equalities*?[81] ([Dedekind and Weber, 1882], 293, transl. slightly altered [Stillwell, 2012], 94)

The formal aspect of the study of the field Ω is a requirement for the assurance of a general ground of Riemann's theory. The next step is, thus, to attribute numerical values (in $\mathbb{C} \cup \{\infty\}$)[82] to the functions, to verify if the "rational relations" investigated in the first part are preserved in the numerical domain. This statement appears to be a conservativity requirement for the relations between functions on the so-called Riemann sphere. "Formal" refers to the fact that all results are inferred from the equation and the manipulation of the four fundamental operations of arithmetic $(+, -, \times, \div)$.[83]

4.1 Point of the Riemann surface

The attribution of numerical values is the core idea for the definition of the point of a Riemann surface. Dedekind and Weber define the point of the surface as a correspondence from the field of functions into the field of numerical constants:

> If *all* the individual elements α, β, γ, ... of the field Ω are replaced by *definite* numerical values α_0, β_0, γ_0, ... in such a way that:
>
> **(I)** $\alpha_0 = \alpha$ if α is constant, and in general,
>
> **(II)** $(\alpha + \beta)_0 = \alpha_0 + \beta_0$
>
> **(III)** $(\alpha - \beta)_0 = \alpha_0 - \beta_0$
>
> **(IV)** $(\alpha\beta)_0 = \alpha_0\beta_0$
>
> **(V)** $(\frac{\alpha}{\beta})_0 = \frac{\alpha_0}{\beta_0}$.
>
> Then, to such a conjunction of definite values must be assigned a point $\mathfrak{P}$ (...) and we say that $\alpha = \alpha_0$ *at* $\mathfrak{P}$ or that α has the value α_0 *at* $\mathfrak{P}$. Two

[81] "*Die bisherigen Betrachtungen über die Funktionen des Körpers Ω, waren rein formaler Natur. (...) Die numerischen Werte dieser Funktionen kamen nirgends in Betracht. (...) Nachdem nun aber der formale Teil der Untersuchung soweit geführt ist, drängt sich die Frage auf, in welchem Umfange es möglich ist, den Funktionen in Ω solche bestimmten Zahlenwerte beizulegen, daß alle zwischen diesen Funktionen bestehenden rationalen Relationen (Identitäten) in richtige Zahlengleichungen übergehen.*"

[82] Dedekind and Weber state that it will be useful to include "∞" as "*one* definite (constant) number". The rules of computation with arithmetic operations can be extended to the domain of numbers to which ∞ has been added, as long as no "indeterminacy" (e.g. $\infty \pm \infty$, $0.\infty$, etc.) is encountered. If such indeterminacy were to appear, Dedekind and Weber affirm that the equation just does not have any truth value.

[83] Stillwell translates "*Spezies*" by "algebraic operations". While it is understandable if one is to look at the 1882 paper as a part of algebraic geometry, I believe it is misleading. Uses of the term "*Spezies*" were essentially made to refer to the four fundamental operations of arithmetic, as a synonym for "*Grundoperationen der Arithmetik*" (see [Müller, 1900]). It is, for example, the term used by Gauss in [Gauss, 1929] to designate the operations of arithmetic.

points are called different if and only if there is a function α in Ω which has a different values for these two points.[84] ([Dedekind and Weber, 1882], 293-294, transl. modified [Stillwell, 2012], 94)[85]

Dedekind and Weber suggest that one might like to consider the point as "represented sensibly somehow located in space" (*den man sich zur Versinnlichung irgendwie im Raume gelegen vorstellen mag*, [Dedekind and Weber, 1882], 294, transl. modified [Stillwell, 2012], 94). However, in a footnote, they immediately advice against it, for such a "geometric representation" is "not necessary and does not make comprehension easier" (*ibid.*). Any geometrical meaning of the word point is to be taken out of Dedekind and Weber's concept of a point, which designates only "the coexistence of values" (*ibid.*, 294). Geyer suggests that by this move Dedekind and Weber are giving up on the "character of manifold" (*Mannigfaltigkeitcharakter*, [Geyer, 1981], 114) of the Riemann surface and only consider it as a system of points. Indeed, Dedekind and Weber want to strip the definition of the Riemann surface of any geometrical, infinitesimal, or (what we would call) topological aspects, so as to concentrate on the algebraico-arithmetical relationships between points – which, as we will see, are related to prime ideals.[86]

