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Abstract

The paper defines a family of nested non-cooperative simultaneous finite games to study coalition structure formation with intra and inter-coalition externalities. Every game has two outcomes - an allocation of players over coalitions and a payoff profile for every player.

Every game in the family has an equilibrium in mixed strategies. The equilibrium can generate more than one coalition with a presence of intra and inter group externalities. These properties make it

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different from the Shapley value, strong Nash, coalition-proof equilibrium, core, kernel, nucleolus. The paper demonstrates some applications: non-cooperative cooperation, Bayesian game, stochastic games and construction of a non-cooperative criterion of coalition structure stability for studying focal points. An example demonstrates that a payoff profile in the Prisoners' Dilemma is non-informative to deduce a cooperation of players.

Keywords: Noncooperative Games
JEL : C71, C72, C73

1 Introduction

There is a conventional dichotomy for game theory applications, divided between a cooperative game theory (CGT) and a non-cooperative game theory (NGT). CGT deals with coalitions as elementary items, NGT deals with strategic individual behavior.

However, the picture is not so smooth. CGT traditionally disregards issues of strategic interactions between individual players, between a player in a coalition and from players in other coalitions. CGT substitutes an individual gain by a value of a coalition, an aggregate gain for everybody, without an explicit individual motivation to obtain it. The consequence is that a value of a coalition does not depend on an environment, i.e. on the whole coalition structure or on allocation of all other players over other coalitions.

From another side NGT with an individual impact on an equilibrium overlooks the structural aspects of player's partition into groups or into coalition structures.[1]

The two theories even seek different goals, their qualities of outcomes are constructed differently and are not-well compatible. CGT concentrates

[1]Coalition structure terminology was used by Aumann and Dreze (1976).

on socially desirable allocations of players with efficient outcomes, while efficiency is defined in aggregate terms, not in explicit individually based, as in the standard definition of Pareto-efficiency. At the same time NGT urges towards equilibrium conditions, which later can/may be Pareto-improved. Thus it is not surprising that there is a neutral zone between CGT and NGT.

This paper concentrates on games, where in an equilibrium players can be allocated in different coalitions, still preserving traditional for NGT strategy and payoff profiles.

Consider two similar examples: a voluntarily division of a group of participants into paintball teams and a voluntarily division of a class into studying groups. In both every a player / a student makes a decision from self-interest considerations, concerning with whom she is and how all others are allocated. Every player makes a decision what to do after she is engaged into a team/ a group, including free-riding. In both cases there are commonly accepted mechanisms, which resolve conflicts between players/students, in which team/group to be. After a team/ a group is formed there is no need that everybody in the a team / a group do the same.

Existence of intra and inter coalition externalities for paintball is clear. Existence of intra-coalition externalities in studying groups is also clear. Inter-coalition externalities for studying groups could be a mutual noise or a slow WiFi connection from an external service.

So the common features of the examples are multi-coalition framework, formed from self-interest behavior of everybody and a presence of two types of externalities. Thus both examples occur in an area between the existing cooperative and non-cooperative game theories.

In this paper "coalition structure", or "partition"[2] for short, is a collection of non-overlapping subsets from a set of players, which in a union make the original set. A group, or a coalition, is an element of a coalition structure or

[2]Existing literature uses both terms.

of a partition.[3]

Nash (1953) suggested that cooperation should be studied within a group and in terms of non-cooperative fundamentals. This suggestion is known now as the Nash Program. Cooperative behavior was understood as an activity inside a group with positive externalities between players.

The best analogy for the difference between Nash Program and the current research is the difference between a partial and a general strategic equilibrium analysis[4] in economics. The former isolates a market and ignores cross-market interactions, the latter explicitly studies cross market interactions from individual strategic actions of self-interested traders.

There are few key points in the suggested model. The first is to expand an individual strategy set to incorporate coalition structures into it. The second is to embed an exogenous mechanism to resolve conflicts between individual choices of coalition structures. The third is a way how to parametrize the final construction by a number of potential deviators.

The resulting contributions of this paper can be summarize as: a construction of a *family* of non-cooperative games and a parametrization of all constructed games by a number of deviators. Every game has an equilibrium in mixed strategies. This allows to reveal new properties of well-known games and to ask new research questions.

Relation of the current approach to existing literature is in Discussion section of the paper.

The paper has the following structure: Section 2 presents an example based on Prisoner's Dilemma and demonstrates that there is actually no explicit cooperation in it. Section 3 contains a formal definition of a sufficient criterion for cooperation, Section 4 contains a discussion of the result and a comparison with existing literature. All further sections deal with reconsidering well-known games and demonstrating their new properties. All the

[3]the same.

[4]meaning, strategic market games of Shapley and Shubik

rest sections present different applications, including a construction of non-cooperative criterion of coalition structure stability and reconsidering focal points of Shelling. The last section is a conclusion.

2 Modified Prisoner's Dilemma (PD)

The example demonstrate that there is no explicit cooperation in the standard PD game . Cooperation as a unique and efficient outcome, when both players are in one coalition requiries additional changes for the game, presented further in this section.

Consider a game of two players $i = 1, 2$, when each can choose only to be alone and has two strategies for this case, L(ow) and H(high). Unless other specified the players can make only one coalition structure $\{\{1\}, \{2\}\}$. Payoffs for this case are in Table 1.

Every cell contains a payoff profile for both players and their allocation over coalitions, i.e. a coalition structure. The unique equilibrium, marked as *, is inefficient. Desirable outcome is $(0; 0)$.

Table 1: Payoff for the standard Prisoners Dilemman

	$L_{2,alone}$	$H_{2,alone}$
$L_{1,alone}$	$\underline{(0;0)}$ $\{\{1\}, \{2\}\}$	(-5;3) $\{\{1\}, \{2\}\}$
$H_{1,alone}$	(3;-5) $\{\{1\}, \{2\}\}$	$(-2; -2)^*$ $\{\{1\}, \{2\}\}$

Let players can choose to be either alone or together, and to have the same set of strategies for every case. Then i has a set of strategies

$$S_i(K = 2) = \{(L_{1,alone}, H_{1,alone}), (L_{1,together}, H_{1,together})\}.$$

Parameter $K = 2$ indicates that a maximum coalition size is 2. It is natural to assume that if at least one of players chooses to be alone, then a coalition of two can not be formed. In other words, the grand coalition of two can be formed only is both choose it. Payoff matrix for this game is in Table 2.

Table 2: Payoff for an extension of Prisoner's Dilemma.

	$L_{2,P_{separ}}$	$H_{2,P_{separ}}$	$L_{2,P_{joint}}$	$H_{2,P_{joint}}$
$L_{1,P_{separ}}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$
$H_{1,P_{separ}}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{*,**}$ $\{\{1\},\{2\}\}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{**}$ $\{\{1\},\{2\}\}$
$L_{1,P_{joint}}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0;0)}$ $\{1,2\}$	$(-5;3)$ $\{1,2\}$
$H_{1,P_{joint}}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{**}$ $\{\{1\},\{2\}\}$	$(3;-5)$ $\{1,2\}$	$(-2;-2)^{**}$ $\{1,2\}$

Every cell in Table 2 contains a payoff profile and a coalition structure. The part of it which corresponds to strategies to be alone coincides with Table 1. The previous equilibrium is marked with *, the newly appeared with **. It is clear that all new equilibria are inefficient, but they belong to different coalition structures.

