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A novel multivariate risk measure: the Kendall VaR

Matthieu Garcin[*], Dominique Guégan[†], Bertrand Hassani[‡]

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Abstract

The definition of multivariate Value at Risk is a challenging problem, whose most common solutions are given by the lower- and upper-orthant VaRs, which are based on copulas: the lower-orthant VaR is indeed the quantile of the multivariate distribution function, whereas the upper-orthant VaR is the quantile of the multivariate survival function. In this paper we introduce a new approach introducing a total-order multivariate Value at Risk, referred to as the Kendall Value at Risk, which links the copula approach to an alternative definition of multivariate quantiles, known as the quantile surface, which is not used in finance, to our knowledge. We more precisely transform the notion of orthant VaR thanks to the Kendall function so as to get a multivariate VaR with some advantageous properties compared to the standard orthant VaR: it is based on a total order and, for a non-atomic and $\mathbb{R}^d$-supported density function, there is no distinction anymore between the d-dimensional VaRs based on the distribution function or on the survival function. We quantify the differences between this new Kendall VaR and orthant VaRs. In particular, we show that the Kendall VaR is less (respectively more) conservative than the lower-orthant (resp. upper-orthant) VaR. The definition and the properties of the Kendall VaR are illustrated using Gumbel and Clayton copulas with lognormal marginal distributions and several levels of risk.[1]

Keywords – Value at Risk, multivariate quantile, risk measure, Kendall function, copula, total order

1 Introduction

For the random variable X, whose realisation is the loss of a unique asset or of a portfolio (so that the realisation of $-X$ is a gain), the Value at Risk at the level α, defined for a given horizon, is the univariate quantile of probability α of X. This metric allows estimating and comparing the risk of portfolios, experienced as the severity of a loss. However, the aggregation of the risks may rely on an arbitrary criterion and be overly simplistic. As a consequence, a multivariate analysis of the risks may be preferred to the risk of the aggregated value of a portfolio. For example, the decomposition of the risk into different asset classes (the loss for stocks, bonds, currencies, etc., expressed separately instead of a sole aggregated loss) can legitimate this approach. The

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computation of capital requirements also leads to the necessity of using a multivariate Value at Risk [23].

In a broader perspective, the risk must be expressed as a vector when the risk components cannot be aggregated. It may be useful in other fields than in finance. For example, in meteorology, the definition of what severe weather events are depends on the combination of unaggregatable parameters such as speed of wind, quantity of precipitation, temperature, cloud cover [31]. It may also be relevant in finance, because assets of different markets can be associated to different liquidity and transaction costs and therefore it might not be possible to aggregate them.

The multivariate analysis of the risk relies on a multivariate probability distribution of losses. A useful tool for representing multivariate distributions is the copula, which is a d-dimensional distribution function $[0,1]^d \rightarrow [0,1]$, where $d \in \mathbb{N}$ is the number of underlying random variables. Copulas are therefore used to describe the dependence between random variables. They have been applied in many fields, mostly in finance, but also in hydrology [18], in astronomy [37, 36] or in telecommunication networks [35, 17].

Embrechts and Puccetti [14] have introduced the notion of multivariate Value at Risk. Let G be the multivariate probability distribution of a random loss vector $X = (X_1, ..., X_d)'$, that is to say the copula-based probability distribution:[2]

$$G : (y_1, ..., y_d) \in \mathbb{R}^d \mapsto \mathbb{P}[X_1 \leq y_1, ..., X_k \leq y_d].$$

If $G_1, ..., G_d$ are the d univariate marginal distribution functions of X, then Sklar's theorem affirms the existence of a copula C such that $G(y_1, ..., y_d) = C(G_1(y_1), ..., G_d(y_d))$ [39]. Embrechts and Puccetti [14] defined the multivariate Value at Risk of X as the set of vectors belonging to the boundary of the α-level set of G. They distinguished the lower-orthant Value at Risk, $\underline{\mathrm{VaR}}_\alpha(G)$, and the upper-orthant Value at Risk, $\overline{\mathrm{VaR}}_\alpha(G)$. The lower-orthant Value at Risk is defined by $\underline{\mathrm{VaR}}_\alpha(G) = \partial\{y \in \mathbb{R}^d | G(y) \geq \alpha\}$, where ∂ denotes the boundary of the mentioned set, whereas the upper-orthant Value at Risk is $\overline{\mathrm{VaR}}_\alpha(G) = \partial\{y \in \mathbb{R}^d | \overline{G}(y) \leq 1 - \alpha\}$, where $\overline{G}$ is the survival function associated to G. In general, the lower-orthant Value at Risk is more conservative than the upper-orthant Value at Risk. If $(y_1, ..., y_d)' \in \underline{\mathrm{VaR}}_\alpha(G)$, then

$$\mathbb{P}[X_1 \leq y_1, ..., X_d \leq y_d] = \alpha.$$

Thus, a partial order is defined for $\mathbb{R}^d$, since only the lower-left quadrant of y is dominated by y according to this order. Consequently, two vectors can be incomparable for this setting. In addition, if y and z belong to $\underline{\mathrm{VaR}}_\alpha(G)$ and y' is in the lower-left quadrant of y, then y and z are equally risky and y is riskier than y'. However, according to this definition of the Value at Risk, it is not enough for asserting that z is riskier than y', if y' is not in the lower-left quadrant of z. For these reasons, another definition of the multivariate Value at Risk inducing a total order of $\mathbb{R}^d$ would be preferable. This is the aim of this paper.

For this purpose, we introduce a new definition of the multivariate Value at Risk, as an alternative to the orthant VaRs, based on another interpretation of what a multivariate Value at Risk should be. We call it the Kendall Value at Risk since it uses the Kendall distribution function: $K : t \in [0,1] \mapsto \mathbb{P}[G(X) \leq t]$, where G is the multivariate probability distribution of the random vector X. Then, the Kendall Value at Risk[3] is $\mathrm{VaR}^K_\alpha(G) = \partial\{y \in \mathbb{R}^d | K(G(y)) \geq \alpha\}$. We will see to which extent it differs from $\underline{\mathrm{VaR}}_\alpha(G)$ and $\overline{\mathrm{VaR}}_\alpha(G)$: in particular, the Kendall VaR is less conservative

[2] In this paper, variables in capital letters are random variables, whereas deterministic variables are in lower case.

[3] We define as well another Kendall Value at Risk similarly to the upper-orthant Value at Risk, as $\overline{\mathrm{VaR}}^K_\alpha(G) = \partial\{y \in \mathbb{R}^d | \overline{K}(\overline{G}(y)) \leq 1 - \alpha\}$, where $\overline{K} : t \in [0,1] \mapsto \mathbb{P}[\overline{G}(X) \leq t]$. We will show that for not too restrictive assumptions, $\overline{\mathrm{VaR}}^K_\alpha(G) = \mathrm{VaR}^K_\alpha(G)$.

than the lower-orthant VaR and more conservative than the upper-orthant VaR. More precisely, our approach unifies the concepts of lower-orthant and upper-orthant VaRs. Both orthant VaRs are in fact bounds of the "true" VaR. These bounds are based on a partial order and nothing indicates which orthant VaR should be prefered. The Kendall VaR provides a rigorous way of defining a consensus for the "true" VaR between both orthant bounds.

The approach of Embrechts and Puccetti [14] has been used as such in the literature [11], or has presented some modifications. For example, Cousin and Di Bernardino [12] have replaced both the sets defined by the lower-orthant and the upper-orthant Value at Risk by their expected value, so that the Value at Risk is only defined by a simple vector instead of an infinite set of vectors. Similarly to the unidimensional Value at Risk, this approach has the advantage to define a total order, but only for $\mathrm{VaR}^{-1}([0,1])$, which is a subset of $\mathbb{R}^d$. The methodological choice of Cousin and Di Bernardino makes it impossible to compare some pairs of vectors: it is hence relevant in the case of the orthant Value at Risk but not in our framework, where total order plays a significant role.

Our total-order multivariate Value at Risk is also linked to alternative definitions of multivariate quantile, which are not used in finance, to our knowledge, such as the quantile surface. Ordering real univariate data is quite easy. However, when dealing with multivariate data, no consensus arises about what should order statistics and quantiles be. In particular, the question of quantiles of multivariate distributions has led to numerous interpretations, often inspired by analogies with different ways of defining the quantiles of a univariate distribution. These interpretations are also thought to fit with a particular framework. In our case, the random vector from which we want to define a quantile is a vector of absolute or relative losses. Such quantiles must enable to define a risk measure, as does the VaR in the univariate case. But the definition of a multivariate quantile will induce some particularities, which are worth noting:

(P1) Contrarily to what happens in the univariate case, a multivariate quantile does not correspond to only one value of the random vector but to a set of values. In addition, when dealing with multivariate quantiles, magnitude is not the only feature to be considered. The direction of vectors is indeed important and several directions are possible. For example, for a bivariate quantile, we may choose the four following directions: $(-1,-1)'$, $(-1,1)'$, $(1,1)'$ and $(1,-1)'$ [4].

