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Jérémy Leymarie, Christophe Hurlin, Antoine Patin

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Abstract

We propose a new approach for comparing loss given default (LGD) models which is based on loss functions defined in terms of regulatory capital charge. These loss functions penalize more the LGD forecasts errors made on credits associated to high exposure and long maturity than the other ones. We also introduce asymmetric loss functions that only penalize the LGD forecasts errors that lead to underestimate the regulatory capital. We show theoretically that the LGD models ranking determined by our approach may differ from the ranking obtained according to the traditional approach that consists in comparing the models according to their LGD forecasts errors. Using an original sample of credits and leasing provided by an international bank, we apply this new approach to compare the LGD forecasts issued from 6 competing models. The empirical results confirm that the ranking based on a naive LGD loss function is generally different from the models ranking obtained with the capital charge symmetric (or asymmetric) loss.

Keywords: loss given default (LGD), credit risk capital requirement, loss function, forecasts comparison

JEL classification: G21, G28 .

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1 Introduction

Since the Basel II agreements, banks have the possibility to develop internal rating models to compute their regulatory capital charge for credit risk, through the internal rating-based approach (IRB). The IRB approach can be viewed as an external risk model (based on the asymptotic single risk factor (ASRF) model) with internal risk parameters, namely the exposure at default (EAD), the probability of default (PD), the loss given default (LGD) and the effective maturity (M). In practice, the Basel Committee on Banking Supervision (BCBS) makes the distinction between two IRB methods. In the foundation IRB (FIRB), banks only estimate the PD and report the EAD, whereas the values of the other risk parameters (i.e. LGD and M) are set by regulators.[1] On the contrary, the advanced IRB (AIRB) allows banks to use their own estimates of PD and LGD, issued from internal risk models.

In this paper, we propose a new approach for comparing LGD models which is based on loss functions defined in terms of regulatory capital charge. Given the importance of the LGD parameter in the Basel risk weight function and the regulatory capital for credit risk, the LGD models comparison is a crucial problematic for banks and regulators. Unlike PD, the LGD estimates enter the capital requirement formula in a linear way and, as a consequence, the estimation errors may have a strong impact on required capital. Furthermore, none benchmark model seems to currently emerge from the "zoo" of LGD models that regulators, banks and academics have to face.[2] Indeed, while on PD models an extensive academic and practitioner literature exists, the literature is more scarce about LGD definition, measurement

[1] In the foundation IRB approach, the LGD is fixed at 45% for senior claims on corporate, sovereigns and banks not secured by recognized collateral, and at 75% for all subordinated claims on corporate, sovereigns and banks. The effective maturity M is fixed at 2.5 years for corporate exposures except for repo-style transactions where the maturity is fixed at 6 months (Roncally, 2014).

[2] By analogy with the "factor zoo" evoked by Cochrane (2011).

and modelling (see Schuermann (2004) for a survey). The LGD can be broadly defined as the ratio of losses that will never be recovered by the bank, to exposure at default, or equivalently by one minus the recovery rate. If this definition is relatively clear, the measurement and the modelling of the LGD raise numerous issues in practice. As concerned the recovery measurement, the BCBS and the regulators (see, for instance, EBA (2016)) made efforts to clarify the notion of default and the scope of losses that should be considered by the banks to measure the *workout* LGD. On the contrary, no particular guidelines have been provided for the LGD models. This may explain why there is a large heterogeneity in the modelling approaches used by AIRB banks and academics. The most often used models range from (1) the simple look-up (contingency) tables to sophisticated machine learning classifications methods (regression tree, bagging, random forests, gradient boosting, artificial neural network, etc.), (2) parametric approaches based on beta, exponential-gamma, inflated beta distributions or on fractional response regression and Tobit models, etc. (3) non-parametric approaches such as kernel density estimators, quantile regressions, multivariate adaptive regression splines, support vector machine and mixture models, etc. Thus, Loterman et al. (2012) evaluate 24 regression techniques for the LGD of six major international banking institutions, whereas Qi and Zhao (2011) compare 6 models which give very different results.

How to compare these LGD models? The benchmarking method currently adopted by banks and academics simply consists in (1) considering a sample of defaulted credits which is split in a training set and a test set, (2) estimating the competing models on the training set and then, (3) evaluating the LGD forecasts on the test set with standard statistical criteria such as the mean square error (MSE), the mean absolute error (MAE), etc. Thus, the LGD model comparison is done independently of the other Basel risk parameters (EAD, PD, M).

The first shortcoming of this approach is the lack of economic interpretability of the loss function applied to the LGD estimates. What is the economic meaning of a MSE of 10% or a MAE of 27% in terms of financial stability? These figures give no information about the estimation error made on the capital charge and the bank's ability to face its unexpected credit losses. The second shortcoming is related to the two-step structure of the AIRB approach. The LGD forecasts produced by the bank's internal models are, in a second step, introduced in the regulatory formula for capital charge. If the LGD models are compared independently of this second step, it leads to give the same weight to a LGD estimation error of 10% made on two contracts with an EAD of 1€ and 100,000€, respectively. Similarly, it leads to give the same weight to a LGD estimation error of 10% made on two contracts, one with a PD of 50%, for which there is only one chance out of two to observe a default within the year, and the other with a PD of 99% for which the default is almost sure.

We propose here a new approach in which the economic losses associated to the LGD models are assessed in terms of regulatory capital and in fine, in terms of bank's capacity to face unexpected losses on its credit portfolio. For that, we define a set of expected loss functions for the LGD forecasts, which are expressed in terms of *capital charge* induced by these forecasts. Hence, these loss functions take into account the exposure, the default probability and the maturity of the loans. For instance, they penalize more the LGD forecasts errors made on credits associated to high exposure and long maturity than the other ones. Besides, we propose asymmetric loss functions that only penalize the LGD forecasts errors that lead to underestimate the regulatory capital. Such asymmetric functions may be preferred by the regulator in order to neutralize the impact of the LGD forecasts errors on the required capital and in fine, to enhance the soundness and stability of the banking system. We show theo-

retically that the model ranking determined by a LGD-based loss function may differ to the ranking based on the capital charge loss function. For that, we demonstrate the conditions under which both ranking are consistent. These conditions have no particular reason to be valid in practice. This theoretical analysis may be related to the notion of model ranking consistency introduced by Hansen and Lunde (2006), Patton (2011) or Laurent, Rombouts and Violante (2013).

Using data for a sample of 9,738 defaulted credits and leasing provided by the bank of a worldwide leader automotive company, we apply our LGD models comparison method. Our dataset is one of the first of its kind to be used in an academic study. Contrary to the existing literature on LGD, which is for the most part related to corporate bonds (given the public availability of data) and market LGDs, our dataset consists in a retail loans portfolio (personal loans and leasing) for which we observe the workout LGDs. The workout LGD empirical distribution reveals interesting features for the modelling. As usual, a significant proportion (10.58%) of the contracts have a loss that exceeds 100% of the EAD, due to the workout costs. Besides, the LGD distribution is bimodal, meaning that the percentage of losses is either relatively high or low. But contrary to Schuermann (2004), we observe that the LGD and the EAD are positively correlated, even if this correlation is relatively small (0.11).

We compare 6 competing LGD models, which are among the most often used in academic and practitioner literature, namely (1) the fractional response regression model, (2) the regression tree, (3) the random forest, (4) the gradient boosting, (5) the artificial neural network and (6) the least square support vector machine. Our results show that the model ranking based on the LGD loss function is generally different from the model ranking obtained with

the capital charge loss functions. Such a difference clearly illustrates that the consistency conditions previously mentioned are not fulfilled, at least for our sample. This result is robust (1) to the choice of the explanatory variables considered in the LGD models, (2) to the introduction (or not) of the EAD as covariate, and (3) to the use of the Basel PDs (collected one year before the default) in the capital charge loss function. Besides, we find that the LGD forecast errors are right-skewed, especially for support vector machine. In this context, the use of asymmetric loss functions provide a model ranking which is very different from the ranking obtained with symmetric loss functions.

The main contribution of this paper is to show, both theoretically and empirically, that the optimal LGD model chosen on the basis of the LGD estimation errors only, may be different to the optimal LGD model determined by the comparison of the capital charge errors. As a consequence, the current model comparison method may lead to inaccurate regulatory capital levels and to weaken bank's solvency. On the contrary, our approach leads to select the optimal LGD model which induces the least important errors on the regulatory capital. Hence, we believe that adopting this new model comparison approach should be of general interest.

The rest of this paper is structured as follows. We discuss in Section 2 the main features of the AIRB approach and the regulatory capital for credit risk portfolios, with a special focus on the LGD models and the LGD model comparison method currently used by banks and academics. In Section 3, we present the capital charge loss function that we propose for LGD models. In Section 4, we describe the dataset and 6 competing LGD models. We then use our capital charge loss function to compare the forecasts produced by the competing models. We summarize and conclude our paper in Section 5.

2 Capital charge for credit risk portfolios

In this section, we propose a brief overview of the role of the LGD in the computation of the regulatory capital under the AIRB approach. Then, we present the main measurement and modelling issues for the LGD, and the method currently used to compare the LGD models.

2.1 Capital requirement, individual risk contributions and LGD

Let us consider a portfolio of n credits indexed by $i = 1, ..., n$. Each credit is characterized by (1) an EAD defined as the outstanding debt at the time of default, (2) a LGD defined as the percentage of exposure at default that is lost if the debtor default, (3) a PD that measures the likelihood of the default risk of the debtor over an horizon of one year and (4) an effective maturity M, expressed in years. The credit portfolio loss is then equal to

$$L = \sum_{i=1}^{n} \text{EAD}_i \times \text{LGD}_i \times D_i \tag{1}$$

where D_i is a binary random variable that takes a value 1 if there is a default before the residual maturity M_i and 0 otherwise.