In the coalition structure $\{\{1\},\{2\}\}$ the players have inter coalition externalities, in the coalition structure $\{1,2\}$, the grand coalition, there are intra-coalition externalities. But payoff profiles are the same in every case.

The game has the important property: an increase in a number of possible coalition structures enriches a set of equilibria. of the game.

Table 2 contains 4 efficient outcomes, located in different coalition structures. Thus only by observing payoff profiles we can not deduce players are in one coalition or not. This differs from the understanding that the Prisoner's Dilemma is an example of inefficiency in cooperation.

An allocation of players in one coalition can be reinstalled with "extrovert" players. Let the same players have preferences over coalition structures and prefer to be together. Let $\epsilon > 0$ is a private corporate gain if the grand coalition $\{1, 2\}$ is formed. Table 3 contains payoffs for such game.

Table 3: Payoff for two extrovert players who, obtain additional payoffs ϵ being in one coalition P_{joint}, when it is realized. Uniqueness of an equilibrium is reinstalled.

	$L_{2,alone}$	$H_{2,alone}$	$L_{2,together}$	$H_{2,together}$
$L_{1,alone}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$
$H_{1,alone}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^*$ $\{\{1\},\{2\}\}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)$ $\{\{1\},\{2\}\}$
$L_{1,together}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0+\epsilon;0+\epsilon)}$ $\{1,2\}$	$(-5+\epsilon;3+\epsilon)$ $\{1,2\}$
$H_{1,together}$	(3;-5) $\{\{1\},\{2\}\}$	(-2;-2) $\{\{1\},\{2\}\}$	$(3+\epsilon;-5+\epsilon)$ $\{1,2\}$	$(-2+\epsilon;-2+\epsilon)^{**}$ $\{1,2\}$

The equilibrium marked with ** is unique and appears when players choose to be together. But still, it is not efficient for the grand coalition. It is important that preferences over coalition structures and a rule to form coalitions change the former equilibrium.

If both players are "introverts", then an equilibrium does not change. Let both players are "introverts" and prefer to be alone. Each has additional markup $\delta > 0$. Payoff matrix for such game is in Table 4.

Consider a case, where players are different: $i = 1$ is extrovert, but $i = 2$ is "introvert". Table 5 is the payoff matrix for this game. If players can not choose a coalition structure, then an equilibrium is the same as above. However, if there is a option to be together this makes a difference for $i = 1$. His equilibrium strategy profile is a mixed strategy with equal weights over two pure strategies $H_{1,alone}$ and $H_{1,together}$. Final coalition structure does not change from this, players have inter-coalition externalities and inefficient out-

Table 4: Payoff for two extrovert players who, obtain additional payoffs ϵ being in one coalition, if it is realized. Uniqueness of an equilibrium is reinstalled.

	$L_{2,alone}$	$H_{2,alone}$	$L_{2,together}$	$H_{2,together}$
$L_{1,alone}$	$(0+\delta; 0+\delta)$ $\{\{1\},\{2\}\}$	$(-5+\delta; 3+\delta)$ $\{\{1\},\{2\}\}$	$(0;0)$ $\{\{1\},\{2\}\}$	$(-5;3)$ $\{\{1\},\{2\}\}$
$H_{1,alone}$	$(3+\delta; -5+\delta)$ $\{\{1\},\{2\}\}$	$(-2+\delta; -2+\delta)^{*,**}$ $\{\{1\},\{2\}\}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)$ $\{\{1\},\{2\}\}$
$L_{1,together}$	(0;0) $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$(0;0)$ $\{1,2\}$	$(-5;3)$ $\{1,2\}$
$H_{1,together}$	(3;-5) $\{\{1\},\{2\}\}$	(-2;-2) $\{\{1\},\{2\}\}$	$(3;-5)$ $\{1,2\}$	$(-2;-2)$ $\{1,2\}$

come. Thus inter-coalition externalities and two coalitions in an equilibrium exist due to mixed strategies.

3 Formal setup of the model

Nash (1950, 1951) suggested a non-cooperative game which consists of a set of players N, with a general element i, sets of individual finite strategies $S_i, i \in N$, and payoffs, defined as a mapping from a set of all strategies into a set of payoff profiles of all players, $\Big(U_i(s)\Big)_{i\in N}$, such that $S = \times_{i\in N} S_i \mapsto \Big(U_i(s)\Big)_{i\in N} \subset \mathbb{R}^{\#N}$, where $\Big(U_i(s)\Big)_{i\in N} < \infty, \forall s \in S$.

The suggested game modifies the mapping by preliminary portioning S into coalition structure specific domains and assigning payoffs for every point in these domains. Every domain is a set of strategy profiles of all players allocated in a coalition structure.

Let there is a set of agents N, with a general element i, a size of N is $\#N$, a finite integer, $2 \leq \#N < \infty$.

There is a parameter K, it can vary from 1 upto $\#N$. It has two interpretations. Let for N agents there is a coalition with a maximum size K. Then no more than K agents are required to dissolve it. The reverse is also true: we need no more than K agents to form this coalition. Closeness of a construction of the object under investigation requires these two simultaneous interpretations for K be equivalent. Thus K is a maximum allowed coalition size.

Every value of K from the set $\{1, \ldots, \#N\}$ induces a family of coalition structures (or a family of partitions) $\mathcal{P}(K)$ over the set of all players N with a general element P:

$$\mathcal{P}(K) = \Big\{P\colon P = \{g_j\colon g_j \subset N; \#g \leq K; \cup_j g_j = N; \forall j_1 \neq j_2 \Rightarrow g_{j_1} \cap g_{j_2} = \emptyset\Big\}.$$

An element P of $\mathcal{P}(K)$ is a partition of players into coalitions, $P = \{g_j\}$, where g_j is a coalition. Every coalition g_j has a size (a number of members) no bigger than K, but can be less. The condition $j_1 \neq j_2 \Rightarrow g_{j_1} \cap g_{j_2} = \emptyset$ is interpreted as that an agent can participate only in one coalition.

If we increase K by one, then we need to add new coalition structures from $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$. This makes families of partitions for different K be nested: $\mathcal{P}(K=1) \subset \ldots \subset \mathcal{P}(K) \subset \ldots \subset \mathcal{P}(K=N)$. The bigger is K, the more coalition structures (or partitions) are involved into consideration. The grand coalition, i.e. a coalition of size N, which includes all players, is present only in the family $\mathcal{P}(K=N)$.

For every partition P an agent i has a finite strategy set $S_i(P)$.[5] It may consist of one strategy. A game is not trivial for $2 \leq K \leq N$. In the dinner game a player had only one strategy per a coalition structure.

