The Value of Biodiversity as an Insurance Device

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Abstract

This paper presents a benchmark endogenous growth model including biodiversity preservation dynamics. Producing food requires land, and increasing the share of total land devoted to farming mechanically reduces the share of land devoted to biodiversity conservation. However, the safeguarding of a greater number of species is associated to better ecosystem services – pollination, flood control, pest control, etc., which in turn ensure a lower volatility of agricultural productivity. The optimal conversion/preservation rule is explicitly characterized, as well as the value of biological diversity, in terms of the welfare gain of biodiversity conservation. The Epstein-Zin-Weil specification of the utility function allows us to disentangle the effects of risk aversion and aversion to fluctuations. A two-player game extension of the model highlights the effect of volatility externalities and the Paretian sub-optimality of the decentralized choice.

KEY WORDS: Biodiversity, stochastic endogenous growth, insurance value, recursive preferences.

JEL CLASSIFICATION: Q56, Q58, Q10, Q15, O13, O20, C73.

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1 Introduction

From 1999 to 2008, 48 000 square kilometers of wildland have been turned into cropland. In 2013, croplands covered 12% of earth ice-free surface; annually, more than 10% of the earth's net primary production is turned into crops (Phalan *et al.* [24]). This destruction of habitats and natural ecosystems for agricultural purposes, useful and inevitable as it may appear, is increasingly questioned. Natural ecosystems provide a wide range of goods and services, such as control of the local climate, clean water provision, flood control, maintenance of soil fertility, pollination, or pest control. The destruction of natural habitats causes species extinctions and thus a loss in biodiversity. In the debate about the trade-off between food production and biodiversity conservation, it is crucial to be able to understand the ecological and economic determinants of the value of ecosystem services. Ironically, a growing body of evidence shows that one of these determinants is the feedback effect that biodiversity destruction exerts on agricultural productivity. Indeed, biodiversity destruction negatively impacts the climatic, hydrological and, more generally, ecological environment, which may in turn affect negatively the mean level and/or the variability of agricultural productivity in time (Fuglie and Nin-Pratt [12], de Mazancourt *et al.* [6]). To put things differently, biodiversity conservation can provide beneficial services that enhance average agricultural productivity and reduce its variability. We are interested in the second of these effects, namely the fact that a high level of biodiversity is able to dampen the fluctuations of agricultural productivity around its trend. Thus, biodiversity acts as a form of insurance, a natural insurance against the fluctuations of agricultural production, and this is likely a main determinant of its overall value.

The insurance value of biodiversity has been analyzed in a series of groundbreaking studies by Baumgärtner [3], Quaas and Baumgärtner [25], [26], Baumgärtner and Quaas [4] and Baumgärtner and Strunz [5]. These studies, however, rely on static models under partial equilibrium.

We provide the first analysis of biodiversity as insurance in a stochastic dynamic setup. Precisely, we study a dynamic problem of optimal land conversion in a stylized stochastic endogenous growth model where (1) increasing the share of land devoted to farming allows to increase agricultural production at the expense of biodiversity; (2) agricultural productivity evolves stochastically around an exogenous deterministic trend, with a volatility that negatively depends on biodiversity.

To better study insurance issues in a dynamic context, we follow the approach pioneered by Epstein and Zin (Epstein and Zin [10], [11], Duffie and Epstein [7]) and we represent preferences by a recursive utility function. In this way, we are able to disentangle aversion to intertemporal fluctuations and relative risk aversion. The risk aversion parameter quantifies the preference for certain rather than uncertain outcomes, and it only makes sense in a stochastic context (but even in static models). Conversely, the aversion to intertemporal fluctuations, i.e. the inverse of the elasticity of intertemporal substitution, measures the propensity to smooth consumption over time, and it is a fundamental parameter also in deterministic dynamic models; in typical endogenous growth models, when the growth rate is positive, it takes the form of the willingness to increase the level of the consumption at the expense of its growth rate. Several studies[1] in the field of natural resources and the environment prove indeed that these two logically distinct concepts cannot satisfyingly be embodied in a single parameter as it happens when the intertemporally additive expected utility is considered.

We develop our approach in two stages. We first characterize the optimal allocation of land to biodiversity conservation. We then consider two farmers exploiting a common land in the absence of well-defined property rights. We study how the equilibrium allocation of land differs from the social optimum.

We show that the optimal share of land devoted to biodiversity conservation is constant over time. It decreases with the social discount rate and increases with risk aversion. It

[1]In a model of reservoir management, Howitt *et al.* [17] show that the intertemporal additive expected utility function does not fit their data, whereas the recursive utility function does. Peltola and Knapp [23] use recursive utility to study forestry management, and Lybbert and McPeak [22] for the trade-off among different livestock among Kenyan pastoralists. They highlight empirically the distinct values that should be taken by the intertemporal elasticity of substitution and risk aversion parameters. Ha-Duong and Treich [16] evaluate policies in a context of global warming; it is shown that the optimal policy responds differently to variation of the intertemporal substitution parameter and the risk aversion one. The same result is observed by Knapp and Olson [18] for rangeland and groundwater management. They consider the effect of both parameters on the optimal decision rule, showing in particular that if intertemporal substitution has a major effect, risk aversion does not impact the optimal policy. The different roles of the two parameters is also proved in Epaulard and Pommeret [9] in the context of extraction of a non-renewable resource in a continuous time framework.

decreases with the aversion to fluctuations when the average trend of agricultural productivity is larger than the discount rate, but increases with the aversion to fluctuations in the opposite case.

We then compute the value of biodiversity, defined in reference to Lucas [20], [21] as the welfare gain of biodiversity conservation in the optimal solution, compared to a solution where total land conversion is achieved. We study the determinants of this particular value of biodiversity, interpreted as an insurance against the volatility of agricultural productivity. We show that it is an increasing function of risk aversion, but that the effect of the aversion to fluctuations is ambiguous. In general, and quite intuitively, the signs of the effects of increasing the aversion to fluctuations on the share of land devoted to biodiversity conservation and on the value of biodiversity are the same. But when the average trend of agricultural productivity is smaller than the discount rate and the intrinsic agricultural volatility as well as risk aversion are high, a counterintuitive result may appear: the two signs may be opposite. In this case, where the prospects of the economy are dull, when society becomes more averse to fluctuations it preserves less and less of something that has more and more value: biodiversity.

In the two-player extension of the model, land is a common-property resource that two farmers want to exploit. The behavior of the two farmers are supposed to coordinate so that the system reaches a Nash equilibrium of the game. We highlight the systematic over-exploitation of the natural resource with respect to the previously described social optimum due to the volatility externalities. We show that the increment in area of land devoted to farming (that is the complement to 1 of the area devoted to biodiversity conservation) does not depend on risk aversion, but is decreasing with the aversion to fluctuations and tends to vanish when the aversion to fluctuations is very high.

Our analysis contributes to three strands of literature. We advance, first, a nascent theoretical literature on the insurance value of biodiversity. As we already emphasized above, we are interested here in the link between biodiversity and destruction of the environment[2], and ecosystem variability (climate, water provision, flood control, maintenance of soil fertility, pollination...). This kind of interaction between biodiversity preservation and uncertainty substantially differs from *portfolio differentiation* arguments used, for instance, by Weitzman

[2]The two are in strict relation, being linked by the species-area relationship, see Rosenzweig [27].

[29], where the word “biodiversity” is intended as the number of cultivated varieties. A series of papers by Baumgärtner, Quaas and Strunz ([3], [25], [26], [4] and [5]) theoretically analyzes the phenomenon in the framework of static models. Of course the static context is a limitation because the effects of biodiversity degradation accumulate and spread over time. We propose the first stochastic dynamic model in this literature. In our model the volatility of the agricultural productivity depends on the whole historical path of the biodiversity conservation decisions. Moreover, the dynamic context is essential to being able to speak about elasticity of intertemporal substitution and aversion to fluctuations.

This remark brings us to a second strand of literature: our paper also contributes to a small but growing literature on natural resources that disentangles intertemporal substitution and risk aversion. In different contexts, Krautkraemer *et al.* [19], Howitt *et al.* [17], Peltola and Knapp [23], Lybbert and McPeak [22], Ha-Duong and Treich [16] and Knapp and Olson [18] show that the recursive utility can better fit the data than the intertemporal additive expected utility because the same parameter cannot be used to represent intertemporal substitution and risk aversion. In the specific case of the relation between optimal growth and biodiversity conservation we show that the optimal allocation of the land responds qualitatively differently to the two parameters.

