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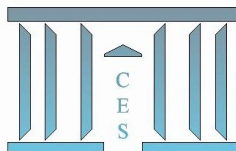
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Fair Utilitarianism

Marc FLEURBAEY, Stéphane ZUBER

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Abstract

Utilitarianism is a prominent approach to social justice that has played a central role in economic theory. A key issue for utilitarianism is to define how utilities should be measured and compared. This paper draws on Harsanyi's approach (Harsanyi, 1955) to derive utilities from choices in risky situations. We introduce a new normalization of utilities that ensures that: 1) a transfer from a rich to a poor is welfare enhancing, and 2) populations with more risk averse people have lower welfare. We propose normative principles that reflect these fairness requirements and characterize fair utilitarianism. We also study some implications of fair utilitarianism for risk sharing and collective risk aversion.

Keywords: Fairness, utilitarianism, risk sharing, collective risk aversion.

JEL Classification numbers: D63, D81.

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1 Introduction

Utilitarianism is a moral philosophical theory according to which social welfare, defined as a sum (or average) of individual utilities, is the ultimate ethical value that should govern collective choices. Utilitarianism played a key role in the development of early neo-classical economic theory and it provided key concepts such as utility and welfare used to analyze individual behaviors and social situations. It continues to be an implicit normative basis for many economic policy judgements (see, e.g., Riley, 2008).

Although utilitarianism has dominated much of economics for more than a century, it is now contested as being insufficiently flexible to incorporate important values that prevail in the public debate, such as fairness and desert.¹ In this paper we put aside these recent controversies and observe that, in spite of the long domination of utilitarianism, one internal issue has never been really settled: How to concretely measure individual utility in an interpersonally comparable way, with a practical method? For lack of a fully applicable recipe, utilitarianism has often been used in a rather abstract way and the economic policy literature has often assumed individual utility functions to be identical.²

Three specific proposals have been made though. One proposal, due to Edgeworth (1881) and recently revived by Argenziano and Gilboa (2015), is to consider that “just noticeable differences” are of equal value across individuals. A second proposal has emerged from the literature on subjective well-being. It consists in using answers to survey questions about happiness or satisfaction as a direct measure of utility (see, e.g., Kahneman, Wakker and Sarin, 1997; Layard, Mayraz and Nickell, 2008). Both approaches rely on the comparability of individual perceptions. But such a comparability is far from obvious and remains much debated.³

¹See Rawls (1971), Sen (1985) and the subsequent literature, as well as Roemer (1996), Mankiw (2010), Fleurbaey and Maniquet (2011) and Suzumura (2016).

²Recently, Boadway (2012) and Piketty and Saez (2012) have criticized this assumption of identical utility functions, which is still prominent in public economics.

³Critical discussions of the “just noticeable differences” approach can be found in Arrow (1951), Svensson (1985) and Hammond (1991). Critical discussions of the happiness approach can be found in Sen (1985), Nussbaum (2008), Graham (2009), Adler (2012) and Fleurbaey

A third proposal to measure individual utilities relates to Harsanyi's (1955) aggregation theorem. This theorem, which involves the Pareto principle applied in risky situations, characterizes utilitarianism as a weighted sum of von Neumann-Morgenstern (henceforth VNM) individual utilities. In that context, the proposal is to set the individual weights equal to the inverse of the utility difference between a "good" option (generally the best) and a "bad" option (generally the worst), and to set the utility of the bad option equal to zero, while the utility of the good option is set equal to one. This has been called "relative utilitarianism" by Dhillon and Mertens (1999). The 0 – 1 normalized utilities have appeared in the literature many times,⁴ and criticized almost as often.⁵

In this paper, we focus on the third proposal and propose an alternative way to define the weights in Harsanyi's theorem, which is a "local" or "marginal" variant of relative utilitarianism. The weights are, in our proposal, the inverse of the individuals' marginal utility at the poverty line. We propose two ways of justifying this third approach. One argument relies on the idea that social priority should be greater for any individual below the poverty line than for any individual above it. Another argument involves the idea that more risk averse individuals have a lower utility level both below and above the poverty line — comparisons of levels, as opposed to differences, do matter for utilitarianism in the evaluation of changes in the composition of the population. In light of such considerations, the 0 – 1

and Blanchet (2013).

⁴The literature is often vague about the origins of the idea. Arrow (1951) attributes the idea of 0 – 1 normalized utility functions to (apparently oral communication by) Abraham Kaplan. Luce and Raiffa (1957, p. 34) refer to the normalization without citing any source. Hammond (1991) cites Isbell (1959) and Schick (1971). Dhillon and Mertens (1999) and Sprumont (2013) refer to Kalai and Smorodinsky (1975) — who equalize these normalized utilities instead of maximizing their sum. Karni (1998) does not refer to earlier sources, and Segal (2000) cites Cao (1981).

Observe that the standard gamble on which the proof of existence of a VNM utility function usually relies (Von Neumann and Morgenstern, 1953, in the appendix written for the 1946 edition) already produces such a 0 – 1 normalization (between two arbitrary options). There remains a leap before such normalization can be used for interpersonal comparisons, but nevertheless, one could argue that the idea has been around since the very advent of VNM utilities.

⁵Arrow (1951), Luce and Raiffa (1957) and Hammond (1991), quoted above, are all critical, as well as Sen (1970), Jeffrey (1971), and Hausman (1995).

normalization appears questionable and leads to unappealing consequences. We name our approach fair utilitarianism to distinguish it from the 0–1 normalization approach (which is sometimes named relative utilitarianism).

We do not claim that our proposal is unquestionably the best, but we would like to argue that it deserves consideration, and we find it surprising that it is not more prominent in the literature — as a matter of fact, we have not found any trace of it. In the next section, we compare the 0 – 1 normalization to the new normalization of utilities that we propose, and provide the intuition for our main results. In Section 3 we introduce a formal framework for our analysis, which involves risk on consumption and on the composition of the population and makes it possible to invoke Harsanyi’s aggregation theorem. In Section 4, a first argument in terms of priority around the poverty line is shown to characterize the utility normalization we propose. In Section 5, an argument involving utility levels and population changes is analyzed and yields another characterization. In Section 6, we discuss some implications of that fair utilitarian approach for risk sharing and collective risk aversion. We show that all individuals must have a consumption level above the poverty level whenever this is feasible, and none should consume more than the poverty level otherwise. It is known that an increase in agents’ relative risk aversion does not necessarily increases collective risk aversion (Mazzocco, 2004; Jouini, Napp and Nocetti, 2013). In the case of the fair utilitarian criterion and individuals having utility functions in a specific class of harmonic absolute risk aversion (HARA) utilities, we show that collective risk aversion always increases around the poverty line. At low levels (resp. high levels) of per capita consumption, collective risk aversion also increases when the risk aversion of the most risk-averse (resp. least risk averse) agent increases. Section 7 contains some brief concluding remarks. An appendix contains the proofs of our results.

2 Comparing utility normalizations

Let x denote an allocation of resources, and $u_i(x_i)$ be the utility of person i with this allocation, where u_i represents individual preferences over resources and x_i are the resources allocated to i . The utilitarian social welfare function W has the well-known form:

$$W(x) = \sum_i u_i(x_i).$$

Suppose that u_i is also a VNM utility function, so that its concavity represents individual i 's risk aversion. The 0 – 1 normalization of utilities sets $u_i(\underline{x}) = 0$ and $u_i(\bar{x}) = 1$ for specific allocations \underline{x}, \bar{x} . It has different implications depending on how the good and bad options are chosen. The most popular approach has been to take the worst option as the bad one, \underline{x} , and the best option as the good one, \bar{x} , thus ensuring that utility ranges over the interval $[0, 1]$. Another approach, proposed by Adler (2012, 2016), consists in picking the worst option as the bad option \underline{x} , and a poverty level as the good option \bar{x} .

The literature quoted in the introduction has focused its criticism of the 0 – 1 normalization on the fact it involves a dubious assumption that the good and the bad option mean the same thing for different people. However, the advocates of this normalization (e.g., Isbell, 1959, and Segal, 2000) have often argued that the normalization did not make such an assumption, but posited, as a matter of fairness, that the best option should be treated as equivalent for everyone in terms of social evaluation, and similarly for the worst option. The debate can of course continue about whether it is really fair to posit such an equivalence when individuals do enjoy these options differently (Luce and Raiffa, 1957, take the example of small gambles over \$1 for a rich and a poor person). Another criticism of relative utilitarianism in its worst-best form (see, e.g., Arrow, 1951) points out that the ranking of prospects may not be invariant to adding feasible options that change the best and worst outcomes for individuals. In an economic context (where x is a distribution of income or consumption), the outcome space is the set of positive real numbers, so that there is no well-defined best outcome and, under non-satiation, individuals have no maximum utility, which is also

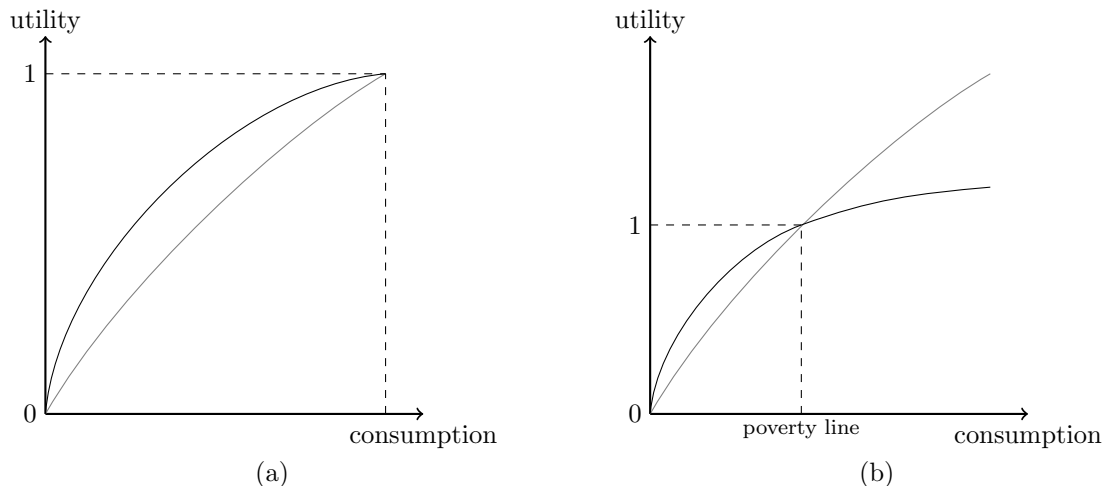


Figure 1: 0 – 1 normalizations of VNM utility functions

problematic for the 0 – 1 normalization.