A set of strategies of agent i for a family of coalition structures $\mathcal{P}(K)$ is

$$S_i(K) = \Big\{s_i(K)\colon s_i(K) \in \{S_i(P)\colon P \in \mathcal{P}(K)\}\Big\}$$

[5]Finite strategies are as in Nash (1950).

with a general element $s_i(K)$. For a given K an agent chooses $s_i(K)$ from $S_i(K)$. All choices are performed simultaneously. *A choice of $s_i(K)$ means a choice of a desirable partition and an action for this partition.*[6] If we increase K by one, then we need to construct additional strategies only for the newly available coalition structures from $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$. This makes strategy sets for different K be nested: $S_i(K=1) \subset \ldots \subset S_i(K) \subset \ldots \subset S_i(K=N)$.

The set of strategies of all players for a fixed K is $S(K) = \times_{i \in N} S_i(K)$, a direct (Cartesian) product of individual strategy sets of all players for the given K. A choice of all players $s(K) = \Big(s_i(K), \ldots, s_N(K)\Big)$ is a point in $S(K)$. For simplicity if there is no ambiguity we will write $s = \Big(s_1, \ldots, s_N\Big) \equiv s(K) = \Big(s_i(K), \ldots, s_N(K)\Big)$. It is clear that an increase in K induces nested strategy sets: $S(K=1) \subset \ldots \subset S(K=N)$.

For every K we define a coalition structure formation mechanism $\mathcal{R}(K)$, (a mechanism or a rule for short). It partitions a set of strategies $S(K)$ into coalition structure specific domains $S(P) = \cup_{i \in N} S_i(P)$:

$$S(K) = \cup_{P \in \mathcal{P}(K)} S(P) = \cup_{P \in \mathcal{P}(K)} \cup_{i \in N} S_i(P).$$

The same in other words, every strategy profile $s = (s_1, \ldots, s_N) \in S(K)$ is assigned only one coalition structure P, $P \in \mathcal{P}(K)$. In the example above we have seen that $S(P)$ does not necessarily be convex. Every coalition structure becomes a standard non-cooperative game: for every P there is a set of players, a non-trivial set of strategies and partition specific payoffs over it's strategy set, defined below.

Definition 1. *For every K* ***a coalition structure formation mechanism*** *$\mathcal{R}(K)$ is a set of measurable mappings such that:*

1. *A domain of $\mathcal{R}(K)$ is a set of all strategy profiles of $S(K)$.*

[6]A desirable partition may not realize due to conflicts in individual choices. A coalition structure formation mechanism resolves conflicts of partition choices between players.

2. *A range of $\mathcal{R}(K)$ is a finite number of subsets $S(P) \subset S(K)$, $P \in \mathcal{P}(K)$. Every $S(P)$ is a strategy set for a coalition structure P.*

3. *$\mathcal{R}(K)$ divides $S(K)$ into coalition structure specific strategy sets, such that the union of all $S(P)$ makes the original set $S(K) = \cup_{P\in\mathcal{P}(K)} S(P) = \cup_{P\in\mathcal{P}(K)} \cup_{i\in N} S_i(P)$.*

4. *Two different coalition structures, $\bar{P}$ and $\tilde{P}$, $\bar{P} \neq \tilde{P}$, have different coalition structure strategy sets $S(\bar{P}) \cap S(\tilde{P}) = \emptyset$.*

Formally the same:

$$\mathcal{R}(K)\colon S(K) = \times_{i\in N} S_i(K) \mapsto\colon \begin{cases} S(K) = \cup_{P\in\mathcal{P}(K)} S(P), \\ \forall s = (s_1, \ldots, s_N) \in S(K) \\ \exists P \in \mathcal{P}(K)\colon s \in S(P), \\ \forall \bar{P}, \tilde{P} \in \mathcal{P}(K), \bar{P} \neq \tilde{P} \Rightarrow S(\bar{P}) \cap S(\tilde{P}) = \emptyset. \end{cases} .$$

Hence there are two ways to construct $S(K)$: in terms of initial individual strategies $S(K) = \times_{i\in N} S_i(K)$, and in terms of realized partition strategies $S(K) = \cup_{P\in\mathcal{P}(K)} S(P)$. Representation of $S(K)$ in terms of coalition structure specific strategy sets may not be a direct product of sets. This can be seen in the examples in the previous section.

If K increases we need to add only a mechanism for strategy sets from $S(K+1) \setminus S(K)$. This supports consistency of coalition structure formation mechanisms for different K. The family of mechanisms is nested: $\mathcal{R}(K = 1) \subset \ldots \subset \mathcal{R}(K) \subset \ldots \subset \mathcal{R}(K = N)$. We assume that a mechanism is given from outside. Discussion of some properties of $\mathcal{R}(K)$ can be found in the next section.

Payoffs in the game are defined like state-contingent payoffs (or payoffs of Arrow-Debreu securities) in finance. For every coalition structure P player i has a payoff function $U_i(P)\colon S(P) \to \mathbb{R}_+$, such that every set $U_i(P)$ for every

$i \in N$ is bounded, $U_i(P) < \infty$. Payoffs are considered as von Neumann-Morgenstern utilities. All payoffs of i for a game with no more than K deviators make a family: $\mathcal{U}_i(K) = \Big\{U_i(P) \colon P \in \mathcal{P}(K), U_i(P) < \infty\Big\}$. Every coalition structure has it's own set of strategies and a corresponding set of payoffs. Thus every coalition structure is a non-cooperative game.

An increase in K increases a number of possible partitions and a set of feasible strategies for every player. We need to add only payoffs for the partitions in $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$. Thus we obtain a nested family of payoff functions:

$$\mathcal{U}_i(K=1) \subset \ldots \subset \mathcal{U}_i(K) \subset \ldots \subset \mathcal{U}_i(K=N).$$

We can easily see that this construction of payoffs allows to obtain both intra and inter coalition (or group) externalities, as payoffs are defined directly over strategy profiles of all players and independently from allocation of players in coalition structures.

Definition 2 (**a simultaneous coalition structure formation game). ***A non-cooperative game for coalition structure formation with a maximum coalition size K is*

$$\Gamma(K) = \Big\langle N, \Big\{K, \mathcal{P}(K), \mathcal{R}(K)\Big\}, \Big(S_i(K), \mathcal{U}_i(K)\Big)_{i \in N}\Big\rangle,$$

where $\Big\{K, \mathcal{P}(K), \mathcal{R}(K)\Big\}$ - coalition structure formation mechanism (a social norm, a social institute), $\Big(S_i(K), \mathcal{U}_i(K)\Big)_{i \in N}$ - properties of players in N, (individual strategies and payoffs), such that:

$$\times_{i \in N} S_i(K) \overset{\mathcal{R}(K)}{\rightarrow} \Big\{S(P) \colon P \in \mathcal{P}(K)\Big\} \rightarrow \Big\{(\mathcal{U}_i(K))_{i \in N}\Big\}.$$

The novelty of the paper is an introduction of coalition structure formation mechanism, which portions the set of all strategies into non-cooperative partition-specific games. If we omit the mechanism part of the game and eliminate restriction on coalition sizes then we obtain the traditional non-cooperative game of Nash: $\times_{i\in N} S_i(K) \to \Big(U_i(s)\Big)_{i\in N}$. Construction of a game $\Gamma(K)$ makes every partition P be a non-cooperative game inside $\Gamma(K)$.

Another novelty of the paper is an introduction of nested games.