In the AIRB approach, the regulatory capital (RC) charge is designed to cover the unexpected bank's credit loss. The unexpected loss is defined as the difference between the 99.9% Value-at-Risk (VaR) of the portfolio loss and the expected loss $\mathbb{E}(L)$. In order to compute the unexpected credit loss, the Basel Committee considers the ASRF model. This model is based on the seminal Merton-Vasicek "model of the firm" (Merton (1974), Vasicek (2002)) and additional assumptions such as the infinite granularity of considered portfolios, the normal distribution of the risk factor and a time horizon of 1 year (BCBS (2004, 2005)). Under these assumptions (cf. appendix A), the unexpected loss of the credit portfolio and hence the regulatory capital, can be decomposed as a sum of independent risk contributions (RC_i)

that only depends on the characteristics of the i^{th} credit (Genest and Brie (2013), Roncally (2014)). The regulatory capital is then equal to

$$\mathrm{RC} = \sum_{i=1}^{n} \mathrm{RC}_i \tag{2}$$

The supervisory formula for the risk contribution RC_i is given by

$$\mathrm{RC}_i \equiv \mathrm{RC}_i\left(\mathrm{EAD}_i, \mathrm{PD}_i, \mathrm{LGD}_i, \mathrm{M}_i\right) = \mathrm{EAD}_i \times \mathrm{LGD}_i \times \delta\left(\mathrm{PD}_i\right) \times \gamma\left(\mathrm{M}_i\right) \tag{3}$$

with

$$\delta\left(\mathrm{PD}_i\right) = \Phi\left(\frac{\Phi^{-1}\left(\mathrm{PD}_i\right) + \sqrt{\rho\left(\mathrm{PD}_i\right)}\Phi^{-1}\left(99.9\%\right)}{\sqrt{1-\rho\left(\mathrm{PD}_i\right)}}\right) - \mathrm{PD}_i \tag{4}$$

where $\Phi\left(.\right)$ denotes the cdf of a standard normal distribution, $\rho\left(\mathrm{PD}\right)$ a parametric decreasing function for the default correlation and $\gamma\left(\mathrm{M}\right)$ a parametric function for the maturity adjustment. The maturity adjustment and the correlation functions suggested by the BCBS depend on the type of exposure: corporate, sovereign or bank exposures, versus residential mortgage, revolving or other retail exposures. The functions $\rho\left(\mathrm{PD}\right)$ and $\gamma\left(\mathrm{M}\right)$ suggested by the BCBS are reported in appendix B.

The Basel II formula (Equations 2, 3 and 4) highlights the importance of the LGD for regulatory capital calculations. Since the LGD enters the capital requirement formula in a linear way, the LGD forecast errors may have a strong impact and, as consequence, the choice of the LGD model is crucial.

2.2 LGD data and models

As previously mentioned, the AIRB approach allows banks to develop internal models for estimating PD, LGD and, in some particular cases, the EAD.[3] However, the LGD definition, measurement and modelling raise numerous practical issues.

[3] For standard credits and loans, the EAD are observed. However, for off-balance sheet EAD, the bank has to estimate a credit conversion factor (CCF). See for instance, Gürtler, Hibbeln and Usselman (2017).

The LGD is defined as the ratio of losses (that will never be recovered by the lender) to exposure at default, or equivalently by one minus the recovery rate. The Basel II Accord requires that all relevant factors that may reduce the final economic value of recovered portion of an exposure must be taken into account into the LGD calculation. These factors correspond to (i) the direct (external) costs associated to the loss of principal and the interest income foregone, (ii) the indirect (internal) costs incurred by the bank for recovery in the form of workout costs (administrative costs associated with collecting information on the exposure, legal costs, etc.) and (iii) the funding costs reflected by an appropriate discount rate tied to the time span between the emergence of default and the actual recovery.[4]

Schuermann (2004) identifies three main ways of measuring LGD. The *market* LGD is calculated as one minus the ratio of the trading price of the asset some time after default to the trading price at the time of default. The *implied market* LGD is derived from risky (but not defaulted) bond prices using a theoretical asset pricing model. As they are based on trading prices, the market and implied market LGDs are generally available only for bonds and loans issued by large firms. On the contrary, the *workout* LGD can be measured for any type of instrument. The workout LGD estimation is based on an economic notion of loss including all the relevant costs tied to the collection process, but also the effect deriving from the discount of cash flows. The scope of data necessary for proper LGD estimation is very broad and entails not only the date of default and all cash flows and events after default but also all relevant information about the obligors and transactions that could be used as risk drivers (collateral, etc.) in the model development. This approach is clearly preferred by the regulators. For instance, in its guidelines on LGD estimation and the treatment of

[4] Grippa et al. (2005) find that workout costs average 2.3% of total operating expenses in their study of Italian bank loans.

defaulted exposures, the EBA (November 2016) states that "*the workout LGD is considered to be the main, superior methodology that should be used by institutions. It is essential that LGD estimates are based on the institutions' own loss and recovery experience in order to make sure that the estimates are adequate for the institutions portfolios and policies and in particular that they are consistent with the recovery processes*" (EBA (2016), page 11). Notice that most empirical academic studies neglect workout costs because of data limitations, even if Khieu et al. (2011) found evidence that market LGDs are biased and inefficient estimates of the workout LGD.[5]

The general purpose of the LGD (internal) models consists in providing an estimate (or a forecast) of the LGD for the credits which are currently in the bank's portfolio and for which the bank does not observe the potential losses induced by a default of the borrower. As a consequence, these models are estimated on a sample of n_d defaulted credits for which the true (ex-post) workout LGD is observed. By identifying the main characteristics of these contracts and the key factors of the recovery rates, it is then possible to estimate (forecast) the LGD for the similar contracts which are currently in the bank's portfolio.

Töws (2016) identifies two major challenges in estimating recovery rates with respect to defaulted bank loans or bonds. First, the LGD theoretically ranges between 0 and 100% of the EAD, meaning that the bank cannot recover more than the outstanding amount and that the lender cannot lose more than the outstanding amount. However, several studies (Schmidt and Stuyck (2002), Schmit (2004), Schuermann (2004), Töws (2016)) show that when workout costs are incorporated, the LGD is sometimes larger than 100%. The second

[5] Conversely, Gürtler and Hibbeln (2013) identify several problems in modeling workout LGDs that can lead to inaccurate LGD forecasts. In particular, the LGDs within the modeling data can be significantly biased downwards if all available defaults with completed workout processes are considered.

challenge in estimating recovery rates is the bimodal nature of the LGD distribution. Indeed, recovery as a percentage of exposure is generally either relatively high (around 70-80%) or low (around 20-30%). Hence, thinking about an "average" LGD can be very misleading.

Because of the specific nature of the LGD distribution, a large variety of LGD models are currently used by academic and practitioners. None benchmark model fully recognized by regulators, banks and academics, seems to currently emerge from this "zoo" of LGD models. This is why the LGD models comparison is so important both for practitioners and academics.[6]

Four categories of LGD models can be identified. The first category corresponds to non-parametric approaches or "models", even if the term "model" is clearly inappropriate in this case. The most simple "models" are the contingency or "look-up" tables containing LGD averages by certain characteristics (segmentation). For example, a cell of this table might include senior unsecured loans for the automotive industry during a recession. An average LGD is computed from the observations of the training set that belong to this cell and then, is applied to the similar credits of the bank's portfolio. These tables have the advantage of being easy to build and easy to use. However, with enough cuts one quickly runs out of data: many cells in this contingency table will likely go unfilled or have only very few observations on which to base an average. The second category corresponds to parametric approaches. Given the fact that LGD is theoretically defined over $[0, 1]$, the parametric models are generally based on beta distribution (Credit Portfolio View of Mc Kinsey), exponential-gamma distribution (Gouriéroux, Monfort and Polimenis (2006)), inflated beta distribution (Ospina and Ferrari

[6] This lack of benchmark model explains the number of benchmarking studies proposed in the academic literature these last years (Bastos (2010), Qi and Zhao (2011), Loterman et al. (2012), Töws (2016) among others).

(2010)) or logistic-Gaussian distribution. In a similar way, the Fractional Response Regression (FRR) model (Papke and Wooldridge (1996)) which keeps the predicted values in the unit interval, have also been used for the LGD estimates by Dermine and Carvalho (2006), Bastos (2010), Qi and Zhao (2011) or Bellotti and Crook (2012). Tanoue, Kawada and Yamashita (2017) propose a parametric multi-step approach for the LGDs of bank loans in Japan. The third category encompasses the kernel density estimators, quantile regressions and mixture models. Calabrese and Zenga (2010) consider a mixture of a Bernoulli random variable and a continuous random variable on the unit interval to model the LGD of a large dataset of defaulted Italian loans. Similarly, Calabrese (2014) suggests a mixture distribution approach for the downturn LGD. Renault and Scaillet (2004) or Hagman, Renault and Scaillet (2005) consider various kernel density estimator of the LGD distribution, whereas Krüger and Rösh (2017) consider quantile regressions for modelling downturn LGD. We can also mention the use of multivariate adaptive regression splines (Loterman et al. (2012)), support vector machine (Yao, Crook and Andreeva (2015)) and least squares support vector machine (Loterman et al. (2012)). These approaches have the common advantage to reveal a number of bumps which can be larger than those obtained with parametric distributions (beta distribution for instance). Finally, the last category includes all the classification and machine learning methods such as regression tree algorithms (Breiman and al. (1984), Hartmann-Wendels, Miller and Töws (2014)), artificial neural networks (Bishop (1995)), random forest (Breiman (2001)), gradient boosting (Friedman (2001)) and many others. Qi and Zhao (2011) and Bastos (2010) compare FRR models to other parametric and nonparametric modeling methods for LGD. They conclude that machine learning methods, such as regression trees and neural networks, perform better than parametric methods when overfitting is properly controlled for. A sim-

ilar conclusion is get by Loterman et al. (2012) who show that non-linear techniques, and in particular support vector machine and neural networks, perform significantly better than more traditional linear techniques.

2.3 LGD models comparison

Choosing the best methodology for fitting the recovery rates curve among the set of potential LGD models, implies to compare the predictive performances of the models according to a suitable framework. In the sequel, we briefly present the comparison method currently used both by academics and banks.

Consider a set of $\mathcal{M}$ LGD models indexed by $m = 1, ..., \mathcal{M}$. The sample of n_d defaulted credits is randomly split into a training set including n_t credits and a test set including n_v credits. In a first step, the models are estimated (for parametric models) or built (for regression trees, neural networks, etc.) on the training set.[7] In a second step, the models are used to produce pseudo out-of-sample forecasts of the LGD for the credits of the test set. The test set is then used solely to assess the prediction performances of the models. Denote by LGD_i the true LGD observed for the i^{th} credit, for $i = 1, ..., n_v$ and by $\widehat{\text{LGD}}_{i,m}$ the corresponding estimate issued from model m.