The definition of the point allows to show the conservation of the rational relations between functions (i.e. the relations given by the rational operations $+, -, \times, \div$) when a numerical value is attributed to it: all rational relations between functions correspond to numerical equalities. It seems to yield the assurance of the consistency of their "formal" results. Indeed, the morphism appears to be giving content to the formal investigations, the equalities translated in numbers granting a certain guarantee of the coherence of the relationships between functions. In fact, it seemed to be a point that particularly pleased Dedekind, as he mentions in a letter from January 1880. Commenting on the "shadowy" (*schattenhafte*) treatment of fields, from which one would "shy away" at a higher degree and to which he has nothing to object but its "vastness" (*Weitläufigkeit*), he confesses to Weber that

the terrible confinement in which the field Ω thus appears, and the complete

[84]"*Wenn* alle *Individuen* α, β, γ, ... *des Körpers* Ω *durch* bestimmte *Zahlwerte* α_0, β_0, γ_0, ... *so ersetzt werden, daß* ***(I)*** $\alpha_0 = \alpha$ *falls* α *konstant is, und allgemein:* ***(II)*** $(\alpha+\beta)_0 = \alpha_0+\beta_0$ ***(III)*** $(\alpha-\beta)_0 = \alpha_0-\beta_0$ ***(IV)*** $(\alpha\beta)_0 = \alpha_0\beta_0$ ***(V)*** $(\frac{\alpha}{\beta})_0 = \frac{\alpha_0}{\beta_0}$ *wird, so soll einem solchen Zusammentreffen bestimmter Werte ein Punkt* $\mathfrak{P}$ *zugeordnet werden (...) und wir sagen,* in $\mathfrak{P}$ *sei* $\alpha = \alpha_0$, *oder* α *habe* in $\mathfrak{P}$ *den Wert* α_0. *Zwei Punkte heißen stets und nur dann verschieden, wenn eine Funktion* α *in* Ω *existiert, die in beiden Punkten verschiedene Werte hat.*"

[85]The evaluation of the functions, here, is thus what we call a field morphism (and Dedekind calls a "substitution"). The mapping from Ω to the "numerical constants" is defined exactly (albeit without alluding to it) as field morphisms were defined by Dedekind in his algebraic number theory (see [Dedekind, 1877], 108-109, [Dedekind, 1879], 470). Note also that Dedekind and Weber are considering an application for a function ω of Ω taken as variable, such that $\omega \mapsto \omega(x)$, with $\omega(x)$ a constant in $\mathbb{C} \cup \{\infty\}$. This move, Bourbaki tells us, has become such a common thing to do in mathematics, that we do not notice anymore how original it is ([Bourbaki, 1984], 134). Geyer considers this definition to be revolutionary and a "provocation by abstraction" ([Geyer, 1981], 114).

[86]See [Geyer, 1981], 119 and [Strobl, 1982], 242-244 for modern reconstructions and explanations of Dedekind and Weber's definition of the point of a Riemann surface and its relation to the modern notion of place.

> rigid determination of each individual entity (*Wesen-Individuum*) ω contained in it pleases me very much. And yet, it is alway nice when this world, by magic, suddenly raises into numerical life (*Zahlen-Leben*)![87] (19/01/1880, [Scheel, 2014], 270)

Weber, who had the same use of the term "formal", wrote, in his 1893 paper on Galois theory, that the only way for the "pure formalism" to "[gain] meaning and life" is by "the substitution of the individual elements with number values" ([Weber, 1893], 521), in reference to the passage from a formal treatment to the attribution of numerical values.

The definition of the point, as Strobl pointed out, was identified early by Weber as being the "main difficulty" of their work (letter from Weber to Dedekind, 18/12/1879, [Scheel, 2014], 265, repr. in [Strobl, 1982], 242). The key role of this definition is also explicitly pointed out in the introduction of their paper, in which they explain that in order to propose new bases for Riemannian function theory, one important task presents itself as essential: to supply the mathematicians with

> a completely precise and rigorous definition of the "point of a Riemann surface" that can also serve as a basis for the investigation of continuity and related questions.[88] ([Dedekind and Weber, 1882], 241, transl. [Stillwell, 2012], 42-43)

Weber also points out, in the letter from 18/12/1879, that in order to obtain a satisfactory definition of the point, "the best could be to return to [Dedekind's] original foundation of the theory of ideals, where then as long as possible, we are not talking about points, but only about prime ideals" ([Scheel, 2014], 265) – as, indeed, is done in [Dedekind and Weber, 1882], in which the points are only introduced halfway through the paper. Dedekind and Weber had been working for a year on setting up their theory of ideals of algebraic functions and their new definition of the Riemann surface, at the time of this suggestion by Weber. In the previous letters, many of the basic ideas for their paper are studied in a more or less rough form, and the theory-to-be is yet to be arranged to have the structure of the published version. Nevertheless, some of the core ideas, such as the idea of a one-to-one correspondence between points and ideals, are already present in Weber's first letter.

The relation between ideals and points is the following: the collection of all the integral functions π of z which have the value 0 in a certain point $\mathfrak{P}$ is a prime ideal $\mathfrak{p}$.