Finally, our paper contributes to the literature on dynamic games in continuous time. Indeed, even if continuous time stochastic game are frequent in various areas of economic theory (for example natural resource exploitation, capital accumulation, oligopoly theory), see for instance Van Long [28] or Haunschmied *et al.* [15], to the best of our knowledge, this is the first continuous time model in the economic theory literature where a model with a Nash equilibrium of a game with Epstein-Zin-Weil preferences is used.

The paper proceeds as follows. In section 2 we present the basics of the model. We explicitly solve it to obtain the optimal land conversion rate in section 3, and we compute the value of biodiversity in section 4. In section 5, we solve the two-player common property resource game and compare its outcome to the optimal solution. Section 6 concludes. We collect all the proofs of the results in the Appendix.

2 The model

We build a highly stylized model, the simplest we can think of that allows us to compute the value of biodiversity as an insurance device against agricultural productivity fluctuations. We first present the model and its optimal solution, and then come to the determination of the value of biodiversity.

We consider an agricultural economy. We describe the problem of a planner or of the unique farmer living in this economy. She owns a stock $L = 1$ of land and she has to decide how to allocate it into two possible intended uses: farming and biodiversity preservation.

For $t \geq 0$, we respectively denote by $f(t) \in [0, 1]$ and $1 - f(t) \in [0, 1]$ the share of land used respectively to farming and to maintain biodiversity. According to the well known species-area curve first proposed in the 20's by O. Arrhenius [2] and H. Gleason [8], as the number of species is constrained by available land, the level of biodiversity $B(t)$ depends on the area of land left undeveloped:

$$B(t) = g(1 - f(t)) \quad \text{with} \quad g'(\cdot) > 0,$$

where the function $g(\cdot)$ is concave and usually specified as a power function.

We assume that agricultural production at time t is given by:

$$Y(t) = f(t)A(t) \tag{1}$$

where $A(t)$ is the productivity of a unit of land devoted to farming at time t, which dynamics is described by a stochastic differential equation (SDE). More precisely, given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a real standard Brownian motion $W\colon [0, +\infty) \times \Omega \to \mathbb{R}$, adapted to some filtration $\mathcal{F}_t$, we assume that $A(t)$ is a solution of the following SDE:

$$\begin{cases} \mathrm{d}A(t) = \alpha A(t)\,\mathrm{d}t + \sqrt{f(t)}\sigma A(t)\,\mathrm{d}W(t) \\ A(0) = A_0. \end{cases} \tag{2}$$

In such an expression, $\alpha \in \mathbb{R}$ represents some (fixed and exogenous) parameter of technological progress in farming activities (it can be equal to 0). The term $\sqrt{f(t)}\sigma$ measures the volatility of agricultural productivity. The exogenous part $\sigma > 0$ represents the intrinsic volatility, due for instance to weather (floods, droughts, etc.). Total volatility decreases as the land devoted to biodiversity preservation, and then biodiversity itself, increases[3]. It is in that sense that biodiversity appears in the model as an insurance against adverse outcomes.

We suppose that at each time $t \geq 0$ all the production is consumed, so that:

$$C(t) = Y(t). \tag{3}$$

This assumption is not innocuous. It implies that there is no precautionary saving: the economy cannot store or save a part of agricultural production to hedge against the risk of a bad future productivity.

The planner maximizes, over the set of the $[0,1]$-valued $\mathcal{F}_t$-adapted processes, an aggregate social welfare criterion in form of an infinite horizon, continuous time, Epstein-Zin-Weil utility function characterized by a constant relative risk aversion θ (positive and different from 1), an intertemporal elasticity of $\phi^{-1} > 0$ and a discount rate $\rho > 0$. Recall that the case $\theta = \phi$ corresponds to the usual time additive expected utility function. The inverse of the intertemporal elasticity of substitution ϕ can also be interpreted as a measure of aversion to fluctuations, as an agent with a high ϕ prefers to smooth consumption over time[4].

As proved in Example 3 page 367 of Duffie and Epstein [7], the corresponding aggregator can be written as:

$$F(C,V) = \frac{\rho}{1-\phi}(1-\theta)V\left[\left(\frac{C}{((1-\theta)V)^{\frac{1}{1-\theta}}}\right)^{1-\phi} - 1\right]. \tag{4}$$

We denote by $V(A_0)$ the value function of the described problem.

Remark 2.1. *In an informal way, as, for instance, in Epaulard and Pommeret [9], we could represent the Epstein-Zin-Weil preferences using an infinitesimal representation, having the advantage of being easily linked to the definition of discrete-time recursive utility. In this context the utility at period t depends on current consumption as well as on the certain equivalent $\overline{U}$ of future utility:*

$$U(t) = \left[\frac{C(t)^{1-\phi}}{1-\phi} + e^{-\rho dt}\frac{\overline{U}(t+dt)^{1-\phi}}{1-\phi}\right]$$

[3] We use a square root function only to simplify the computations, but the method we use works for more general concave functions.

[4] If $\theta > \phi$, individuals are more risk averse than they are concerned about consumption smoothing. It is said in this case (Gollier [13]), that an agent has Preferences for an Early Resolution of Uncertainty (PERU), which is in particular the case among poor vulnerable populations (Lybbert and MacPeak [22]).

being $\overline{U}(t+dt)$ *the quantity:*

$$\overline{U}(t+dt) = \left[E\left(U\left(t+dt\right)^{1-\theta}\right)\right]^{\frac{1}{1-\theta}}.$$

3 The optimal land conversion rate

The following conditions, that we will always suppose to be verified in the following, will be shown to be necessary to ensure that the value function remains finite and to express explicitly its value.

Hypothesis 3.1. *The parameters satisfy the following conditions:*

$$\phi \neq 1, \qquad \rho > \alpha(1-\phi), \qquad \frac{1-\theta}{1-\phi} > 0. \tag{5}$$

Observe in particular that the second inequality in (5) is always satisfied if $\phi > 1$, and that the third inequality requires that either $\theta, \phi < 1$ or $\theta, \phi > 1$. If $\theta = \phi$, as in the expected utility case, this third condition is always satisfied. Under Hypothesis 3.1 the value of the positive constant

$$\frac{\rho - \alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}$$

will be important to distinguish between interior and corner solutions. In the two following propositions we will see what happens when this constant is greater or smaller than 1. We begin by describing the dynamics of the system in the interior solution case.

Proposition 3.2. *Let Hypothesis 3.1 be satisfied. Assume that:*

$$\frac{\rho - \alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} < 1. \tag{6}$$

Then the value function of the problem can be written explicitly. It is equal to:

$$V(A) = \frac{1}{1-\theta}\beta A^{1-\theta} \tag{7}$$

where

$$\beta = \left[\frac{\rho}{\frac{\sigma^2}{2}\theta}\left(\frac{\rho - \alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right)^{-\phi}\right]^{\frac{1-\theta}{1-\phi}}. \tag{8}$$

The optimal control is constant and deterministic, and it is given by:

$$f^*(t) = f^* := \frac{\rho - \alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}, \qquad \textit{for any } t \geq 0. \tag{9}$$

Finally, (5) guarantees the respect of the transversality condition.

Proof. See Appendix. □

Lemma 3.3. *The optimal conversion rate f^* is an increasing function of the discount rate ρ, a decreasing function of the intrinsic volatility of agricultural productivity σ, and a decreasing function of risk aversion θ. It is also a decreasing function of aversion to fluctuations ϕ if $\alpha - \rho < 0$, but an increasing function of aversion to fluctuations if $\alpha - \rho > 0$.*

Proof. Straightforward derivations of (9) give the results. □

The first three results are consistent with intuition. The higher the discount rate that is the more impatient society is, the less it cares about the future and the less it wants to ensure against future uncertainty. Such a society has a strong incentive to convert a large part of land to agriculture to enjoy present food consumption. Likewise, the higher intrinsic volatility and the more risk averse society is, the more it wants to ensure against future uncertainty.

Things are more complex regarding the effect of society's aversion to fluctuations. When the difference between the trend of agricultural productivity and the discount rate is negative, future prospects are – on average – rather poor and at the same time society is impatient. Both effects lead to better present outcomes than future outcomes. A society averse to fluctuations logically wants to counteract these forces, and is thus willing to ensure against adverse outcomes in the future by conserving more biodiversity. The opposite occurs when $\alpha - \rho > 0$.

Notice than when $\alpha > \rho$, increasing both risk aversion and aversion to fluctuations has an ambiguous effect on the optimal conversion rate, since the two parameters characterizing preferences play in opposite directions.

In the situation described in Proposition 3.2 a complete description of the optimal dynamics of the system can be provided. We have indeed the following corollary of the previous result.