The assessment of the prediction performances of the LGD models is generally based on an expected loss $\mathcal{L}$ defined as

$$\mathcal{L}_m \equiv \mathcal{L}\left(\text{LGD}_i, \widehat{\text{LGD}}_{i,m}\right) = \mathbb{E}\left(L\left(\text{LGD}_i, \widehat{\text{LGD}}_{i,m}\right)\right) \tag{5}$$

where $L(.,.)$ is an integrable loss function, with $L : \Omega^2 \to \mathbb{R}^+$, that satisfies the main standard

[7] For the classification methods (regression trees, neural networks, etc.), the training set is further split into training and validation subsets. The validation set is used to select the criterion for evaluating the candidate splitting rules, the depth of the tree or any other parameter required by these methods.

properties discussed in Granger (1999).[8] Since the LGD is a continuous variable defined over a subspace Ω of $\mathbb{R}^+$ (typically $[0,1]$ or $[0,\delta]$ with $\delta > 1$), the loss function is similar to those generally used for any standard regression models. In the literature (Gupton and Stein (2002), Caselli and Querci (2009), Matuszyk, Mues and Thomas (2010), Loterman et al. (2012), etc.), the loss functions generally used for the LGD estimates are the quadratic loss function $L(x,\widehat{x}) = (x-\widehat{x})^2$ or the absolute loss function $L(x,\widehat{x}) = |x-\widehat{x}|$. In practice, the prediction performances of the LGD models are compared through the empirical mean of their losses computed on the test set, defined as

$$\widehat{\mathcal{L}}_m = \frac{1}{n_v}\sum_{i=1}^{n_v} L\left(\text{LGD}_i, \widehat{\text{LGD}}_{i,m}\right) \tag{6}$$

Given the functional form of the loss function, the empirical mean $\widehat{\mathcal{L}}_m$ corresponds to a common measure of predictive accuracy, for instance the MSE, MAE or RAE defined as

$$\text{MSE: } \widehat{\mathcal{L}}_m = \frac{1}{n_v}\sum_{i=1}^{n_v}\left(\text{LGD}_i - \widehat{\text{LGD}}_{i,m}\right)^2$$

$$\text{MAE: } \widehat{\mathcal{L}}_m = \frac{1}{n_v}\sum_{i=1}^{n_v}\left|\text{LGD}_i - \widehat{\text{LGD}}_{i,m}\right|$$

$$\text{RAE: } \widehat{\mathcal{L}}_m = \sum_{i=1}^{n_v}\left|\text{LGD}_i - \widehat{\text{LGD}}_{i,m}\right| \Big/ \sum_{i=1}^{n_v}\left|\text{LGD}_i - \overline{\text{LGD}}_i\right|$$

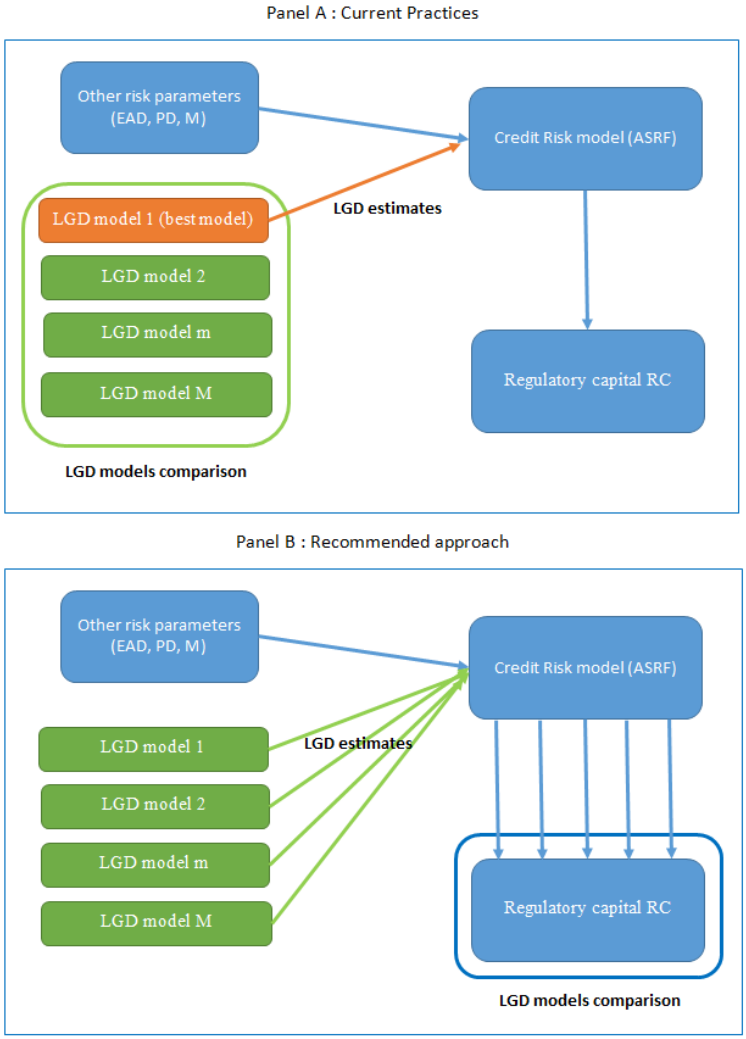
The LGD models are compared and ranked according to the realization of the statistic $\widehat{\mathcal{L}}_m$ on the test set. A model m is preferred to a model m' as soon as $\widehat{\mathcal{L}}_m < \widehat{\mathcal{L}}_{m'}$. Denote by $\widehat{m}^*$ the model associated to the minimum empirical mean $\widehat{\mathcal{L}}_m$ for $m = 1, ..., \mathcal{M}$. Under some regularity conditions, $\widehat{\mathcal{L}}_m$ converges to $\mathcal{L}_m$, and the model $\widehat{m}^*$ corresponds to the optimal model m^* defined as

$$m^* = \underset{m=1,...,\mathcal{M}}{\arg\min}\, \mathbb{E}\left(L\left(\text{LGD}_i, \widehat{\text{LGD}}_{i,m}\right)\right) \tag{7}$$

[8] If we denote by $e = x - \widehat{x}$ the error and rewritte the loss function as a function of e, these required properties can be summarized as follows: (i) $L(0) = 0$, (ii) $\min L(e) = 0$ so that $L(e) \geq 0$, (iii) $L(e)$ is monotonically non-decreasing as e moves away from zero so that $L(e_1) \geq L(e_2)$ if $e_1 > e_2 > 0$ and if $e_1 < e_2 < 0$.

This general approach has three main shortcomings. The first one, which is not specific to the context of LGD models, is that the models are compared on the basis of the empirical means of the loss function. Nothing guarantee that the observed differences are statistically significant. A simple solution consists in testing the null hypothesis of no difference in the accuracy of two competing forecasts with a DM-type test (Diebold and Mariano (1995)) or to identify a Model Confidence Set (Hansen, Lunde and Nason (2011)) that contains the "best" forecasting models given a level of confidence. The second drawback of the current approach is the lack of economic interpretability of the loss function applied to the LGD estimates. What is the economic meaning of a MSE of 10% or a MAE of 27% in terms of regulatory capital perspective? These figures give no information about the estimation error made on the capital charge and, in fine, on the ability of the bank to face unexpected losses. The last pitfall is related to the two-step structure of the AIRB approach. Indeed, the output of the bank's internal models, including the LGD models, are Basel risk parameter estimates which are, in a second step, introduced in the ASRF model to compute the capital charge for each credit. As shown in the top panel of Figure 1, the LGD models comparison is currently done independently of this second step and, as a consequence, of the ASRF model and the other risk parameters (EAD, PD, etc.). What are the consequences of this comparison scheme? It leads to give the same weight to a LGD estimation error of 10% made on two contracts with an EAD of 1€ and 100,000€, respectively. Similarly, this approach leads to give the same weight to a LGD estimation error of 10% made on two contracts, one with a PD of 1% and the other with a PD of 99% for which the default is almost sure.

Figure 1: Comparison of LGD models in the regulatory framework



3 Capital charge loss functions for LGD models

"*Of great importance, and almost always ignored, is the fact that the economic loss associated with a forecast may be poorly assessed by the usual statistical metrics. That is, forecasts are used to guide decisions, and the loss associated with a forecast error of a particular sign and size is induced directly by the nature of the decision problem at hand.*". Diebold and Mariano (1995), page 2.

This quotation issued from the seminal paper of Diebold and Mariano (1995), perfectly illustrates the drawbacks of the current practices of LGD models comparison. In the Basel II perspective, the LGD estimates are only *inputs* of the ASRF model which produces the key estimates, namely the capital charge for credit risk. So, the economic loss associated to the LGD models has to be assessed in terms of regulatory capital and in fine, in terms of bank's capacity to face unexpected losses on its credit portfolio. The bottom panel of Figure 1 summarizes the alternative approach that we recommend for LGD models comparison. The LGD forecasts issued from the competing models and other risk parameters (EAD, PD, etc.) are used to compute the corresponding capital charges. Then, our approach consists in comparing the LGD models not in terms of forecasting abilities for the LGD itself, but in terms of forecasting abilities for the regulatory capital charge. The main advantage of this approach is that it favors the LGD model that leads to the lowest estimation errors for the loans with the highests EAD and PD. This approach requires an (regulatory) expected loss expressed in terms of *capital charge* to assess the LGD estimates.