[87] *"die furchtbare Abgeschlossenheit, in welcher der Körper Ω so erscheint, und die vollständige starre Bestimmtheit jedes einzelnen in ihm enthaltenen Wesen-Individuums ω mir sehr wohl gefällt; und schön ist es doch, wenn diese Welt durch einen Zauberschlag plötzlich zum Zahlen-Leben erweckt wird!"*
It seems that, here, "*furchtbare*" (literally horrible, terrible) is used in a slightly ironic sense, as denoting a sense of wonder. Indeed, the closeness of the concept of *Körper* is an aspect which Dedekind values, as he explains in [Dedekind, 1894]: "This name [*Körper*] should, just as in the natural sciences, in geometry and in the life of human society, designate here too a system which possesses a certain completeness, perfection, closure, whereby it appears as an organic whole, as a natural unity" ([Dedekind, 1894], 20).

[88] "*(...) eine vollkommen präzise und strenge Definition des 'Punktes der Riemannschen Fläche' (...), welche auch als Basis für die Untersuchung der Stetigkeit und der damit zusammenhängenden Fragen dienen kann.*"

The point $\mathfrak{P}$ is said to "generate" the prime ideal $\mathfrak{p}$, and there is always one and only one point $\mathfrak{P}$ generating the prime ideal $\mathfrak{p}$. This point is called the "null point" (*Nullpunkt*) of the ideal $\mathfrak{p}$ (*ibid.*, 95). The idea for the null points is present in Weber's first letter on the subject (letter from the 02/02/1879, in [Scheel, 2014], 220-224):

> The collection of all the functions of ν which vanish at a determinate number of fixed points with a determinate order obviously form an ideal.[89] ([Scheel, 2014], 222)

Weber initially proposed to call "this null point, (...) the fundamental point of the ideal" (*Grundpunkte des Ideals*, *ibid.*, 222). It is not clear why they preferred the name *Nullpunkt* in the final version of their theory. A likely possibility is that the implication that the null point is somehow *fundamental* to the ideal it generates, which is suggested in the name *Grundpunkte des Ideals*, could have been an issue for Dedekind. Indeed, ideals do not depend on the null points in any way – as the so-called "formal" first part shows. In addition, the name *Nullpunkt* – which was a fairly common expression – points to the defining property of this notion, which also would seem preferable from Dedekind's viewpoint.

In the same letter, Weber also gives a first proof of the existence of a one-to-one correspondence between null points and prime ideals (*ibid.*, 223). This result is the key point for the correspondence between the field Ω and Riemann's notion of class of algebraic functions. In [Dedekind and Weber, 1882], the characterization of the concept of point is completed by an indication on how to obtain "all the existing points $\mathfrak{P}$ exactly once", that is to form a collection of all the points $\mathfrak{P}$ at which the variable z remains finite, and all the "complementary points" $\mathfrak{P}'$ at which it is infinite.[90] The collection of all the points is said to be the "Riemann surface". Yet this Riemann surface is a mere collection of points, a "simple totality" in which each point appears only once: it does not have any structure, doesn't describe the singularities of the functions, nor the multiplicity and ramification of the surface. As such, the surface as such is not fully described.

4.2 Polygons

The definition of a point is an "invariant concept" of the field: it does not depend on the choice of a variable z, but only on the functions of Ω. The importance of founding the definition of the Riemann surface on an invariant concept of the field was pointed out by Dedekind in an undated fragment of letter to Weber:

> I believe that the whole theory must be built from the beginning even more on the quest for invariant concepts, and in doing so, I always come back

89 "*Der Inbegriff aller Functionen aus ν welche in einer bestimmten Anzahl fester Punkte verschwinden in bestimmter Ordnung bildet offenbar ein Ideal.*"

90 Rather unsurprisingly, Dedekind and Weber do not give an effective procedure to construct the null points.

again to Riemann.[91] ([Scheel, 2014], 12)

Here, coming back to Riemann seems to be coming back to a characterization of functions by their points, their multiplicities, and the singularities (zeros and poles). To do so, Dedekind and Weber start by introducing the notion of "order number" (*Ordnungszahl*). An order number $r > 0$ is attributed to the functions π of Ω vanishing in $\mathfrak{P}$, and $r < 0$ to the functions which become infinite.[92] The order of a function π is defined as the positive integer r such that $\frac{\pi}{\varrho^r}$ is neither 0 nor ∞ at $\mathfrak{P}$. Such a number r always exists for any function π. The same is valid for the quotient $\frac{\pi}{\varrho'^r}$ and thus

> π receives the order r or is said to be infinitely small of order r at $\mathfrak{P}$. We also shall say that π is 0^r at $\mathfrak{P}$ or that π is 0 at $\mathfrak{P}^r$.[93] ([Dedekind and Weber, 1882], 297, transl. [Stillwell, 2012], 97)