(P2) The notion of aggregation has to be questioned, since aggregation smooths micro-structure of losses. Should the vectors of relative losses $(-1\%, 1\%, -1\%, 2\%, -1\%)'$ and $(0\%, 0\%, 0\%, 0\%, 0\%)'$ be aggregated in 0%, and therefore be considered as equal-risk vectors? Or is it possible to have a vectorial interpretation of losses, that is not to be limited to the sum of the coordinates?

(P3) As there is no consensus about what a multidimensional quantile is, several approaches can be considered. However, a good method must enable an easy comparison of d-dimensional vectors in order to determine which of two vectors is riskier. If not, the *quantile* can lead to probabilistic misinterpretations, mainly if $d \geq 2$.

In Section 2, we present various interpretations of multivariate quantiles. In Section 3, we focus on the new Value at Risk that we introduce in this paper: the Kendall Value at Risk. We compare it theoretically with lower- and upper-orthant VaRs. In Section 4, we present the results of an application to simulated data. Section 5 concludes.

2 Multivariate quantiles

Among the possible definitions of a multivariate quantile, we can cite the following ones. Further details can be found in the survey of Serfling [38].

1. *Spatial quantile.* This approach was introduced by Abdous and Theodorescu [1]. They extended to the multivariate case the fact that the p-quantile of a univariate random variable X is any solution of the minimization problem [15]:

$$\inf_{y \in \mathbb{R}} \mathbb{E} \left\{ \frac{|X-y| + (2p-1)(X-y)}{2} - \frac{|X| + (2p-1)X}{2} \right\}. \tag{1}$$

Then, they define a p-quantile of a real multivariate random variable $X = (X_1, ..., X_d)'$ as any solution of the minimization problem:

$$\inf_{y \in \mathbb{R}^d} \mathbb{E} \left\{ \|X - y\|_{n,p} - \|X\|_{n,p} \right\}, \tag{2}$$

where $\|.\|_{n,p}$ has some norm-like properties, while not being a norm, and is defined by:

$$\|X\|_{n,p} = \left(\sum_{i=1}^{d} \left| \frac{|X_i| + (2p-1)X_i}{2} \right|^n \right)^{1/n}.$$

In the context of Value at Risk, such a quantile definition has the advantage of being different from a simple aggregation. However, the interpretation of what the quantile means in terms of ordering losses does not seem straightforward. In addition, the calculation takes much time if the dimension is high because of the intricate estimation of multidimensional densities. This method has been applied for quantile prediction and Value at Risk for financial data [22]. However, this method leads to a probabilistic misinterpretation (P3).

2. *Geometric quantile.* The multidimensional extension of equation (1) in equation (2) can lead to another interpretation than the spatial quantile. In the geometric quantile [8], the probability p is not a simple real number but a vector, $\tilde{p}$, so that the quantity $\tilde{u} = 2\tilde{p} - 1$ belongs to the unit ball $B^d = \{u | u \in \mathbb{R}^d, |u| < 1\}$, which is a direct extension of the unidimensional case $u = 2p - 1 \in (-1, 1)$. As a consequence, the u-quantile, for $u \in B^d$, is any solution of the minimization problem:

$$\inf_{y \in \mathbb{R}^d} \mathbb{E} \left\{ \|X - y\|_u - \|X\|_u \right\},$$

where $\|.\|_u$ is defined by:

$$\|X\|_u = \|X\| + \langle u, X \rangle,$$

with the help of the Euclidean inner product $\langle ., . \rangle$ and Euclidean norm $\|.\|$. Then, two readings of the minimization problem are possible. On the one hand, we get rid of the difficult interpretation of a multidimensional probability, $\tilde{p}$, by only considering its magnitude. In this case, the p-quantile , where $p \in (0, 1)$, is described by the set of geometric u-quantiles, where $u \in B^d$ and $|u| = p$. But this center-outward approach is not well adapted to our financial risk-measure framework. On the other hand, we can favour a particular direction and choose a single vector of norm 1, $u^\star$. It represents either the ideal asset return or the most feared loss. Then, a p-quantile can be defined as a geometric $pu^\star$-quantile. The latter interpretation allows easier ordering of the vectors than the spatial quantile does. However, the same curse of dimensionality and probabilistic misinterpretation (P3) as in the spatial-quantile approach arise. We find several applications of this method in the literature, for example in medicine [9] or in ecology [7].

3. *Quantiles based on inversions of mappings.* In the unidimensional framework, a quantile is defined as the generalized inverse of the cumulated distribution function. If one defines a mapping G from $\mathbb{R}^d$ to $\mathbb{R}^d$ or from $\mathbb{R}^d$ to $\mathbb{R}$, then inversions can also define a quantile [30]. For some choice of G, probabilistic misinterpretations (P3) may arise. This method has been applied to Value at Risk by Embrecht and Puccetti [14] as cited above. For example, if the copula and therefore the joint distribution of the coordinates is known, the function G for the p-quantile $y = (y_1, ..., y_d) \in \mathbb{R}^d$ is simply defined as:

$$G(y) = \mathbb{P}(X_1 \leq y_1, ..., X_d \leq y_d) = p,$$

where $X = (X_1, ..., X_d)'$ is a real random vector. It has to be noted that, in this copula-inversion-based approach, the probability associated to a quantile consists in the weight of the lower-left quadrant of the quantile. Then, for two distinct vectors y and z having the same image p by G, the lower-left quadrants are different and the probabilistic interpretation of what a Value at Risk is becomes complicated. Indeed, the union of both lower-left quadrants provide less risky vectors than y and z, but its probability measure is strictly higher than p.

4. *Center-outward quantile surface.* If one is given a statistical depth function[4], the center of the distribution is defined as the maximal-depth vector[5]. Then, a natural ordering arises based on the depth. Thus, quantiles are concentric regions around the center [32, 40]. More precisely, given a probability $p \in (0, 1)$, the p-quantile of a distribution G is the set of vectors of depth α_p, which is defined such that the probability associated to G of the region of vectors of depth D higher than α_p is p: $p = \mathbb{P}_G(D(X, G) \geq \alpha_p)$, where X is a random vector of distribution G. Thanks to this definition, no probabilistic misinterpretation (P3) is possible. It is an advantage on the methods previously presented. However, this center-outward approach is not adapted to our financial risk-measure purpose. Indeed, with such a method, a vector containing only negative coordinates may be in the same quantile as a vector containing only positive coordinates, whereas in finance the direction of the vectors matters (P1).

5. *Modified quantile surface.* We propose to modify the center-outward quantile surface so as to focus rather on gains and losses than on the center of the distribution. Instead of determining first a spatial median, we associate a metric for each vector. Vectors with the same metric are gathered in an equivalence class: they are supposed to be equivalently risky. We can then compare each pair of classes and define an order resulting in the definition of a multivariate probability distribution and thus of quantiles. We propose two methods to achieve this goal, depending on the definition of the equivalence classes.

   ▷ We choose a vector $u^\star$ corresponding to the ideal gain, such that, in the empirical distribution, no vector of returns will provide higher utility to the portfolio manager than $u^\star$. We then define a function of dissimilarity with the ideal gain. Contrarily to a depth function, a dissimilarity function gives the lowest value to the vector of interest, $u^\star$. For example, we can use, $d^M(y, G, u^\star)$, the squared Mahalanobis distance to $u^\star$ for a vector y with respect to the multivariate probability distribution G, defined by

$$d^M(y, G, u^\star) = (y - u^\star)' \Sigma_G^{-1} (y - u^\star),$$

---

[4] For example, if we consider the so-called likelihood depth, the depth function is simply the probability density [16]. Otherwise, the Mahalanobis depth [33], $D^M(y, G)$, of a vector y with respect to a multivariate probability distribution G is defined by

$$D^M(y, G) = \left[1 + (y - \mu_G)' \Sigma_G^{-1} (y - \mu_G)\right]^{-1},$$

where μ_G and Σ_G are respectively the mean vector and the covariance matrix of a random vector of distribution G.

[5] The center is also sometimes called median of the distribution, but it is quite different from a statistical median in the sense of a 0.5-quantile.

where Σ_G is the covariance matrix of a multivariate random variable of probability distribution G.

▷ The Kendall probability distribution

$$K : t \in [0,1] \mapsto \mathbb{P}[G(X) \leq t],$$

where G is the multivariate probability distribution of the random vector X, defines natural equivalence classes [34]. If vectors y and y' are such that $G(y) = G(y')$, then y is as risky as y' and these vectors are riskier than every vector y'' such that $G(y'') < G(y)$. Contrarily to the orthant Value at Risk, we affirm that y is a vector of the set of the Value at Risk of probability $K(G(y))$, rather than of probability $G(y)$, which is lower than $K(G(y))$ by construction. We base our new definition on the Kendall stochastic ordering [34] instead of the traditional product ordering. The first one is a total order, whereas the second one is only partial. An explanatory illustration is provided in Figure 1.