Definition 3 (**family of games**). *A family of games is* ***nested*** *if :*

$$\mathcal{G} = \Gamma(K=1) \subset \ldots \subset \Gamma(K) \subset \ldots \subset \Gamma(K=N).$$

The suggested games allow to study intra and inter coalition externalities for players and avoid coalition-as-one-unit approach of cooperative game theory, study individual motivations and individual payoff allocations.

Let $\Sigma_i(K)$ be a set of all mixed strategies of i, i.e. a space of probability measures over $S_i(K)$, $\Sigma_i(K) = \Big\{\sigma_i(K)\colon \int_{S_i(K)} d\sigma_i(K) = 1\Big\}$, with a general element $\sigma_i(K)$, where an integral is Lebegue-Stiltjes integral. Sets of mixed strategies for all other players are defined in the standard way $\Sigma_{-i}(K) = \Big\{\Big(\sigma_j(K)\Big)_{j\neq i} : \forall j \neq i \text{ there is } \int_{S_j(K)} d\sigma_j(K) = 1\Big\}$.

Expected utility can be defined in terms of strategies the players choose or in terms of final partition-specific strategies. Expected utility of i in terms of individual strategies is:

$$EU_i^{\Gamma(K)}\Big(\sigma_i(K), \sigma_{-i}(K)\Big) = \int_{S(K)=\times_{i\in N} S_i(K)} U_i(s_i, s_{-i}) d\sigma_i(K) d\sigma_{-i}(K)$$

or in terms of partition-specific strategies is

$$EU_i^{\Gamma(K)}\Big(\sigma_i(K), \sigma_{-i}(K)\Big) = \sum_{P\in\mathcal{P}(K)} \int_{S(P)} U_i\Big(S(P)\Big)(s_i, s_{-i}) d\sigma_i(K) d\sigma_{-i}(K).$$

Expected utilities are constructed in the standard way.

Definition 4 (**an equilibrium in a game** $\Gamma(K)$). *A mixed strategies profile* $\sigma^*(K) = (\sigma_i^*(K))_{i\in N}$ *is an equilibrium strategy profile for a game*

$\Gamma(K)$ *if for every* $i \in N$ *and every* $\sigma_i(K) \neq \sigma_i^*(K)$ *holds true the inequality:*

$$EU_i^{\Gamma(K)}\Big(\sigma_i^*(K), \sigma_{-i}^*(K)\Big) \geq EU_i^{\Gamma(K)}\Big(\sigma_i(K), \sigma_{-i}^*(K)\Big).$$

Equilibrium in the game $\Gamma(K)$ is defined in a standard way. It's existence is just an expansion of Nash theorem or an application of Banach-Shauder fixed point theorem. However this result for non-cooperative games with coalition structure formation is different from the results of cooperative games, where an equilibrium for a deviation of more than one player may not exist. For example, in coalition form games with empty cores. Another outcome of the model is that there is no need to introduce additional properties of games, like transferable/non-transferable utilities, axioms on a system of payoffs, super-additivity, or weights. The equilibrium also captures multi-coalition formation, including inside coalition activities in equilibrium. All these properties can not be present the Shapley value, based on super-additivity. More examples of games with coalition structure formation can be found in the rest of the paper. Equilibrium existence result can be generalized for the whole family of games.

Theorem 1. *The family of games* $\mathcal{G} = \{\Gamma(K), K = 1, 2, \ldots, N\}$ *has an equilibrium in mixed strategies,* $\sigma^*(\mathcal{G}) = (\sigma^*(K = 1), \ldots, \sigma^*(K = N))$, $\Big(\sigma^*(K)\Big)_{i \in N}$.

Technical side off the result is obvious. The theorem expands the classic Nash theorem.

The interesting question is how robust is an equilibrium to an increase in K. In the previous section we have seen that two "extroverts" change an equilibrium, when have an option be together, but this is not true for two "introverts".

An equilibrium in the game can also be characterized by equilibrium partitions.

Definition 5 (equilibrium coalition structures or partitions). *A set* *of partitions* $\{P^*\}(K)$*,* $\{P^*\}(K) \subset \mathcal{P}(K)$*, of a game* $\Gamma(K)$*, is a set of equilibrium partitions, if it is induced by an equilibrium strategy profile* $\sigma^*(K) = (\sigma_i^*(K))_{i \in N}$*.*

In the same way we can define equilibrium payoffs for the whole family of games. This consideration is important for construction of a non-cooperative stability criterion presented further in this paper.

4 Discussion

At the moment there are two game theories: a non-cooperative and a cooperative. A non-cooperative is based on a mapping of a strategy profile of all players (a subset from $\mathbb{R}^N$) into a payoff profile for all players (a bounded subset from $\mathbb{R}^N$). A cooperative game theory is based on a mapping of a subset of integer numbers (a subset from $\mathbb{N}$) into a subset of real numbers (a subset from $\mathbb{R}$).

A background assumption of the present paper is that an individual action of every self-interest agent, located at some coalition, can have a non-negligible effect for payoffs for every other, located at some different coalitions. Construction of a non-cooperative games presented in this paper disregards possible negligibility of these effects. A possibility of negligible effects leaves a space for a co-existence of non-cooperative and cooperative game theories.

Insufficiency of cooperative game theory to study coalitions and coalition structures was earlier noted by many authors. Maskin (2011) wrote that "features of cooperative theory are problematic because most applications of game theory to economics involve settings in which externalities are important, Pareto inefficiency arises, and the grand coalition does not form". Myerson (p.370, 1991) noted that "we need some model of cooperative behavior that does not abandon the individual decision-theoretic foundations of game theory".

Thus there is a demand for a specially designed non-cooperative game to study coalition structures formation along with an adequate equilibrium concept for this game.

4.1 Referring to existing literature

There is a voluminous literature on coalitions, a list of authors is far from complete: Aumann, Hart, Holt, Maschler, Maskin, Moulen, Myerson, Peleg, Roth, Serrano, Shapley, Schmeidler, Weber, Winter, Wooders and many others. Many authors address and solve many important questions. However answering a question is not resolving a general problem. Solution to a problem is not in a volume of literature: one need certain answers for the questions:

Problem Identification: What are properties of problem which make it special?

Solution Existence: Does the problem so far has a solution?

Choice of a tool: Shall we borrow some existing tool or to construct a new one to solve the problem after it is well-defined?

4.1.1 Identification of a problem

There are different views on complications appearing with non-cooperative formation of coalition structures. Two opinions are presented above. There are at least two more recent.

Serrano (2014) wrote on usage non-cooperative, but not cooperative game theory: "the axiomatic route find difficulties identifying solutions", and that for studying coalition formation *"it may be worth to use strategic-form games, as proposed in the Nash Program"*.

Ray and Vohra (2015) wrote on complexity and contradictions in approaches, offer a systematic view on the area, based on "collection of coalitions", or a modified cooperative game theory. "*Yet as one surveys the landscape of this area of research, the first feature that attracts attention is* ***the fragmented nature of the literature***[7]*... The literature on coalition formation embodies two classical approaches that essentially form two parts of this chapter: (i) The blocking approach, in which we require the immunity of a coalitional arrangement to blocking, perhaps subcoalitions or by other groups which intersect the coalition in question... (ii) Noncooperative bargaining, in which individuals make proposals to form a coalition, which can be accepted or rejected...*

After all, the basic methodologies differ apparently at ***an irreconcilable level***[8] *over cooperative and noncooperative game approaches...* "

Every view describes the problem partially differently and suggests different ways of solution. Existing variability of views does not let to conclude that the problem is still well-identified. Not well-identified problem can not be well-solved.