Dedekind and Weber are using Riemann's vocabulary to designate the singularities of the function. They also adopt the convention that if a zero is of order r, then it is the same as having the function vanishing at r superposed points, introducing the idea of multiple points. But their adopting Riemann's "infinitely small of order r" does not mean that their approach involves uses of infinitesimals such as one can find in Riemann's works. In the undated fragment of letter from Dedekind to Weber mentioned above, Dedekind makes the first step towards the characterization of singularities that is given in [Dedekind and Weber, 1882]. He writes:

> Above all, when the algebraic equation between s and z is given (from which the field Ω is produced), each point must be defined clearly and the collection (*Inbegriff*) $\mathcal{T}$ of all these points exactly described in such a way that really all the functions in Ω appear as univalent local functions (*Ortsfunctionen*) in $\mathcal{T}$. Then, it seems advisable that the systems of m points (m-gons) be described as products of m points, and again multiplied with each other. Each system of point is the product of powers of points.[94] ([Scheel, 2014], 12)

Here, the discrete definition of point as the coincidence of values acts as the characterization of functions as univalent and local. $\mathcal{T}$ is the simple totality of all points. Insofar as they wish to characterize the functions by their 0 and poles, they want to be able to work with the collection of all the points in a way that takes into account their orders.

[91] "*Ich denke mir, die ganze Theorie müßte von Anfang an noch mehr mit dem Streben nach invarianten Begriffen aufgebaut werden, und dabei komme ich immer mehr wieder auf Riemann zurück*"

[92] A function η has the value ∞ in $\mathfrak{P}$, when $\frac{1}{\eta}$ vanishes at $\mathfrak{P}$. See [Stillwell, 2012], 98.

[93] "*π erhält die Ordnungszahl r oder heißt unendlich klein in der Ordnung r im Punkte $\mathfrak{P}$. Wir werden auch sagen, π ist 0^r in $\mathfrak{P}$ oder π is 0 in $\mathfrak{P}^r$.*"

[94] "*Vor Allem müßte, wenn die algebraische Gleichung zwischen s und z gegeben ist (aus der sich der Körper Ω entwickelt), jeder Punct deutlich charakterisirt und der Inbegriff $\mathcal{T}$ aller dieser Puncte genau beschrieben werden, in der Weise, daß wirklich alle Functionen in Ω als einwerthige Ortsfunctionen in $\mathcal{T}$ erscheinen. Dann scheint es zweckmäßig, Systeme von m Puncten (m-ecke) wie Producte von m Puncten zu bezeichnen und wieder mit einander zu multipliciren; jedes Punctsystem ist Product von Punctpotenzen.*"

Such collections are named "polygons"[95] in [Dedekind and Weber, 1882], and will be of great use in the rest of the paper:

> We give the name *polygons* to complexes of points, which may contain the same point more than once and denote them by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, ...
>
> We also let $\mathfrak{A}\mathfrak{B}$ denote the polygon obtained from the points of the polygons $\mathfrak{A}$ and $\mathfrak{B}$ put together, in such a way that a point $\mathfrak{P}$ that appears r-tuply in $\mathfrak{A}$ and s-tuply in $\mathfrak{B}$, appears $(r+s)$-tuply in $\mathfrak{A}\mathfrak{B}$.[96] ([Dedekind and Weber, 1882], 299, transl. modified [Stillwell, 2012], 98)

From the definition of the product of polygons, one can deduce the "meaning" (*Bedeutung*) of the power of a point $\mathfrak{P}^r$, which was used in the definition of the order number: it is a point that appears n times in the polygon, and thus is repeated n times on itself. Here, the multiplicity of the surface starts to appear. It is possible, therefore, to decompose a polygon into the product of all its points, each with its own power (the number of times it appears in the polygon): $\mathfrak{A} = \mathfrak{P}^r\mathfrak{P}_1^{r_1}\mathfrak{P}_2^{r_2}...$ – which is basically a decomposition into prime elements. We have, here, the product of points, mentioned by Dedekind in his letter, to define the collection of points and their orders. Dedekind and Weber call the number of points in a polygon its order, and a polygon of order n is called an n-gon.

The approach proposed by Dedekind, here, relies again on an arithmetical treatment of the points and will allow to have a perfect correspondence with ideals. Indeed, from the decomposition of a polygon into the product of its points, one deduces that polygons are in correspondence with the composite (non prime) ideals:

$$\mathfrak{A} = \mathfrak{P}^r\mathfrak{P}_1^{r_1}\mathfrak{P}_2^{r_2}...$$

corresponds to the ideal

$$\mathfrak{a} = \mathfrak{p}^r\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}...$$

whose prime factors correspond to the points of the polygon. The ideal $\mathfrak{a}$ is said to be generated by the polygon $\mathfrak{A}$. Dedekind and Weber call this polygon $\mathfrak{A}$ the "*Nullpolygon*" of $\mathfrak{a}$. This allows Dedekind and Weber to study the *laws of divisibility of polygons*, established by the one-to-one correspondence between (composite) ideals and polygons:

> the laws of divisibility of polygons agree completely with the laws of divisibility of integers and of ideals. Points play the role of prime factors.[97] ([Dedekind and Weber, 1882], 299, transl. slightly altered [Stillwell, 2012], 98)

[95] A polygon corresponds to what is today called a positive divisor, see [Dieudonné, 1974] or [Stillwell, 2012] note 39, page 98.