3.1 Capital charge expected loss

The capital charge expected loss $\mathcal{L}_{CC,m}$ is simply defined as the expected loss defined in terms of regulatory capital charge, which is associated to the LGD estimates issued from a LGD model m. Formally, we have

$$\mathcal{L}_{CC,m} \equiv \mathcal{L}\left(\mathrm{RC}_i, \widehat{\mathrm{RC}}_{i,m}\right) = \mathbb{E}\left(L\left(\mathrm{RC}_i, \widehat{\mathrm{RC}}_{i,m}\right)\right) \tag{8}$$

where $L(.,.)$ is an integrable *capital charge* loss function with $L : \mathbb{R}^{+2} \to \mathbb{R}^{+}$, and

$$\mathrm{RC}_i = \mathrm{EAD}_i \times \mathrm{LGD}_i \times \delta(\mathrm{PD}) \times \gamma(\mathrm{M}_i)$$

$$\widehat{\mathrm{RC}}_{i,m} = \mathrm{EAD}_i \times \widehat{\mathrm{LGD}}_{i,m} \times \delta(\mathrm{PD}) \times \gamma(\mathrm{M}_i)$$

The variable RC_i denotes the risk contribution of the i^{th} credit, defined by the regulatory formula (Equation 3). This risk contribution only depends on the risk parameters associated to the credit i, namely EAD_i, LGD_i and M_i.[9] Notice that the PD is not indexed by i, meaning that we consider the same default probability for all the credits in the portfolio. As we only consider defaulted credits, the PD value has not to be estimated and should set to an arbitrary value, typically close to 1. Similarly, $\widehat{\mathrm{RC}}_{i,m}$ denotes the estimated risk contribution for credit i, which is based on the individual risk parameters (EAD_i and M_i), the common value for the PD and the LGD estimates issued from the model m.

Given the functional form of the capital charge loss function $L(.,.)$, the empirical counterpart $\widehat{\mathcal{L}}_{CC,m}$ can be defined in terms of MSE, MAE, RAE or any usual model comparison criteria. For instance, we can consider the following losses

$$\text{Capital Charge MSE: } \widehat{\mathcal{L}}_{CC,m} = \frac{1}{n_v}\sum_{i=1}^{n_v}\left(\mathrm{RC}_i - \widehat{\mathrm{RC}}_{i,m}\right)^2$$

$$\text{Capital Charge MAE: } \widehat{\mathcal{L}}_{CC,m} = \frac{1}{n_v}\sum_{i=1}^{n_v}\left|\mathrm{RC}_i - \widehat{\mathrm{RC}}_{i,m}\right|$$

$$\text{Capital Charge RAE: } \widehat{\mathcal{L}}_{CC,m} = \sum_{i=1}^{n_v}\left|\mathrm{RC}_i - \widehat{\mathrm{RC}}_{i,m}\right| / \sum_{i=1}^{n_v}\left|\mathrm{RC}_i - \overline{\mathrm{RC}}_i\right|$$

where n_v denotes the size of the test set of defaulted credits. Other comparison criteria, especially designed for the financial regulation purpose, can be defined. For instance, regulator may prefer to penalize more the capital charge underestimates rather than the overestimates. For that, we propose asymmetric loss functions defined as

$$\text{Asymmetric MSE: } \widehat{\mathcal{L}}_{CC,m} = \frac{1}{n_v^+}\sum_{i=1}^{n_v^+}\left(\mathrm{RC}_i - \widehat{\mathrm{RC}}_{i,m}\right)^2 \times \mathbb{I}_{\left(\mathrm{RC}_i > \widehat{\mathrm{RC}}_{i,m}\right)}$$

[9] For simplicity, we assume that there is no off-balance sheet exposure and that the exposure at default is not estimated, but observed ex-post. The maturity is also observed.

$$\text{Asymmetric MAE: } \widehat{\mathcal{L}}_{CC,m} = \frac{1}{n_v^+} \sum_{i=1}^{n_v^+} \left| \text{RC}_i - \widehat{\text{RC}}_{i,m} \right| \times \mathbb{I}_{\left(\text{RC}_i > \widehat{\text{RC}}_{i,m}\right)}$$

where $\mathbb{I}_{(.)}$ denotes the indicator function that takes a value 1 when the event occurs and 0 otherwise, and n_v^+ is the number of defaulted contracts for which we observe $\text{RC}_i > \widehat{\text{RC}}_{i,m}$. These loss functions are particularly suitable to compare LGD models which tend to produce skewed LGD estimation errors (cf. Section 4).

Whatever the choice of the loss function, the comparison rule for the LGD models is the same as before. A model m is preferred to a model m' as soon as $\widehat{\mathcal{L}}_{CC,m} < \widehat{\mathcal{L}}_{CC,m'}$. Denote by $\widehat{m}^*_{CC}$ the model associated to the minimum empirical mean $\widehat{\mathcal{L}}_{CC,m_{CC}}$ among the set of $\mathcal{M}$ models. Under some regularity conditions, $\widehat{\mathcal{L}}_{CC,m_{CC}}$ converges to $\mathcal{L}_{CC,m_{CC}}$, and allows to identify the optimal model in terms of capital charge expected loss.

As previously mentioned, the expected loss expressed in terms of capital charge depend on the value of the PD chosen for the defaulted credits that belong to the test set.

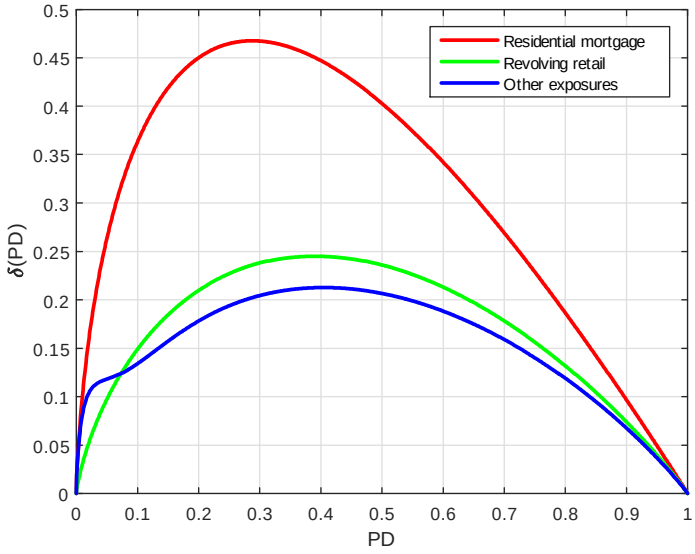
Proposition 1 (PD value) *The ranking of the LGD models based on the capital charge expected loss, does not depend on the choice of the PD value.*

The proof of proposition 1 is straightforward. Since $\delta\,(\text{PD})$ is a constant term that does not depend on the contract i or the model m, the choice of PD does not affect the *relative* values of the expected losses observed for two alternative models m and m'. This choice only affects the *absolute* value of the expected losses $\mathcal{L}_{CC,m}$ and $\mathcal{L}_{CC,m'}$.

Equation 4 implies that $\delta\,(1) = 0$ and $\delta\,(0) = 0$. As a consequence, the PD value has to be chosen on the interval $]0, 1[$. Here, we recommend to use the value PD* that maximizes the value of $\delta\,(\text{PD})$ and hence, the regulatory capital (since RC_i is an increasing function of

$\delta(\text{PD})$). The profile of the capital charge coefficient $\delta(\text{PD})$ depends on the type of exposure (cf. appendix B) and is displayed on Figure 2. The capital charge coefficient increases with PD until an inflexion point, which then causes the capital charge to decrease to 0 when the PD tends to 1. This profile is explained by the fact that once that inflexion point is reached, losses are no longer absorbed by the regulatory capital which is designed to cover the unexpected bank's credit loss, but by the provisions done for the expected credit losses $\mathbb{E}(L)$ (Genest and Brie (2013)). The maximum of the $\delta(.)$ function is reached for a PD value of $\text{PD}^* = 28.8\%$ in the case of residential mortgage, $\text{PD}^* = 39\%$ for revolving retail and $\text{PD}^* = 40.45\%$ for other exposure.

Figure 2: δ function for different types of retail exposure



The first advantage of the expected loss expressed in terms of capital charge is that it has a direct economic interpretation. A capital charge MAE of 800€ means that the average absolute estimation error observed between the capital charge estimates (associated to the LGD estimates issued from a given model) and the true ones (based on the observed LGD

for the defaulted credit) is equal to 800€. Similarly, the asymmetric MSE corresponds to the variance of the capital charge underestimates produced by a given LGD model. A second advantage of the regulatory expected loss is that it corresponds to a weighted average of the loss defined in terms of LGD estimates. Indeed,

$$\mathcal{L}_{CC,m} = \mathbb{E}\left(L\left(\mathrm{RC}_i, \widehat{\mathrm{RC}}_{i,m}\right)\right) = \mathbb{E}\left(L\left(\omega_i \mathrm{LGD}_i, \omega_i \widehat{\mathrm{LGD}}_{i,m}\right)\right)$$

where the weight $\omega_i = \mathrm{EAD}_i \times \delta(\mathrm{PD}) \times \gamma(\mathrm{M}_i)$ depends on the exposure and the maturity of the credit. As a consequence, a LGD estimation error of 10% made on a contract with an EAD of 1€ has less impact on the expected loss than a LGD estimation error of 10% made on a contract with an EAD of 100,000€.

3.2 Ranking consistency

One crucial question is to know if the model ranking determined by a LGD-based loss function may differ to the ranking based on the capital charge loss function. If both rankings are similar, our approach has no interest. Theoretically, such a situation may occur but only only under very particular conditions. The aim of this section is to determine these conditions and to evaluate their plausibility. This analysis may be related to the notion of ranking consistency introduced by Hansen and Lunde (2006), Patton (2011, 2016) or Laurent, Rombouts and Violante (2013) in another context.[10] Here, we state that the ranking between any two LGD models is consistent when it is the same whether it is based on the LGD or regulatory capital based estimation errors. Consider the following assumption on the LGD loss functions.

Assumption A1: $L(x, \widehat{x}) = g(x - \widehat{x})$ with $g : \mathbb{R} \to \mathbb{R}^+$, a continuous and integrable function.

[10] In the context of Patton (2011) or Laurent, Rombouts and Violante (2013), a volatility model ranking is said consistent when it is the same whether it is based on the true conditional variance or a conditionally unbiased proxy.

Assumption A2: The function $g(.)$ is multiplicative, meaning that $\forall k \in \mathbb{R}$, $g(k(x-\widehat{x})) = g(k)\,g(x-\widehat{x})$.