We would like to introduce another set of worries, having to do with the heterogeneity of risk attitudes, and the implications of the 0 – 1 normalization for two types of social decisions: 1) the allocation of consumption across individuals (which depends on marginal utilities), and 2) the addition of new members to the population (which depends on utility levels).

Consider Figure 1. On panel (a), the normalization spans the worst-best interval. On panel (b), utility is equal to 1 at the poverty line. The figures show the VNM utility functions of two individuals with different levels of risk aversion, when resources are measured by a level of consumption.

On panel (a), the more risk averse person (black curve) has greater marginal utility up to some point and then lower marginal utility. Depending on the exact shape of the curves, the relative priority of the two agents, when considering allocations of resources between them, may be reversed at any point in the interval, and this point may vary for different pairs of individuals. As a consequence, it may happen that for one pair of individuals, the more risk averse has greater priority for most of the interval, whereas for another pair, the less risk averse has greater priority for most of the interval. This is odd, since it seems that greater

risk aversion should modify the relative priority for the allocation of resources in a more systematic way.

Now, if one considers adding a new person to the society at a given level of consumption, on panel (a) the more risk averse is everywhere considered better-off in utility, and therefore it is always better to add more risk averse rather than less risk averse agents. This is not very intuitive, since one rather tends to view risk aversion as a burden.

On panel (b), a highly risk averse person (black curve) who is just below the poverty line has less priority for the allocation of resources than a less risk averse person who is just above the poverty line. It seems odd to penalize the risk averse in this way. In contrast, when the problem is to add new members to the society, a highly risk averse person is preferred to a less risk averse person when they are at the same level of consumption below the poverty line, but the preference is reversed when they are above. It is hard to make sense of this reversal around the poverty line. In particular, it seems counterintuitive to deem the more risk averse person's utility to be higher than a less risk averse person's when hit by the bad luck of poverty.

Figure 2 illustrates the alternative normalization that we propose in this paper. The level and slope of everyone's utility function at the poverty line is the same. This produces a very clear pattern for the two social decisions discussed before. For the allocation of resources, the more risk averse individuals have greater priority below the poverty line and lower priority above the poverty line. This makes sense if one views risk aversion as triggering a greater loss in utility under bad luck and a lower gain under good luck. Moreover, with the proposed normalization, every individual below the poverty line has greater priority than every individual above the poverty line, independently of risk aversion, which seems consistent with the standard understanding of a poverty line as identifying those who should be targeted the most in social policy.

As far as adding new members to the population is concerned, this normalization treats risk aversion as a burden and therefore systematically prefers to add less risk averse rather than more risk averse newcomers to the population.

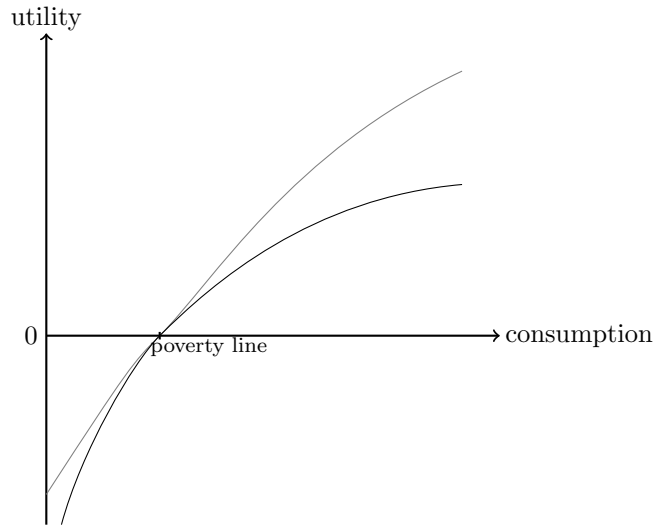


Figure 2: Normalizing the level and slope of VNM utility functions at the poverty level

Observe that this new normalization can be viewed as a local variant of the $0 - 1$ normalization, since it is the limit of a $0 - 1$ normalization when the good and bad options come closer together and the corresponding utility levels are kept in proportion.

In the literature, two kinds of axioms have been proposed to justify the $0 - 1$ normalization. First, impartiality axioms stipulate that it is indifferent to favor one individual or the other if the utility gains are the same (in a specific sense, see Karni, 1998; Segal, 2000). Second, restricted independence axioms, inspired by Arrow's Independence of Irrelevant Alternatives, refer to changes in available intermediate options (Dhillon and Mertens, 1999; Sprumont, 2013). In this paper, we suggest two new principles that reflect the considerations presented in this section about relative priorities for the allocation of resources and for the addition of new agents.

3 The framework

To characterize our normalization, we develop a framework making possible to consider that diverse populations consisting of people with different risk preferences may exist. We do so to be able to represent our intuitions regarding the addition of more or less risk averse new agents to a population. The framework can also be justified by the interest in developing *universal social orderings* to evaluate and compare the welfare (living standards) for populations of possibly different sizes and preferences (across time and space for instance, see Sen, 1976, or Fleurbaey and Tadenuma, 2014).

We let \mathbb{N} denote the set of positive integers, \mathcal{N} the set of non-empty finite subsets of \mathbb{N} , \mathbb{R} the set of real numbers and \mathbb{R}_+ (resp. \mathbb{R}_{++}) the set of non-negative (resp. positive) real numbers. For a set D and any $n \in \mathbb{N}$, D^n is the n -fold Cartesian product of D . Also, for two sets D and E , D^E denotes the set of mappings from E into D .

The set of *potential* individuals who may or may not exist is \mathbb{N} . In an alternative, only a subpopulation $N \in \mathcal{N}$ exists. Each potential individual $i \in \mathbb{N}$ is associated with a preference ordering, so that considering different subpopulations N makes it possible to study the impact of changes in individual preferences. We study the allocation of resources among the individuals composing the subpopulation that exist. To simplify, we assume that a single good, labelled as consumption, has to be distributed. It can be a summary statistics of all the resources available (for instance an equivalent consumption level).

We let $X = \bigcup_{N \in \mathcal{N}} (\mathbb{R}_+)^N$ denote the set of possible alternatives when at least one individual exists. An alternative then is a function assigning a positive consumption level to each individual living in that particular alternative, where N is the subset of such individuals living. Hence the size of the population may vary from one alternative to another and it is important to keep track of the population in an alternative. For any $x \in X$, we let $N(x) \in \mathcal{N}$ be the set of individuals in the alternative and $n(x) = |N(x)|$ be the number of individuals in the alternative. We assume that there always are finitely many people in any

alternative.

We also assume that it is not always known for sure what the final allocation of consumption will be nor what set of individuals will eventually exist. To model this, we assume that there exists an infinite set of states of the world S , with typical element $s \in S$. We denote Σ a σ -algebra over S , so that (S, Σ) is a measurable space. There is a probability measure P on the measurable space (S, Σ) . We assume that the measure space (S, Σ, P) satisfies a property of convexe-rangedness:

For any $A \in \Sigma$ and $\kappa \in [0, 1]$, there exists $A' \in \Sigma$ such that $A' \subset A$ and $P(A') = \kappa P(A)$.

A prospect f is a function from S to X , which is supposed to be Σ -measurable. For $s \in S$, $f(s)$ is therefore the alternative induced by the prospect f in state s . We assume that prospects are finitely-valued, in the sense that there exists a finite partition (A_1, \dots, A_m) of S for which $f(s) = f(s')$ for all $s, s' \in A_k$ and all $k = 1, \dots, m$. We denote F the set of all such prospects. For any random variable K , i.e., any Σ -measurable function $K : S \rightarrow \mathbb{R}$, its expected value is $\mathbb{E}[K] = \int_S K(s) dP(s)$.

Although our framework seems similar to Savage's (1954) model of decision under uncertainty (our prospects are similar to Savage's acts), we want to model a situation of risk. Indeed, we assume that all individuals consider P as the correct belief function, which is an objective probability measure. For our purpose, this framework is more convenient than the usual framework for objective risk that deals with lotteries, because it makes it easy to express individual prospects and redistribution of resources.