Then, whatever the methodological choice for the equivalence classes, the definition of the quantiles follows the center-outward quantile-surface method, which provides a consistent probabilistic interpretation (P3).

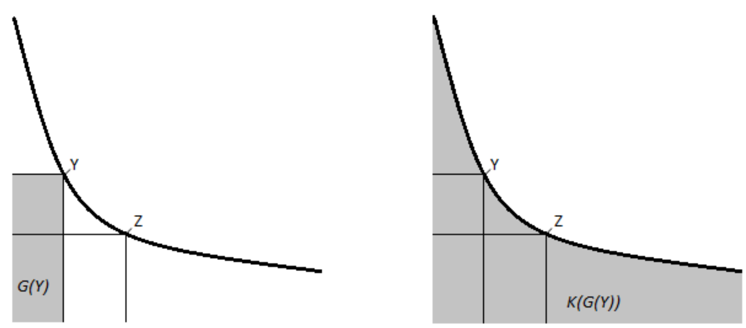

Figure 1: On the left, the thick line is a set of two-dimensional vectors having the same lower-left cumulated probability. In particular, $G(y) = G(z)$, which is the probability measure of the lower-left quadrant of y or z. However, some vectors of the lower-left quadrant of z are not in the lower-left quadrant of y and therefore cannot be compared to y. On the right, the multivariate probability distribution only leads to the definition of equivalence classes. Therefore, every vector in the grey zone is less risky than every vector on the thick line. The vectors dominated by z are the same than those dominated by y. The probability associated with y and z is therefore the probability measure of all the grey zone, that is $K(G(y))$, which is equal to $K(G(z))$ and which is greater than $G(y)$.

Among all the presented notions of multivariate quantile, the modified quantile surface is the most relevant one in the perspective of Value at Risk. In particular, the modified quantile surface based on Kendall function is interesting since it is a refinement of the orthant Value at Risk. We call it *Kendall Value at Risk*. We make the choice to focus on the Kendall Value at Risk in the rest of the paper.

3 The Kendall Value at Risk

3.1 Definition

As introduced above, the Kendall distribution function is $K : t \in [0,1] \mapsto \mathbb{P}[G(X) \leq t]$, where G is the multivariate probability distribution of the random vector X, associated to a given copula. The Kendall function has been used for example to estimate Archimedean copulas [19], or for creating hierarchical Kendall copulas, which deal with high-dimension problems [5]. Thanks to this function, we define the Kendall Value at Risk.

Definition 1. *For a random vector X of dimension d, the Kendall Value at Risk of probability $\alpha \in [0,1]$, denoted VaR^K_α, is the boundary set of the set of vectors $y \in \mathbb{R}^d$ such that $K(G(y)) \geq \alpha$, where G is the multivariate distribution of X and K the corresponding Kendall function:*

$$\mathrm{VaR}^K_\alpha(G) = \partial\{y \in \mathbb{R}^d | K(G(y)) \geq \alpha\}.$$

Similarly to the distinction between lower-orthant and upper-orthant Value at Risk, we can make a distinction between two kinds of Kendall Value at Risk, based either on the multivariate distribution function G or the corresponding survival function $\bar{G}$.

Definition 2. *For a random vector X of dimension d, the survival Kendall Value at Risk of probability $\alpha \in [0,1]$, denoted $\overline{\mathrm{VaR}}^K_\alpha$, is the boundary set of the set of vectors $y \in \mathbb{R}^d$ such that $\overline{K}(\overline{G}(y)) \leq 1-\alpha$, where $\overline{G}$ is the survival function associated to the multivariate distribution G of X and $\overline{K}$ is the Kendall function[6] corresponding to $\overline{G}$, that is $\overline{K} : t \in [0,1] \mapsto \mathbb{P}[\overline{G}(X) \leq t]$:*

$$\overline{\mathrm{VaR}}^K_\alpha(G) = \partial\{y \in \mathbb{R}^d | \overline{K}(\overline{G}(y)) \leq 1-\alpha\}.$$

In fact these different definitions of a multivariate Value at Risk are linked, as exposed in the following proposition. Indeed, when G has advantageous albeit not too restrictive features, both definitions are strictly equivalent.

Proposition 1. *Let $\alpha \in [0,1]$, G be a non-atomic multivariate distribution function of dimension $d \in \mathbb{N}$ having a density function whose support is $\mathbb{R}^d$, with K the Kendall function and $\overline{G}$ the survival distribution, both associated to G. Then:*

$$\overline{\mathrm{VaR}}^K_\alpha(G) = \mathrm{VaR}^K_\alpha(G).$$

Proof. If G is non-atomic and if the support of the density function is $\mathbb{R}^d$, then $K \circ G$ is strictly monotonic and continuous. Thus

$$\mathrm{VaR}^K_\alpha(G) = \partial\{y \in \mathbb{R}^d | K(G(y)) \geq \alpha\} = \{y \in \mathbb{R}^d | K(G(y)) = \alpha\}.$$

Moreover, with the same assumptions, $\overline{K} \circ \overline{G}$ is strictly monotonic and continuous as well and is equal to $1 - K \circ G$. As a consequence:

$$\begin{aligned}
\overline{\mathrm{VaR}}^K_\alpha(G) &= \partial\{y \in \mathbb{R}^d | \overline{K}(\overline{G}(y)) \leq 1-\alpha\} \\
&= \{y \in \mathbb{R}^d | \overline{K}(\overline{G}(y)) = 1-\alpha\} \\
&= \{y \in \mathbb{R}^d | 1 - K(G(y)) = 1-\alpha\} \\
&= \mathrm{VaR}^K_\alpha(G).
\end{aligned}$$

□

[6] We stress the fact that $\bar{K}$ is not the survival Kendall function associated to G but the standard Kendall function associated to $\bar{G}$.

For clarity, we focus on the first definition of the Kendall VaR, but Proposition 1 allows generalizing the following results to the survival Kendall VaR in many cases. In particular, in the simulations made at the end of the paper, in which the marginal distributions are lognormal and the copula is a Gumbel or a Clayton, both definitions of Kendall VaR are identical. For this reason, we can write with no ambiguity "*the* Kendall VaR". Incidentally, the orthant VaRs based on G and $\overline{G}$ do not have this property and we must mention which of the upper- and lower-orthant VaRs is used. The nature of the difference between upper- and lower-orthant VaRs, compared to the similitude between both Kendall VaRs, is illustrated in Figure 2, where this fact is related to a problem of cover of $\mathbb{R}^d$, when $d = 2$.

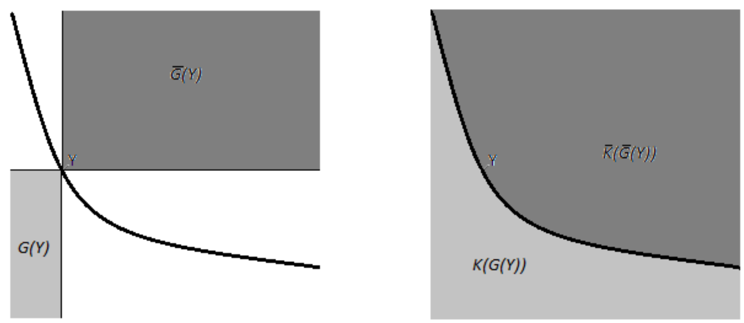

Figure 2: On the left, the thick line is a set of two-dimensional vectors having the same lower-left cumulated probability. In particular, $G(y)$ is the probability measure of the lower-left quadrant of y. The probability associated to the lower-orthant (respectively upper-orthant) VaR is the measure of the light-grey zone (resp. of the complement set of the dark-grey zone). On the right, the probability associated with y in the Kendall approach is the probability measure of all the light-grey zone under the equivalence class, that is $K(G(y))$. It is also equal to the measure of the complement set of the dark-grey zone, which is the probability associated to to the survival Kendall VaR. Moreover, in this example, $\{K(G(y)), \overline{K}(\overline{G}(y))\}$ is a cover of $\mathbb{R}^2$, whereas $\{G(y), \overline{G}(y)\}$ is not.