Corollary 3.4. *Let the assumptions of Proposition 3.2 be satisfied. Then the optimal evolution of $A(t)$ and $C(t)$ are respectively:*

$$A(t) = A_0 \exp\left[\left(\alpha - \frac{\sigma^2}{2} f^*\right) t + \sqrt{f^*}\sigma W(t)\right] \tag{10}$$

and:

$$C(t) = f^* A(t).$$

In particular

$$\mathbb{E}\left[A(t)\right] = A_0 e^{\alpha t}, \qquad \mathbb{V}ar\left[A(t)\right] = A_0^2 e^{2\alpha t}\left(e^{\sigma^2 f^* t} - 1\right). \tag{11}$$

The dynamics of the optimal land productivity described in (10) is a geometric Brownian motion, so that at any time t the distribution of $A(t)$ is log-normal and has, respectively, the expected value and the variance described in (11). Given the expression of the dynamics of A in (2), the growth rate of the expected value of $A(t)$ only depends on the parameter α, while f^* positively impacts its variance.

We may now come back to the results of lemma 3.3 and go deeper into their signification. First, the effect appearing when risk aversion is increased can be seen as a form of precautionary saving effect. Indeed, increasing θ has the consequence of reducing f^* which, on the one hand, decreases the (certain) consumption level today but, on the other hand, increases the average value of the consumption growth rate[5] that is given by $\mathbb{E}[g] = \alpha - \frac{\sigma^2}{2} f^*$.

Second, increasing society's aversion to fluctuations also generates a level effect and a growth effect. Things are similar to what we have in the standard deterministic benchmark AK growth model (see for instance Acemoglu [1], Section 11.1). There we have a linear production function characterized by a technological level A, that corresponds to the parameter α of our model, and a discount rate ρ. In the case of the AK model, the parameter ϕ appearing in the instantaneous utility $u = \frac{C(t)^{1-\phi}}{1-\phi}$ cannot be interpreted as risk aversion since the model is deterministic. ϕ is indeed the inverse of the elasticity of intertemporal substitution, that is the aversion to fluctuations. In that context, the effect of increasing the aversion to

[5]According to equation (10), the average value of the growth rate is given by $\frac{1}{t}\mathbb{E}\left[\left(\alpha - \frac{\sigma^2}{2} f^*\right) t + \sqrt{f^*}\sigma W(t)\right]$ and it has not to be confused with the the value of the growth rate of the average value of $C(t)$ that is α, as shown in (11).

fluctuations parameter ϕ depends on the sign of $A-\rho$: if $A-\rho>0$ then increasing ϕ increases the initial consumption $C(0)$ and reduces the positive growth rate, while the opposite happens when $A-\rho<0$. In both cases the effect of increasing ϕ is that of equalizing the consumption over time[6]. Our model embodies the same mechanism. Indeed the effect of ϕ depends on the value of $\alpha-\rho$. When it is positive, the bigger ϕ is, the bigger the initial consumption $C(0)=f*A(0)$ is, and the smaller the average value of the growth rate (one can easily see that, as far as $\alpha-\rho>0$, $\frac{d\mathbb{E}[g]}{d\phi}=\frac{d}{d\phi}\left(\frac{\alpha(\theta-1}{\theta}+\frac{\alpha-\rho}{\phi\theta}\right)<0$). The opposite happens when $\alpha-\rho<0$.

We now characterize the value function in the corner solution case.

Proposition 3.5. *Let Hypothesis 3.1 be satisfied. Assume that*

$$\frac{\rho-\alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\geq 1. \tag{12}$$

Then the value function of the problem can be written explicitly. It is equal to:

$$V_C(A)=\frac{1}{1-\theta}\beta_C A^{1-\theta}, \tag{13}$$

where

$$\beta_C:=\left[\frac{1}{\rho}\left(\rho-\alpha(1-\phi)+\frac{\sigma^2}{2}\theta(1-\phi)\right)\right]^{-\frac{1-\theta}{1-\phi}}. \tag{14}$$

Moreover the optimal control is constant and deterministic, and it is given by:

$$f^*(t)=f^*:=1, \qquad \textit{for any } t\geq 0.$$

Proof. See Appendix. □

As underlined by the previous results, the structure of the value function is the same in the two cases (i.e. both are homogeneous of degree $1-\theta$) but, of course the multiplicative constants differ. A corollary similar to Corollary 3.4 can be obtained in the corner case: the optimal dynamics of $A(t)$ is described by (10) where, instead of f^*, we have 1.

[6]Being the optimal consumption in the AK model just exponential, the unique "fluctuation" that can be reduced in order to homogenize consumption over time is measured by the absolute value of its growth rate.

4 The value of biodiversity

In our model, the value of biodiversity comes from its ability to provide society with an insurance against the fluctuations of agricultural productivity. We want to make here this property more precise, so that the value of biodiversity can be properly quantified. To do so, we build on the famous works by Lucas [20], [21] on the welfare cost of fluctuations.

In reference to Lucas [20], [21] we define and compute the value of biodiversity as the welfare gain of biodiversity conservation. According to Lucas, the welfare cost of fluctuations is the percentage increase in consumption needed at all dates to compensate the representative agent for the presence of fluctuations, i.e. to make him indifferent between the actual consumption path, subject to fluctuations, and the corresponding deterministic consumption path. In the same spirit, we define here the welfare gain of biodiversity conservation as the percentage increase in consumption society is willing to accept to be in the case where all land is used for farming and biodiversity is nil, compared to the optimal case where the biodiversity level is $1 - f^*$. It is thus defined as follows:

Definition 4.1. *The welfare gain of biodiversity conservation is the percentage increase in consumption society is willing to accept at all dates to give up the optimal level of biodiversity at the benefit of no biodiversity at all.*

The value function in the no-biodiversity case is denoted by $V_B(A)$ and is characterized in the following proposition. Let λ be the welfare cost defined above. According to Definition 4.1, λ satisfies:

$$V\,(A) = V_B((1+\lambda)\,A). \tag{15}$$

Observe that when (12) is satisfied i.e. when we are at the optimum in the corner case $f^* = 1$, the optimal and the no-biodiversity solution are equivalent so in this section we suppose that (6) is verified, i.e. that we are in the interior case at the optimum. We will also have a technical assumption to be able to characterize the explicit form of the welfare in the no-biodiversity case.

Proposition 4.2. *Let Hypothesis 3.1 and Assumption (6) be satisfied and suppose that:*

$$\rho - \alpha(1-\phi) + \frac{\sigma^2}{2}\theta(1-\phi) > 0. \tag{16}$$

Then the welfare in the no-biodiversity case is given by:

$$V_B(A) = \frac{1}{1-\theta}\beta_B A^{1-\theta} \tag{17}$$

where

$$\beta_B := \left[\frac{1}{\rho}\left(\rho - \alpha(1-\phi) + \frac{\sigma^2}{2}\theta(1-\phi)\right)\right]^{-\frac{1-\theta}{1-\phi}}. \tag{18}$$

Proof. See Appendix. □

Remark 4.3. *Observe that, since (5) holds and we supposed that (6) is verified, $V_B(A)$ is always smaller than $V(A)$.*[7] *Indeed, if we denote by $a := \rho - \alpha(1-\phi) > 0$ and $b := \frac{\sigma^2}{2}\theta\phi > 0$, thanks to (6) we have $a < b$. We distinguish two cases: $\phi > 1$ and $\phi < 1$ (we cannot have $\phi = 1$ because of Hypothesis 3.1).*

If $\phi > 1$ the function $(a,b) \mapsto a^\phi b^{1-\phi}$ is convex and then

$$\beta^{-\frac{1-\phi}{1-\theta}} = \frac{b}{\rho\phi}\left(\frac{a}{b}\right)^\phi = \frac{1}{\rho\phi}a^\phi b^{1-\phi} > \frac{1}{\rho\phi}(\phi a + (1-\phi)b) = \frac{1}{\rho}\left(a + (\frac{1}{\phi} - 1)b\right) = \beta_B^{-\frac{1-\phi}{1-\theta}}. \tag{19}$$

So, thanks to (5) we can conclude that $\beta < \beta_B$ (both are positive). Thanks to (5), when $\phi > 1$, we have $\theta > 1$ thus the factor $\frac{1}{1-\theta}$ is negative and, from the previous relation between β and β_B, we can conclude that $V_B(A) < V(A) < 0$ for any positive A.

Conversely, if $\phi < 1$ the function $(a,b) \mapsto a^\phi b^{1-\phi}$ is concave and then $\beta^{-\frac{1-\phi}{1-\theta}} < \beta_B^{-\frac{1-\phi}{1-\theta}}$. So $\beta > \beta_B$ and (now $1-\theta > 0$) $0 < V_B(A) < V(A)$ for any positive A.