The suggested paper has encompassed all the mentioned details into one, and the problem can be pronounced as: how to construct coalition structures from actions of self-interest agents, when co-exist intra and inter coalition externalities.

4.1.2 Existence of a solution

A diversity of approaches in the voluminous literature means the problem is not well-identified so far. Not well-identified problem can not be well-solved. This is the reason for a diversity of contradicting views on coalition (structure) formation from individual behavior. "A problem can not be solved if it's bound are unknown" (A. Tchekmarev[9])

The current paper suggests an identification for the problem and a consistent, natural solution for it. The proposed game generalizes the non-

[7] stressed by DL
[8] stressed by DL
[9] Personal communication on engineering design.

cooperative game of Nash for coalition structure formation.

4.1.3 Tool

How to solve the problem, which was not solved for so long? The answer comes from Albert Einstein: "The significant problems we have cannot be solved at the same level of thinking with which we created them". This means that to solve the problem one needs to reconsider a basic tool of non-cooperative game theory analysis. The current paper dares to suggest such a tool.

4.2 Comparison with existing approaches

4.2.1 A threat

A popular approach to use a "threat "as a basic concept for coalition formation analysis was suggested by Nash (1953). Consider a strategy profile from a subset of players. Let this profile be a threat to someone, beyond this subset. The threatening player**s** may produce externalities for each other (and negative externalities not excluding!). How credible could be such threat and how to describe it?

At the same time there may be some other player beyond the subset of players, who may obtain a bonanza from this threat. But this beneficiant may not join the group due to expected intra-group negative externalities for members or from members of this group. Thus a concept of a threat can not serve as an elementary concept.

There is a parallel argument against using the concept of a "threat" as an elementary tool. Assume there are several agents, who individually can not make harm to some others, possibly allocated in different coalitions. And the threat targets not whole coalitions, but only to some members. But if these small agents make a threat together, it can be credible. Does this mean that the small agents unite in one coalition? They may have there own contradictions to join in one coalition. How to describe affect for those, who are not a target for the threat? May be a formal example will be more illustrative here, but the volume of the paper does not let us present it here.

4.2.2 Usage of the non-cooperative approach

The justification of a chosen tool, a non-cooperative game, comes from Maskin (2011) and a remark of Serrano (2014), that for studying coalition formation "it may be worth to use strategic-form games, as proposed in the Nash Program". This paper explicitly works with a non-cooperative approach.

However, there is the difference of the research agenda in this paper from the Nash Program (Serrano 2004). Nash program aims to study a non-cooperative formation of *one* coalition, this paper aims to study non-cooperative formation of coalition structures, which may include more than one coalition.

The constructed finite non-cooperative game allows to study what can be a cooperative behavior, when the individuals "rationally further their individual interests" (Olson, 1971).

4.2.3 Novelties of the paper

Nash (1950, 1951) suggested to construct a non-cooperative game as a mapping of a set of strategies into a profile of payoffs, $\times_{i\in N} S_i \to (U_i)_{i\in N}$.

This paper has two contributions in comparison to his paper: a construction of a non-cooperative game with an embedding coalition structure formation mechanism, and a parametrization of games by a number of deviators K: $\times_{i\in N} S_i(K) \overset{\mathcal{R}(K)}{\to} \{S(P) \colon P \in \mathcal{P}(K)\} \to \{(\mathcal{U}_i(K))_{i\in N}\}$, where $K \in \{1, \ldots, \#N\}$. The game suggested by Nash becomes a partial case for these games, when coalition structures do not matter.

Every outcome of every game consists of two items 1/ a payoff profile; 2/ an allocation of players over coalitions. Equilibrium in mixed strategies always exists and may be not efficient like in traditional non-cooperative games.

We can take $\mathcal{R}(K), K = 1, \ldots, \#N$ as a social institute with a trust from every player. This means that no one has an incentive to avoid it. Application of $\mathcal{R}(K)$ separates two issues: how to construct an equilibrium (what is done in this paper) and how to improve quality of a result or how to achieve social desirability (an efficiency of $\mathcal{R}$, not studied in this paper).

To study an efficiency one need to describe a set of all rules $\mathcal{R}(K)$, for every $K = 1, \ldots, \#N$. Then one need to define social desirability in terms of final payoffs. This is the big research program left for the future.

Further development of this program leads to a non-cooperative welfare theory with reconsideration of the first and the second welfare theorems and Arrow's impossibility theorem. These projects can be applied to social and political theories, where disagreement on rules of coalition structure formation can be interpreted as social/political conflicts. All such issues are left for the future. Author believes that this new research direction will enrich non-cooperative game theory and provide non-cooperative fundamentals for social and mechanism design.

4.2.4 Differences from cooperative game theory concepts

There is the great literature on cooperative game theories, with the specially designed equilibrium concepts: the strong Nash, the coalition-proof, the nucleolus, the kernel, the bargaining set.

The suggested equilibrium concept has the common differences with the listed CGT equilibrium concepts: every player makes an individual choice, and obtains an individual payoff for every player. In equilibrium the player has an impact from intra- and inter- coalition externalities. Thus a coalition may have different values (as a sum of individual payoffs of a member of a coalition) in different coalition structures. Such cases are beyond a definition of a game in cooperative form.

There are other more specific differences with every existing cooperative game theory equilibrium concept. Differences from the core approach of Aumann (1960) are clear: a presence of externalities, no restrictions that only one group deviates, no restrictions on the direction of a deviation (inside or outside), and a construction of individual payoffs from a strategy profile of all players. There is no need to assume transferable/non-transferable utilities for players. The suggested approach allows to study coalition structures, which differ from the grand coalition. Finally the introduced concepts enables to offer a non-cooperative necessary stability criterion based only on an equilibrium of a game, what is presented in further in this paper.

The approach does not need a concept of a blocking coalition. Blocking coalition approach does not specify externalities between a coalition and the original set of players, it does not allow simultaneous deviation of more than one coalition. However construction of a sequential development of the current game will follow in the next paper.

Multi-coalition approach let us ask another question. If there is one coalition, which can not block, but there are two, separate coalition, which do can block. How to describe actions of these coalitions from individual actions of their members? This is the same argument, as why not to use a threat as an elementary concept. This situation can be studied by the suggested model.

The suggested approach is different in a role for a central planner offered by Nash, who "argued that cooperative actions are the result of some process of bargaining" Myerson (p.370, 1991). The central planner offers a predefined family of coalition structure formation mechanisms, a family of eligible partitions and a family of rules to construct these partitions from individual strategies of players.

5 Formal definition of cooperation

In the example of Section 2 we have seen that we can not deduce existence of a cooperation only from observing a payoff profile. This section formalizes an idea of cooperation presented in the example above. A suggested cooperation criterion is sufficient and can be relaxed in many ways.