3.2 Kendall Value at Risk for an Archimedean copula

In this section, we assume that the multivariate distribution of the random vector X of dimension d is provided by an Archimedean copula C of generator ϕ:

$$C : (u_1, ..., u_d) \in [0,1]^d \mapsto \phi^{-1}\left(\sum_{j=1}^{d} \phi(u_j)\right).$$

It is a wide class of copulas which includes the following copulas: independent, Gumbel, Clayton, Frank, Joe and Ali-Mikhail-Haq, among others. Moreover, this framework leads to simple expressions for the Kendall function, so that it is an interesting illustration to our theory.[7] We make some assumptions on ϕ:

- $\triangleright$ $\phi : (0,1] \to [0, \infty)$,
- $\triangleright$ $\phi(1) = 0$,

[7] It is known that Archimedean copulas can be difficult to use in high dimension for estimation purpose, nevertheless the vine approach permits to bypass this problem. Vine copulas are indeed based on nested bivariate copulas instead of a sole high-dimension copula [10, 23, 26, 27]. Statistical selection techniques may help to truncate the vine so as to reduce the dimension of the problem in a relevant way [6].

▷ $(-1)^i(\phi^{-1})^{(i)}(x) > 0$ for all $1 \leq i \leq d$ and all $x \geq 0$,

▷ $\lim_{t\to 0^+} \phi(t)^i(\phi^{-1})^{(i)}(\phi(t)) = 0$ for all $1 \leq i \leq d-1$.

Then, the Kendall distribution function can be derived as:

$$K : t \in (0,1] \mapsto t + \sum_{i=1}^{d-1} \frac{(-\phi(t))^i}{i!}(\phi^{-1})^{(i)}(\phi(t)), \quad (3)$$

where $f^{(i)}$ denotes the i-th derivative of f [3, 20]. We now apply this formula in two examples.

Example 1. *The Gumbel copula is an Archimedean copula of parameter $\theta \geq 1$, generated by the function*

$$\phi : t \mapsto (-\log(t))^\theta.$$

When $\theta = 1$, the Gumbel copula is equal to the independent copula. The inverse generator is $\phi^{-1}(x) = \exp(-x^{1/\theta})$. According to equation (3), if we consider the bivariate case, the Kendall function is:

$$K : t \mapsto t - \frac{t\log(t)}{\theta}.$$

We show in Figure 3 how the Kendall function behaves when θ changes: the greater θ, the closer the Kendall function and the identity. In particular, when θ tends toward infinity, K converges toward the identity, so that the Kendall VaR and the lower-orthant VaR are equal in this limit case.

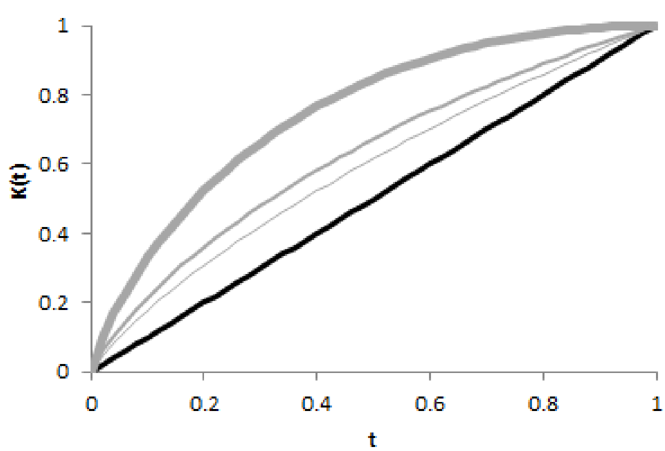

Figure 3: Kendall function (in grey) of the Gumbel copula for $\theta = 1$ (thick line), $\theta = 2$ (medium line) and $\theta = 3$ (thin line). The greater the difference between the Kendall function and the identity (in black, corresponding to $\theta \to \infty$), the greater the difference between the Kendall VaR and the lower-orthant VaR.

Example 2. *The independent copula leads to easy formulas in higher dimension. It is a particular case of the Gumbel copula with $\theta = 1$. According to equation (3), for a dimension $d \geq 2$, we get the following formula for K:*

$$K : t \mapsto t\left(1 + \sum_{i=1}^{d-1} \frac{(-\log(t))^i}{i!}\right).$$

When d goes to infinity, $\sum_{i=1}^{d-1} \frac{(-\log(t))^i}{i!}$ tends toward $-1 + e^{-\log(t)} = -1 + 1/t$, for every $t \neq 0$. Therefore, the limit behaviour of K, for $d \to \infty$ is a discontinuous function, equal to 0 for $t = 0$ and equal to 1 everywhere else. It leads to the maximal difference possible between the Kendall VaR and the lower-orthant VaR. We see in Figure 4 the Kendall function for various values of d.

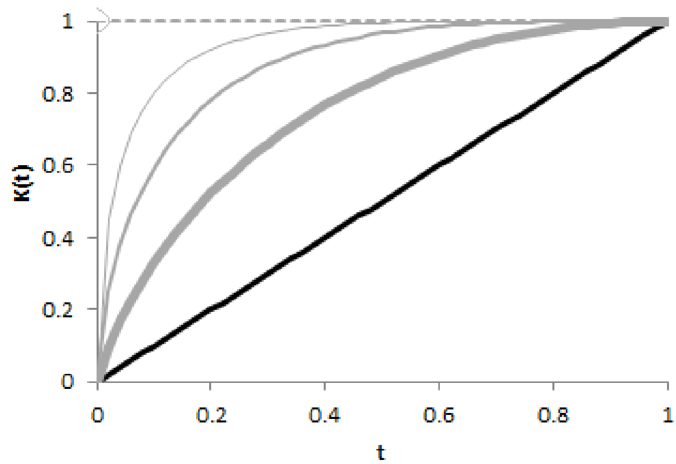

Figure 4: Kendall function (in grey) of the independent copula for $d = 2$ (thick line), $d = 3$ (medium line), $d = 4$ (thin line) and the limit case $d \to \infty$ (dotted line). The greater the difference between the Kendall function and the identity (in black), the greater the difference between the Kendall VaR and the lower-orthant VaR.

3.3 Properties

In this section, the copulas are not assumed to be Archimedean anymore.

It is well known that the univariate Value at Risk is – in general – not a coherent risk measure since it is not sub-additive [2, 28]. Nevertheless, as this property depends on the underlying distribution, sub-additivity can arise [13, 24, 25]. Thus, the absence of sub-additivity can be a limitation for the use of the VaR in certain cases. With the multivariate approach, the question is opened. A definition of what a consistent risk measure is in the multivariate case can be provided based on the choice of a partial order [29]. As the question of multivariate sub-additivity is open and beyond this article's scope, we think that it is here more fruitful to focus on other more specific properties of the Kendall VaR. In particular, it is interesting to understand the difference between the Kendall Value at Risk and the lower-orthant Value at Risk.

As we mentioned, the probability associated to a vector by the Kendall function is higher (respectively lower) than in the approach of the lower-orthant (resp. upper-orthant) Value at Risk. It can be shown by the Fréchet-Hoeffding bounds that $t \leq K(t) \leq 1$ [19] and that $t \leq \overline{K}(t) \leq 1$ as well. For a vector y and a multivariate probability distribution G, the lower-orthant approach links y to the level of probability $G(y)$, whereas the Kendall approach associates it to a probability $K(G(y))$, which is therefore in the interval $[G(y), 1]$. In other words, the Kendall VaR is less conservative than the lower-orthant VaR. We can compare both VaRs in the following manner:

Proposition 2. *Let K be strictly monotonic on a neighbourhood of a given probability $\alpha \in [0, 1]$. Then, the Kendall Value at Risk and the lower-orthant Value at Risk are linked by:*

$$\underline{\mathrm{VaR}}_{\alpha}(G) = \mathrm{VaR}^{K}_{K(\alpha)}(G).$$

Proof. Let $A = \{y \in \mathbb{R}^d | G(y) \geq \alpha\}$ and $B = \{y \in \mathbb{R}^d | K(G(y)) \geq K(\alpha)\}$.

- ▷ K is an increasing function, since it is a probability distribution function. Therefore, it comes immediately that $A \subset B$.
- ▷ The reciprocal inclusion does not hold in general. However, with the assumption of strict monotonicity of K in a neighbourhood V_α of α, the restriction of K to V_α is invertible. Let $y \in B$ and $\alpha' \in V_\alpha$ such that $\alpha' < \alpha$ (it does not exist if $\alpha = 0$ but this case is besides trivial), so that $K(\alpha') < K(\alpha)$. Let assume $y \notin A$. Then $G(y) < \alpha$. Two cases arise. First, if $G(y) \leq \alpha'$, then $K(G(y)) \leq K(\alpha') < K(\alpha)$, what is contradictory with the assumption $y \in B$. Second, if $G(y) \in (\alpha', \alpha)$, then $G(y)$ is in V_α, so that $K(G(y)) < K(\alpha)$: the contradiction also holds. Therefore the assumption $y \notin A$ was absurd and we can conclude that $B \subset A$.
- ▷ Finally $A = B$.