Remark 4.4. *From Remark 2, since we suppose that (5) and (6) are verified, both V and V_B are increasing functions of A, and as we showed that $V_B(A)$ is always smaller than $V(A)$ then λ is always positive.*

Proposition 4.5. *The welfare cost of biodiversity losses is:*

$$\lambda = f^*\left(\phi + \frac{1-\phi}{f^*}\right)^{\frac{1}{1-\phi}} - 1. \tag{20}$$

Proof. See Appendix. □

[7]This result follows already from the fact that $V(A)$ is the maximum of the welfare varying the control among a set containing in particular the control $f \equiv 1$ that is the control chosen in the benchmark. We give in any case a direct argument because the expressions of β and β_B are not, at a first look, immediately comparable.

Lemma 4.6. *The value of biodiversity λ is an increasing function of the intrinsic volatility of agricultural productivity and of risk aversion. However, the effect of aversion to fluctuations on the value of biodiversity is ambiguous.*

Proof. See Appendix. □

The first two results are intuitive and fit well with the effects of intrinsic volatility and risk aversion on the conversion rate. Indeed, more intrinsic volatility of agricultural productivity and more risk aversion result in a lower optimal conversion rate i.e. more insurance, and a higher value of biodiversity, i.e. a higher welfare cost of biodiversity losses.

To investigate further the role of aversion to fluctuations it is useful to look at the case where the optimal conversion rate is very high (f^* close to 1).

Lemma 4.7. *For f^* close to 1,*

$$\lambda \simeq \frac{\phi}{2}\left(1-f^*\right)^2 \tag{21}$$

and

$$\frac{d\lambda}{d\phi} \simeq \frac{1}{2}\frac{1-f^*}{\frac{\sigma^2}{2}\theta\phi}\left[\rho-\alpha+\left(\frac{\sigma^2}{2}\theta-\alpha\right)\phi\right]. \tag{22}$$

Proof. See Appendix. □

When the optimal conversion rate is close to 1, the value of biodiversity is proportional to the square of the share of land optimally devoted to maintain biodiversity.

According to Assumption (6), we have: $\left(\frac{\sigma^2}{2}\theta-\alpha\right)\phi > \rho-\alpha$; likewise, according to Assumption (16), we have: $\left(\frac{\sigma^2}{2}\theta-\alpha\right)\phi < \rho-\alpha+\frac{\sigma^2}{2}\theta$. Then:

$$\frac{1-f^*}{\frac{\sigma^2}{2}\theta\phi}\left(\rho-\alpha\right) < \frac{d\lambda}{d\phi} < \frac{1-f^*}{\frac{\sigma^2}{2}\theta\phi}\left[\rho-\alpha+\frac{\sigma^2}{4}\theta\right].$$

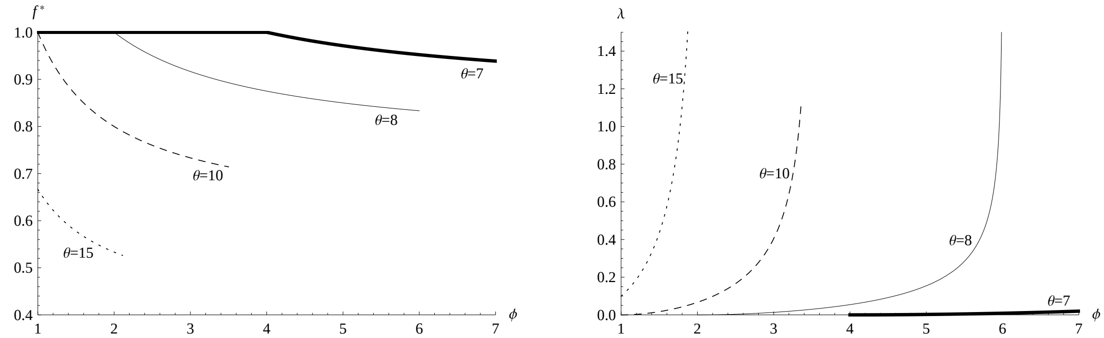
The first inequality implies that if $\rho-\alpha>0$, then $\frac{d\lambda}{d\phi}>0$. The second one implies that if $\rho-\alpha+\frac{\sigma^2}{4}\theta<0$, then $\frac{d\lambda}{d\phi}<0$; this requires $\rho-\alpha<0$ and, besides, is all the more likely since intrinsic volatility σ and risk aversion θ are small.

At a first look one could imagine that the signs of the effect of increasing the aversion to fluctuations (and in fact any other parameter) on the value of biodiversity and on the share of land devoted to biodiversity conservation should be the same. It is actually what happens when the discount rate ρ is higher than the trend of productivity α. Then, increasing aversion to fluctuations increases both the share of land devoted to biodiversity conservation and the value of biodiversity. It is also what happens when the discount rate ρ is lower than the trend of productivity α, that is when society is patient and has on average good economic prospects, and intrinsic volatility and risk aversion are small. Now, increasing the aversion to fluctuations decreases both the share of land devoted to biodiversity conservation and the value of biodiversity. The need for insurance is low in these circumstances. Nevertheless, if we look at equation (21), we can see that two effects are at work: on the one hand a direct effect (the term $\phi/2$) through which increasing ϕ increases λ, on the other hand an indirect effect (the term $(1-f^*)^2$) of the same sign as the effect of ϕ on the share of land devoted to biodiversity conservation. When the trend of productivity α is higher than the discount rate ρ, the two effects are discordant. The first effect is all the more likely to prevail since intrinsic volatility and risk aversion are big, as it becomes clear if we write equation (22) as:

$$\frac{d\lambda/\lambda}{d\phi/\phi} \simeq 1 - \frac{2(\alpha-\rho)}{(\alpha-\rho)\left(\frac{\sigma^2}{2}\theta-\alpha\right)\phi}.$$

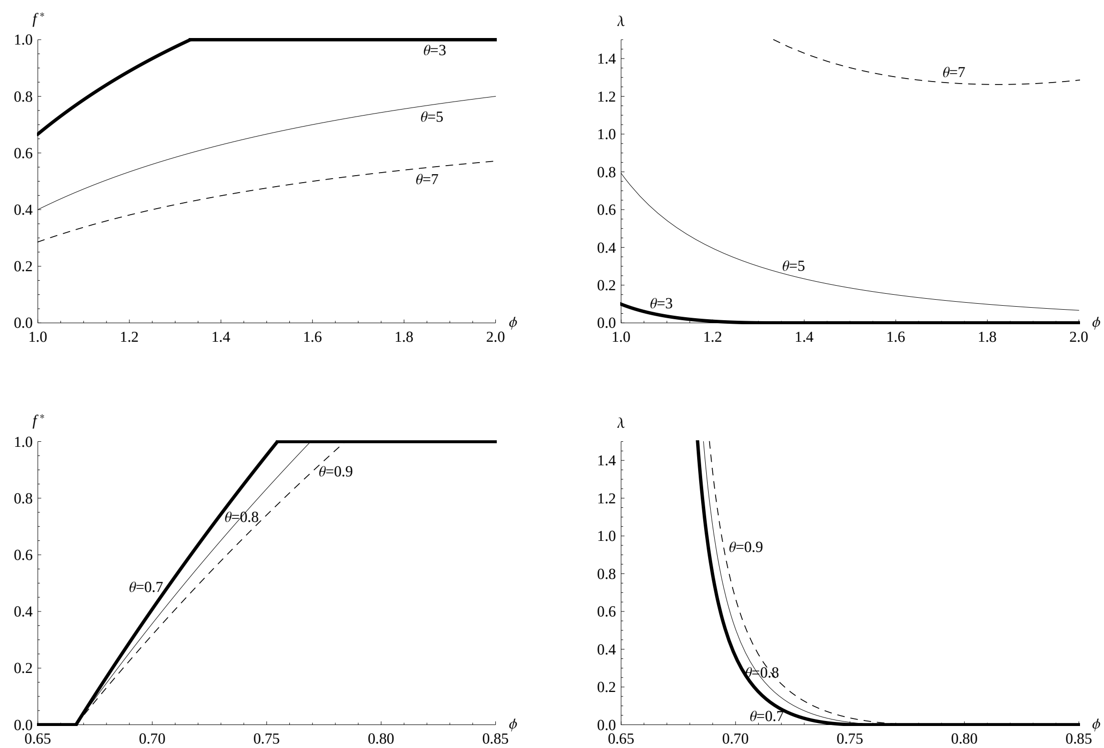
In this case, increasing the aversion to fluctuations decreases the share of land devoted to biodiversity conservation and at the same time increases the value of biodiversity.