[96] "*Komplexe von Punkten, welche denselben Punkt auch mehrmals enthalten können, nennen wir* Polygone *and bezeichnen dieselben mit* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, ... *Es bedeute ferner* $\mathfrak{A}\mathfrak{B}$ *das aus den Punkten von* $\mathfrak{A}$ *und von* $\mathfrak{B}$ *zusammengesetzte Polygon in der Weise, daß, wenn ein Punkt* $\mathfrak{P}$ *r-mal in* $\mathfrak{A}$, *s-mal in* $\mathfrak{B}$ *auftritt, er* $(r+s)$*-mal in* $\mathfrak{A}\mathfrak{B}$ *vorkommt.*"

[97] "*[D]ie Gesetze der Teilbarkeit der Polygone in vollkommener Übereinstimmung mit denen der Teilbarkeit der ganzen Zahlen und der Ideale. Die Rolle der Primfaktoren übernehmen dabei die Punkte.*"

If one wants to have a unit, for this new notion of divisibility, one has to admit the polygon containing 0 points, denoted by $\mathfrak{O}$ (called "*Nulleck*").

These developments lead Dedekind and Weber to define the GCD and the LCM of two or more polygons:

> The *greatest common divisor* of two polygons $\mathfrak{A}$ and $\mathfrak{B}$ is the polygon that contains each point the *least* number of times it occurs in $\mathfrak{A}$ and in $\mathfrak{B}$. If it is [the empty polygon] then $\mathfrak{A}$, $\mathfrak{B}$ are called *relatively prime.*
>
> The *least common multiple* of two polygons $\mathfrak{A}$ and $\mathfrak{B}$ is the polygon that contains each point the *greatest* number of times it occurs $\mathfrak{A}$ and in $\mathfrak{B}$. If $\mathfrak{A}$, $\mathfrak{B}$ are relatively prime, then $\mathfrak{A}\mathfrak{B}$ is their least common multiple.[98] ([Dedekind and Weber, 1882], 299, transl. [Stillwell, 2012], 99)

Note that this corresponds exactly to the definition of the GCD and the LCM by powers of prime numbers, in number theory – which can also be given for ideals. The GCD (resp. the LCM) of two ideals is generated by the GCD (resp. the LCM) of the polygons which generate the said two ideals. The null-gon $\mathfrak{O}$ generates the ideal $\mathfrak{o}$. These notions are used in later paragraphs in [Dedekind and Weber, 1882] as tools for the proofs.

This way, Dedekind and Weber write, is established "the complete coincidence between ideals and the collection of integral functions equal to zero at the same fixed points" (*ibid.*).

Dedekind and Weber only defined polygons with positive powers, which implies that only positive order numbers (i.e. when the functions vanish) had been taken into account. To characterize the negative order numbers (i.e. when the functions become infinite), Dedekind and Weber introduce quotients of polygons, rather than defining an arbitrary polygon. In fact, they appear to be following their arithmetical approach and the correspondence between polygons and ideals, and propose to "symbolically set" the function $\eta = \frac{\mathfrak{A}}{\mathfrak{B}}$, in which $\mathfrak{A}$ is called the upper polygon and describes the 0 of the function,[99] and $\mathfrak{B}$ is the lower polygon and describes the poles.[100] Thus, Dedekind and Weber introduce negative powers – those of the lower polygon – without adopting an approach which would not agree with the arithmetic of ideals as developed in the first part. In fact, by adopting the presentation of negative powers as the powers of the denominator of a quotient, they seem to lean even more towards an arithmetical treatment of the theory.

[98] "*Der* größte gemeinschaftliche Teiler *zweier Polygone $\mathfrak{A}$, $\mathfrak{B}$ ist dasjenige Polygon, welches jeden Punkt so oft enthält, als er in $\mathfrak{A}$ und $\mathfrak{B}$* mindestens *vorkommt. Ist dies $\mathfrak{O}$, so heißen $\mathfrak{A}$, $\mathfrak{B}$* relativ prim. *Das* kleinste gemeinschaftliche Vielfache *von $\mathfrak{A}$ und $\mathfrak{B}$ ist dasjenige Polygon, welches jeden Punkt so oft enthhält, als er in $\mathfrak{A}$ und $\mathfrak{B}$* höchstens *vorkommt. Sind $\mathfrak{A}$, $\mathfrak{B}$ relativ prim, so ist $\mathfrak{A}\mathfrak{B}$ ihr kleinstes gemeinschaftliches Vielfache.*"

[99] And its degree is the number of sheets of the surface, see [Geyer, 1981], 120.