For an alternative $x \in X$, whenever $i \in N(x)$, $x_i \in \mathbb{R}_+$ denotes the consumption of individual i . For a prospect $f \in F$, whenever $i \in N(f(s))$, $f_i(s)$ denotes the consumption of individual i in state of the world $s \in S$. We denote $A_i(f) = \{s \in S \mid i \in N(f(s))\}$ the event in which individual i exists and $p_i(f) = P(A_i(f))$ the probability that he exists. We also let f_i denote the mapping $f_i \in \mathbb{R}_+^{A_i(f)}$ determining individual i 's consumption in each state of the

world in which i exists, under the social prospect f . The mapping f_i represents i 's individual prospect.

For a subpopulation $N \in \mathcal{N}$, let F_N denote the set of prospects f such that, for every $s \in S$, $N(f(s)) = N$. These are the prospects such that the same individuals are present in all states of the world, so that there is no risk on individual preferences. By convention, for any $x \in X$, we let x denote the prospect $f \in F$ such that $f(s) = x$ for all $s \in S$. These are sure prospects yielding the same allocation in all states of the world. Similarly, let $X \subset F$ also denote the set of sure prospects and $X_N \subset F_N$ be the set of sure prospects such that the population is N .

It remains to specify how individuals rank different prospects and alternatives. We retain the prominent assumption in the literature dealing with risk, namely, we assume that individuals are expected utility maximizers. One difficulty is that, for a given social prospect, an individual $i \in \mathbb{N}$ may not exist in all states of the world. We hence need to define individual preferences conditional on their existence in an event $A \in \Sigma$. To do so, for any $A \in \Sigma$ we let $\Sigma_A = \{B \in \Sigma, B \subset A\}$ be a sigma-algebra over A . For any Σ_A -measurable function $K : A \rightarrow \mathbb{R}$, its conditional expected value is $\mathbb{E}_A[K] = \frac{1}{P(A)} \int_A K(s) dP(s)$. Hence, the set \mathbb{R}_+^A of mappings that are Σ_A -measurable represents individual prospects conditional on existence in A . We then assume that individual preferences are conditional expected utilities:

For each $i \in \mathbb{N}$ and $A \in \Sigma$, individual i 's preferences are represented by a complete reflexive and transitive relation \succsim_i^A over \mathbb{R}_+^A . Furthermore, there exists an increasing, continuously differentiable and concave function $u_i : \mathbb{R}_{++} \rightarrow \mathbb{R}$ such that, for all $f_i, g_i \in \mathbb{R}_{++}^A$,

$$f_i \succsim_i^A g_i \iff \mathbb{E}_A[u_i \circ f_i] \geq \mathbb{E}_A[u_i \circ g_i].$$

In order to stick to the utilitarian tradition and in particular to Harsanyi's (1955) approach, we also assume that the social observer is an expected utility maximizer:

The social ordering \succsim is a complete reflexive and transitive relation over F . Furthermore, there exists a continuous function $U : X \rightarrow \mathbb{R}$ such that, for all $f, g \in F$:

$$f \succsim g \iff \mathbb{E}[U \circ f] \geq \mathbb{E}[U \circ g].$$

4 Fair social choice under risk for fixed populations

Let us first consider the case in which there is no uncertainty about the composition of the population, so that N is fixed. Then, the well-known Pareto principle can be expressed in the following way.⁶

Same-population Pareto. For all $N \in \mathcal{N}$, for all $f, g \in F_N$, if $f_i \succsim_i^S g_i$ for all $i \in N$, then $f \succsim g$. If furthermore $f_i \succ_j^S g_i$ for some $j \in N$, then $f \succ g$.

As discussed in Section 2, we also want social orderings to satisfy principles of fairness regarding relative priorities for the allocation of resources. In this respect, a standard principle in the literature is the Pigou-Dalton principle that describes social-welfare-enhancing transfers of resources. It is however known that the Pigou-Dalton principle may conflict with the Pareto principle (Fleurbaey and Trannoy, 2003; Brun and Tungodden, 2004). We thus focus on a weak fairness requirement that applies only when transfers are made from a non-poor to a poor.

Pigou-Dalton transfer for the poor. There exists a poverty level $z_p \in \mathbb{R}_{++}$ such that, for all $N \in \mathcal{N}$, for all $x, y \in X_N$, if there exist $i, j \in N$ such that

1. $y_i < x_i < z_p < x_j < y_j$;
2. $x_i - y_i = y_j - x_j$;
3. $x_k = y_k$ for all $k \in N \setminus \{i, j\}$;

then $x \succsim y$.

⁶Our version of the Pareto principle is a version of the so-called “Strong Pareto” principle.

Let \succsim_N denote the restriction of the social welfare ordering \succsim to prospects in F_N . The next proposition characterizes fair utilitarian criteria.

Theorem 1 *For all $N \in \mathcal{N}$, the social ordering \succsim_N satisfies Same-population Pareto and Pigou-Dalton transfer for the poor if and only if there exists a poverty level $z_p \in \mathbb{R}_+$ such that, for all $f, g \in F_N$,*

$$f \succsim_N g \iff \mathbb{E} \left[\sum_{i \in N} \frac{u_i(f_i)}{u'_i(z_p)} \right] \geq \mathbb{E} \left[\sum_{i \in N} \frac{u_i(g_i)}{u'_i(z_p)} \right].$$

Fair utilitarian criteria give equal weight to individual utilities that are normalized so that marginal utility is the same for all individuals at the poverty level. As noted in Section 2, it also has the following attractive property: The more risk averse a person is, the greater marginal utility it has in poverty and the lower marginal utility it has out of poverty. In other words, a greater risk aversion implies that one gets greater social priority in poverty and less priority out of poverty.

Observe that this result does not say anything about utility levels, since only marginal utility matters for utilitarianism dealing with fixed populations. The question of utility levels comes up in the next section.

5 Fair criteria for populations of variable sizes and compositions

As discussed in Section 2, it seems intuitive to deem a population facing a risk to be less well-off if it contains more risk averse individuals. To make such judgments, we must be able to compare populations with different risk attitudes. In this section, we incorporate uncertainty about the preferences or existence of people composing the population into our analysis. This also makes more general evaluations possible, allowing for instance a possible uncertainty about what the preferences of people will be in the future.

However, this also raises the issue of how the Pareto principle should be ap-

plied. Indeed, individual preferences are normatively relevant only for states of the world in which the preference bearers exist. We thus introduce a general version of the Pareto principle, such that individual preferences are respected whenever individuals live in the same states of the world in the two prospects that are being compared.

Pareto. For all $f, g \in F$, if for all $i \in \mathbb{N}$, $A_i(f) = A_i(g) := A_i$, and if $f_i \succsim_i^{A_i} g_i$ for all $i \in \mathbb{N}$ such that $P(A_i) > 0$, then $f \succsim g$. If furthermore $f_i \succ_i^{A_i} g_i$ for some $j \in \mathbb{N}$ such that $P(A_j) > 0$, then $f \succ g$.

The next two principles describe how populations of different sizes and composition are being compared. The first principle introduces the notion of a critical level of consumption, i.e., a level such that adding a person at this level is a matter of social indifference.

Critical level. There exists $z_c \in \mathbb{R}_{++}$ such that for all $x, y \in X$, if $N(y) = N(x) \cup \{j\}$, with $j \in \mathbb{N} \setminus N(x)$, $y_i = x_i = z_c$ for all $i \in N(x)$ and $y_j = z_c$ then $x \sim y$.

Our critical-level principle is related to other principles proposed in the literature on population ethics (see Blackorby, Bossert and Donaldson, 2005, for a presentation of this literature). The main difference is that our principle is expressed in terms of resource and not in terms of welfare. Our principle is weak because it does not require the critical-level to be the same for all possible allocations, instead it posits that there is a level at which population expansion is indifferent. The level of z_c can be thought of as a subsistence level below which life is not worth living, so that a smaller population is preferable when everyone is below this level, whereas a greater population is desirable when everyone is at a level above z_c .

A second principle compares populations of same size facing an egalitarian prospect, i.e., a prospect in which income varies across states but individuals always end up in equal positions. The principle represents the argument made in Section 2 that risk aversion is a burden that reduces social welfare. The reason

why this should be so is that a more risk-averse individual always has a lower certainty-equivalent, and is therefore willing to pay more to avoid the aggregate risk on income. Thus, when facing a given egalitarian risky prospect, a population with less risk-averse agents should be considered better off.

To state our principle, we need the following definition. We say that an individual $j \in \mathbb{N}$ is more risk averse than another individual $i \in \mathbb{N}$ if there exists a continuously differentiable, increasing and concave function ψ such that $u_j = \psi \circ u_i$.

Dominance of more risk averse populations. For all $f, g \in F$, if there exists a subpopulation $N \in \mathcal{N}$ and two individuals $i, j \in \mathbb{N} \setminus N$, with j more risk averse than i , such that:

1. $N(f(s)) = N \cup \{i\}$ and $N(g(s)) = N \cup \{j\}$ for all $s \in S$,
2. $f_i(s) = g_j(s) = f_k(s) = g_k(s)$ for all $k \in N$ and $s \in S$,

then $f \succsim g$.

To make sure that the above principle is not empty, we make the following assumption, that guarantees the existence of an individual that is comparable to others in terms of risk aversion.