As a consequence, when considering the definition of both VaRs, we get:

$$\begin{aligned} \underline{\mathrm{VaR}}_\alpha(G) &= \partial\{y \in \mathbb{R}^d | G(y) \geq \alpha\} \\ &= \partial\{y \in \mathbb{R}^d | K(G(y)) \geq K(\alpha)\} \\ &= \mathrm{VaR}^K_{K(\alpha)}(G). \end{aligned}$$

□

Similarly, we can show that the survival Kendall VaR (which is often equal to the standard Kendall VaR as stated in Proposition 1.) is more conservative than the upper-orthant VaR. It is the meaning of the next proposition, since $\overline{K}$ being a growing function we have $1 - \overline{K}(1-\alpha) \leq \alpha$.

Proposition 3. *Let K be strictly monotonic on a neighbourhood of a given probability $\alpha \in [0,1]$. Then, the survival Kendall Value at Risk and the upper-orthant Value at Risk are linked by:*

$$\overline{\mathrm{VaR}}_\alpha(G) = \overline{\mathrm{VaR}}^K_{1-\overline{K}(1-\alpha)}(G).$$

The proof is similar to the one of Proposition 2. The message conveyed by both Propositions 2 and 3 is that the Kendal VaR is a compromise between both orthant VaRs: it is more conservative than one and less conservative than the other.

The interesting metric for comparing the Kendall VaR and the lower-orthant VaR is given by the positive function $r : \alpha \in [0,1] \to K(\alpha) - \alpha$. It is the difference of risk associated to a same vector, between the Kendall VaR and the lower-orthant VaR, for a given level of risk. In other words, for a probability α, $\underline{\mathrm{VaR}}_\alpha(G)$ is a set of vectors corresponding to this risk α; for the same set of vectors, the Kendall Value at Risk associates another level of risk, which is $K(\alpha)$ according to Proposition 2; $r(\alpha)$ denotes this difference of the estimated risk. For example, the Gumbel copula in example 1 leads to $r(\alpha) = -\frac{\alpha \log(\alpha)}{\theta}$. More generally, r can be linked to the Kendall rank correlation coefficient, known as Kendall's tau coefficient.

Proposition 4. *The average difference between the probabilities associated to the Kendall function and to the sole copula, for d-dimensional vectors and a continuous copula, is:*

$$\int_0^1 r(\alpha)d\alpha = (1-\tau)\left(\frac{1}{2} - \frac{1}{2^d}\right),$$

where τ is the Kendall rank correlation coefficient.

Proof. Kendall's tau and the Kendall function are linked by the following relation, for a continuous copula [20]:

$$\tau = \frac{2^d - 1 - 2^d \int_0^1 K(\alpha)d\alpha}{2^{d-1} - 1}.$$

Therefore:

$$\begin{array}{rcl} \int_0^1 r(\alpha)d\alpha & = & \int_0^1 (K(\alpha) - \alpha)d\alpha \\ & = & \frac{2^d-1-(2^{d-1}-1)\tau}{2^d} - \frac{1}{2} \\ & = & (1-\tau)\left(\frac{1}{2} - \frac{1}{2^d}\right). \end{array}$$

□

In the bivariate case, this average difference is $(1-\tau)/4$, which belongs to $[0, 1/2]$ due to the fact that $\tau \in [-1, 1]$. When d tends toward infinity, the average difference increases concomitantly with the dimension d, up to $(1-\tau)/2 \in [0, 1]$. The case of the independent copula, for which $\tau = 0$, leads to an average r of $(1/2) - (1/2)^d$, whose value, $1/4$ for $d = 2$, progressively increases with the dimension up to $1/2$. It confirms the analysis made in example 2. If we consider comonotonic coordinates, then $\tau = 1$ and the average r is equal to zero, whatever the dimension d. Graphically, it corresponds to a case where all the vectors dominated by a reference vector belong to the lower-left quadrant of this reference vector: the order implied by the orthant VaRs, which is partial in general, is total in this particular case and there is no difference between the orthant and the Kendall VaRs. On the opposite, if the coordinates are countermonotonic, then $\tau = -1$ and the average r reaches its maximum: $1 - (1/2)^{d-1}$, which goes from $1/2$, for $d = 2$, to 1, when d goes toward infinity.

We can quantify as well the difference between the probability associated to a vector by the upper-orthant method and by the survival Kendall method: $\overline{r} : \alpha \in [0, 1] \to \alpha - (1 - \overline{K}(1-\alpha))$, which is a positive function. Proposition 5 states that the average twist of the risk level between the upper-orthant VaR and the survival Kendall VaR is, in absolute value, exactly the same as the average twist between the lower-orthant VaR and the standard Kendall VaR. In the framework of Proposition 1 where the survival and the standard Kendall VaRs are equal, the unified Kendall VaR can thus be seen as a balanced compromise between lower- and upper-orthant VaRs, as it twists as much, in absolute value, the risk associated to both in average over all the possible risk levels. Nevertheless, for a particular level of risk, the Kendall VaR can be closer to the one or to the other.

Proposition 5. *The average difference between the probabilities associated to the sole survival copula and to the Kendall function of the survival copula, for d-dimensional vectors and a continuous copula, is:*

$$\int_0^1 \overline{r}(\alpha)d\alpha = (1-\tau)\left(\frac{1}{2} - \frac{1}{2^d}\right),$$

where τ is the Kendall rank correlation coefficient.

Proof. By a change of variable, we have:

$$\begin{array}{rcl} \int_0^1 \overline{r}(\alpha)d\alpha & = & \int_0^1 (\overline{K}(\alpha) - \alpha)d\alpha \\ & = & \int_0^1 \overline{K}(\alpha)d\alpha - \frac{1}{2}. \end{array}$$

Moreover, what we note $\overline{K}$ is, according to Definition 2, the Kendall function corresponding to the survival distribution function. It can thus be written in terms of the Kendall's tau of the survival copula, $\overline{\tau}$:

$$\int_0^1 \overline{K}(\alpha)d\alpha = \frac{2^d - 1 - (2^{d-1} - 1)\overline{\tau}}{2^d}.$$

Besides, we know that the Kendall's tau of the survival copula is equal to the Kendall's tau of the copula itself [21], so that $\overline{\tau} = \tau$. This leads immediately to the result stated in the proposition. $\square$

4 Application and Results

In this section we apply the methodology presented above and evaluate the Kendall VaR using various copulas. This section is divided in three subsections. The first illustrates the characteristics of the copulas we are working with. In a second subsection, we illustrate the probability transformation implied by the Kendall distribution. Finally, in a third subsection we present and compare orthant and Kendall VaRs.

Indeed, in order for us to properly analyse the results of the Kendall VaR, it is necessary to analyse them with respect to the form of the copula underlying the Kendall distribution used to transform the percentile. Note that in the following applications we focus on Archimedean copulas, as their properties are of particular interest; for instance, these are non-linear, may capture different dependence patterns and may behave as extreme-value copulas.

4.1 Archimedean Copula

In this first section we briefly illustrate the characteristics of Archimedean copulas, in particular their non-linearity. This non-linearity is visible for the four bivariate Archimedean copulas presented in Figure 5, for instance the Gumbel which is upper-tail dependent, i.e. extreme positive events have a tendency to occur simultaneously while others are independent, the Clayton copula which is lower-tail dependent, i.e. extreme negative events have a tendency to occur simultaneously while others are independent, the Frank copula, which is more body-centered, i.e. events present in the body are more dependent than those present in the tails, and the Joe copula, which is also upper-tail dependent but with a dependency magnitude far steeper than in the Gumbel copula case.

Figures 6 and 7 illustrate both the Clayton and Gumbel copulas density and distribution functions exhibiting asymmetric behavior. The asymmetry is of particular importance to capture asymmetric shocks in financial markets as we will demonstrate in a companion paper.

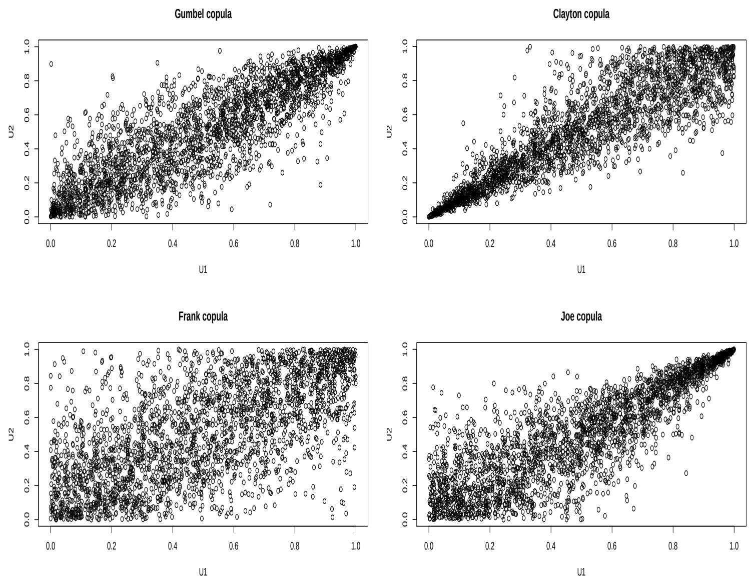

Figure 5: Archimedean copula illustrations: This figure represents four Archimedean copulas, the Gumbel which is upper-tail dependent, the Clayton which is lower-tail dependent, the Frank copula which is more body centered and the Joe which is also upper-tail dependent.