[100] A similar notion of quotient was defined for ideals, see [Stillwell, 2012], 86-89.

4.3 Definition of the Riemann surface

From there, Dedekind and Weber can define the Riemann surface with its structure and not as a mere collection of points. They call the Riemann surface with ramification the "'absolute' Riemann surface".

The Riemann surface is defined as the product of all polygons "moving through" (*sich bewegen*), i.e., taking successively all the values in $\mathbb{C} \cup \{\infty\}$. As they proved just before, for any numerical value c, there is a polygon $\mathfrak{A}$ with n points (not necessarily distincts) for which the variable z takes the value c. These n points are said to be "conjugate" for z. This entails that z can take continuously all the values in $\mathbb{C} \cup \{\infty\}$. This is what Dedekind and Weber designate as the polygon $\mathfrak{A}$ "moving through" all the possible values "and indeed in such a way that all its points change simultaneously." This allows for the discrete characterization used so far to adequately describe the surface associated to the field Ω. In this way, one obtains all the existing points, including several times the points for which $z - z_0$ or $\frac{1}{z}$ vanishes[101] for an order higher than 1 (which always happens only a finite number of times for algebraic functions). Hence, if one takes the product of all these polygons, one obtains:

$$\prod \mathfrak{A} = T\mathfrak{Z}_z$$

where T is the "simple totality" formed by collecting *all* the points *once*, and $\mathfrak{Z}_z$ is "a particular finite polygon called the ramification or winding polygon of T in z" ([Dedekind and Weber, 1882], 301, transl. [Stillwell, 2012], 100). The Riemann surface is an invariant concept of the field, since it is defined as a product of polygons (hence a product of products of points) which are themselves invariant concepts of the field.

The ramification polygon $\mathfrak{Z}_z$ is a *finite* polygon whose points describe the ramification of the Riemann surface. These points are called winding points or ramification points (*Verzweigungs- oder Windungspunkt*) of T in z, and each point is said to be "of order s if it appears s-tuply in $\mathfrak{Z}_z$." (*ibid.*, 100). If $z - z_0$ or $\frac{1}{z}$ is infinitely small of degree e, then $s = e - 1$. The order of the ramification polygon is called "the ramification or winding number w_z"[102] Moreover, as its name suggests:

> The points of the ramification polygon at which z has a finite value together generate the *ramification ideal* in z.[103] ([Dedekind and Weber, 1882], 301, transl. [Stillwell, 2012], 100)

The correspondence between surface and field implies that the singularities of the surface can be expressed in terms of (divisibility of) ideals: the ramification of the surface is thus translated into what we would now call the ramification in the field Ω, and which is characterized for Dedekind and Weber by the "ramification ideal" $\mathfrak{z}$.

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[101] Dedekind and Weber define the points at which the function of z becomes infinite as the points at which $\frac{1}{z}$ vanishes.

[102] Further down the paper, Dedekind and Weber also show that the ramification number is necessarily an even number (*ibid.*, 113). See also [Geyer, 1981], 121 and [Strobl, 1982], 242-243.

[103] *"Diejenigen Punkte des Verzweigungspolygons, in welchen z einen endlichen Wert hat, erzeugen zusammen das Verzweigungsideal in z."*

The change of framework of Dedekind and Weber's rewriting of Riemann's theory implies a reformulation of all the notions of Riemann's theory: genus, ramification number, number of double points, etc., all are rewritten in the algebraico-arithmetical framework using polygons. The definition of the "'absolute' Riemann surface" given by Dedekind and Weber has no geometrical component and its link with the Riemannian idea of a Riemann surface seems difficult to grasp. Dedekind and Weber mention that to "pass from this definition of the 'absolute' Riemann surface to the well-known Riemannian conception", it suffices to "think the surface spread over a z-plane which is then covered n-tuply everywhere except at the ramification points" ([Dedekind and Weber, 1882], 301, transl. [Stillwell, 2012], 100). But they don't explore this possibility further, which appeals at least to a geometrical representation – if not to spatial intuition. The Riemann surface defined by Dedekind and Weber is rather detached from any spatial consideration. By relying exclusively on rational relations between functions, Dedekind and Weber develop their theory in such a way that (what we would call) topological notions are not *needed* in the definition but should eventually follow from the notions introduced. Dedekind and Weber, however, do not go as far as developing the "investigation of continuity and other related questions", among which one counts Abelian functions, but hope to do so "on another occasion" ([Dedekind and Weber, 1882], 241).

Let me close this paper with a few word about the rest of Dedekind and Weber's paper. Indeed, for several more paragraphs, Dedekind and Weber continue to redefine, in their new framework, the basic notions of Riemann's theory such as the genus of the surface, and give new proofs of several important results such as Abel's theorem or the Riemann-Roch theorem.