Simulations allow us to check that these results hold when f^* is not supposed to be close to 1 (see Figure 1 for a case where $\alpha < \rho$, Figures 2 and 3 for $\alpha > \rho$). They also allow us to exhibit sets of parameters σ, θ and ϕ satisfying the constraints imposed in the model, such that, when $\alpha > \rho$, we have $\frac{d\lambda}{d\phi} > 0$ (see Figure 3). A general comment of these simulations is that the value of biodiversity is very sensitive to the aversion to fluctuations, and can become very high when f^* is not close to one, that is when it is optimal to allocate a significant share of land to biodiversity conservation.



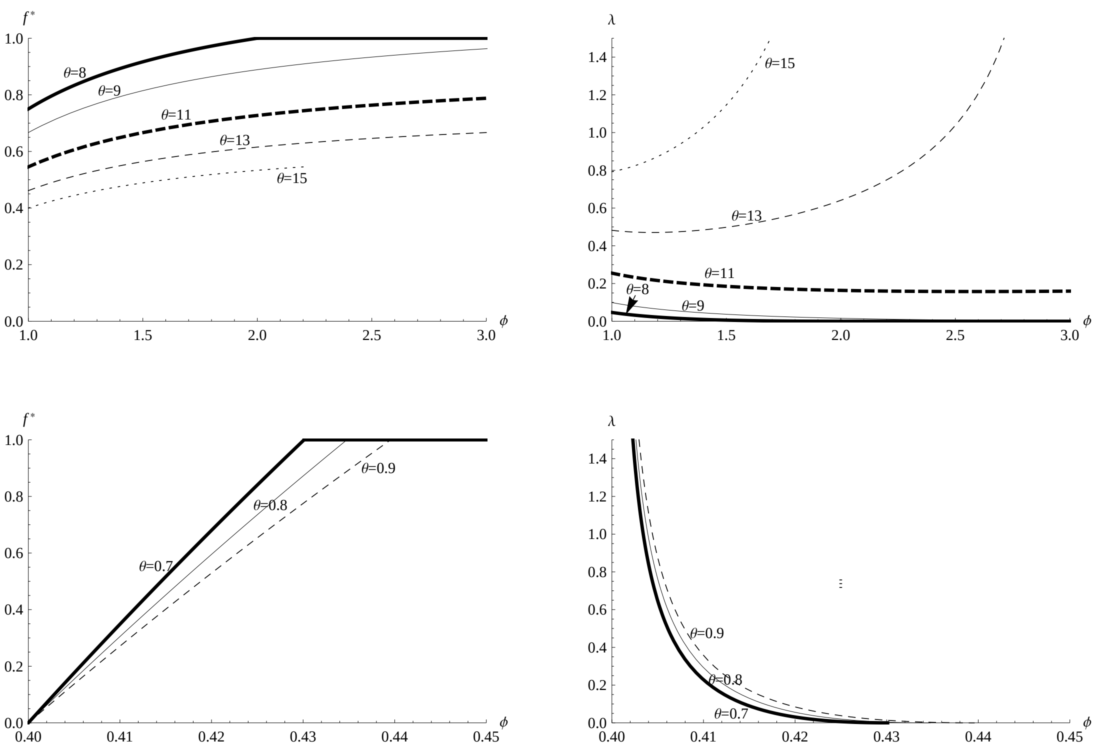
(a) Optimal conversion rate (b) Value of biodiversity

Figure 1: Optimal conversion rate and biodiversity value as a function of aversion to fluctuations for $\alpha = 0.03$, $\rho = 0.05$, $\sigma = 0.1$.



(a) Optimal conversion rate (b) Value of biodiversity

Figure 2: Optimal conversion rate and biodiversity value as a function of aversion to fluctuations for $\alpha = 0.03$, $\rho = 0.01$, $\sigma = 0.1$.



(a) Optimal conversion rate

(b) Value of biodiversity

Figure 3: Optimal conversion rate and biodiversity value as a function of aversion to fluctuations for $\alpha = 0.05$, $\rho = 0.03$, $\sigma = 0.1$.

5 The conversion of a common-property resource: volatility externalities at work

We now consider a decentralized version of the model where two farmers can appropriate land, a common property resource, for farming purposes. The two farmers are indexed by $i \in \{1,2\}$. The total amount of land available is still normalized to 1. Farmer i ($i \in \{1,2\}$) may appropriate some share $f_i(t)$ of this total amount, knowing that the following constraint needs to be verified:

$$f_1(t) + f_2(t) \leq 1, \qquad t \geq 0.$$

So the set of admissible strategies of player i ($i \in \{1,2\}$) depends on the strategy chosen by the other player (denoted with "f_{-i}"). More precisely, given $f_{-i}(t)$, it has the following expression

$$\mathcal{U}^i_{f_{-i}} := \left\{ f_i(\cdot)\colon [0,+\infty)\times\Omega \to [0,1] \ : f_i(\cdot) \text{ is } \mathcal{F}_t\text{−progr. meas. and } f_1(t)+f_2(t) \leq 1 \right\}. \tag{23}$$

Definition 5.1. *A couple* $(f_1(\cdot), f_2(\cdot))$ *of* $[0,1]$*-valued* $\mathcal{F}_t$*-adapted processes is said to be an* admissible couple of strategies *if, for any* $i \in \{1,2\}$, $f_i \in \mathcal{U}^i_{f_{-i}}$.

Given an admissible couple of strategies $(f_1(\cdot), f_2(\cdot))$ the total share of land devoted to farming at time t is $f_1(t) + f_2(t)$ while the share of land used to preserve biodiversity at time t is $(1 - f_1(t) - f_2(t))$.

We assume that farmers 1 and 2 are potentially heterogeneous according to their farming activities (we will assume $\alpha_1 \leq \alpha_2$), and according to their discount rates denoted ρ_1 and ρ_2. The productivity of a unit of land appropriated by farmer i for farming at time $t \geq 0$ is given by:

$$\begin{cases} \mathrm{d}A_i(t) = \alpha_i A_i(t)\,\mathrm{d}t + \sqrt{f_1(t) + f_2(t)}\,\sigma_i A_i(t)\,\mathrm{d}W(t) \\ A_i(0) = A^i_0. \end{cases} \tag{24}$$

The volatility externality comes from the fact that the conversion decisions of the two farmers affect the volatility of the agricultural productivity of each of them.

Given an $\mathcal{F}_t$-adapted $[0,1]$-valued strategy $\overline{f}_2(\cdot)$ for the player 2 we will say that $\overline{f}_1(\cdot) \in \mathcal{U}^1_{\overline{f}_2}$ is a best response of player 1 to $\overline{f}_2(\cdot)$ if it is an optimal strategy (among all the strategies of $\mathcal{U}^1_{\overline{f}_2}$) for the optimization problem characterized by the state equation (24) (where we consider

$i = 1$ and $f_2(\cdot) = \overline{f}_2(\cdot)$) and Epstein-Zin-Weil utility function with a constant relative risk aversion θ, an elasticity of intertemporal substitution ϕ^{-1} and a discount rate ρ. Similarly we define a best response of player 2 to some strategy $\overline{f}_1(\cdot)$ of player 1. A couple of $[0,1]$-valued, $\mathcal{F}_t$-adapted processes $(f_1(\cdot), f_2(\cdot))$ is said to be a Nash equilibrium if $f_1(\cdot)$ is a best response to $f_2(\cdot)$ and $f_2(\cdot)$ is a best response to $f_1(\cdot)$.

Proposition 5.2. *Provided that* $\frac{1-\phi}{1-\theta} > 0$, *and that*

$$\rho_1 > \max\left\{\alpha_1(1-\phi), \left(\alpha_1 - \frac{\sigma^2}{2}\theta\right)(1-\phi)\right\}, \tag{25}$$

$$\rho_2 > \max\left\{\alpha_2(1-\phi), \left(\alpha_2 - \frac{\sigma^2}{2}\theta\right)(1-\phi)\right\}, \tag{26}$$

the best response of farmer 1 for a constant and deterministic strategy $f_2(t) = f_2 \in (0,1)$, $t \geq 0$ *of farmer 2 is the constant and deterministic strategy with value*

$$f_1 = \min\left(\frac{\rho_1 - \left(\alpha_1 - \frac{\sigma^2}{2}\theta f_2\right)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}, 1 - f_2\right) \tag{27}$$

and similarly for farmer 2.