Definition 6 (complete cooperation in a coalition)**.** *In a game* $\Gamma(K)$ *with an equilibrium* $\sigma^*(K) = (\sigma_i^*)_{i \in N}$ *a set of players* g, ***cooperate completely in the coalition*** g *if for every player* $i \in g$ *there is*

ex ante *: for every* i *in* g*, a desirable coalition* g *always belongs to a chosen coalition structure, i.e such if* s_i *is chosen by* i*, then* $s_i \in S_i(P_i)$, $g \in P_i$*, where* P_i *is a coalition structure chosen by* i*.* [10]

ex post *: every realized coalition structure contains* g*, i.e.* $g \in \forall P^*$*, where* P^* *is a formed equilibrium partition of* $\Gamma(K)$*,*

First of all, cooperation is defined for a game $\Gamma(K)$. If a game changes due to a change in the number of deviators K, the cooperation may evaporate.

Every player chooses partitions, every chosen partition contains the desirable coalition g. Individually chosen coalition structures by all players may be different.

After the game is over the coalition g always belongs to every final equilibrium coalition structure, disregard allocation of players in other coalitions. A final partition may differ from an individually chosen one, but will contain the desirable coalition. We may suggest that the definition above is a definition of non-cooperative cooperation. There are many important details in cooperation, which are omitted in the presented definition.

[10] However coalition structures chosen by different players may be different.

6 Application: Bayesian games

In this section we demonstrate how intra-coalition externalities of the grand coalition may happen from equilibrium mixed strategies. In order to show that, a standard Battle of Sexes game is modified.

There are two players, Ann and Bob. Each has two options: to be together with another or to be alone. In every option each can choose where to go: to Box or to Opera. Hence every player has four strategies. A set of strategies of the game leads to 16 outcomes. Every outcome consists of a payoff profile and a partition (or a coalition structure). Both players have preferences over coalition structures: they prefer to be together, then be separated.

The rules of coalition structure formation mechanism are:

1. If they both choose to be together, i.e. both choose the coalition structure $P_{alone} = \{Ann, Bob\}$ then:
   (a) if both choose the same action for $P_{together}$ (i.e. both choose Box or both choose Opera), then they go to where they both choose to go,
   (b) otherwise they do not go anywhere, but enjoy just being together;
2. if at least one of them chooses to stay alone, i.e. chooses a partition $P_{alone} = \{\{Ann\}, \{Bob\}\}$, then each goes alone to where she/he chooses, may be to a different Opera or to a different Box performances.

Formally the rules are:

$$\mathcal{R}(K=1)\colon S(K=1) \mapsto S(\{\{1\},\{2\}\}),$$
$$\forall s \in S_i(K=1) = \{O_{Ann,alone}, B_{Ann,alone}\} \times \{O_{Bob,alone}, B_{Bob,alone}\}$$

and

Table 5: Payoff for two extrovert players who, obtain additional payoffs ϵ being in one coalition P_{joint}, when it is realized. Equilibrium uniqueness is reinstalled.

	$L_{2,alone}$	$H_{2,alone}$	$L_{2,together}$	$H_{2,together}$
$L_{1,alone}$	$\underline{(0;0+\delta)}$ $\{\{1\},\{2\}\}$	$(-5;3+\delta)$ $\{\{1\},\{2\}\}$	$\underline{(0;0+\delta)}$ $\{\{1\},\{2\}\}$	$(-5;3+\delta)$ $\{\{1\},\{2\}\}$
$H_{1,alone}$	$(3;-5+\delta)$ $\{\{1\},\{2\}\}$	$(-2;-2+\delta)^{*,**}$ $\{\{1\},\{2\}\}$	$(3;-5+\delta)$ $\{\{1\},\{2\}\}$	$(-2;-2+\delta)$ $\{\{1\},\{2\}\}$
$L_{1,together}$	$\underline{(0;0-5+\delta)}$ $\{\{1\},\{2\}\}$	$(-5;3+\delta)$ $\{\{1\},\{2\}\}$	$\underline{(0+\epsilon;0)}$ $\{1,2\}$	$(-5+\epsilon;)$ $\{1,2\}$
$H_{1,together}$	$(3;-5+\delta)$ $\{\{1\},\{2\}\}$	$(-2;-2+\delta)^{**}$ $\{\{1\},\{2\}\}$	$(3+\epsilon;-5)$ $\{1,2\}$	$(-2+\epsilon;-2)$ $\{1,2\}$

Table 6: Payoff for the Bach-or-Stravinsky game. B is for Box, O is for Opera. If the players choose to be together, and it is realized due to the rule of coalition structure formation, then each obtains an additional fixed payoff $\epsilon > 0$.

	$B_{Bob,alone}$	$O_{Bob,alone}$	$B_{Bob,together}$	$O_{Bob,together}$
$B_{Ann,alone}$	$(2;1)^*$ $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$	(2;1) $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$
$O_{Ann,alone}$	(0;0) $\{\{1\},\{2\}\}$	$(1;2)^*$ $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$	(1;2) $\{\{1\},\{2\}\}$
$B_{Ann,together}$	(2;1) $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$	$(2+\epsilon;1+\epsilon)^{**}$ $\{1,2\}$	$(\epsilon;\epsilon)$ $\{1,2\}$
$O_{Ann,together}$	(0;0) $\{\{1\},\{2\}\}$	(1;2) $\{\{1\},\{2\}\}$	$(\epsilon;\epsilon)$ $\{1,2\}$	$(1+\epsilon;2+\epsilon)^{**}$ $\{1,2\}$

$$\mathcal{R}(K=2)\colon S(K=2) \mapsto \begin{cases} S(\{1,2\}), \\ \text{ if } s \in \{O_{Ann,alone}, B_{Ann,alone}\} \times \{O_{Bob,alone}, B_{Bob,alone}\} \\ S(K=2) \setminus S(\{1,2\}\}), \quad \text{otherwise} \end{cases}$$

Table 6 corresponds to the game with $K = 2$, where the game for $K = 1$ is a nested component. If Ann and Bob play the game with $K = 1$ with a single coalition structure $\{\{Ann\}, \{Bob\}\}$, then the payoffs for this game are in the two-by-two top-left corner of Table 6. If Ann and Bob are together, then each obtains an additional payoff ϵ, and the corresponding cells make a two-by-two bottom-right corner.

Every game with $K = 1$ and $K = 2$ has only one mixed strategies equilibrium and only one equilibrium partition. Mixed strategies equilibrium for $K = 1$ is described in every textbook. A change in K from $K = 1$ to $K = 2$ results in a changes of an equilibrium strategy profile and an equilibrium partition.

For $K = 2$ the game still has a mixed strategies equilibrium as for $K = 1$, but there are differences: another domain of pure strategies,another coalition structure and another payoff profile. Mixed strategies equilibrium for Ann is: $\sigma^*(B_{Ann,together}) = (1 + \epsilon)/(3 + 2\epsilon)$, $\sigma^*(O_{Ann,together}) = (2 + \epsilon)/(3 + 2\epsilon)$, $i \in \{\text{Ann}, \text{Bob}\}$.