The attempts to prove the Riemann-Roch theorem occupy a considerable place in Dedekind and Weber's correspondence,[104] and seem to allow them to test, so to speak, the conceptual framework they are setting up. After having met, in April 1879 in Braunschweig ([Scheel, 2014], 238), Dedekind and Weber start discussing at length the proof of the Riemann-Roch theorem, exchanging tentative proofs and correcting each other's mistakes. It is a very slow process, which reinterprets the theorem in a new framework, and attempts to provide a rigorous proof of the result. By 1879, however, the conceptual framework of [Dedekind and Weber, 1882] is not completely developed, and the proof(s) (or the attempts to give a proof) largely differ from that given in the paper.

The proof of the Riemann-Roch theorem in [Dedekind and Weber, 1882] illustrates well how Dedekind and Weber are able to rewrite and prove important results using solely the arithmetico-algebraic arsenal developed in their paper. The Riemann-Roch theorem has been described as the most striking example of the intimacy of the relationships between topological notions and properties of functions.[105] This seems to be lost in Dedekind and Weber's version of it. Their treatment of the Riemann-Roch theorem

[104] [Scheel, 2014], 239-251.

[105] See the Introduction of [Stillwell, 2012], as well as [Bourbaki, 1984] and [Bottazzini and Gray, 2013], 337-339, [Gray, 1987], [Gray, 1998].

involves manipulations of (quotients of) polygons, classes of polygons[106] and families of polygons (*Polygonscharen*) – an equivalent of our modern vector space. For each notion introduced (class of polygons, family of polygons, ...) – which are used in the definitions and proofs – Dedekind and Weber introduce notions of product, divisibility and/or divisor, so as to handle them with the same arithmetical approach they used throughout their paper.

At no point do Dedekind and Weber venture out of the conceptual framework they set up. Since their notion of Riemann surface lacks the possibility to set up the appropriate structure for an integration theory, Dedekind and Weber introduce notions related to integration theory by purely algebraic means, without any recourse to "continuity". Dedekind and Weber define (algebraically) differentials of different "kinds" (*Gattung*) according to the classification given by Riemann (following Legendre).[107] Dedekind and Weber also shed further light on differentials of second and third kind, using the Riemann-Roch theorem and relating them to the singularities of the surface. They define the residues of an algebraic differential. They are able to prove, with the conceptual apparatus developed in the previous paragraphs, the theorem stating that the sum of the residues of an algebraic function is zero.

5 Conclusion

Dedekind and Weber's paper had a sparse and delayed reception. Dedekind himself saw the risk that his contemporaries might not react as well as he hoped to their paper:

> It shall bring me so much joy, if the thing finds some success, which for the time being, I am not really counting on, because many will shy away from the tedious modules.[108] (30/10/1880, in [Scheel, 2014], 271)

In fact, a few years later, Weber himself departed considerably from the approach they used in 1882. Rather than using modules and ideals, Weber used the notion of "*Functionale*"[109] for the theory of algebraic functions and for the theory of algebraic integers. For algebraic integers, the introduction of Dedekind's concept of ideal is followed immediately by the properties relating it to functionals:

> One can refer functionals and ideals to each other in such a way that the following rules prevail:
> 1. Any integral function corresponds to a determinate ideal, and associate functionals correspond to the same ideal.

[106] Two polygons $\mathfrak{A}, \mathfrak{A}'$ with the same number of points are called *equivalents* if there is a function η in Ω such that $\eta = \frac{\mathfrak{A}}{\mathfrak{A}'}$ ([Stillwell, 2012], 105). All the polygons equivalent to a given polygon $\mathfrak{A}$ constitute a *polygon class A* (*ibid.*, 105).

[107] For more details, see [Stillwell, 2012], 110-133 and [Haffner, 2014b].

[108] "*Es soll mich nun auch herzlich freuen, wenn die Sache einigen Beifall finden wird, worauf ich aber vorläufig nicht allzu sehr baue, weil die langweiligen Moduln gewiß manchen zurückschrecken werden...*"

[109] A functional is an algebraic function that can be written as the quotient of two algebraic integral functions.

2. Any ideal corresponds to infinitely many but only to associate integral functionals.
3. The product of two or more integral functionals corresponds to the product of the corresponding ideals.
4. An integral number corresponds to a principal ideal.
5. Units correspond to the ideal $\mathfrak{o}$.[110] ([Weber, 1896] II, 548)

In particular, the prime ideals correspond to prime functionals, and "the decomposition of ideals into prime factors, and actually the laws of divisibility of ideals, ensue in complete agreement from the corresponding theorems of the theory of functionals" [111] (*ibid.*, 550). All considerations of divisibility are thus completely taken care of with the notion of *Functionale* and Dedekind's arithmetic of ideals is not developed.