Assumptions A1 and A2 are satisfied by all the usual loss functions generally considered in the LGD literature, for instance, the quadratic loss function $L(x,\widehat{x}) = (x-\widehat{x})^2$ with $g(y) = y^2$ and

$$L(kx, k\widehat{x}) = L(g(k(x-\widehat{x}))) = k^2(x-\widehat{x})^2 = g(k)\,g(x-\widehat{x})$$

or the absolute loss function $L(x,\widehat{x}) = |x-\widehat{x}|$ with $g(y) = |y|$, for which we have

$$L(kx, k\widehat{x}) = L(g(k(x-\widehat{x}))) = |k|\,|x-\widehat{x}| = g(k)\,g(x-\widehat{x})$$

Consider a set of $\mathcal{M}$ LGD models, indexed by $m = 1, \cdots, \mathcal{M}$. We refer to the ordering based on the expected loss as the true ranking and we assume that LGD-based expected losses are ranked as follows

$$\mathcal{L}_1 < \mathcal{L}_2 < ... < \mathcal{L}_{\mathcal{M}} \tag{9}$$

with $\mathcal{L}_m = \mathbb{E}(g(\varepsilon_{i,m}))$ and $\varepsilon_{i,m} = \text{LGD}_i - \widehat{\text{LGD}}_{i,m}$, $\forall m = 1, ..., \mathcal{M}$. Now, define the corresponding capital charge expected loss, $\mathcal{L}_{CC,m}$, for the model m as

$$\mathcal{L}_{CC,m} = \mathbb{E}\left(g\left(\eta_{i,m}\right)\right) \tag{10}$$

with $\eta_{i,m} = \text{RC}_i - \widehat{\text{RC}}_{i,m}$. By definition of the regulatory capital charge, we have[11]

$$\eta_{i,m} = \text{EAD}_i \times \delta(\text{PD}) \times \gamma(\text{M}) \times \varepsilon_{i,m} \tag{11}$$

Our goal is to determine under which conditions, the model ranking is consistent in the sense that

$$\mathcal{L}_{CC,1} < \mathcal{L}_{CC,2} < ... < \mathcal{L}_{CC,\mathcal{M}} \tag{12}$$

[11] For simplicity, we assume that the credits have the same maturity M. In the general case, the consistency condition of the model rankings can be easily deduced from the formula given in this benchmark case.

Proposition 2 (Model ranking consistency) *The model rankings based on the LGD and capital charge expected losses are consistent, i.e.* $\mathcal{L}_1 < \mathcal{L}_2 < ... < \mathcal{L}_{\mathcal{M}}$ *and* $\mathcal{L}_{CC,1} < \mathcal{L}_{CC,2} < ... < \mathcal{L}_{CC,\mathcal{M}}$, *as soon as,* $\forall m = 1, ..., \mathcal{M} - 1$

$$cov\left(g\left(EAD_i\right), g\left(\varepsilon_{i,m}\right)\right) - cov\left(g\left(EAD_i\right), g\left(\varepsilon_{i,m+1}\right)\right) < \mathbb{E}\left(g\left(EAD_i\right)\right)\left(\mathcal{L}_{m+1} - \mathcal{L}_m\right) \quad (13)$$

The proof is reported appendix C. Since $\mathcal{L}_m < \mathcal{L}_{m+1}$, the consistency condition of proposition 2 is satisfied as soon as $cov\left(g\left(\text{EAD}_i\right), g\left(\varepsilon_{i,m}\right)\right) < cov\left(g\left(\text{EAD}_i\right), g\left(\varepsilon_{i,m+1}\right)\right)$. Thus, the use of capital charge expected loss function does not change the LGD models ranking as soon as the covariances of the LGD estimation errors with the exposures are ranked in the same manner as the models themselves. In the simple case of two models, if the LGD model 1 has a smaller LGD-based MSE than a model 2, it will have also a smaller MSE in terms of capital charge, if its squared LGD estimation errors are less correlated to the squared EAD than the errors of model 2. For instance, if the model 2 produces large LGD estimation errors for high exposures and low LGD errors for low exposures, whereas it is not the case for model 1, both model comparison approaches will provide the same rankings. Obviously, this condition is very particular and in the general case, the two comparison approaches are likely to provide inconsistent LGD models rankings.

Proposition 2 has a direct interpretation in the special case where the exposures are independent from the estimation errors of the LGD models.

Corollary 3 *As soon as the* EAD_i *and the LGD estimation errors* $\varepsilon_{i,m}$ *are independent, the model rankings based on the LGD and capital charge expected losses are consistent, i.e.* $\mathcal{L}_1 < \mathcal{L}_2 < ... < \mathcal{L}_{\mathcal{M}}$ *and* $\mathcal{L}_{CC,1} < \mathcal{L}_{CC,2} < ... < \mathcal{L}_{CC,\mathcal{M}}$.

The proof is provided in the appendix D. This corollary implies that when the credit

exposures and the LGD estimation errors $\varepsilon_{i,m}$ are independent, it is useless to compute the capital charge losses associated to the LGD models, since the model rankings will be similar. Thus, the current comparison model approach that consists to compare the MSE, MAE, MAPE or RAE of the competing LGD models is sufficient. However, this independence hypothesis is unlikely in practice. First, Schuermann (2004) states that the size of exposure seems to have no strong effect on losses.[12] But, even if the variables EAD_i and the LGD_i are independent, it does not necessarily implies that EAD_i and the errors $\text{LGD}_i - \widehat{\text{LGD}}_{i,m}$ are independent. Second, it is important to notice that the introduction of the EAD as an explanatory variable in a LGD model, does not necessarily guarantee that the variables EAD and the estimation errors are independent. The independence property depends on the model (linear or not) and the estimation method used. For instance, it is the case for linear regression model estimated by OLS. On the contrary, for nonlinear models or classification method such a regression tree, a SVM or a random forest, the errors may be correlated with the explanatory variables.

4 Empirical application

In this section, we propose an empirical application of our comparison approach for LGD models. The objective is to compare the models ranking obtained with our approach, to the ranking that we would obtain with a comparison of the LGD-based expected losses.

[12] On the contrary, Eales and Bosworth (1998) in their study devoted to Australian small business and larger consumer loans, conclude that size does matter. They find that loss recovery is U-shaped with a maximum at \$100-500k. Besides, they note that business bankruptcy almost always results in higher severity than consumer bankruptcies.

4.1 Data description

Our dataset, one of the first of its kind to be used in an academic study, was provided by an international bank specialized in financing, insurance and related activities for a worldwide leader automotive company. Contrary to the existing literature on LGD, which is for the most part related to corporate bonds (given the public availability of data) and market LGDs, our dataset consists in a retail loans portfolio (personal loans and leasing) for which we observe the workout LGDs.[13]

The initial sample includes 23,933 loans. We limit our analysis to the 9,738 closed recovery processes for which we observe the final losses. The corresponding sample covers 6,946 credit and 2,792 leasing contracts granted to individual (6,521 contracts) and professional (3,217 contracts) Brazilian customers that defaulted between January 2011 and November 2014. For each contract, we observe the characteristics of the loan (e.g. type of contract, leasing - credit, interest rate, duration, etc) and of the borrower (professional, individual, etc.), as well as the LGD and EAD. All the contracts are in default, so by definition their PD is equal to 1 (certain event). However, we collect for each contract the PD calculated by the internal bank's risk model one year before the default occurs. For the contracts that entered in default in less than one year, the PD is set to the value determined by the internal bank's risk model at the granting date. Finally, we complete the database with the Brazilian GDP growth rate, unemployment rate and interest rate. These three macroeconomic variables will be introduced in the LGD models in order to take into account the influence of the economic cycles on the recovery rate (as in Bellotti and Crook (2012)). The name and the description of the dataset variables are reported in Table 6 in appendix E.

[13] Workout recoveries are also used by Khieu et al. (2011), Dermine and Neto de Carvalho (2005) or Töws (2016). See Miller and Töws (2017) for workout LGDs for lease.

Table 1 displays some descriptive statistics about the LGD, PD and EAD by year, by exposure and customer type. For confidentiality reasons, we do not report the average values. The number of defaulted contracts per year ranges between 1,573 and 2,789. The maximum and the average (not reported) losses tend to decrease between 2011 and 2014. Credit and leasing have approximately the same average and maximum loss rate. The EAD ranges from less than 1 BRL to 123,550 BRL. The PD ranges from less than 1% to 71%, but almost 3/4 of the PD values are below 10%.

Table 1: Descriptive statistics on LGD, PD and EAD

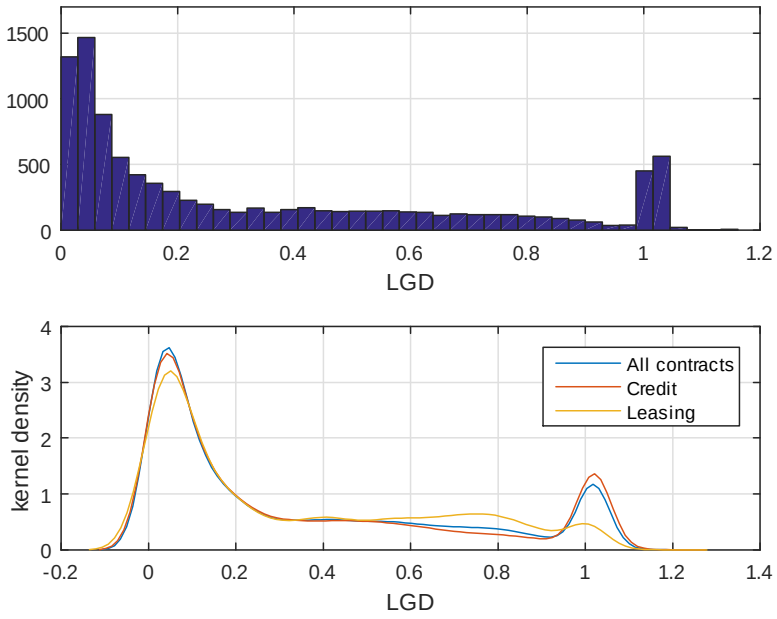
	Nb of obs	LGD(%)		PD (%)		EAD (BRL)	
Panel A. By year							
	–	min	max	min	max	min	max
2011	1573	0.00	116.14	0.06	71.03	0.01	104,959
2012	2430	0.00	114.65	0.09	61.50	383	123,551
2013	2946	0.00	102.97	0.10	61.26	0.18	114,858
2014	2789	0.00	101.59	0.06	70.53	350	92,595
Panel B. By exposure							
	–	min	max	min	max	min	max
Credit	6946	0.00	116.14	0.06	71.03	0.01	118,284
Leasing	2792	0.00	114.33	0.06	71.03	411	123,551
Panel C. By customer type							
	–	min	max	min	max	min	max
Individuals	6521	0.00	115.35	0.06	53.61	0.18	114,858
Professionals	3217	0.00	116.14	0.11	71.03	0.01	123,551

These figures hide a great heterogeneity of the recovery rates. The empirical distribution of the 9,738 workout LGDs is displayed on the top panel of Figure 3. Three remarks should be made here. First, 10.58% of all defaulted contracts have a recovery rate that exceeds 100%, with a maximum value of 116.14%, due to the workout costs.[14] Second, the kernel density

[14] Notice that this percentage is smaller than those generally observed in the litterature. For instance, in the

estimate of the LGD distribution is bimodal (bottom panel of Figure 3), meaning that the percentage of exposure is either relatively high or low. This finding confirms the property largely documented in the literature (Schuermann (2004)). Finally, the LGD distributions for the credit and leasing contracts are relatively close, except for the right part of the distribution. The probability to observe a high loss is less important for the leasing contracts than for the credits. This difference illustrates the role of the collateral in the recovery processes (in the case of leasing, the vehicle belongs to the bank and plays the same role as a collateral).