Equilibrium mixed strategies may appear in any coalition structure, for every $K = 1, 2$. When they appear in the grand coalition, then there is an additional equilibrium intra-coalition activity. Such games are not described in literature.

The presented game allows to construct intra-coalition externalities from mixed strategies within one partition. Mixed intra-coalition externality means that players are exposed to equilibrium fluctuations from strategic actions of another player.

A game as above can not be constructed within any cooperative game equilibrium concept. It is impossible to construct Shapley value (1953) here even if the grand coalition is in the equilibrium: players have equilibrium mixed strategies inside it. Value of an equilibrium coalition is constant, but allocation of payoffs varies.

7 Application: Stochastic games

Shapley (1953) defined stochastic games as " the play proceeds by steps from position to position, according to transition probabilities controlled jointly by the two player". This section demonstrates how this type of games with coalition structures as states of a game may appear. The example differs from example above as a set of the equilibrium mixed strategies induces more than one equilibrium coalition structure.

7.1 Corporate lunch game

There is a set of four identical players $N = \{A, B, C, D\}$. An individual strategy is a coalition structure, or an allocation of all players across tables during lunch. A coalition structure is an allocation of players over no more than four tables, where possibly empty tables do not matter. A rule of coalition formation is a unanimous agreement to form a coalition. If player a chooses a coalition, but other members of the coalition did not choose him, the player eats alone.

A player has preferences over coalition structures: she/he prefers to eat with someone, and other players eat individually. A motivation could be a possible dissipation of information or gossips. From another side, if one eats alone he is hurt by a possible formed coalition of others. Coalition structures, and payoff profiles for the highest cases payoffs are presented in Table 7:

Payoffs in Table 7 are organized in the way that formation of a coalition by other players deteriorates payoffs for the rest. Thus the same coalition may

Table 7: Office lunch game: strategies and payoff profiles. Full set of equilibrium mixed strategies are indicated only for player A.

num	Coalition structure	Payoff profile U_A, U_B, U_C, U_D	Coalition values as in cooperative game theo
1*	$\{A,B\},\{C\},\{D\}$: $\sigma^*_A = \sigma^*_B = 1/3$	(10,10,3,3)	$20_{A,B}$, 3_C, 3_D
2*	$\{A,C\},\{B\},\{D\}$: $\sigma^*_A = \sigma^*_C = 1/3$	(10,3,10,3)	$20_{A,C}$, 3_B, 3_D
3*	$\{A,D\},\{C\},\{B\}$: $\sigma^*_A = \sigma^*_D = 1/3$	(10,3,3,10)	$20_{A,C}$, 3_C, 3_B
4	$\{A\},\{B\},\{C,D\}$	(3,3,10,10)	3_A, 3_B, $20_{C,D}$
5	$\{A\},\{D\},\{B,C\}$	(3,10,10, 3)	3_A, 3_D, $20_{C,B}$
6	$\{A\},\{C\},\{B,D\}$	(3,10,3,10)	3_A, 3_C, $20_{B,D}$
7	$\{A\},\{B\},\{C\},\{D\}$	(3,3,3,3)	3_A, 3_B, 3_C, 3_D
8	$\{A,B\},\{C,D\}$	(3,3,3,3)	$6_{A,B}$, $6_{C,D}$
9	$\{A,C\},\{B,D\}$	(3,3,3,3)	$6_{A,C}$, $6_{B,D}$
10	$\{A,D\},\{B,C\}$	(3,3,3,3)	$6_{A,D}$, $6_{B,C}$
11	all other with $K = 3,4$	(0,0,0,0)	$= 0$

have different values in different coalition structures, compare payoff for the coalition $\{A, B\}$ in different coalition structures: line 1, $(\{A, B\}, \{C\}, \{D\})$ and line 8, $(\{A, B\}, \{C, D\})$.

If there are two 2-player coalitions every player receives only three units of payoffs, while if a player is in a coalition of two, while others are separate the player obtains a ten units of payoff. An increase in a size of a maximum coalition only decreases payoffs for all players.

Outcomes of the game are coalition structures or states of a stochastic game. An increase of K from 2 to 3 and 4 does not change an equilibrium in mixed strategies, hence we can speak about robustness of an equilibrium for $K = 2$ to an increase in K. This issue is addressed later in this paper.

It is clear that the game does not have an equilibrium in pure strategies. This is a Bayesian game, with a probability distribution of equilibrium mixed strategies. Equilibrium mixed strategies are indicated only for player A in the first three lines of the Table 7. States of the game are coalition structures appearing due to equilibrium mixed strategies, but not due to exogenous shocks.

7.2 A formal definition of a stochastic game of coalition structure formation

Let $\Gamma(K)$ be a non-cooperative game as defined above.

Definition 7. *A game $\Gamma(K)$ is a stochastic game if a set of equilibrium partitions is bigger than two, $\#(\{P^*\}(K)) \geq 2$, where a state is an equilibrium partition $P^* \in \{P^*\}(K)$.*

Studying properties of stochastic games with non-cooperative coalition structure formation and families of such games are left for the future.

8 Application: non-cooperative criterion for stability

Nash equilibrium, as in Nash (1950, 1951), is significantly based on a deviation of one player. This leads to the problem in literature, when a socially desirable outcome can not be supported by a Nash equilibrium. Sometimes such points are called "focal points" (Shiller).

Aumann (1990) demonstrates absence of a self-enforcement property of Nash equilibrium for a focal point in "a stag and a hare game". Deviation of more than one players demands a reconstruction for a set of individual strategies. This is done in the example below, a focal point becomes an equilibrium of some non-cooperative game from the current paper. Then we demonstrate how to construct a non-cooperative coalition structure criterion, i.e. how to measure stability of a coalition structure from an increase in a number of deviators. Then we construct a non-cooperative coalition structure stability criteria.

There are two hunters $i = 1, 2$. If players can hunt only individually, then $K = 1$, and the only partition is $P_{alone} = \{\{1\}, \{2\}\}$. An individual strategy set of i is $S_i(K = 1) = \Big\{(P_{alone}, hare), (P_{alone}, stag)\Big\}$ with a general element $s_i \equiv s_i(K)$. Every s_i consists of two terms: who is a hunting partner and what is an animal to hunt. For example, a strategy $s_i = (P_{separ}, hare)$ is interpreted as player i chooses to hunt alone for a hare.

A set of a corresponding strategies for the game with $K = 1$ is $S(K = 1) = S_1(K = 1) \times S_1(K = 1)$.

For a game with $K = 2$ every hunter can choose either to hunt alone, in a coalition structure $P_{alone} = \{\{1\}, \{2\}\}$, or together, $P_{together} = \{1, 2\}$. For every hunting partition a player chooses a target for hunting: a hare or a stag, as in the game for $K = 1$. A set of strategies of i is

$$S_i(K = 2) = \Big((P_{alone}, hare), (P_{alone}, stag), (P_{together}, hare), (P_{together}, stag)\Big),$$

where a strategy consists of two terms. A set of strategies of the game is a direct (Cartesian) product, $S(K=2) = S_1(K=2) \times S_2(K=2)$.

We do not rewrite the rules for coalition structure formation, as they are the same as in the BoS game above. The difference is in renaming strategies.