Proposition 5.3. *If, in addition,*

$$\phi > \frac{1}{2} \tag{28}$$

and

$$\frac{\phi}{2\phi - 1}\frac{(\rho_1 + \rho_2 - (\alpha_1 + \alpha_2)(1-\phi))}{\frac{\sigma^2}{2}\theta\phi} \in (0,1) \tag{29}$$

then the couple of constant strategies $(f_1(t), f_2(t)) = (\widetilde{f}_1, \widetilde{f}_2)$, *for any* $t \geq 0$, *where*

$$\widetilde{f}_1 = \frac{\phi}{2\phi-1}\left[\phi\frac{\rho_1 - \alpha_1(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} + (1-\phi)\frac{\rho_2 - \alpha_2(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right] \tag{30}$$

$$\widetilde{f}_2 = \frac{\phi}{2\phi-1}\left[\phi\frac{\rho_2 - \alpha_2(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} + (1-\phi)\frac{\rho_1 - \alpha_1(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right] \tag{31}$$

is a Nash equilibrium. It is the unique Nash equilibrium in constant and deterministic strategies.

Proof. See Appendix. □

When the assumptions of Proposition 5.2 are verified the total amount of land devoted to farming activity is:

$$\widetilde{f} = \frac{\phi}{2\phi - 1} \frac{(\rho_1 + \rho_2 - (\alpha_1 + \alpha_2)(1 - \phi))}{\frac{\sigma^2}{2}\theta\phi} \tag{32}$$

Denoting by ρ the average discount rate of the two players ($\rho = (\rho_1 + \rho_2)/2$) and α the average trend of productivity in the economy ($\alpha = (\alpha_1 + \alpha_2)/2$) allows to write the total amount of land devoted to farming as:

$$\widetilde{f} = \frac{2\phi}{2\phi - 1} \frac{\rho - \alpha(1 - \phi)}{\frac{\sigma^2}{2}\theta\phi}. \tag{33}$$

If we compare this expression with (9) we observe that (since we suppose that condition (28) is verified) the total area of land devoted to farming in the 2-player case is larger than the optimal level:

$$\widetilde{f} = \frac{2\phi}{2\phi - 1} f^* > f^*. \tag{34}$$

Notice that the factor $\frac{2\phi}{2\phi-1}$ measuring the increment in area devoted to farming is decreasing with the aversion to fluctuations ϕ and tends to 1 when ϕ is very high. Hence a high aversion to fluctuations tends to overcome the incentive that the two farmers face of appropriating too much land for their farming activity.

If we extend the analysis to the corner solution case where condition (29) is violated and $\frac{\phi}{2\phi-1} \frac{\rho_1+\rho_2-(\alpha_1+\alpha_2)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi} \geq 1$, a continuous of deterministic and constant Nash equilibria arises and in correspondence of any of them the land used to maintain biodiversity is zero.

An analogous N-player version of the game can be studied. In this case, for any choice of parameters, letting N to infinity induces a no-biodiversity preservation outcome. It can be seen as the usual situation when a large number of agents interact. This fact suggests that the reference level $f = 1$ studied in Section 4 is indeed a good benchmark.

6 Conclusion

This paper presents a stylized dynamic model including a particular motive for biodiversity conservation: its insurance value against fluctuations of agricultural productivity. Producing food requires land, and increasing the share of total land devoted to farming mechanically reduces the share of land devoted to biodiversity conservation. However, the safeguarding of a greater number of species is associated to better ecosystem services – pollination, flood control, pest control, etc., which in turn ensure a lower volatility of agricultural productivity. The optimal conversion/conservation rule is explicitly characterized, as well as the value (in terms of the welfare cost of biodiversity losses) of biological diversity. The Epstein-Zin-Weil specification of the utility function allows us to disentangle the effects of risk aversion and aversion to fluctuations. A two-player game extension of the model allows to highlight the effect of volatility externalities and the Paretian sub-optimality of the decentralized choices.

We identify at least two interesting extensions of our work.

First, we do not enter into the well known debate on rent sparing versus rent sharing initiated by Green *et al.* [14]. The debate aims at determining whether agriculture should be concentrated on intensively farmed land in order to conserve more natural spaces for biodiversity, or should be extensive, less productive, and wildlife-friendly. Our framework can encompass both cases. It would nevertheless be interesting to distinguish between the two management practices, which do not have the same consequences in terms of average agricultural productivity and of volatility.

Second, we consider here that the economy does not have access to financial insurance and that there are no savings/storage possibilities. Indeed, if a financial insurance system and/or a storage device were available, farmers could insure against adverse outcomes by other means than biodiversity conservation. Quaas and Baumgartner [26] study this problem and show in a static framework that both types of insurance (natural and financial) are substitutes. It would be interesting to see whether their result holds in a dynamic framework, and how the arbitrage depends on risk aversion and aversion to fluctuations, separately.

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Appendix: Proof of the results

Proof of Proposition 3.2. By Proposition 9 and Appendix C of Duffie and Epstein [7]) the value function V can be characterized as the solution of the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \sup_{f\in[0,1]} \left(\alpha A V'(A) + F(fA, V) + \frac{1}{2} f \sigma^2 A^2 V''(A) \right), \tag{35}$$

where V' and V'' are the first and the second derivative of $V(A)$ and $F(C,V)$ is the aggregator defined in (4). This expression can be rewritten as:

$$\rho\frac{1-\theta}{1-\phi}V(A) = \sup_{f\in[0,1]}\left(\alpha AV'(A)+\frac{\rho}{1-\phi}(1-\theta)V(A)\left(\frac{fA}{((1-\theta)V(A))^{\frac{1}{1-\theta}}}\right)^{1-\phi}+\frac{1}{2}f\sigma^2A^2V''(A)\right). \tag{36}$$

We want to prove that the function defined in (7) is a solution of such an equation. We try to find a solution of the form

$$V(A) = \frac{1}{1-\theta}\beta A^{1-\theta} \tag{37}$$

for some $\beta > 0$. We have

$$V'(A) = \beta A^{-\theta} \tag{38}$$

and

$$V''(A) = -\theta\beta A^{-\theta-1}.$$

So V of the prescribed form is a solution if and only if:

$$\begin{aligned} 0 &= \sup_{f\in[0,1]}\left(-\frac{\rho}{1-\phi}\beta A^{1-\theta}+\frac{\rho}{1-\phi}\beta A^{1-\theta}\left(\frac{fA}{\left(\beta A^{1-\theta}\right)^{\frac{1}{1-\theta}}}\right)^{1-\phi}+\alpha A\beta A^{-\theta}-\frac{1}{2}f\sigma^2A^2\theta\beta A^{-\theta-1}\right)\\ &= \beta A^{1-\theta}\sup_{f\in[0,1]}\left(-\frac{\rho}{1-\phi}+\frac{\rho}{1-\phi}\left(\frac{fA}{\left(\beta A^{1-\theta}\right)^{\frac{1}{1-\theta}}}\right)^{1-\phi}+\alpha-f\frac{\sigma^2}{2}\theta\right). \end{aligned} \tag{39}$$

The f that maximizes this Hamiltonian is given by

$$f^* = \left(\frac{1}{\rho}\frac{\sigma^2}{2}\theta\beta^{\frac{1-\phi}{1-\theta}}\right)^{-1/\phi} \tag{40}$$

(after finding the expression of β we will be able to show that this expression is indeed always in $(0,1)$). Using this expression in (39) and simplifying $\beta A^{1-\theta}$ we can see that a function of the form (37) is a solution of the HJB equation if and only if:

$$0 = -\frac{\rho}{1-\phi}+\frac{\rho}{1-\phi}\left(\frac{\left(\frac{1}{\rho}\frac{\sigma^2}{2}\theta\beta^{\frac{1-\phi}{1-\theta}}\right)^{-1/\phi}}{\beta^{\frac{1}{1-\theta}}}\right)^{1-\phi}+\alpha-\left(\frac{1}{\rho}\frac{\sigma^2}{2}\theta\beta^{\frac{1-\phi}{1-\phi}}\right)^{-1/\theta}\frac{\sigma^2}{2}\theta, \tag{41}$$

so (after some computations) if and only if:

$$\beta = \left(\frac{\rho\left(\frac{\rho-\alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right)^{-\phi}}{\frac{\sigma^2}{2}\theta}\right)^{\frac{1-\theta}{1-\phi}}.$$

Using this expression and (6) one can easily see that the expression of f^* given in (40) is always in $(0,1)$ and then the control $f(t) \equiv f^*$ is admissible. Thanks to the general theory (see again Proposition 9 and Appendix C of Duffie and Epstein [7]), since it is obtained as the feedback provided by a solution of the HJB of the problem it is the optimal control of the problem.