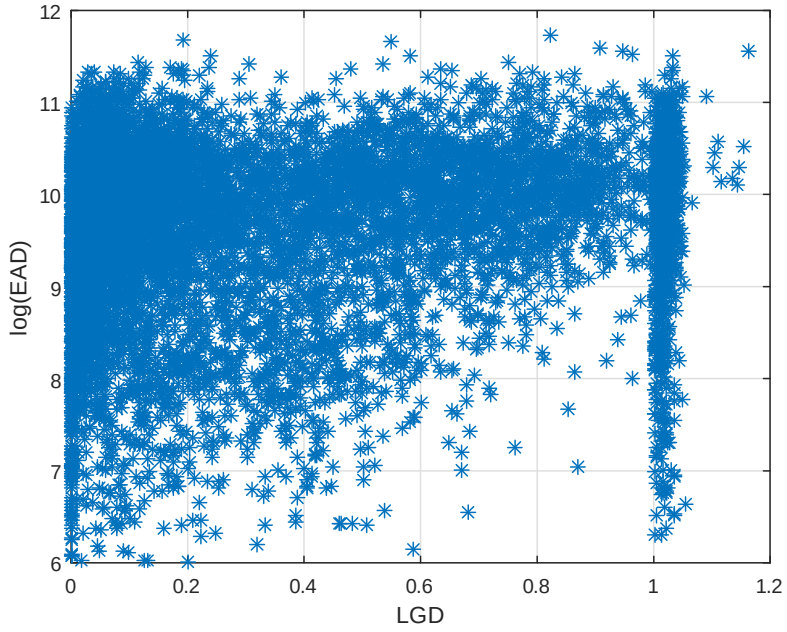
Figure 3: Empirical distribution of the LGDs



Finally, Figure 4 shows that there is a positive correlation between the LGD and the EAD. This correlation is relatively small (0.11), but significant. This observation justifies the introduction of the exposure as explanatory variable in our LGD models.

leasing industry, Schmidt and Stuyck (2002) or Schmit (2004) report that up to 59% of all defaulted contracts in their sample have a recovery rate that exceeds 100%.

Figure 4: Scatter plot of LGD versus log(EAD)



4.2 Competing LGD Models

For our comparison, we consider 6 competing LGD models, which are often used in academic and practitioner literature (see for instance Bastos (2010), Qi and Zhao (2011), Loterman et al. (2012), etc.), namely (1) the fractional response regression model, (2) the regression tree, (3) the random forest, (4) the gradient boosting, (5) the artificial neural network and (6) the least square support vector machine. In the sequel, we briefly present these competing models and mention the main references for further details.

4.2.1 Fractional response regression

The fractional response regression (FRR) model, proposed by Papke and Wooldridge (1996), allows to model the conditional mean of continuous variable defined over $[0, 1]$. The FRR

specification is defined as

$$\mathbb{E}\left(\mathrm{LGD}_i \middle| X_i\right) = G\left(X_i'\beta\right) \tag{14}$$

where X_i is a k-vector of explanatory variables for the i^{th} loan, β a k-vector of parameters and $G(.)$ a link function, with $G : \mathbb{R} \to [0, 1]$. A natural choice for the link function is the logistic function with

$$G\left(X_i'\beta\right) = \frac{1}{1 + \exp\left(-X_i'\beta\right)} \tag{15}$$

The model parameters are estimated by quasi-maximum likelihood (QML), where the quasi likelihood is defined as a modified Bernouilli likelihood. If we denote by $\widehat{\beta}$ the QML estimator of β, the LGD estimator is then given by $\widehat{\mathrm{LGD}}_i = G(X_i'\widehat{\beta})$.

4.2.2 Regression tree

The regression tree (RT), initially introduced by Breiman et al. (1984), is a machine-learning forecasting method. For a continuous variable, the tree is obtained by recursively partitioning the covariates space according to a prediction error (defined as the squared difference between the observed and predicted values) and then, by fitting a simple mean prediction within each partition.

The sketch of a regression tree algorithm is the following. The algorithm starts with a root node gathering all observations. For each covariate X, find the set R that minimizes the sum of the node impurities in the two child nodes and choose the split that gives the minimum overall X and R. The splitting procedure continues until no significant further reduction of the sum of squared deviations is possible. At the end of the procedure, we get a partition into K regions $R_1, \ldots, R_K$, also called terminal nodes or leaves. For each terminal node k, the LGD forecast is then given by the average LGD, denoted $\overline{\mathrm{LGD}}_k$, estimated from all the

contracts that belong to the region R_k.

$$\widehat{\text{LGD}}_i = \sum_{k=1}^{K} \overline{\text{LGD}}_k \times \mathbb{I}_{(X_i \in R_k)} \tag{16}$$

There exist many algorithms for regression tree regressions. Here, we consider the CART algorithm (Breiman et al. (1984)).

4.2.3 Random forest

Random forest (RF), introduced by Breiman (2001), is a boostrap aggregation method of regression trees, trained on different parts of the same training set, with the goal of reducing overfitting (or, equivalently estimator variance). Random forest generally induces a small increase in the bias compared to regression trees and a loss of interpretability, but generally greatly boosts the performance of the model. In addition to constructing each tree using a different bootstrap sample of the data as in bagging approaches, random forests change how the regression trees are constructed. Indeed, each node is split using the best among a subset of covariates randomly chosen at that node. Assume that B boostraped regression trees are combined and denote by $\widehat{\text{LGD}}_{i,b}$ the prediction of the b^{th} tree, then the random forest prediction is defined as:

$$\widehat{\text{LGD}}_i = \frac{1}{B} \sum_{b=1}^{B} \widehat{\text{LGD}}_{i,b} \tag{17}$$

4.2.4 Gradient boosting

Gradient boosting (GB) is an iterative aggregation procedure that consecutively fits new models (typically regression trees) to provide a more accurate estimate of the dependent variable (Friedman (2001)). The general feature of this algorithm consists in constructing for each iteration, a new base-learner which is maximally correlated with the negative gradient of a loss function, evaluated at the previous iteration over the whole sample. In general,

the choice of the loss function is up to the researcher, but most of the studies consider the quadratic loss function.[15]

The gradient boosting algorithm can be summarized as follows. A first regression tree is built on the LGD training set. Denote by $f_0(X_i)$ the prediction for the i^{th} loan and define the corresponding residuals $r_{i0} = \text{LGD}_i - f_0(X_i)$ for $i =, .., n_v$. At the first iteration, a new regression tree is applied to the residuals r_{i0}. The LGD predictions are then updated using the iterative formula $f_1(X_i) = f_0(X_i) + r_{i1}$, where r_{i1} denotes the adjusted residuals issued from the regression tree. After M iterations, algorithm stops and the final LGD predictions are given by

$$\widehat{\text{LGD}}_i = f_0(X_i) + \sum_{m=1}^{M} r_{im} \tag{18}$$

4.2.5 Artificial neural network

Artificial neural networks (ANN) are a class of flexible non-linear models, initially introduced by Bishop (1995). It produces an output value by feeding inputs through a network whose subsequent nodes apply some chosen activation function to a weighted sum of incoming values. The type of ANN considered in this study is a multilayer perceptron similar to that used by Qi et Zhao (2011) for the LGD forecasts. It consists in a three-layer network based on input-layer units, hidden-layer units, and output-layer units. The central idea of the algorithm is (1) to extract linear combinations of the covariates from the input-layer units to the hidden-layer units and (2) to apply nonlinear function on these derived features in the output-layer units to predict the dependant variable.

Let f be the unknown underlying function, through which a vector of input variables X

[15] Another possibility would consist to use our capital charge loss function for the gradient boosting algorithm. But, this new estimation method for LGD is beyond the scope of this paper.

explains LGD, i.e. $LGD_i = f(X_i)$. Derived features Z_m are created using linear combinations of the covariates such as

$$Z_{im} = G(\alpha'_m X_i), \quad \forall m = 1, \ldots, M \tag{19}$$

where M is the number of hidden layer units, α_m a vector of coefficients (including a constant term) from the input-layer units to the hidden-layer units and $G(.)$ the logistic function, which is the common activation function used in neural network. The LGD are then modeled as a function of these linear combinations such that

$$f(X_i) = \beta_0 + \sum_{m=1}^{M} \beta_m Z_{im} + \varepsilon_i \tag{20}$$

where β_m are coefficients from the hidden-layer units to the output-layer units. The LGD forecasts are then given by $\widehat{\text{LGD}}_i = f(X_i)$.

4.2.6 Support vector machine

Initially introduced by Vapnik (1995), support vector machine (SVM) is a machine learning tool for classification and regression. SVM has become popular for its ability to deal with large data, its small number of meta-parameters, and its good results in practice. In the following, we consider SVM regression method (as in Yao, Crook and Andreeva (2015)) due to the continuous nature of the LGD variable. The objective in a SVM regression consists in approximating the response variable y_i by a function $f(.)$ that has not to deviate from y_i more than a margin value ε for each observation of the sample, while simultaneously controlling for the model complexity. The details of the method are explained in appendix F.