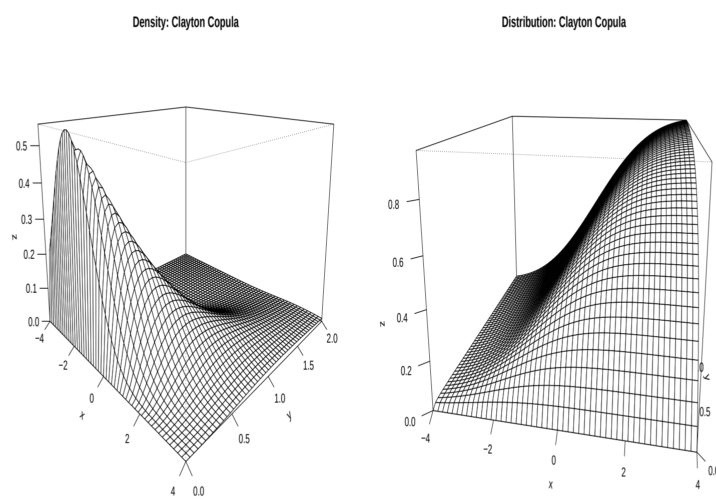

Figure 6: This figure presents the density and distribution functions of a Clayton copula whose margins are a Gaussian distribution N(0,2) and an exponential distribution with parameter 2. We observe that the functions are asymmetric.

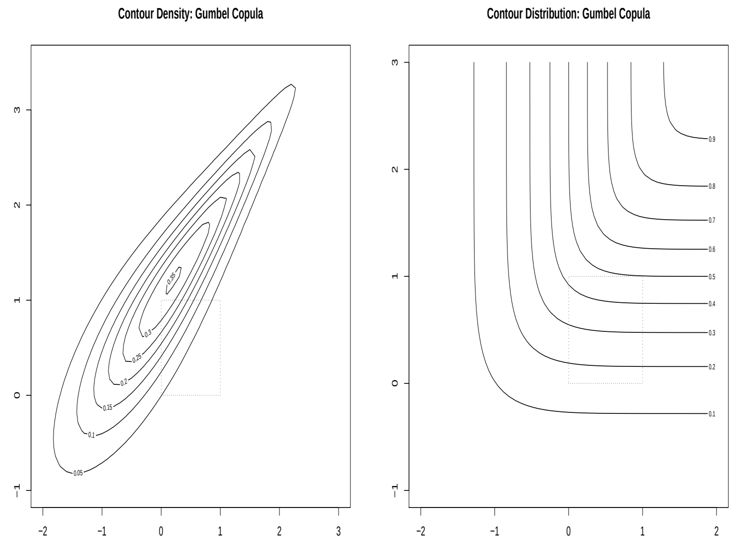

Figure 7: This figure presents the density and distribution functions of a Gumbel copula whose margins are Gaussian, N(0,1) and N(1,1) respectively. Here, a contour function has been used to transform the figure in three dimensions in two. Note that the distribution function on the right is a first representation of the lower-orthant quantiles presented below.

4.2 Kendall Distribution

In Section 3.1, we introduce the Kendall distribution which relies on an underlying copula. Figure 8 shows how the Kendall distribution evolves with the type of copula we are using. Each Kendall distribution link corresponds to a set of percentiles and the bissector represents the percentiles of the orthant approach. It is interesting to note that the shape of the Kendall distribution obtained with a particular copula is consistent with the scatter plots of the copulas presented in Figure 5. In other words, if the copula captures an upper-tail dependance behaviour, the Kendall distribution inflexion point is located in the left tail of the distribution (see Kendall distribution relying on a Gumbel or a Joe copula). However, if the copula captures a lower-tail dependance behaviour, the Kendall distribution inflexion point is located in the right tail of the distribution (see Kendall distribution relying on a Clayton copula). The Kendall transformation breaks the linearity provides inside the the orthant approach.

Our interpretation of the behaviour of the Kendall distribution is that it captures the properties of the copula in its construction and subsequently in the calculation of the corresponding quantile while the standard orthant methodology loses a part of the information calculating the quantiles, as the orthant methodology partially suppress the effect of the non-linear dependency.

In Figure 9, we present the dynamic twist of Kendall distributions built using the four copulas presented in Figure 5. Here, the parameters of the four copulas evolved from 1 to 25, and the Kendall distributions obtained are respectively plotted on four figures. As discussed earlier, we see that the Kendall distribution corresponds to a probability twist.

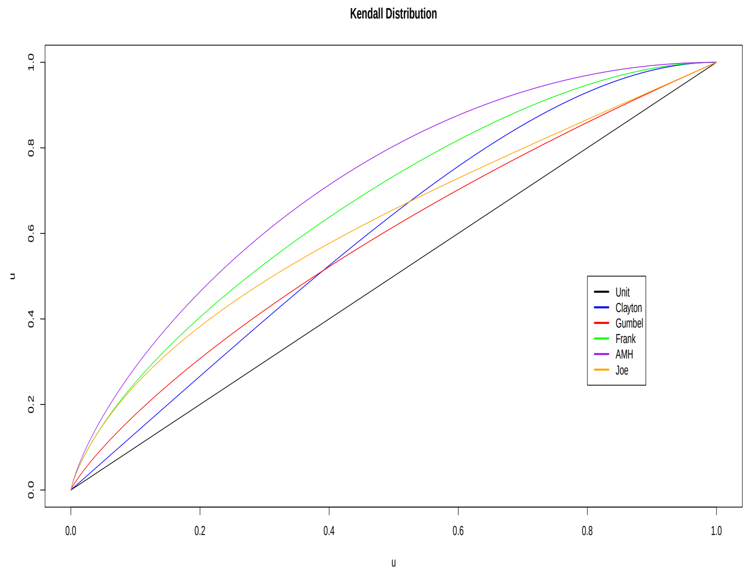

Figure 8: This figure presents the various Kendall distributions forms obtained using multiple Archimedean copulas, for instance the Clayton, the Gumbel, the Frank, the Ali-Mikhail-Haq and the Joe copulas. The transformations are compared to the bisector of the unit square representing the percentile of the orthant approach.

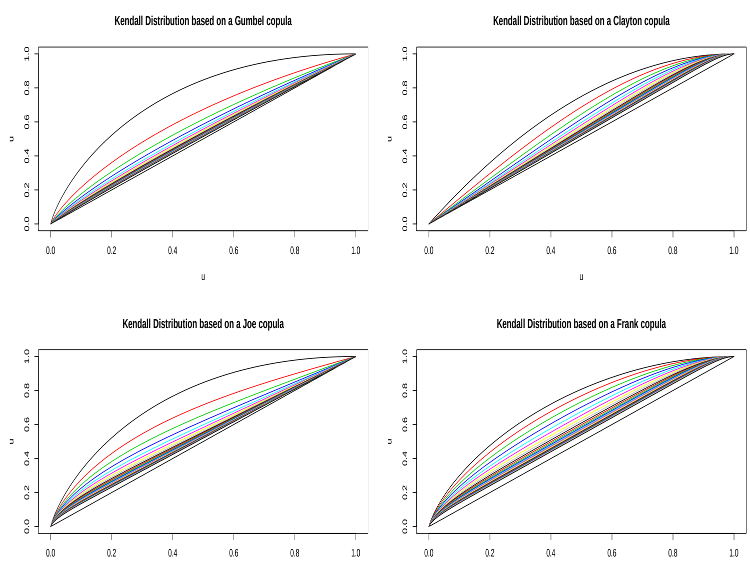

Figure 9: This figure presents the Kendall distributions obtained considering four Archimedean copulas, for instance Clayton, Gumbel, Frank and Joe. The parameters of each copula are slowly varying from 1 to 25. The transformation are compared to the bisector of the unit square representative of the percentile of orthant approach.

4.3 Multivariate VaR representation and calculation

In this section, we compare and discuss the sets of VaRs obtained from both the lower-orthant strategy and the Kendall approach. Recall that multivariate VaRs will be represented by vectors. To initiate our experimentation, we build two copula functions, for instance the Clayton and the Gumbel copulas, with parameters equal to 3. Both have been constructed using two lognormal marginal distributions, with the following sets of parameters ($\mu = 5$, $\sigma = 2$) and ($\mu = 8$, $\sigma = 1.2$).