While Hilbert did use some of their ideas for his *Nullstellensatz*, the adoption of the ideal theory as a basis for a treatment of Riemannian function theory did not stir up much enthusiasm. A first significant response can be found in Hensel and Landsberg's 1902 book (dedicated to Dedekind, [Hensel and Landsberg, 1902]). However, Hensel and Landsberg change the basis of the theory from ideal theory to function theory, so as to be able to penetrate deeper into Riemann's theory, in particular as regards Abelian functions. In [Brill and Noether, 1892], Brill and Noether explicitly exclude Dedekind and Weber's "theory relying on number theory" (as well as Kronecker's) from their report on algebraic functions.[112] Dedekind and Weber's theory is considered as mainly an application of number theory to function theory. However, Dedekind and Weber do not, themselves, present their treatment of the theory of functions as a mere "application" of number theoretical methods.

Rather, as I tried to show in this paper, their theory is an arithmetical rewriting of the basic concepts of Riemann's theory of functions – a conceptual reconstruction of Riemann's ideas using elementary arithmetical notions such as divisibility for new objects (ideals, polygones, ...). Both the arithmetical aspect of their work, and the fact that it is, indeed, a rewriting of Riemann's theory are essential to understand their paper. Indeed, the relationship to Riemann's theory allows to understand the path followed by Dedekind and Weber, and their desire to re-define the concept of Riemann surface and the related notions. As such, it allows to see that their work is not about investigating fields for themselves. The arithmetical aspect of their work, on the other hand, highlights how

[110] *"Man kann nun die Ideale und Functionale in der Weise auf einander beziehen, dass dabei folgende Gesetze obwalten: 1. Jedem ganzen Functional entspricht ein bestimmtes Ideal, und associirten Functionalen entspricht dasselbe Ideal. 2. Jedem Ideal entsprechen unendlich viele, aber nur associirte ganze Functionale. 3. Dem Producte zweier oder mehrerer ganzer Functionale entsprechen die Producte der den Factoren entsprechenden Ideale. 4. Einer ganzen Zahl entspricht ein Hauptideal. 5. Den Einheiten entspricht das Ideal $\mathfrak{o}$."*

[111] *"die Zerlegung der Ideale in Primfactoren und überhaupt die Gesetze der Theilbarkeit der Ideale ergeben sich in völliger Uebereinstimmung mit den entsprechenden Sätzen aus der Theorie der Functionale."*

[112] An additional reason for them do so, they explain, is the fact that Klein's lectures on Riemann surfaces in 1892 did give a detailed account of their work ([Brill and Noether, 1892], v) – which does not mean that Klein adopted their approach.

their work is part, at the same time, of a larger effort to provide a 'better' presentation for Riemann's theory and to arithmetize parts of mathematics. Finally, it also is essential to understand the strategy chosen for their rewriting and brings out specific elements of practice and the key role played by arithmetic. The core of the "arithmetical" approach proposed by Dedekind and Weber, here, is the definition of new arithmetical operations, relationships between objects reinterpreted into arithmetical terms. Such arithmetical terminology and the design of arithmetical notions and methods of proofs are used throughout all of Dedekind and Weber's paper up to the proof of the theorems such as the Riemann-Roch theorem.

Here, and in many other works, Dedekind seems to suggest, by his practice itself, that to give oneself the possibility of using only the "simplest principles of arithmetic" justifies abstract and technical detours. For that purpose, arithmetic operations are performed at several levels, as if there were several layers of arithmetic(s). For Dedekind, numbers and arithmetical operations defined for them can be considered and used as "auxiliary means", as helps for rigorous reasonings. They can, as is clearly illustrated in the strategies implemented in [Dedekind and Weber, 1882], be used as epistemic tools involved in the production of new mathematical knowledge, in the design of new concepts and new methods of proof, and/or to strengthen the foundation of certain theories.

This specific use of arithmetic is intimately related to Dedekind's conception of number and his foundational works. Arithmetic, intrinsically linked to the nature of human mind, possesses a character of "of intrinsic necessity" which Dedekind saw in [Dedekind and Weber, 1882] ([Scheel, 2014], 271). From this viewpoint, underlining the inherently arithmetical nature of [Dedekind and Weber, 1882] also gives us the possibility to show that Riemann's function theory influenced Dedekind's foundational works, and thus stimulated foundational reflections outside of the sole well-known criticism of the Dirichlet principle. Indeed, a close study of the texts suggests that the possibility to develop arithmetical methods of proofs in several different domains modulated Dedekind's conception of arithmetic and through this of number, and led to the very general and abstract definition of number given in [Dedekind, 1888]. Dedekind indeed spent almost two decades working on his definition of natural numbers[113] and this long genesis suggests that his mathematical works had a strong impact on his foundational views.

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[113] See [Dugac, 1976].

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