4.3 Empirical results

The competing models are estimated on a training set of 7,791 credits (80% of the sample) and the pseudo out-of-sample forecasts are evaluated on test set of 1,947 credits. For each

Table 2: Descriptive statistics on the LGD and regulatory capital forecast errors

	FRR	ANN	TREE	SVM	RF	GB
LGD errors						
minimum	-0.662	-0.681	-0.475	-0.511	-0.626	-0.505
maximum	0.888	1,001	0.874	1.058	0.990	0.890
mean	0.011	0.011	0.010	0.156	0.009	0.010
median	-0.136	-0.122	-0.142	0.005	-0.114	-0.138
variance	0.117	0.116	0.118	0.119	0.117	0.116
skewness	0.824	0.804	0.817	0.843	0.791	0.830
excess kurtosis	-0.618	-0.525	-0.605	-0.606	-0.473	-0.626
Regulatory capital errors						
minimum	-11,689	-12,027	-8,279	-9,018	-10,135	-8,910
maximum	8,588	8,786	8,973	10,213	10,054	8,367
mean	60	44	66	732	38	66
median	-257	-231	-256	13	-232	-257
variance	3,813,596	3,799,691	3,730,561	4,071,567	3,737,503	3,717,518
skewness	0.55	0.41	0.80	1.36	0.61	0.79
excess kurtosis	2.52	2.83	2.01	2.84	2.73	1.91

model, we consider the same set of explanatory variables including the exposure at default, the contract duration, the time to default, the interest or renting rate, the type of exposure (credit versus leasing), the customer type (individual or professional), the brand of the car and the state of the car (new or second-hand).

Table 2 displays some figures about LGD and regulatory capital forecast errors, respectively defined by $\mathrm{LGD}_i - \widehat{\mathrm{LGD}}_{i,m}$ and $\mathrm{RC}_i - \widehat{\mathrm{RC}}_{i,m}$. Notice that, given this notation, a positive error implies an underestimation of the true value. The regulatory capital charges are computed with a PD value set to $\mathrm{PD}^* = 40.45\%$, which corresponds to the maximal charge for the retail exposures. We observe that the empirical means of the LGD and RC forecast errors are slightly positive, whereas the medians are generally negative. This feature

is due to the positive skewness observed for the errors of all models (in particular for the SVM model). The kernel density estimates of the forecast errors distributions displayed in Figure 5, show that one can frequently observe capital requirement underestimates larger than 4,000 BRL, whereas similar overestimates are more rarer. Such a feature is problematic within a regulatory perspective, and justifies the use of asymmetric loss functions for comparing LGD models. We also observe in Table 2 that the excess kurtosis for the RC are positive, indicating fat tails for the error distributions. As concerned the LGD errors, the artificial neural network and the gradient boosting have the smallest variance. However, it is no longer the case for the artificial neural network when one considers the RC forecast errors. This result clearly illustrates the usefulness of our comparison approach based on the ASRF model.

Figure 5: Kernel density estimate of the estimation error

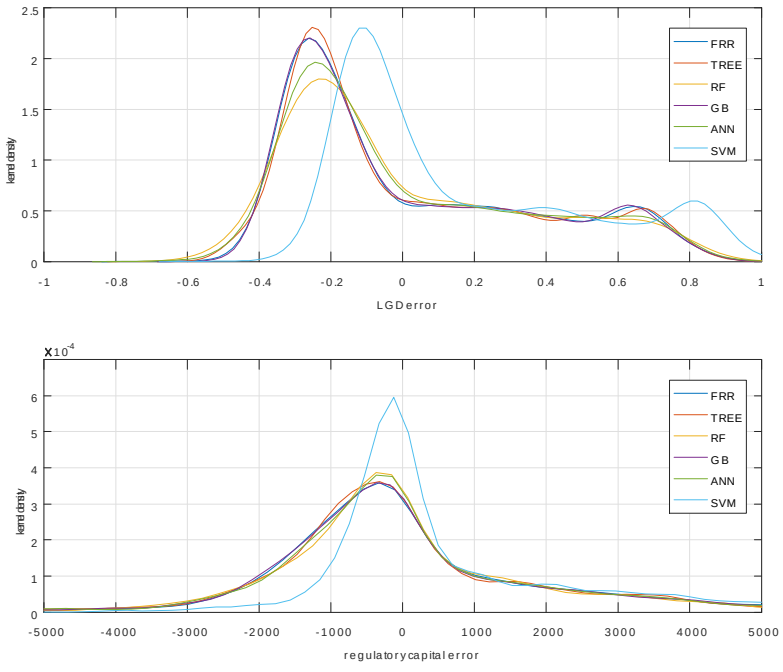


Figure 6 displays the scatter plot of the LGD forecast errors (x-axis) and the RC forecast

errors (y-axis), obtained with the support vector machine. Each point represents a contract (credit or leasing). This plot shows the great heterogeneity that exists between both type of errors. Due to the differences in exposure at default across borrowers, the magnitudes of the RC errors can drastically differ for the same level of LGD forecast error. Consider the two credits represented by the symbols A and B, with an EAD equal to 51,983 BRL and 6,024 BRL, respectively. For the same level of LGD forecast error (74.4%), the support vector machine slightly underestimates the capital requirement (953 BRL, i.e. 72% of the required capital charge) in the case of the credit B, whereas the underestimation reaches 8,231 BRL (86% of the required capital charge) in the case of credit A. Obviously, from a regulatory perspective, the second LGD error should be more penalized than the first one, as its consequence on the RC estimates are more drastic. The dispersion of the observations within the y-axis fully justifies our comparison approach for LGD models, based on expected loss functions expressed in terms of capital charge. Furthermore, the scatter plot confirms the asymmetric pattern of the errors distribution associated to the support vector machine model. This model leads to relatively few overestimates (negative errors), both for LGD and RC, while it leads to large underestimates (positive errors). Thus, any competing LGD model that leads to less underestimates should be preferred from a regulatory perspective. This is why, we recommend the use of asymmetric loss functions for the RC errors.

These features (heterogeneity and asymmetry) are not specific to the SVM model, even if the skewness of the errors is more pronounced for this model compared to the other ones. Figure 7 shows that the profile of the scatter plots of the LGD and RC errors are quite similar for the 6 competing models. This similarity is due to the fact that we use the same set of covariates for all the models.

Figure 6: Scatter plot of the LGD errors and regulatory capital errors for the support vector machine (SVM) model

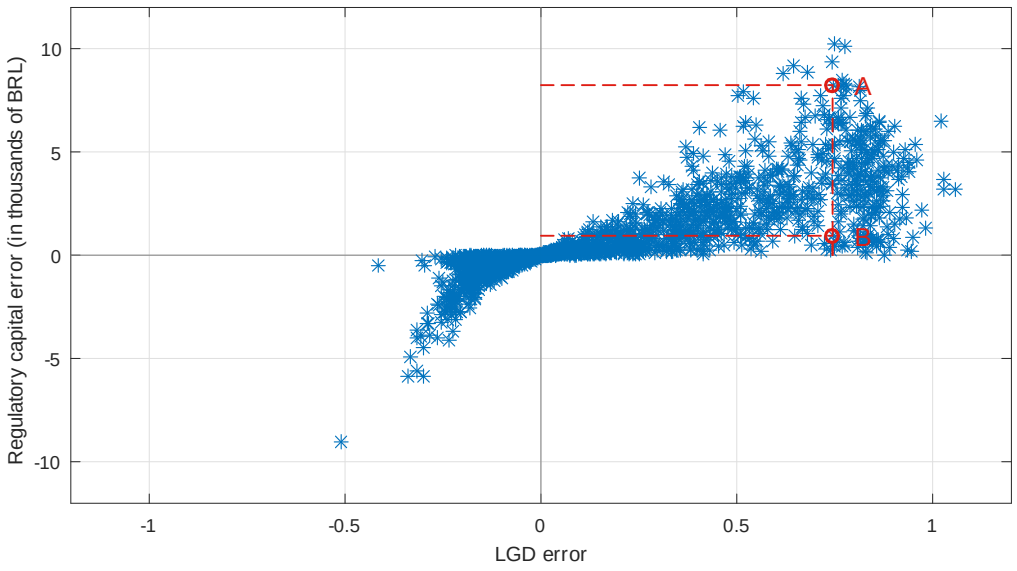
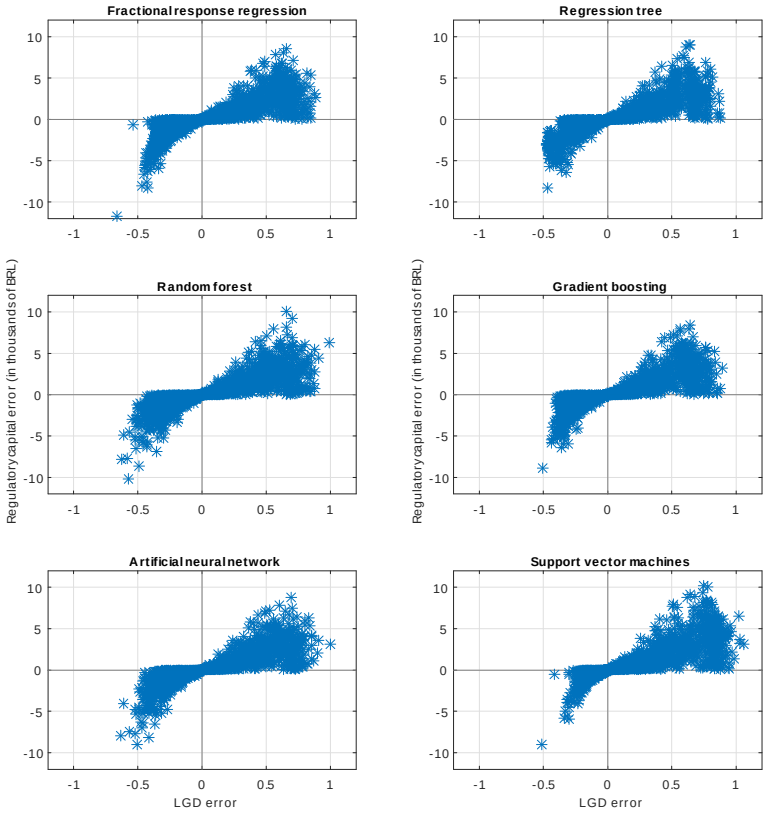


Table 3 displays the model rankings issued from two usual LGD and RC-based loss functions, namely the MSE and the MAE.[16] We also report the rankings issued from the corresponding asymmetric expected losses. Regarding the MSE, the gradient boosting is ranked as the best model, either for LGD or RC symmetric losses. But, when one considers asymmetric losses, it collapses to the penultimate rank and the random forest exhibits the best forecasting abilities. Two comments should be made here. First, in this empirical application, the best model identified by the LGD and RC-based approaches is the same. Nevertheless, this result should not be generalized. As we can observe, the rest of the LGD model rankings are not consistent. For instance, artificial neural network is identified as the second best model with the LGD loss while it holds the fourth rank with the capital charge loss. Conversely, regression tree is ranked second with the capital charge loss while it is ranked at the penultimate posi-

[16] The values of the losses are displayed in Table 7 in appendix G.