There exists an individual $i \in \mathbb{N}$ such that, for all $j \in \mathbb{N} \setminus \{i\}$, either i is more risk averse than j , or j is more risk averse than i .

It is then possible to characterize the class of fair utilitarian criteria for variable populations.

Theorem 2 *The social ordering \succsim satisfies Pareto, Critical-level and Dominance of more risk averse populations if and only if there exists a critical level $z_c \in \mathbb{R}_+$ such that, for all $f, g \in F$,*

$$f \succsim g \iff \sum_{i \in \mathbb{N}} p_i(f) \times \mathbb{E} \left[\frac{u_i(f_i) - u_i(z_c)}{u'_i(z_c)} \right] \geq \sum_{i \in \mathbb{N}} p_i(g) \times \mathbb{E} \left[\frac{u_i(g_i) - u_i(z_c)}{u'_i(z_c)} \right].$$

A noticeable consequence of Theorem 2 is that these fair utilitarian criteria satisfy Pigou-Dalton transfer for the poor, although it is not an assumption of their characterization. In addition, the income poverty level is shown to be exactly equal to the critical-level. This result comes from the fact that the normalization of marginal utilities at z_c is the only normalization that guarantees that, given that utility levels are equalized at z_c (which is implied by Critical level), a more risk averse individual is worse-off than a less risk averse individual when facing a small risk around z_c (see Fig. 2).

6 Optimal risk sharing and social risk aversion

In this section, we explore some of the implications of fair utilitarian criteria. To do so, we consider a simple static economy with no production and a given population N of size $n \in \mathbb{N}$.⁷ We assume that the only risk is a risk on per capita endowment in the economy, described by a finitely-valued random variable ω on the measure space (S, Σ, P) . Thus per capita endowment in state $s \in S$ is $\omega(s)$. A prospect f is *feasible* if for all $s \in S$:

$$\frac{1}{n} \sum_{i \in N} f_i(s) \leq \omega(s).$$

6.1 Fair risk sharing

Let us first look at fair optimal risk sharing is defined as the solution of the same optimization problem but using a fair utilitarian criterion:

$$\begin{aligned} \max_{f \in F_N} \quad & \sum_{i \in N} \mathbb{E} \left[\frac{u_i(f_i) - u_i(z_c)}{u'_i(z_c)} \right] \\ \text{s.t.} \quad & \frac{1}{n} \sum_{i \in N} f_i(s) \leq \omega(s) \quad \forall s \in S. \end{aligned} \tag{1}$$

⁷We will compare different populations of the same size, but we only consider situations where there is no risk on the composition of the group.

We want to compare the fair optimal risk sharing rule with the optimal risk sharing rule using 0 – 1 normalized utilitarian criteria. Optimal risk sharing is defined as the solution of the following first-best optimization problem:

$$\begin{aligned} \max_{f \in F_N} \quad & \sum_{i \in N} \mathbb{E} \left[\frac{u_i(f_i) - u_i(0)}{u_i(\bar{x}) - u_i(0)} \right] \\ \text{s.t.} \quad & \frac{1}{n} \sum_{i \in N} f_i(s) \leq \omega(s) \quad \forall s \in S, \end{aligned} \tag{2}$$

where \bar{x} is either the best available option ($n \times \sup_{s \in S} \omega(s)$), like in “relative utilitarianism” (Dhillon and Mertens, 1999), or an exogenous poverty line, as suggested by Adler (2012).

Let us denote $f_i^*(\omega(s))$ the optimal allocation for problem (1) and $f_i^{**}(\omega(s))$ the fair optimal risk sharing rule, defined as the solution of problem (2). The next proposition contrasts the two risk sharing rules. It highlights that, while the standard 0 – 1 normalized approach can generate large inequalities, the fair risk sharing rule induces bounds on what agents may receive.

Proposition 1

1. *The optimal allocation $f_i^*(\omega(s))$ for problem (1) is such that:*
 - (a) *For all $s \in S$ and all $i, j \in N$, if i is more risk-averse than j and $\omega(s) \leq z_c$, then $z_c \geq f_i^*(\omega(s)) \geq f_j^*(\omega(s))$.*
 - (b) *For all $s \in S$ and all $i, j \in N$, if i is more risk-averse than j and $\omega(s) \geq z_c$, then $z_c \leq f_i^*(\omega(s)) \leq f_j^*(\omega(s))$.*
2. *For any risk on per capita endowment, and for any $0 < \varepsilon < 1$, there exists a population N and an individual i in this population with preferences such that the optimal allocation for problem (2) yields $f_i^{**}(\omega(s)) \geq (1 - \varepsilon)n\omega(s)$ and $f_k^{**}(\omega(s)) \leq \varepsilon\omega(s)$ for all $k \in N \setminus \{i\}$.*

The first part of Proposition 1 shows that the fair utilitarian sharing necessarily imposes bounds to the inequality in the allocation of resources. When

resource are low, so that it is impossible to provide everyone with a consumption level above the poverty line, then nobody should have a consumption level above the line. It can be viewed as a fairness requirement that poverty should be shared by everyone in poor societies. Moreover, when resources are low, the more risk averse agents should have relatively more. On the other hand, when resource are sufficiently large, so that it is possible to raise everyone's consumption above the poverty line we should do so. Again, this can be viewed as a fairness requirement that prosperity should be shared. In this case, the more risk averse agents should have relatively less.

The first part of Proposition 1 also implies that, when $\omega(s) = z_p$, the optimal consumption level is perfectly egalitarian (at z_p for all agents). The existence of such an efficient and egalitarian optimum was assumed by Jouini, Napp and Nocetti (2013). It is an implication of fair utilitarian criteria.

The second part of Proposition 1 shows that the risk sharing rule derived from the 0 – 1 normalized utilitarian criteria can create large inequalities, where one individual gets almost all the aggregate endowment $n\omega(s)$, while the others have a consumption level close to zero. In particular, even when there are enough resource to bring everyone above the poverty level, it is optimal to have only one individual well above the poverty level while others remain in deep poverty. This seems to be a quite extreme distributive conclusion, which is avoided whatever the preferences of the population by the fair utilitarian rule.

6.2 Collective risk aversion

One might think that populations composed of more risk averse agents are more prone to undertake risks when acting collectively. However, Mazzocco (2004) and Jouini, Napp and Nocetti (2013) showed that is not always the case, even when there are only two agents with Constant Relative Risk Aversion (CRRA) utility functions. In this section, we study how the risk aversion associated with a fair utilitarian criterion changes when the composition of population $N \in \mathcal{N}$ changes.

Let us assume that all (potential) individuals have utility functions of the

following specific HARA form:⁸ for all $i \in \mathbb{N}$, there exists $\tau_i \in \mathbb{R}_{++}$ such that $u_i(x_i) = (\kappa + x_i)^{1-1/\tau_i}/(1 - 1/\tau_i)$ for all $x_i \in \mathbb{R}_{++}$, where κ is a positive real number. Parameter τ_i determines whether individual i is more or less risk averse. Indeed, relative risk tolerance of individual i at the consumption level $x_i \in \mathbb{R}_+$ is $t_i(x_i) = -u'_i(x_i)/[x_i u''_i(x_i)] = \tau_i(1 + \kappa/x_i)$. At any consumption level $x_i \in \mathbb{R}_+$, an individual whose utility has a higher parameter τ_i is more risk tolerant. Relative risk tolerance is the inverse of relative risk aversion, so that the higher τ_i the less risk averse individual i is.

For any population $N \in \mathcal{N}$ and for any level of per capita resource $z \in \mathbb{R}_+$, let us denote:

$$W_N(z) := \max_{x \in \mathbb{R}^n} \sum_{i \in N} \frac{u_i(x_i) - u_i(z_c)}{u'_i(z_c)}$$

$$s.t. \quad \frac{1}{n} \sum_{i \in N} x_i \leq z.$$

$W_N(z)$ is the indirect social welfare for the fair utilitarian criterion when per capita endowment is equal to z .

To characterize collective risk attitudes, we need to study the level of risk aversion associated to function W_N . The *collective relative risk tolerance* for a population $N \in \mathcal{N}$ at the per capita resource level $z \in \mathbb{R}_{++}$ is $t_N(z) := -W'_N(z)/[zW''_N(z)]$. The *collective relative risk aversion* for a population $N \in \mathcal{N}$ at the per capita resource level $z \in \mathbb{R}_{++}$ is $R_N(z) := 1/t_N(z) = -zW''_N(z)/[W'_N(z)]$.

The next proposition highlights situations in which we are sure that an increase in an agent's relative risk aversion increases collective relative risk aversion. To present the result, denote for any $N \in \mathcal{N}$ $\overline{\tau}_N := \max_{i \in N} \tau_i$ and $\underline{\tau}_N := \min_{i \in N} \tau_i$. Also denote $\underline{z}_N := (\kappa/(z_c + \kappa))^{\overline{\tau}_N/\underline{\tau}_N} (z_c + \kappa) - \kappa$. The level of resources \underline{z}_N guarantees that the optimal sharing rule induces strictly positive consumption levels

⁸HARA utility functions have the form $u_i(x_i) = \frac{\zeta}{1-1/\tau_i} (\eta_i + \tau_i x_i)^{1-1/\tau_i}$, with $\tau_i, \zeta > 0$. Our assumption is to set $\eta_i = \kappa \tau_i$ so that comparative risk aversion is determined only by parameter τ_i . The CRRA can be seen as a limit case when $\kappa \rightarrow 0$. The advantage of our formulation is to have well-defined utility functions on \mathbb{R}_+ , which is not the case of CRRA utility functions when $\tau_i < 1$ and $x_i = 0$.

for all individuals.