Figures 10[8] and 11 represent the lower-orthant VaRs obtained using the Clayton copula presented in the previous paragraph. Lower-orthant VaRs are obtained calculating all the combinations of all pairs of margins providing an identical bivariate probability.

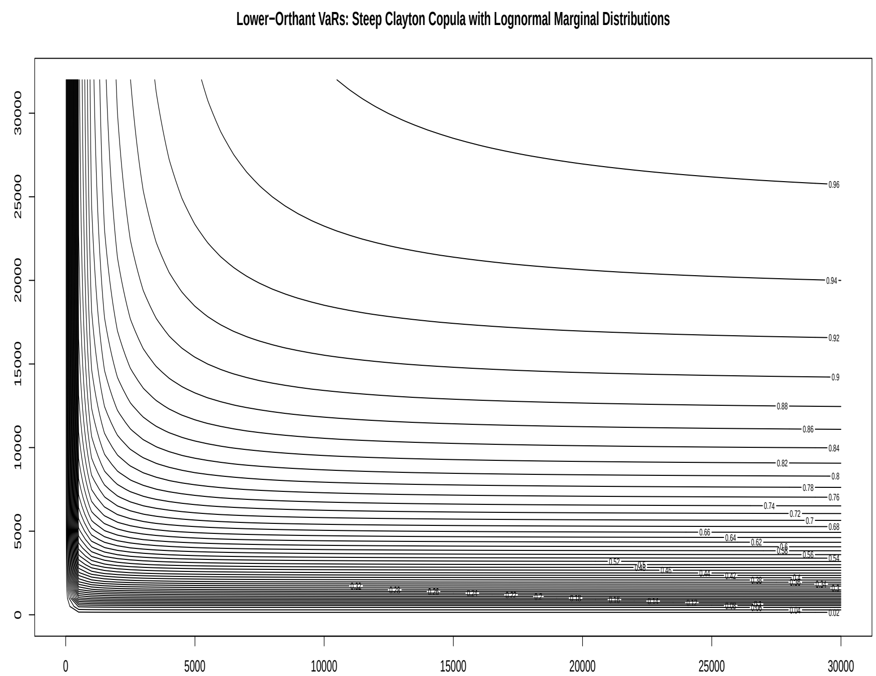

Figure 10: This figure presents the quantile function of a steep Clayton copula with two different lognormal marginal distributions. Here each quantile is represented by a vector. Each axis represents the projection of the multivariate quantile on each marginal distribution. Each line of the graph represent a percentile, going from 0 to 96%.

As analyzed in the previous subsection, the Kendall distribution transforms the natural probabilities taking into account the shape of the copula. This transformation allows us to calculate in a similar fashion the lower-orthant VaR, the Kendall VaR transforming the lower-orthant percentile into the Kendall one.

Figures 12, 13 and 14 compare lower-orthant, upper-orthant and Kendall VaRs using the Clayton and the Gumbel copulas built as explained in the first paragraph of this subsection. In these figures,

[8]We have observed that the result is similar with a lot of copulas.

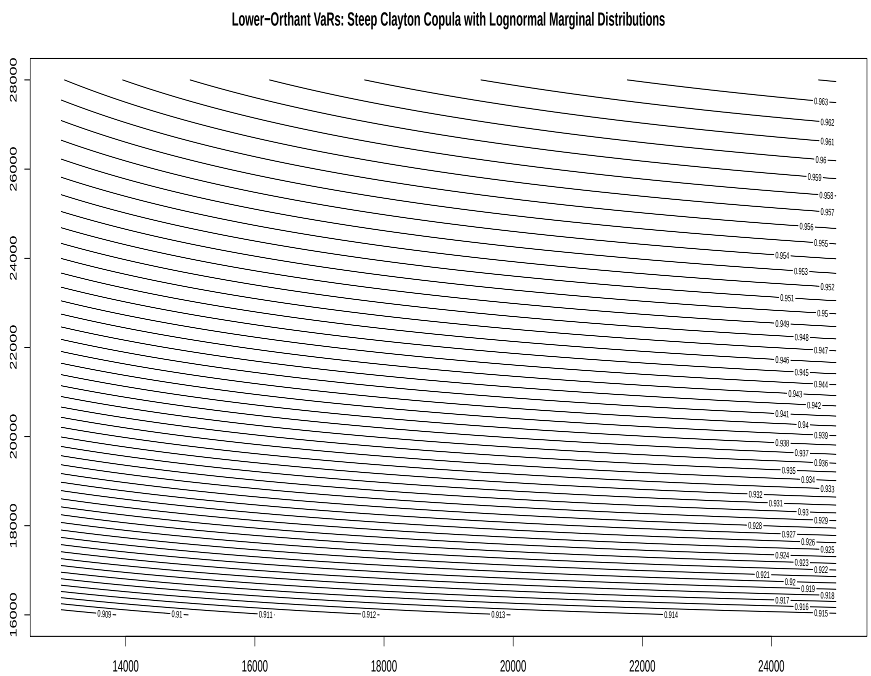

Figure 11: This figure presents the quantile function of a steep Clayton copula with two different lognormal marginal distributions. Here each quantile is represented by a vector. In this figure, we focus on the highest quantiles. Each axis represents the projection of the multivariate quantile on each marginal distribution. Each line of the graph represent a percentile, going from 90.9% to 96.4%.

the dotted lines represent the Kendall VaR, the continuous lines located above[9] the dotted lines represent the lower-orthant VaR, and the continuous lines located below, the upper-orthant VaR. On each figure, the VaRs are given at the same percentile, but we see that the Kendall VaR is not equivalent to the lower-orthant as the Kendall distributions twist the probabilities. As a result, in our case the lower-orthant equivalent of the Kendall VaRs given a percentile α is much lower than the lower-orthant ones obtained at the same percentile α. On the contrary, upper-orthant VaRs are much less conservative. Indeed, Figures 12, 13 and 14 compare lower-orthant VaRs obtained for α respectively equal to 86%, 98% and 71% (approximately) with their Kendall equivalent, i.e. for $K(\alpha)$ also respectively equal to 86%, 98% and 71%. To obtain the previous $K(\alpha)$, α has to respectively be equal to 80%, 89% and 56%. Besides, Figures 12 and 13 provide a comparison of the Kendall VaR respectively obtained for $K(\alpha)$ equal to 62% and 65%, with upper-orthant VaRs obtained for $1-\alpha$ respectively equal to 38% and 35%. Figure 13 provides upper-orthant, Kendall and lower-orthant VaR on the same graph, but here the upper-orthant is obtained for $1-\alpha$ equal to 29%.

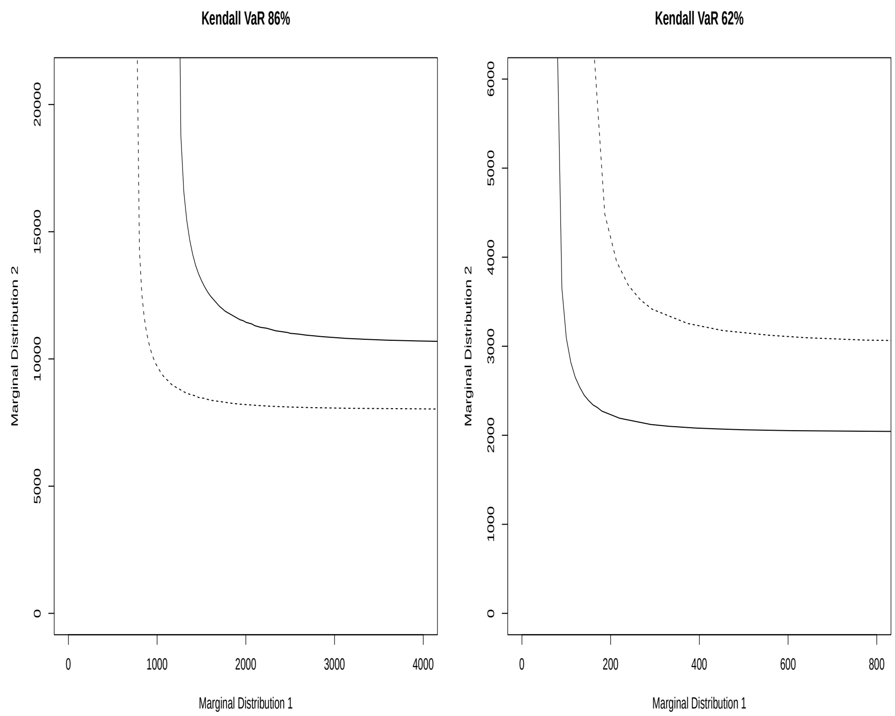

Figure 12: This figure represents the Kendall VaR obtained using a Gumbel copula with two different lognormal marginal distributions. The figure on the left compares the 86% Kendall VaR to its lower-orthant equivalent. We observe that the Kendall VaR is inferior to the lower-orthant one. The figure on the right compares the 62% Kendall VaR to the upper-orthant VaR computed from the survival copula of the Gumbel. Here, we see that the Kendall VaR is superior to the upper-orthant one.