Every player knows, which game is played, either with $K = 1$ or with $K = 2$, and this is a common knowledge.

Payoffs for the both games $\Gamma(K = 1)$ and $\Gamma(K = 2)$ are presented in Table 8. Some payoffs outcomes have the special interpretation: $(8; 8)$ every hunter obtains a hare, $(4; 4)$ hunters obtain one hare for two, $(100; 100)$ both hunters obtain one stag.

If hunters play a game with $K = 1$, then a maximum achievable payoff is $(8, 8)$, when each hunts individually for a hare. An an equilibrium strategy profile is $\Big((P_{alone}, hare), (P_{alone}, hare)\Big)$ with the payoff $(8; 8)$. In the game $\Gamma(K = 1)$ the players can not reach the efficient outcome $(100, 100)$. It is available only if of both hunters decide to hunt together. This focal point (in terminology of Schelling) can be reached only within the game $\Gamma(K = 2)$. This is the explanation for the problem posed by Aumann - there is a point, which seems to be attractive, but one can not describe it in terms of a Nash equilibrium of a traditional non-cooperative game. The focal point is an equilibrium in the game $\Gamma(K = 2)$, but not in the game $\Gamma(K = 1)$.

Self-enforcing property of the equilibrium is that both players realize individual gain from a change of a game from $K = 1$ to $K = 2$ and neither can deviate. But players can not reach the outcome $(100; 100)$ without a change in a game.

If there is an uncertainty, which game is played, either $\Gamma(K = 1)$ or $\Gamma(K = 2)$, then players will randomize between two strategies: $(P_{alone}, hare)$ and $(P_{separ}, stag)$. In this case the game becomes a stochastic game as described earlier.

Table 8: Expanded stag and hare game

	P_{alone}, hare	P_{alone}, stag	$P_{together}$,hare	$P_{together}$, stag
P_{alone},hare	$(8;8)^*$; $\{\{1\},\{2\}\}$	(8;0); $\{\{1\},\{2\}\}$	(8;8);$\{\{1\},\{2\}\}$	(8;0); $\{\{1\},\{2\}\}$
P_{alone}, stag	(0;8); $\{\{1\},\{2\}\}$	(0;0); $\{\{1\},\{2\}\}$	(0;8); $\{\{1\},\{2\}\}$	(0;0); $\{\{1\},\{2\}\}$
$P_{together}$, hare	(8;8); $\{\{1\},\{2\}\}$	(8;0); $\{\{1\},\{2\}\}$	(4;4); $\{1,2\}$	(8;0); $\{1,2\}$
$P_{together}$, stag	(0;8); $\{\{1\},\{2\}\}$	(0;0) ;$\{\{1\},\{2\}\}$	(0;8);$\{1,2\}$	$(100;100)^{**}$; $\{1,2\}$

8.1 Criterion of coalition structure (a partition) stability

There is a nested family of games

$$\mathcal{G} = \{\Gamma(K=1),\ldots,\Gamma(K),\ldots\Gamma(K=N)\} \colon \Gamma(K=1) \subset \Gamma(K) \subset \Gamma(K=N).$$

Every game $\Gamma(K)$, $K = 1,\ldots,\#N$ has an equilibrium may be in mixed strategies,

$$\Big(\sigma^*(1),\ldots,\sigma^*(K),\ldots,\sigma^*(K=N)\Big),$$

where $\sigma^*(K) = (\sigma_i^*(K))_{i\in N}$ and equilibrium coalition structures (partitions):

$$\Big(\{P^*\}(1),\ldots,\{P^*\}(K),\ldots,\{P^*\}(K=N)\Big),$$

$\{P^*\}(K) \subset \mathcal{P}(K)$.

The family of games has an equilibrium expected payoff profiles:

$$\Big((EU_i^{\Gamma(1)})^*_{i\in N},\ldots,(EU_i^{\Gamma(K)})^*_{i\in N},\ldots,(EU_i^{\Gamma(K=N)})^*_{i\in N}\Big),$$

where $(EU_i^{\Gamma(K)})^*_{i\in N} \equiv (EU_i^{\Gamma(K)}(\sigma^*))_{i\in N}$.

Let a game $\Gamma(K_0) \in \mathcal{G}$ has $\sigma^*(K_0)$ as an equilibrium mixed strategy set. The question is: what is a condition when an equilibrium strategy profile does not change with an increase in a maximum coalition size K_0?

The criterion is based on the idea that a set of mixed strategies should not change with an increase in K, i.e. when a game $\Gamma(K_0)$ is sequentially changed for $\Gamma(K+1), \ldots, \Gamma(N)$. The criterion is a sufficient criterion and defines a maximum coalition size, when an equilibrium strategy profile for a less K_0 still supports an equilibrium for a bigger K.

Definition 8. ***Coalition structure (partition) stability criterion*** *for a game $\Gamma(K_0)$ is a maximum coalition size K^*, when an equilibrium still holds true, i.e. for all $i \in N$ there is a number K^* such that*

1.

$$K^* = \max_{\substack{K=K_0,\ldots,N \\ \Gamma(K_0)\ldots,\Gamma(K=N)}} \left\{ EU_i^{\Gamma(K_0)}\Big(\sigma_i^*(K_0), \sigma_{-i}^*(K_0)\Big) \geq EU_i^{\Gamma(K)}\Big(\sigma_i^*(K), \sigma_{-i}^*(K)\Big) \right\},$$

2. Dom $\sigma^*(K^*) =$ Dom $\sigma^*(K_0)$

where $\sigma^*(K_0)$ is an equilibrium in the game $\Gamma(K_0)$, $\sigma^*(K)$ is an equilibrium in a game $\Gamma(K)$, $K = K_0, \ldots, N$, and Dom is a domain of equilibrium mixed strategies set.

The definition is operational, it can be constructed directly from a definition of a family of games. This definition guarantees stability of both payoffs and partitions, and is a sufficient criterion of stability. Some applications may require weaker forms of the criterion. We have seen in Prisoner's Dilemma example ("non-extroverts", "non-introverts"), that an increase in K may result in an increase in number of equilibrium coalition structures without a change in payoffs. The criterion defines $K^* = 2$ for this game.

The example supports Aumann (1990), why Nash equilibrium is generally not self-enforcing for $K = 1$. In the extended version of stag and hare game an increase in K along with an adequate modification in strategy sets changed an equilibrium. The same took place in a variation of Battle of Sexes game. However this did not happen in the Corporate Lunch game.

The proposed criterion may serve as a measure of trust to an equilibrium or as a test for self-enforcing of an equilibrium. This criterion can be applied to study opportunistic behavior in coalition partitions. If players in a coalition g of a game $\Gamma(K_1)$ have perfect cooperation, this does not mean that in a wider game $\Gamma(K_2)$, $K_1 < K_2$, they will still cooperate.

Studying stability is tightly connected to studying focal points. This is another big area for research left for the future.

9 Conclusion

The current paper presents a family of non-cooperative games for coalition structure formation from self-interest fundamentals. The paper offers also a non-cooperative criterion to measure stability of coalition structures families of nested games. Examples demonstrate how the constructed family can be used for reconsidering well-known games and suggests new directions of research. Application of the game for network construction comes in an accompaning paper.

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