We now show that condition (5) guarantees the respect of the transversality condition.

The term $\frac{\rho(1-\theta)}{1-\phi}V(A)$ appearing in the HJB (36) for the recursive utility corresponds to the standard term $\rho V(A)$ appearing in the standard HJB arising for separable expected utility functionals so the counterpart of the standard discount $e^{-\rho t}$ is, in the recursive utility setting, given by $e^{-\rho\frac{1-\theta}{1-\phi}t}$ (recall that, as already observed, if we choose $\phi = \theta$, the recursive utility case reduces to the separable expected utility).

Indeed to prove the verification result for our infinite horizon case one has to argue as in the proof of Proposition 9 of Duffie and Epstein [7] using the function $\tilde{V}(t,A) = e^{-\rho\frac{1-\theta}{1-\phi}t}V(A)$ (being $V(A)$ the value function of our problem, characterized in Proposition 3.2), letting the final time T tend to infinity. Given the sign of $V(A)$ (positive when $\theta \in (0,1)$ and negative is $\theta > 1$) we need to distinguish two cases:

(i) *if* $\theta \in (0,1)$ we need to show that $\lim_{T\to+\infty} \mathbb{E}\left[\tilde{V}(T,A(T))\right] = 0$ for any admissible trajectory $A(\cdot)$. We have that $\mathbb{E}\tilde{V}(T,A(T)) = e^{-\rho\frac{1-\theta}{1-\phi}T}\frac{1}{1-\theta}\beta\mathbb{E}\left[(A(T))^{1-\theta}\right]$. One can easily realize that the maximum possible value for $\mathbb{E}\left[(A(T))^{1-\theta}\right]$ is attained when we choose $f(t)$ always equal to 0. In this case we have:

$$e^{-\rho\frac{1-\theta}{1-\phi}T}\frac{1}{1-\theta}\beta\mathbb{E}\left[(A(T))^{1-\theta}\right] = \frac{1}{1-\theta}\beta A_0^{1-\theta}e^{-\rho\frac{1-\theta}{1-\phi}T+\alpha(1-\theta)T} = \frac{1}{1-\theta}\beta A_0^{1-\theta}e^{\left(\frac{1-\theta}{1-\phi}(-\rho+\alpha(1-\phi))\right)T}.$$

So, since $\theta \in (0,1)$, its limit for $T \to \infty$ is equal to zero if and only if $\frac{1-\theta}{1-\phi}(-\rho+\alpha(1-\phi)) < 0$. This condition (that corresponds to the usual condition $\rho > (1-\theta)\alpha$ that we have in the separable expected utility case) is implied by (5) (by multiplying the two conditions).

(ii) *if* $\theta > 1$ we need to show that there exists at least an admissible trajectory $A(\cdot)$ satisfying $\lim_{T\to+\infty} \mathbb{E}\left[\tilde{V}(T,A(T))\right] \neq -\infty$. We consider, as an admissible control, $f(t) = f^*$ for any $t \geq 0$. We have then:

$$A(T) = A_0 e^{\alpha - \frac{\sigma^2}{2}f^*T + \sqrt{f^*}\sigma W(T)}$$

and then (using the expression for the moments of the log-normal distributions):

$$\mathbb{E}[A(T)^{1-\sigma} = A_0^{1-\sigma}e^{(1-\theta)(\alpha-\theta f^*\frac{\sigma^2}{2}}$$

so that (by writing explicitly the value of f^* given in the text of the proposition):

$$\begin{aligned}\mathbb{E}\left[\tilde{V}(T, A(T))\right] &= e^{-\rho\frac{1-\theta}{1-\phi}T}\frac{1}{1-\theta}\beta\mathbb{E}\left[(A(T))^{1-\theta}\right] = \frac{1}{1-\theta}\beta A_0^{1-\theta}e^{-\rho\frac{1-\theta}{1-\phi}T}e^{\frac{(\alpha-\rho)(1-\theta)}{\phi}T} \\ &= \frac{1}{1-\theta}\beta A_0^{1-\theta}e^{\frac{1-\theta}{1-\phi}(-\rho+\alpha(1-\phi))T}.\end{aligned}$$

Its limit for $T \to \infty$ is equal to zero if and only if $\frac{1-\theta}{1-\phi}(-\rho+\alpha(1-\phi)) < 0$. As before, this condition is implied by (5).

□

Proof of Proposition 3.5. The proof follows the lines of that of Proposition 3.2. We have again to find a solution of the HJB (35) so we need to prove that, if the condition (12) is verified, the following equation is satisfied:

$$0 = \beta_C A^{1-\theta} \sup_{f\in[0,1]}\left(-\frac{\rho}{1-\phi}+\frac{\rho}{1-\phi}\left(\frac{fA}{\left(\beta_C A^{1-\theta}\right)^{\frac{1}{1-\theta}}}\right)^{1-\phi}+\alpha-f\frac{\sigma^2}{2}\theta\right), \tag{42}$$

that, using the expression of β_C given in (14) and simplifying the term $\beta_C A^{1-\theta}$, becomes,

$$\begin{aligned}0 &= \sup_{f\in[0,1]}\left[-\frac{\rho}{1-\phi}+\frac{\rho}{1-\phi}\left(\frac{f}{\left(\left(\frac{\rho-\alpha(1-\phi)}{\rho}+\frac{1}{\rho}\frac{\sigma^2}{2}\theta(1-\phi)\right)^{-\frac{1-\theta}{1-\phi}}\right)^{\frac{1}{1-\theta}}}\right)^{1-\phi}+\alpha-f\frac{\sigma^2}{2}\theta\right] \\ &= \sup_{f\in[0,1]}\left[-\frac{\rho}{1-\phi}+\left(f^{1-\phi}\left(\frac{\rho}{1-\phi}-\alpha+\frac{\sigma^2}{2}\theta\right)\right)+\alpha-f\frac{\sigma^2}{2}\theta\right].\end{aligned}$$

One can easily see that the function $f \mapsto \left(f^{1-\phi}\left(\frac{\rho}{1-\phi}-\alpha+\frac{\sigma^2}{2}\theta\right)\right)-f\frac{\sigma^2}{2}\theta$, defined on $(0,+\infty)$ is increasing in the interval $[0,\bar{f}]$ where

$$\bar{f} := \left(\frac{\frac{\sigma^2}{2}\theta}{(\rho-\alpha(1-\phi))+\frac{\sigma^2}{2}\theta(1-\phi)}\right)^{-\theta},$$

and then it decreasing. Since (12) is verified, $\bar{f} \geq 1$ so the maximum of the function restricted to the interval $[0,1]$ is in $f^* = 1$. Using this fact we can rewrite (43) as:

$$0 = -\frac{\rho}{1-\phi}+\left(\frac{\rho}{1-\phi}-\alpha+\frac{\sigma^2}{2}\theta\right)+\alpha-\frac{\sigma^2}{2}\theta$$

that can easily be seen to be true. Concluding as in the proof of Proposition 3.2, we prove that the function defined in (13) is indeed the value function of the problem and that $f^*(t) \equiv f^* := 1$ is the optimal control of the problem. □

Proof of Proposition 4.2. The argument is exactly the same we used to prove Proposition 3.2 but here the only possible choice of f is 1 so the HJB equation becomes:

$$\rho\frac{1-\theta}{1-\phi}V(A) = \left[\alpha A V'(A) + \frac{\rho}{1-\phi}(1-\theta)V(A)\left(\frac{fA}{\left((1-\theta)V(A)\right)^{\frac{1}{1-\theta}}}\right)^{1-\phi} + f\frac{\sigma^2}{2}A^2V''(A)\right]_{f=1}.$$

The explicit solution of this equation (that we find by inspection as in the proof Proposition 3.2) is the welfare in the benchmark (it is the function of the problem where one can only choose $f = 1$). Not surprisingly, its form is the same of the value function of the corner case (Proposition 3.5) where the optimal control was always chosen equal to 1. □

Proof of Proposition 4.5. We use the expressions of V and V_C to compute $V(A) = V_C((1+\lambda)A)$. We obtain:

$$\left(\frac{\rho\left(\frac{\rho-\alpha(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right)^{-\phi}}{\frac{\sigma^2}{2}\theta}\right)^{\frac{1-\theta}{1-\phi}} A^{1-\theta} = \left(\frac{\rho-\alpha(1-\phi)}{\rho} + \frac{1}{\rho}\frac{\sigma^2}{2}\theta(1-\phi)\right)^{-\frac{1-\theta}{1-\phi}}(1+\lambda)^{1-\theta}A^{1-\theta},$$

i.e.