With respect to the marginal distribution selected and the copulas chosen, the difference between the lower-orthant quantile and the Kendall one is not linear, i.e. the quantile may be closer to each

[9] The comparison between these sets of vectors must be understood in the sense of the lexicographical order for each pair of vectors.

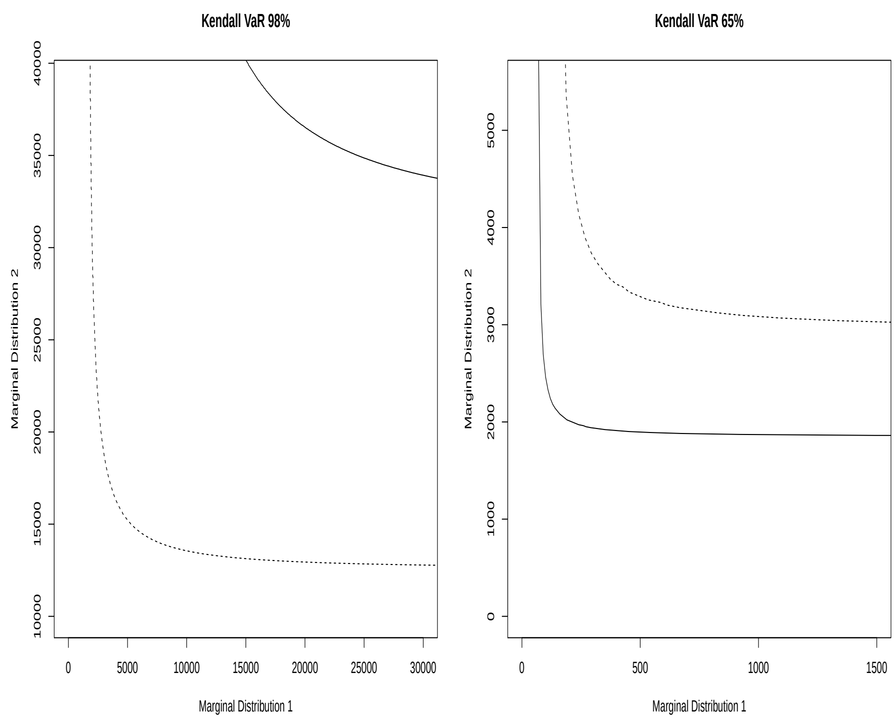

Figure 13: This figure represents the Kendall VaR obtained using a Clayton copula with two different lognormal marginal distributions. The figure on the left compares the 98% Kendall VaR to its lower-orthant equivalent. We observe that the Kendall VaR is inferior to the lower-orthant one. The figure on the right compares the 65% Kendall VaR to the upper-orthant VaR computed from the survival copula of the Clayton. Here, we see that the Kendall VaR is superior to the upper-orthant one.

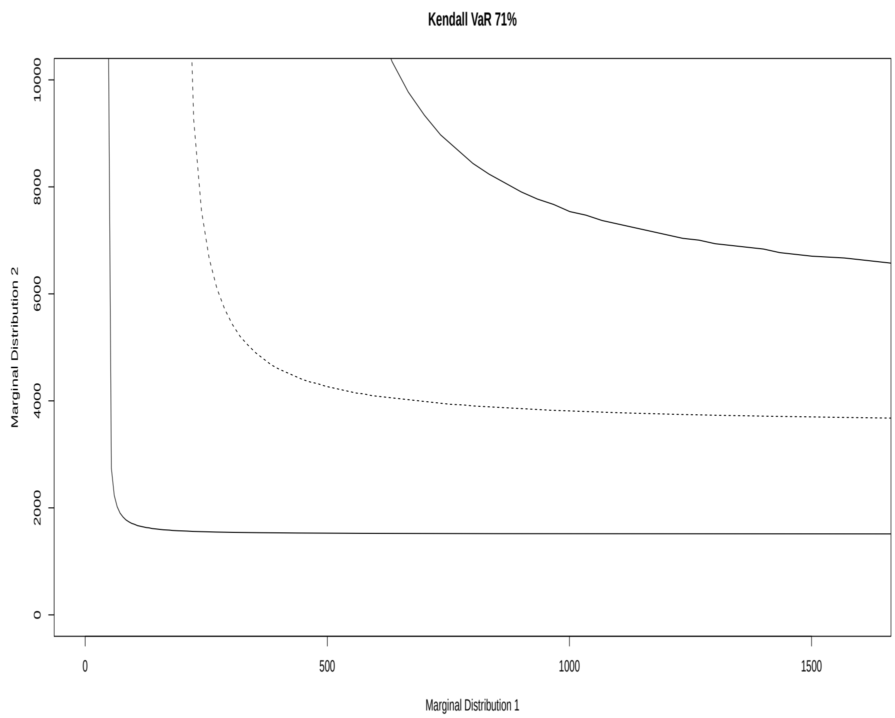

Figure 14: This figure represents the Kendall VaR obtained using a Clayton copula with two different lognormal marginal distributions. The figure compares the 71% Kendall VaR to its lower-orthant equivalent and to the upper-orthant obtained using the survival copula of the Clayton used here. We observe that the Kendall VaR is inferior to the lower-orthant one and superior to the upper-orthant.

other depending on the dependence structure selected (both in terms of copula and parameters). This statement is consistent with what we observed in Figure 8.

The next statement is valid for both the Kendall and the orthant VaRs. On the last five figures presented, for instance Figures 10, 11, 12, 13 and 14, it is very important to note that taking each point of any multivariate quantile represented by the lines on the figures above and summing each couple of projections, this sum is not constant as it evolves along the line. This phenomenon is a consequence of the non-linearity of the dependence structure captured by the copula used. Indeed, if we consider the Gumbel copula which is upper tail dependent, (i.e. extreme positive events have a tendency to occur simultaneously), the bivariate VaR at a given percentile (or multivariate VaR in a dimension superior to 2) is represented by a matrix where each line represents different combinations having the same probability[10] but not necessarily the same projected values. In other words, for a given percentile, a bivariate VaR can be represented by a very large projection on a first marginal distribution axis and a very low one on the second marginal distribution axis, or two projections of approximately the same magnitude, but the sum of these two different combinations will be completely different. This particular phenomenon will have to be bore in mind in case either of the methodologies proposed here is implemented. Indeed, if we take the example of the use of multivariate risk measures for financial institutions' regulatory capital, then these methodologies will lead to both capital calculation and allocation issues as all the combination obtained at a given percentile are equally valid.

5 Conclusion

In this paper, introducing the Kendall VaR, the issues related to the formalisation and the calculation of a multivariate VaR have been addressed. Indeed, the Kendall VaR allows the calculation of a multivariate VaR using the probability transformation implied by the Kendall distribution. For instance, the Kendall distribution captures the intrinsic characteristics of the dependence architecture represented by the selected copula (non-linearity, upper- or lower-tail dependence etc.) and transfers it in one dimension. Therefore the Kendall distribution allows operating a percentile transformation. The Kendall VaR is more conservative than the upper-orthant VaR and less conservative than the lower-orthant VaR.

In a first part of the paper, after introducing this new approach to build a mutivariate risk measure, that we call the Kendall VaR, we show that it is equivalent to work, inside this framework and with a non-atomic $\mathbb{R}^d$-supported density function, with a multivariate distribution G or its survival representation. Then we provide a simple relationship between the Kendall VaR and the orthant VaRs, which allows us defining the Kendall VaR as a unique compromise between the bounds represented by both orthant VaRs. Finally we quantify, the difference between the Kendall VaR and the orthant VaRs.

In a second part, we illustrate, with different examples the fact that for any multivariate distribution based on a copula, the Kendall VaR can be constructed and plotted gathering all the sets of values of identical probability.

Besides, the Kendall VaR created is compared to the orthant VaRs. The probability of the latter has not suffered any modifications and therefore partially loses the information captured by the copula. Indeed, the Kendall VaR is more appropriate than the orthant VaRs when copulas cap-

[10] A point of each line.

turing non-linear dependencies are used, as the Kendall VaR does not lose any information with respect to the property of interest. We notice that the Kendall VaR and the orthant VaRs are equal in case of comonotonicity.

Building and comparing the two approaches, we observed that the non-linearity of the copulas implies that the sums of each set representing a given percentile are not constant. This phenomenon will have an important impact if any of these methodologies is used within financial institutions (for instance banks or insurance companies), as if these approaches are used to evaluate the diversified capital pertaining to the various risks faced by them, the accurate value of the capital as well as the allocation of this one will be problematic. Indeed, multiple sets of values will be representative of the same level of risks going from one end to the other. In terms of applications, this result provides a variety of possible interpretations, it will be the purpose of a companion paper.

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