Proposition 2 *For all $N \in \mathcal{N}$ and all $i, j \in \mathbb{N} \setminus N$, if individual i is more risk averse than individual j then collective relative risk aversion is higher in population $N \cup \{i\}$ than in population $N \cup \{j\}$ at $z \geq \underline{z}_M$ (where $M = N \cup \{i, j\}$) in the following cases:*

1. *when $e^{-\rho}z_p - (1 - e^{-\rho})\kappa \leq z \leq e^{\rho}z_p + (e^{\rho} - 1)\kappa$, with $\rho = \frac{\tau_M}{\tau_M - \tau_M}$.*
2. *when $z \leq z_p$ (resp. $z \geq z_p$), provided j is more risk averse than any individual in N (resp. provided i is less risk averse than any individual in N).*

Proposition 2 implies that, around the poverty level, an increase in an agent's relative risk aversion always increases collective relative risk aversion (Case 1). This is true for a large interval around the poverty level when there is not too much heterogeneity in relative risk tolerance and when the most risk averse agents (with lowest risk tolerance τ_i) is not too risk averse. In particular, a marginal change from a perfectly homogenous population always changes collective risk aversion in the expected direction.

Proposition 2 also provides sufficient conditions for changes in the risk aversions of individuals at the extreme of the distribution to affect collective risk aversion in the expected direction: when resources are low for the most risk averse agents; when resources are high for the least risk averse agent.

7 Conclusion

In this paper, we have proposed a new normalization of utilities in the context of Harsanyi's (1955) aggregation theorem. This normalization equalizes utility and marginal utility across all individuals at the poverty line, and we have shown that this normalization is the only one that respects either a basic fairness axiom around the poverty line, or a preference against risk-aversion when adding newcomers to the population.

We have proposed this normalization for utilitarianism, but observe that non-utilitarian criteria can also use it. For instance, the generalized utilitarian (also called “prioritarian”) criterion that maximizes a sum of concave transforms of normalized utilities, with the same transform for all individuals, can also satisfy the fairness and the anti-risk-aversion principles —though not the Pareto principle *ex ante*.

This analysis emphasizes that normalizing utilities for utilitarianism is not just a matter of convenience but involves serious fairness considerations. The fairness considerations underlying the standard 0 – 1 normalization have to do with impartiality in extreme situations (Segal, 2000) or in situations which have similar roles in their contribution to relative utility (Karni, 1998). The normalization proposed here is incompatible with these impartiality principles. In particular, it implies that, in extreme situations, the more risk averse individuals are worse-off than the less risk averse because they suffer more from losing and benefit less from winning. This suggests that the impartiality principles may actually not be as compelling as one might think. Risk aversion is a burden that should perhaps be recognized as such and that justifies a differential treatment based on individual risk attitudes in the situations depicted in the impartiality axioms.

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Appendix

Proof of Theorem 1

Proving that social ordering \succsim characterized in Theorem 1 satisfies Same-population Pareto and Pigou-Dalton transfer for the poor is straightforward. We show that only this social ordering satisfies the two principles.

Step 1: The social VNM function is an affine combination of individuals ones

Let first remark that social choices on F_N can be translated as choices on the set of simple lotteries over X_N . To do so, define $\Delta(X_N)$ the set of simple lotteries, that is mapping $p : X_N \rightarrow [0, 1]$ such that $\sum_{x \in X_N} p(x) = 1$ and $p(x) > 0$ for a finite number of elements in X_N . For any prospect $f \in F_N$, there exists a finite partition (A_1, \dots, A_m) for which, whatever $k = 1, \dots, m$, $f(s) = f(s') \neq f(s'')$ for all $s, s' \in A_k$ and $s'' \in S \setminus A_k$. Define $p^f \in \Delta(X_N)$ such that, for any $x \in X$, $p^f(x) = P(A_k)$ if $f(s) = x$ for all $s \in A_k$ and $p^f(x) = 0$ otherwise. Conversely, for any $p \in \Delta(X_N)$, we can define an associated prospect $f^p \in F_N$ such that $f^p(s) = x$ for all $s \in A_k$, with $P(A_k) = p(x)$ when $p(x) > 0$.

Hence, we can define the following orderings on $\Delta(X_N)$:

1. For all $i \in N$, for all $p, q \in \Delta(X_N)$,

$$p \tilde{\succ}_i q \iff V_i(p) = \mathbb{E}[u_i(f_i^p)] \geq \mathbb{E}[u_i(f_i^q)] = V_i(q).$$

2. for all $p, q \in \Delta(X_N)$,

$$p \tilde{\succ}_0 q \iff V_0(p) = \mathbb{E}[U(f^p)] \geq \mathbb{E}[U(f^q)] = V_0(q).$$

Each V_i is linear in the sense of Fishburn (1984) and, by Same-population Pareto, for all $p, q \in \Delta(X_N)$, $V_0(p) \geq V_0(q)$ whenever $V_i(p) \geq V_i(q)$ for all $i \in N$ (with a strict inequality if $V_j(p) > V_j(q)$ for some $j \in N$). Hence, by Theorem 2 in Fishburn (1984), there exist positive real numbers $(\alpha_i^N)_{i \in N} \in \mathbb{R}_{++}^N$ and a real number β^N such that

$$V_0(p) = \sum_{i \in N} \alpha_i^N V_i(p) + \beta^N.$$

The weights $(\alpha_i^N)_{i \in N} \in \mathbb{R}_{++}^N$ are furthermore unique because utility is self-centered (it depends only on individual consumption), so that the utility functions u_i are linearly independent.

Given the correspondence between preferences over prospects and preferences over lotteries, this implies that:

$$U(x) = \sum_{i \in N} \alpha_i^N u_i(x_i) + \beta^N, \quad (3)$$

where U is the function used to compute the social expected utility.

Step 2: Characterizing the weights on utilities

Assume now that there are two individuals $i, j \in N$ such that $\alpha_i^N u'_i(z_p) < \alpha_j^N u'_j(z_p)$, where z_p is the poverty level. By continuity of u'_i and u'_j , this implies that there exist $z < z_p < z'$ such that $\alpha_j^N u'_j(z') > \alpha_i^N u'_i(z)$. For any $\varepsilon \in \mathbb{R}_{++}$ such that $0 < \varepsilon < \max\{z' - z_p, z_p - z\}$, define $x, y \in X$ in the following way:

1. $y_i = z, x_i = z + \varepsilon, x_j = z' - \varepsilon$ and $y_j = z'$;
2. $x_k = y_k$ for all $k \in N \setminus \{i, j\}$.

By Step 1, $x \succeq_N y \iff \alpha_i^N u_i(z + \varepsilon) + \alpha_j^N u_j(z' - \varepsilon) > \alpha_i^N u_i(z) + \alpha_j^N u_j(z')$. But, by concavity of the functions u_i and u_j , we have $u_i(z + \varepsilon) - u_i(z) < u'_i(z) \varepsilon$ and $u_j(z') - u_j(z' - \varepsilon) > u'_j(z') \varepsilon$. We thus have

$$\alpha_i^N (u_i(z + \varepsilon) - u_i(z)) < \alpha_i^N u'_i(z) \varepsilon < \alpha_j^N u'_j(z') \varepsilon < \alpha_j^N (u_j(z') - u_j(z' - \varepsilon))$$

this contradicts $x \succ_N y$ and hence Pigou-Dalton transfer for the poor.

Therefore, there must exist $\kappa^N > 0$ such that, for all $i \in N$, $\alpha_i^N u'_i(z_p) = \kappa^N$. This implies that $\alpha_i^N = \frac{\kappa^N}{u'_i(z_p)}$ for all $i \in N$. By expected utility and Eq. (3), this means that for all $f, g \in F_N$,

$$f \succsim_N g \iff \mathbb{E} \left[\sum_{i \in N} \frac{u_i(f_i)}{u'_i(z_p)} \right] \geq \mathbb{E} \left[\sum_{i \in N} \frac{u_i(g_i)}{u'_i(z_p)} \right].$$

Proof of Theorem 2

Proving that social ordering \succsim characterized in Theorem 2 satisfies Pareto, Critical-level and Dominance of more risk averse populations is straightforward. We show

that only this social ordering satisfies the three principles.

By the first step of the proof of Theorem 1 (Eq. (3)), Pareto (which implies Same-population Pareto) implies that for all $x \in X$:

$$U(x) = \sum_{i \in N(x)} \alpha_i^{N(x)} u_i(x_i) + \beta^{N(x)},$$

where the weights $(\alpha_i^N)_{i \in N} \in \mathbb{R}_{++}^N$ are unique positive numbers and $\beta^{N(x)}$ is a real number.

Now consider any $i \in \mathbb{N}$ and any $N \in \mathcal{N}$ such that $i \in N$. Let $A, B, C \in \Sigma$ such that A, B, C form a partition of S and $P(A) = P(B) = P(C) = 1/3$. For any $z_1, z_2, z_3, z_4 \in \mathbb{R}_+$, define $f, g \in F$ in the following way:

For all $s \in A$, $N(f(s)) = N(g(s)) = N$, $f_j(s) = z_1$ and $g_j(s) = z_3$ for all $j \in N$;

For all $s \in B$, $N(f(s)) = N(g(s)) = \{i\}$, $f_i(s) = z_2$ and $g_i(s) = z_4$;

For all $s \in C$, $N(f(s)) = N(g(s)) = N \setminus \{i\}$, $f_j(s) = z_3$ and $g_j(s) = z_1$ for all $j \in (N \setminus \{i\})$.