$$\left(f^{*-\phi}\left[f^*\phi + (1-\phi)\right]\right)^{\frac{1-\theta}{1-\phi}} = (1+\lambda)^{1-\theta},$$

i.e. (20). □

Proof of Lemma 4.6.

$$\frac{d\lambda}{df^*} = \phi\left(\phi + \frac{1-\phi}{f^*}\right)^{\frac{\phi}{1-\phi}}\left(1-\frac{1}{f^*}\right) < 0.$$

Thus:

$$\begin{aligned}
\frac{d\lambda}{d\theta} &= \frac{d\lambda}{df^*}\frac{df^*}{d\theta} = -2\frac{d\lambda}{df^*}\frac{\rho-\alpha(1-\phi)}{\sigma^2\theta^2\phi} > 0\\
\frac{d\lambda}{d\sigma^2} &= \frac{d\lambda}{df^*}\frac{df^*}{d\sigma^2} = -2\frac{d\lambda}{df^*}\frac{\rho-\alpha(1-\phi)}{\sigma^4\theta\phi} > 0\\
\frac{d\lambda}{d\phi} &= \frac{d\lambda}{df^*}\frac{df^*}{d\phi} + \frac{d\lambda}{d\phi} = -2\frac{d\lambda}{df^*}\frac{\rho-\alpha}{\sigma^2\theta\phi^2} +\\
&\quad f^*\left(\phi+\frac{1-\phi}{f^*}\right)^{\frac{1}{1-\phi}}\left[\frac{1}{(1-\phi)^2}\ln\left(\phi+\frac{1-\phi}{f^*}\right) + \frac{1}{1-\phi}\frac{1-\frac{1}{f^*}}{\phi+\frac{1-\phi}{f^*}}\right].
\end{aligned}$$

□

Proof of Lemma 4.7. One has only to use the Taylor expansion of the expression of λ. The first order term vanishes while the second gives the claim. □

Proof of Proposition 5.2. We start by characterizing the optimal response of player 1 given a certain fixed deterministic constant strategy f_2 of player 2. To do this we solve the dynamic optimization problem where the value of f_2 is considered fixed (and deterministic). We use again Proposition 9 and Appendix C of Duffie and Epstein [7]) the value function V of the player 1's problem can be characterized as the solution of the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \sup_{f_1 \in [0,1-f_2]} \left(\alpha_1 A_1 V'(A_1) + F(f_1 A_1, V) + \frac{1}{2} (f_1 + f_2) \sigma^2 A_1^2 V''(A_1) \right). \tag{43}$$

where V' and V'' are the first and the second derivative of $V(A)$ and $F(C, V)$ is the aggregator defined in (4). This expression can be rewritten as

$$\begin{aligned} 0 &= \sup_{f_1 \in [0,1-f_2]} \left(\alpha_1 A_1 V'(A_1) + \frac{\rho_1}{1-\phi}(1-\theta)V(A_1) \left[\left(\frac{f_1 A_1}{((1-\theta)V(A_1))^{\frac{1}{1-\theta}}} \right)^{1-\phi} - 1 \right] \right. \\ &+ \left. (f_1 + f_2) \frac{\sigma^2}{2} A_1^2 V''(A_1) \right). \end{aligned}$$

We want to prove that the function defined in (7) is a solution of such an equation. We try to find a solution of the form

$$V(A) = \frac{1}{1-\theta} \beta_1 A^{1-\theta} \tag{44}$$

for some $\beta_1 > 0$.

So V of the prescribed form is a solution if and only if:

$$\begin{aligned} 0 &= \sup_{f_1 \in [0,1-f_2]} \left(\frac{\rho_1}{1-\phi} \beta_1 A_1^{1-\theta} \left(\frac{f_1}{(\beta)^{\frac{1}{1-\theta}}} \right)^{1-\phi} - f_1 \frac{\sigma^2}{2} A_1^2 \theta \beta_1 A_1^{-\theta-1} \right) \\ &+ \alpha_1 A \beta_1 A^{-\theta} - \frac{\rho_1}{1-\phi} \beta_1 A_1^{1-\theta} - f_2 \frac{\sigma^2}{2} A_1^2 \theta \beta_1 A_1^{-\theta-1} \\ &= \beta_1 A_1^{1-\theta} \left[\sup_{f_1 \in [0,1-f_2]} \left(\frac{\rho_1}{1-\phi} \left(\frac{f_1}{(\beta)^{\frac{1}{1-\theta}}} \right)^{1-\phi} - f_1 \frac{\sigma^2}{2} \theta \right) + \alpha_1 - \frac{\rho_1}{1-\phi} - f_2 \frac{\sigma^2}{2} \theta \right]. \end{aligned}$$

The interior f_1 that maximizes this Hamiltonian is given by:

$$f_1^* = \left(\frac{1}{\rho_1} \frac{\sigma^2}{2} \theta \beta_1^{\frac{1-\phi}{1-\theta}} \right)^{-1/\phi}. \tag{45}$$

We then replace to find β_1:

$$0 = \frac{\rho_1}{1-\phi} \left(\frac{\left(\frac{1}{\rho_1} \frac{\sigma^2}{2} \theta \beta_1^{\frac{1-\phi}{1-\theta}} \right)^{-1/\phi}}{\beta_1^{\frac{1}{1-\theta}}} \right)^{1-\phi} - \left(\frac{1}{\rho_1} \frac{\sigma^2}{2} \theta \beta_1^{\frac{1-\phi}{1-\theta}} \right)^{-1/\phi} \frac{\sigma^2}{2} \theta + \alpha_1 - \frac{\rho_1}{1-\phi} - \frac{\sigma^2}{2} \theta, \tag{46}$$

i.e.
$$\frac{\phi}{1-\phi}\left(\frac{1}{\rho_1}\right)^{-1/\phi}\left(\frac{\sigma^2}{2}\theta\right)^{-\frac{(1-\phi)}{\phi}}\left(\beta_1^{\frac{1-\phi}{1-\theta}}\right)^{-1/\phi}=\frac{\rho_1-[\alpha_1-f_2\theta\theta](1-\phi)}{1-\phi}, \tag{47}$$
so (after some computations) V of the prescribed form is a solution if and only if:
$$\beta_1=\left(\frac{\rho_1\left(\frac{\rho_1-\left(\alpha_1-f_2\frac{\sigma^2}{2}\theta\right)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right)^{-\phi}}{\frac{\sigma^2}{2}\theta}\right)^{\frac{1-\theta}{1-\phi}}.$$
Observe that the condition (25) guarantees that the quantity $\rho_1-\left(\alpha_1-f_2\frac{\sigma^2}{2}\theta\right)(1-\phi)$ is positive for any choice of $f_2\in[0,1]$

In a similar way, we can find the optimal response of player 2 given a certain fixed deterministic constant strategy f_1 of player 1. We find:
$$f_2^*=\left(\frac{1}{\rho_2}\frac{\sigma^2}{2}\theta\beta_2^{\frac{1-\phi}{1-\theta}}\right)^{-1/\phi} \tag{48}$$
with
$$\beta_2=\left(\frac{\rho_2\left(\frac{\rho_2-\left(\alpha_2-f_1\frac{\sigma^2}{2}\theta\right)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi}\right)^{-\phi}}{\frac{\sigma^2}{2}\theta}\right)^{\frac{1-\theta}{1-\phi}}.$$
Thus if we find a couple of values $(f_1,\ f_2)$ that solves
$$\begin{aligned} f_1^* &= \min\left(\frac{\rho_1-\left(\alpha_1-f_2^*\frac{\sigma^2}{2}\theta\right)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi},1-f_2^*\right)\\ f_2^* &= \min\left(\frac{\rho_2-\left(\alpha_2-f_1^*\frac{\sigma^2}{2}\theta\right)(1-\phi)}{\frac{\sigma^2}{2}\theta\phi},1-f_1^*\right)\end{aligned}$$
the couple of constant deterministic strategies with vales $(f_1,\ f_2)$ is a Nash equilibrium.

In particular, if (28) and (29) are verified we find the internal solution characterized by:
$$\begin{aligned} f_1^*\frac{\sigma^2}{2}\theta &= \frac{1}{2\phi-1}\left[\phi(\rho_1-\alpha_1(1-\phi))+(1-\phi)(\rho_2-\alpha_2(1-\phi))\right]\\ f_2^*\frac{\sigma^2}{2}\theta &= \frac{1}{2\phi-1}\left[\phi(\rho_2-\alpha_2(1-\phi))+(1-\phi)(\rho_1-\alpha_1(1-\phi))\right].\end{aligned}$$

□