Denote $\alpha_i := \alpha_i^{\{i\}}$, that is α_i is the multiplicative constant transforming individual utility into social utility U when the population is reduced to the singleton $\{i\}$. Because the social ordering is an expected utility and by Eq. (3), $f \succsim g \iff \frac{1}{3}\alpha_i u_i(z_1) + \frac{1}{3}\alpha_i^N u_i(z_2) \geq \frac{1}{3}\alpha_i u_i(z_3) + \frac{1}{3}\alpha_i^N u_i(z_4)$. But by Pareto, it is also the case that $f \succsim g \iff \frac{1}{2}u_i(z_1) + \frac{1}{2}u_i(z_2) \geq \frac{1}{2}u_i(z_3) + \frac{1}{2}u_i(z_4)$, because individuals other than i are indifferent (they face a fifty-fifty chance of receiving either z_1 or z_3). The reasoning is true for any $z_1, z_2, z_3, z_4 \in \mathbb{R}_+$, so there must exist a continuous increasing function Ψ such that for all $z_1, z_2 \in \mathbb{R}_+$:

$$\frac{1}{3}\alpha_i u_i(z_1) + \frac{1}{3}\alpha_i^N u_i(z_2) = \Psi\left(\frac{1}{2}u_i(z_1) + \frac{1}{2}u_i(z_2)\right). \quad (4)$$

Denote G, H the real-valued functions such that $G(v_1) = \frac{2\alpha_i}{3}v_1$ and $H(v_2) = \frac{2\alpha_i^N}{3}v_2$, and $V = \{(v_1, v_2) \in \mathbb{R}^2 | \exists (z_1, z_2) \in \mathbb{R}_{++}^2, v_1 = \frac{1}{2}u_i(z_1), v_2 = \frac{1}{2}u_i(z_2)\}$, a rectangular connected subset of \mathbb{R}^2 . Eq.(4) implies that for any $(v_1, v_2) \in V$,

$G(v_1) + H(v_2) = \Psi(v_1 + v_2)$. Given that Ψ , G and H are continuous and increasing, it must be the case that there exists $c \in \mathbb{R}_+$, $a, b \in \mathbb{R}$ such that $G(v_1) = cv_1 + a$ and $H(v_2) = cv_2 + b$ (Aczél, 1966, Chap. 3, Cor. 1). Hence we need $\frac{2\alpha_i}{3} = \frac{2\alpha_i^N}{3}$ so that $\alpha_i = \alpha_i^N$.

To sum up, there exist weights $(\alpha_i)_{i \in \mathbb{N}} \in \mathbb{R}_{++}^{\mathbb{N}}$ and real numbers $\beta^N \in \mathbb{R}$ such that, for all $x \in X$:

$$U(x) = \sum_{i \in N(x)} \alpha_i u_i(x_i) + \beta^{N(x)}.$$

By Critical-level, there exists $z_c \in \mathbb{R}_{++}$ such that, for any $i \in \mathbb{N} \setminus \{1\}$, $\alpha_i u_i(z_c) + \beta^{\{i\}} = \alpha_i u_i(z_c) + \alpha_1 u_1(z_c) + \beta^{\{1, i\}}$. But, symmetrically, $\alpha_1 u_1(z_c) + \beta^{\{1\}} = \alpha_i u_i(z_c) + \alpha_1 u_1(z_c) + \beta^{\{1, i\}}$, so that $\alpha_i u_i(z_c) + \beta^{\{i\}} = \alpha_1 u_1(z_c) + \beta^{\{1\}}$ for all $i \in \mathbb{N}$. By repeated application of Critical-level, it is also the case that, for any $N \in \mathcal{N}$ and any $i \in N$, $\sum_{i \in N} \alpha_i u_i(z_c) + \beta^N = \alpha_i u_i(z_c) + \beta^{\{i\}}$. Hence, for all $N \in \mathcal{N}$, $\beta^N = -\sum_{i \in N} \alpha_i u_i(z_c) + \alpha_1 u_1(z_c) + \beta^{\{1\}}$.

To sum up, and denoting $\Lambda = \alpha_1 u_1(z_c) + \beta^{\{1\}}$, for all $f, g \in F$,

$$\begin{aligned} f \succsim g &\iff \mathbb{E} \left[\sum_{i \in N(f)} \alpha_i u_i(f_i) + \beta^{N(f)} \right] \geq \mathbb{E} \left[\sum_{i \in N(g)} \alpha_i u_i(g_i) + \beta^{N(g)} \right] \\ &\iff \mathbb{E} \left[\sum_{i \in N(f)} \alpha_i (u_i(f_i) - u_i(z_c)) + \Lambda \right] \geq \mathbb{E} \left[\sum_{i \in N(g)} \alpha_i (u_i(g_i) - u_i(z_c)) + \Lambda \right] \\ &\iff \mathbb{E} \left[\sum_{i \in N(f)} \alpha_i (u_i(f_i) - u_i(z_c)) \right] \geq \mathbb{E} \left[\sum_{i \in N(g)} \alpha_i (u_i(g_i) - u_i(z_c)) \right] \\ &\iff \mathbb{E} \left[\sum_{i \in N(f)} \tilde{u}_i(f_i) \right] \geq \mathbb{E} \left[\sum_{i \in N(g)} \tilde{u}_i(g_i) \right], \end{aligned} \tag{5}$$

with $\tilde{u}_i(x_i) = \alpha_i (u_i(x_i) - u_i(z_c))$. Function \tilde{u}_i is such that $\tilde{u}_i(z_c) = 0$ and $\tilde{u}_i'(x_i) = \alpha_i u_i'(x_i)$.

By assumption, there exist an individual i such that any other individual is either more risk averse or less risk averse. Let $j \in \mathbb{N}$ be less risk averse than i (the reasoning is similar when j is more risk averse than i). Let $N \in \mathcal{N}$ be such that $i, j \notin N$. Assume that $\tilde{u}_i'(z_c) \neq \tilde{u}_j'(z_c)$:

Case 1: $\tilde{u}_i'(z_c) > \tilde{u}_j'(z_c)$. Then there exists $z > z_c$ such that $\tilde{u}_i(z) > \tilde{u}_j(z)$, be-

cause $\tilde{u}_i(z_c) = \tilde{u}_j(z_c)$. Let $x, y \in X^e$ be such $N(x) = N \cup \{i\}$ $N(y) = N \cup \{j\}$, and $x_i = y_i = z$ for all $i \in N$. Then, $x \succ y$ (by Eq. (5)), because $\sum_{k \in N(x)} \tilde{u}_k(x_k) = \sum_{k \in N} \tilde{u}_k(z) + \tilde{u}_i(z) > \sum_{k \in N} \tilde{u}_k(z) + \tilde{u}_j(z) = \sum_{k \in N(y)} \tilde{u}_k(y_k)$. But this contradicts Dominance of more risk averse populations given that i is more risk averse than j .

Case 2: $\tilde{u}'_i(z_c) < \tilde{u}'_j(z_c)$. Then there exists $z < z_c$ such that $\tilde{u}_i(z) > \tilde{u}_j(z)$, because $\tilde{u}_i(z_c) = \tilde{u}_j(z_c)$. Thus, by the same reasoning as above, $x \succ y$, which is a contradiction of Dominance of more risk averse populations.

Hence, for any $j \in \mathbb{N} \setminus \{i\}$, we must have $\tilde{u}'_j(z_c) = \tilde{u}'_i(z_c)$. Denote $K = \tilde{u}'_i(z_c)$. We obtain that for all $j \in \mathbb{N}$ $\tilde{u}'_j(z_c) = \alpha_j u'_j(z_c) = K$, so that $\alpha_j = K/u'_j(z_c)$. Hence, by Eq. (5) and the definition of functions \tilde{u}_i , for all $f, g \in F$:

$$f \succsim g \iff \mathbb{E} \left[\sum_{i \in N(f)} \frac{u_i(f_i) - u_i(z_c)}{u'_i(z_c)} \right] \geq \mathbb{E} \left[\sum_{i \in N(g)} \frac{u_i(g_i) - u_i(z_c)}{u'_i(z_c)} \right].$$

Proof of Proposition 1

Proof of part 1

Let us first focus on interior solutions of Problem (1). Given that the utility functions u_i are concave, the first-order conditions are both necessary and sufficient for an optimum. These conditions are that, for all $s \in S$ and all $i \in N$,

$$\frac{u'_i(f_i^*(\omega(s)))}{u'_i(z_c)} = \lambda(s), \quad (6)$$

where $\lambda(s) > 0$ is the multiplier associated with the resource constraint. This implies for all $s \in S$ and all $i, j \in N$,

$$\frac{u'_i(f_i^*(\omega(s)))}{u'_i(z_c)} = \frac{u'_j(f_j^*(\omega(s)))}{u'_j(z_c)}. \quad (7)$$

Assume that $\omega(s) = z_p$. Because of the resource constraint $\frac{1}{n} \sum_{i \in N} f_i(s) \leq z_p$, we cannot have $f_i^*(z_p) \geq z_p$ for all $i \in N$ and $f_j^*(z_p) > z_p$ for some $j \in N$. On the

other hand, $f_i^*(z_p) \leq z_p$ for all $i \in N$ and $f_j^*(z_p) < z_p$ for some $j \in N$ would be inefficient (we can increase the consumption of j without violating the resource constraint), and thus not optimal. There are only two possibility for an optimum:

1. $f_i^*(z_p) = z_p$ for all $i \in N$;
2. there exists $i, j \in N$ such that $f_i^*(z_p) < z_p < f_j^*(z_p)$.

In case 2, by concavity of the utility functions, we would have

$$\frac{u'_i(f_i^*(\omega(s)))}{u'_i(z_c)} < \frac{u'_i(z_c)}{u'_i(z_c)} = 1 = \frac{u'_j(z_c)}{u'_j(z_c)} < \frac{u'_j(f_j^*(\omega(s)))}{u'_j(z_c)},$$

which is a violation of the first-order condition (7). Thus only case 1 is possible.

This proves that there exists an efficient and egalitarian allocation when $\omega(s) = z_p$. Then, in the case of an interior solution, we can proceed like in the proof of Proposition 3 in Jouini, Napp and Nocetti (2013), which is straightforward.

Let us now consider the possibility of a corner solution where $f_i^*(\omega(s)) = 0$ for some individual $i \in N$. For this to be the case, we would need $\frac{u'_i(0)}{u'_i(z_c)} \leq \lambda(s)$.

Assume that $\omega(s) < z_c$, we do have $f_i^*(\omega(s)) < z_c$, as required by statement (a) in Part 1 of the proof. Also, if i is more risk-averse than another individual j , we have by our normalization that $\frac{u'_i(0)}{u'_i(z_c)} > \frac{u'_j(0)}{u'_j(z_c)}$, and therefore that $\frac{u'_i(0)}{u'_i(z_c)} < \lambda(s)$. Thus $f_j^*(\omega(s)) = 0 \leq f_i^*(\omega(s))$, as required.

Assume that $\omega(s) \geq z_c$. By the same reasoning as above, we know that there exists an individual j such that $f_j^*(\omega(s)) \geq z_c$ and $\frac{u'_j(f_j^*(\omega(s)))}{u'_j(z_c)} = \lambda(s)$. But, by concavity of the utility functions, we necessarily have:

$$\lambda(s) = \frac{u'_j(f_j^*(\omega(s)))}{u'_j(z_c)} \leq 1 < \frac{u'_i(0)}{u'_i(z_c)} \leq \lambda(s),$$

which is contradiction. An corner solution is never possible in that case.

Proof of part 2

Assume that the population N is composed of an individual i with preferences $u_i(z) = 1 - e^{-\alpha z}$, for all $z \in \mathbb{R}$, and $n - 1$ other individuals with the same preferences $u_k(z) = 1 - e^{-\theta\alpha z}$, for all $z \in \mathbb{R}$, where $\alpha > 0$ and $\theta > 0$.⁹

We will focus without loss of generality on cases with interior solutions. Given that the utility functions u_i are concave, Problem (2) is convex and the following first-order conditions are both necessary and sufficient for an optimum for all $j \in N$ and $s \in S$,

$$\frac{u'_j(f_j^{**}(\omega(s)))}{u_j(\bar{x}) - u_j(0)} = \lambda(s). \quad (8)$$

Given the utility function we have chosen, this yields:

$$\frac{\alpha e^{-\alpha f_i^{**}(\omega(s))}}{1 - e^{-\alpha \bar{x}}} = \frac{\theta \alpha e^{-\theta \alpha f_k^{**}(\omega(s))}}{1 - e^{-\theta \alpha \bar{x}}},$$

for all $s \in S$ and $k \in N \setminus \{i\}$. This can be written in the following way:

$$f_i^{**}(\omega(s)) = \theta f_k^{**}(\omega(s)) - \frac{1}{\alpha} \ln \left(\theta \frac{1 - e^{-\alpha \bar{x}}}{1 - e^{-\theta \alpha \bar{x}}} \right).$$

For a given a $\theta > 0$, this means that for α high enough we have $f_i^{**}(\omega(s)) \approx \theta f_k^{**}(\omega(s))$ for all $s \in S$ and $k \in N \setminus \{i\}$. Obviously, for any $\epsilon > 0$, when θ is high enough, the resource constraints $\sum_{j \in N} f_j^{**}(\omega(s)) = n\omega(s)$ implies that $f_i^{**}(\omega(s)) \geq (1 - \epsilon)n\omega(s)$ and $f_k^{**}(\omega(s)) \leq \epsilon\omega(s)$ for all $k \in N \setminus \{i\}$.

Proof of Proposition 2

Collective relative risk tolerance

Let N be any population in \mathcal{N} with size $n = |N|$. Denote $x_i^{N*}(z)$ the optimal allocation sharing rule when per capita resources are equal to $z \in \mathbb{R}_{++}$. For any $i \in N$, we know that, for interior solutions, there exists $\lambda \in \mathbb{R}_{++}$ such that (see

⁹We could take people with different preferences of the CARA type exhibited here, provided they have similar enough preferences.

Equation 6): $\frac{u'_i(x_i^{N^*}(z))}{u'_i(z_c)} = \lambda$. Given the HARA form of the utility function u_i with parameter τ_i , this means that $x_i^{N^*}(z) = e^{-\tau_i \lambda}(z_c + \kappa) - \kappa$. The resource constraint yields:

$$\frac{1}{n} \sum_{i \in N} e^{-\tau_i \lambda} = \frac{z + \kappa}{z_c + \kappa}.$$

Define implicitly function $\phi_N : \mathbb{R}_{++} \rightarrow \mathbb{R}$ by $\frac{1}{n} \sum_{i \in N} e^{-\tau_i \phi_N(y)} = y$ for all $y \in \mathbb{R}_{++}$. For any $z \in \mathbb{R}_{++}$, also define the function y such that $y(z) = \frac{z + \kappa}{z_c + \kappa}$. We obtain $x_i^{N^*}(z) = e^{-\tau_i \phi_N(y(z))}(z_c + \kappa) - \kappa$.

Remark that $\phi_N(y) \leq 0$ if $y \geq 1$ and $\phi_N(y) \geq 0$ if $y \leq 1$. Furthermore, recalling that $\overline{\tau_N} := \max_{i \in N} \tau_i$ and $\underline{\tau_N} := \min_{i \in N} \tau_i$, $e^{-\overline{\tau_N} \phi_N(y)} \geq y \geq e^{-\underline{\tau_N} \phi_N(y)}$ when $y \geq 1$ (so that $\phi_N(y) \leq 0$), and in that case $-\frac{1}{\underline{\tau_N}} \ln(y) \leq \phi_N(y) \leq -\frac{1}{\overline{\tau_N}} \ln(y)$. Similarly, when $y \leq 1$, $-\frac{1}{\overline{\tau_N}} \ln(y) \leq \phi_N(y) \leq -\frac{1}{\underline{\tau_N}} \ln(y)$.

When $z \leq z_c$, we know by Proposition 1 (first part) that $x_i^{N^*}(z) \geq e^{-\overline{\tau_N} \phi_N(y(z))}(z_c + \kappa) - \kappa$ for all $i \in N$. And, because $-\phi_N(y) \geq \frac{1}{\underline{\tau_N}} \ln(y)$ when $y \leq 1$, we obtain $x_i^{N^*}(z) \geq e^{\frac{\overline{\tau_N}}{\underline{\tau_N}} \ln(y(z))}(z_c + \kappa) - \kappa$ for all $i \in N$. When $z > z_N$, we have $x_i^{N^*}(z) > 0$ for all $i \in N$ and thus interior solutions. Note that, when $z \geq z_c$, we know by Proposition 1 (first part) that we necessarily have interior solutions.

Hara, Huang and Kuzmics (2007) proved the following result that relates the collective relative risk tolerance to the individual ones through the efficient sharing rule:

$$t_N(z) = \frac{1}{n} \sum_{i \in N} \frac{x_i^{N^*}(z)}{z} \times t_i(x_i^{N^*}(z)).$$

In the case of our HARA utility functions, $t_i(x_i) = \tau_i(1 + \kappa/x_i)$. Using the optimal sharing rule for the fair utilitarian social ordering, we thus have:

$$t_N(z) = \frac{z_c + \kappa}{nz} \sum_{i \in N} \tau_i e^{-\tau_i \phi_N(y(z))}. \quad (9)$$

Change in relative risk aversion

Consider any population $N \in \mathcal{N}$, with size $n = |N|$, and any $i, j \in (N \setminus N)$, such that individual i is more risk averse than individual j (so that $\tau_i < \tau_j$). Let

$\tilde{\tau} \in \mathbb{R}_{++}$ be such that $\tau_j \geq \tilde{\tau} \geq \tau_i$: this would be the risk tolerance of a fictitious individual, with relative risk aversion that is intermediate between the one of i and j .

Let us fix $z \in \mathbb{R}_{++}$ the per capita endowment of the economy. We define function $\gamma_{y(z),\tilde{\tau}}^N(\ell)$ implicitly by:

$$\frac{1}{n+1} \left(\sum_{k \in N} e^{-\tau_k \gamma_{y(z),\tilde{\tau}}^N(\ell)} \right) + \frac{1}{n+1} e^{-(\tilde{\tau}+\ell) \gamma_{y(z),\tilde{\tau}}^N(\ell)} = y(z)$$

Differentiating with respect to ℓ (while keeping $y(z)$ constant), we obtain:

$$\frac{\partial \gamma_{y(z),\tilde{\tau}}^N(\ell)}{\partial \ell} = -\gamma_{y(z),\tilde{\tau}}^N(\ell) \frac{e^{-(\tilde{\tau}+\ell) \gamma_{y(z),\tilde{\tau}}^N(\ell)}}{\sum_{k \in N} \tau_k e^{-\tau_k \gamma_{y(z),\tilde{\tau}}^N(\ell)} + (\tilde{\tau}+\ell) e^{-(\tilde{\tau}+\ell) \gamma_{y(z),\tilde{\tau}}^N(\ell)}}.$$

We can also introduce the function $\xi_{y,\tilde{\tau}}^N(\ell)$ implicitly by:

$$\xi_{y(z),\tilde{\tau}}^N(\ell) = \frac{z_c + \kappa}{nz} \left(\sum_{k \in N} \tau_k e^{-\tau_k \gamma_{y(z),\tilde{\tau}}^N(\ell)} + (\tilde{\tau} + \ell) e^{-(\tilde{\tau}+\ell) \gamma_{y(z),\tilde{\tau}}^N(\ell)} \right).$$

Again differentiating with respect to ℓ , we obtain:

$$\begin{aligned} \frac{\partial \xi_{y(z),\tilde{\tau}}^N(\ell)}{\partial \ell} &= \frac{z_c + \kappa}{nz} \left[-\frac{\partial \gamma_{y(z),\tilde{\tau}}^N(\ell)}{\partial \ell} \left(\sum_{k \in N} \tau_k^2 e^{-\tau_k \gamma_{y(z),\tilde{\tau}}^N(\ell)} + (\tilde{\tau} + \ell)^2 e^{-(\tilde{\tau}+\ell) \gamma_{y(z),\tilde{\tau}}^N(\ell)} \right) \right. \\ &\quad \left. - \gamma_{y(z),\tilde{\tau}}^N(\ell) (\tilde{\tau} + \ell) e^{-(\tilde{\tau}+\ell) \gamma_{y(z),\tilde{\tau}}^N(\ell)} + e^{-(\tilde{\tau}+\ell) \gamma_{y,\tilde{\tau}}^N(\ell)} \right] \\ &= \Xi_{y,\tilde{\tau}}^N(\ell) \times \left[\left(\sum_{k \in N} \left[1 + \gamma_{y(z),\tilde{\tau}}^N(\ell) (\tau_k - \tilde{\tau} - \ell) \right] \tau_k e^{-\tau_k \gamma_{y(z),\tilde{\tau}}^N(\ell)} \right) \right. \\ &\quad \left. + (\tilde{\tau} + \ell) e^{-(\tilde{\tau}+\ell) \gamma_{y(z),\tilde{\tau}}^N(\ell)} \right], \end{aligned}$$

$$\text{with } \Xi_{y,\tilde{\tau}}^N(\ell) = \frac{e^{-(\tilde{\tau}+\ell) \gamma_{y,\tilde{\tau}}^N(\ell)} (z_c + \kappa)}{nz \left(\sum_{k \in N} \tau_k e^{-\tau_k \gamma_{y,\tilde{\tau}}^N(\ell)} + (\tilde{\tau} + \ell) e^{-(\tilde{\tau}+\ell) \gamma_{y,\tilde{\tau}}^N(\ell)} \right)} > 0.$$

The quantity $\frac{\partial \xi_{y(z),\tilde{\tau}}^N}{\partial \ell}(0)$ measures the impact on collective relative risk toler-

ance of a small increase in the relative risk tolerance of a person with risk tolerance $\tilde{\tau}$ when the society is composed of this person and the population N , and when the per capita endowment of the economy is z . Clearly, if we have $\frac{\partial \xi_{y(z),\tilde{\tau}}^N}{\partial \ell}(0) \geq 0$ for all $\tau_j \geq \tilde{\tau} \geq \tau_i$, then collective relative risk aversion is higher in population $N \cup \{i\}$ than in population $N \cup \{j\}$.

Given that $\Xi_{y,\tilde{\tau}}^N(0) > 0$ and

$$\frac{\partial \xi_{y(z),\tilde{\tau}}^N}{\partial \ell}(0) = \Xi_{y,\tilde{\tau}}^N(0) \times \left[\left(\sum_{k \in N} \left[1 + \gamma_{y(z),\tilde{\tau}}^N(0)(\tau_k - \tilde{\tau}) \right] \tau_k e^{-\tau_k \gamma_{y(z),\tilde{\tau}}^N(0)} \right) + \tilde{\tau} e^{-\tilde{\tau} \gamma_{y(z),\tilde{\tau}}^N(0)} \right],$$

a sufficient condition for collective relative risk aversion to be higher in population $N \cup \{i\}$ than in population $N \cup \{j\}$ is that $1 + \gamma_{y(z),\tilde{\tau}}^N(0)(\tau_k - \tilde{\tau}) \geq 0$ for all $k \in N$. Denote $\bar{\tau} = \overline{\tau_{N \cup \{i,j\}}}$ and $\underline{\tau} = \underline{\tau_{N \cup \{i,j\}}}$.

Case 1: $z \geq z_p$. If $z \geq z_p$, then $y = (z + \kappa)/(z_p + \kappa) \geq 1$ and for any population $M \in \mathcal{N}$ $\phi_M(y) \leq 0$ and $-\frac{1}{\tau_M} \ln(y) \leq \phi_M(y) \leq -\frac{1}{\tau_M} \ln(y)$. We have $\gamma_{y(z),\tilde{\tau}}^N(0) = \phi_M(y)$ where M is composed of population N and a fictitious individual with relative risk tolerance $\tilde{\tau}$ so that $\underline{\tau}_M \geq \underline{\tau}$ and $\overline{\tau}_M \leq \bar{\tau}$. Hence $-\frac{1}{\underline{\tau}} \ln(y) \leq \gamma_{y(z),\tilde{\tau}}^N(0) \leq -\frac{1}{\bar{\tau}} \ln(y) \leq 0$.

If $\tau_i \geq \tau_k$ for all $k \in N$, then $\tilde{\tau} \geq \tau_k$ and $\gamma_{y,\tilde{\tau}}^N(0)(\tau_k - \tilde{\tau}) \geq 0$ for all $k \in N$, which is sufficient to have $\frac{\partial \xi_{y(z),\tilde{\tau}}^N}{\partial \ell}(0) \geq 0$ for all $\tau_j \geq \tilde{\tau} \geq \tau_i$.

For all $k \in N$, $\gamma_{y(z),\tilde{\tau}}^N(0)(\tau_k - \tilde{\tau}) \geq \gamma_{y(z),\tilde{\tau}}^N(0)(\bar{\tau} - \underline{\tau}) \geq -\frac{\bar{\tau} - \underline{\tau}}{\bar{\tau}} \ln(y)$. If $\frac{z + \kappa}{z_p + \kappa} = y \leq e^{\frac{\underline{\tau}}{\bar{\tau} - \underline{\tau}}}$, then $1 + \gamma_{y(z),\tilde{\tau}}^N(0)(\tau_k - \tilde{\tau}) \geq 0$ for all $k \in N$, which is sufficient to have $\frac{\partial \xi_{y(z),\tilde{\tau}}^N}{\partial \ell}(0) \geq 0$ for all $\tau_j \geq \tilde{\tau} \geq \tau_i$.

Case 2: $z \leq z_p$. If $z \leq z_p$, then $y = (z + \kappa)/(z_p + \kappa) \leq 1$. Like above, we can show that $-\frac{1}{\underline{\tau}} \ln(y) \geq \gamma_{y,\tilde{\tau}}^N(0) \geq -\frac{1}{\bar{\tau}} \ln(y) \geq 0$.

If $\tau_j \leq \tau_k$ for all $k \in N$, then $\tilde{\tau} \leq \tau_k$ and $\gamma_{y,\tilde{\tau}}^N(0)(\tau_k - \tilde{\tau}) \geq 0$ for all $k \in N$, which is sufficient to have $\frac{\partial \xi_{y(z),\tilde{\tau}}^N}{\partial \ell}(0) \geq 0$ for all $\tau_j \geq \tilde{\tau} \geq \tau_i$.

For all $k \in N$, $\gamma_{y(z),\tilde{\tau}}^N(0)(\tau_k - \tilde{\tau}) \geq -\gamma_{y(z),\tilde{\tau}}^N(0)(\bar{\tau} - \underline{\tau}) \geq \frac{\bar{\tau} - \underline{\tau}}{\underline{\tau}} \ln(y)$. If $\frac{z + \kappa}{z_p + \kappa} = y \geq e^{-\frac{\underline{\tau}}{\bar{\tau} - \underline{\tau}}}$, then $1 + \gamma_{y(z),\tilde{\tau}}^N(0)(\tau_k - \tilde{\tau}) \geq 0$ for all $k \in N$, which is sufficient to have $\frac{\partial \xi_{y(z),\tilde{\tau}}^N}{\partial \ell}(0) \geq 0$ for all $\tau_j \geq \tilde{\tau} \geq \tau_i$.