Figure 7: Scatter plot of the LGD errors and regulatory capital errors (all models)



tion with the LGD-based loss. Similar results are obtained when one compares the rankings associated to the asymmetric LGD and RC-based loss functions. These inversions prove that the ranking consistency condition of proposition 2 is not valid, at least in our sample, for some couples of models. Second, our results highlight the usefulness of asymmetric loss functions. These functions penalize the models with the largest positive errors (underestimates), as the gradient boosting for instance. Notice that the support vector machine is the worst model, no matter if the loss is symmetric or not. As concerned the MAE criteria, we get similar conclusions, except for the support vector machine, which is ranked as the best model when one

Table 3: Model rankings based on LGD and capital charge expected loss functions

Ranking	LGD Loss	CC Loss	Asym. LGD Loss	Asym. CC Loss
		Mean squared error		
1.	GB	GB	RF	RF
2.	ANN	TREE	FRR	ANN
3.	FRR	RF	ANN	FRR
4.	RF	ANN	TREE	TREE
5.	TREE	FRR	GB	GB
6.	SVM	SVM	SVM	SVM
		Mean absolute error		
1.	SVM	SVM	RF	RF
2.	RF	RF	FRR	ANN
3.	ANN	ANN	ANN	FRR
4.	GB	TREE	TREE	TREE
5.	FRR	GB	GB	GB
6.	TREE	FRR	SVM	SVM

considers symmetric LGD-based loss. Indeed, this model generates relatively few, but large, errors. As a consequence, it is less penalized by the MAE criteria than by the MSE, which is more sensitive to extreme values. However, the support vector machine remains the worst model when one considers asymmetric loss functions, due to the large skewness of its forecast errors. Finally, we also observe some differences between the LGD-based and the RC-based rankings, even if these changes are less frequent than with the MSE criteria, confirming the inconsistency of both rankings.

5 Robustness checks

Our empirical results are robust to a variety of robustness checks. Firstly, instead of considering a common PD value for all the credits in the computation of the capital charges, we use the individual PD calculated by the internal bank's risk model one year before the default

occurs. The corresponding LGD model rankings are reported in Table 4. The corresponding values of the losses are displayed in the bottom part of Table 7 in appendix G. The rankings based on the MSE are similar to that obtained with a common PD (cf. Table 3). The only change concerns the gradient boosting and the classification tree models in the case of asymmetric capital charge loss function. There is no change for the symmetric capital charge loss function. As a consequence, we still observe model ranking inversions compared to the ranking based on the LGD loss functions.

Table 4: Model rankings based on LGD and capital charge (with Basel PD) expected loss functions

Ranking	LGD loss	CC loss	Asym. LGD loss	Asym. CC loss
		Mean squared error		
1.	GB	GB	RF	RF
2.	ANN	RF	FRR	ANN
3.	FRR	TREE	ANN	FRR
4.	RF	ANN	TREE	GB
5.	TREE	FRR	GB	TREE
6.	SVM	SVM	SVM	SVM
		Mean absolute error		
1.	SVM	SVM	RF	RF
2.	RF	RF	FRR	ANN
3.	ANN	ANN	ANN	FRR
4.	GB	TREE	TREE	TREE
5.	FRR	GB	GB	GB
6.	TREE	FRR	SVM	SVM

Secondly, we extend the set of explanatory variables considered for the 6 competing LGD models. As several studies show that recoveries in recessions are lower than during expansions (Schuermann (2004), Bellotti and Crook (2012)), we introduce three additional macroeconomic variables in order to take into account the influence of the economic cycles on the

recovery rate, namely the Brazilian GDP growth, unemployment and interest rates. Table 5 displays the corresponding LGD model rankings obtained for a common PD value. Similar (not reported) results are obtained when one considers the Basel PD estimates. For the MSE criterion, the random forest model outperforms all the competing models whatever the loss function considered. It is also the case for the asymmetric MAE criterion based on the regulatory capital. As in the previous cases, we observe a ranking inconsistency for other models, meaning that the condition of proposition 2 is not valid for these couples of models. The loss values reported in Table 8 (appendix G) are generally smaller than those obtained without macroeconomic variables, confirming the influence of the business cycle on the recovery rates.

Table 5: Model rankings based on LGD and capital charge expected loss functions: LGD models with macroeconomic variables and common PD

Ranking	LGD Loss	CC Loss	Asym. LGD Loss	Asym. CC Loss
		Mean squared error		
1.	RF	RF	RF	RF
2.	GB	TREE	ANN	ANN
3.	ANN	GB	TREE	TREE
4.	TREE	ANN	GB	GB
5.	FRR	FRR	FRR	FRR
6.	SVM	SVM	SVM	SVM
		Mean absolute error		
1.	SVM	SVM	RF	RF
2.	RF	RF	ANN	ANN
3.	ANN	ANN	TREE	TREE
4.	GB	TREE	GB	GB
5.	TREE	GB	FRR	FRR
6.	FRR	FRR	SVM	SVM

Finally, we also consider the same type of regressions by excluding the exposure at default from the set of explanatory variables. The qualitative results (not reported) remain the same:

we observe a global inconsistency of the LGD model rankings based on the LGD estimates or on the capital charge estimates. So, include (or exclude) the EAD as explanatory variable in the LGD models, has no consequence on the validity of the condition of proposition 2, as soon as we consider non-linear LGD models.

6 Conclusion

The LGD is one of the key modeling components of the credit risk capital requirements. In the advanced IRB (AIRB) adopted by most of the major international banks, the LGD forecasts are issued from internal risk model. However, forecasting the credit loss incurred if an obligor of the bank defaults raises numerous issues as regards the definition, the measurement and the modeling of this loss. While professional and academic practices seem to be well established for the PD modeling, no particular guideline has been proposed concerning how LGD models should be compared, selected and evaluated. As a consequence, the model benchmarking method generally adopted by banks and academics simply consists in evaluating the LGD forecasts on a test set, with standard statistical criteria such as MSE, MAE, MAPE, etc., as for any continuous variable. Thus, the LGD models comparison is done regardless of the other Basel risk parameters (EAD, PD, M) and by neglecting the impact of the LGD forecast errors on the regulatory capital.

In this paper, we propose an original comparison methodology for the LGD models, which is based on expected loss functions expressed in terms of regulatory capital charge. These loss functions allow to more penalize the LGD forecast errors associated to large exposure or to long credit maturity. We also define asymmetric loss functions that only penalize the LGD models that lead to regulatory capital underestimates, since these underestimates weaken the

bank's ability to face unexpected credit losses. We theoretically demonstrate that, except under specific conditions, the LGD model rankings either based on the LGD or regulatory capital forecast errors, are not similar. Thus, the current comparison methodology may lead to select a LGD model that has the smallest MSE among all the competing models, but that induces small errors on small exposures, but large errors on large exposures. In this context, our comparison methodology will lead to select another LGD model. Using a sample of credits provided by an international bank, we illustrate the interest of our method by comparing the rankings of 6 competing LGD models. Our empirical results confirm that the ranking based on a naive LGD loss function is generally different from the models ranking obtained with the capital charge symmetric (or asymmetric) loss.

A natural extension of this work will consist to propose statistical tests designed to compare the expected capital charge losses for a pair of LGD models (DM-type test) or to identify a model confidence set (Hansen, Lunde and Nason (2011)) that contain the "best" LGD models, for a given level of confidence.

A Asymptotic Single Risk Factor model

Here, we detail the sketch of the proof of the regulatory formula for the credit capital charge (for more details, see Roncalli (2014) or Gouriéroux and Tiomo (2007)). Let us consider a portfolio of n credits indexed by $i = 1, ..., n$. The portfolio loss is equal to

$$L = \sum_{i=1}^{n} \text{EAD}_i \times \text{LGD}_i \times D_i$$

where EAD_i is the exposure at default for the i^{th} credit (assumed to be constant), LGD_i is the loss given default (random variable) and D_i is a binary random variable that takes a value 1 if there is a default before the residual maturity M_i and 0 otherwise. Formally, $D_i = 1_{(\tau_i \leq M_i)}$ where τ_i is the default time (random variable).

Assumption A1: *The default depends on a set of factors X and we denote by x the realization of X.*

Assumption A2: *The loss given default LGD_i is independent from the default time τ_i.*

Assumption A3: *The default times τ_i, $i = 1, ..., n$ are independent conditionally to the X factors*

Assumption A4: *The portfolio is infinitely fine-grained, which means that there is no concentration*

$$\lim_{n \to \infty} \max \frac{\text{EAD}_j}{\sum_{i=1}^{n} \text{EAD}_i} = 0 \quad \forall j$$

Under assumptions A1-A4, it is possible to show that the conditional distribution of L given X degenerates to the conditional expectation $\mathbb{E}_X(L) = \mathbb{E}(L|X = x)$ and we get

$$L|X \xrightarrow{p} \mathbb{E}_X(L) = \sum_{i=1}^{n} \text{EAD}_i \times \mathbb{E}(\text{LGD}_i) \times p_i(x)$$

where $p_i(x) = \mathbb{E}_X(D_i) = \mathbb{E}(D_i = 1|X = x)$ is the conditional default probability. Notice that under assumption A2, $\mathbb{E}_X(\text{LGD}_i) = \mathbb{E}(\text{LGD}_i)$. As a consequence, the portfolio loss has a marginal distribution given by

$$L \xrightarrow{d} g(X) = \sum_{i=1}^{n} \underbrace{\text{EAD}_i}_{\text{constant term}} \times \underbrace{\mathbb{E}(\text{LGD}_i)}_{\text{constant term}} \times \underbrace{p_i(X)}_{\text{random var.}}$$

Denote by F_L the cdf of L such that $F_L(l) \equiv \Pr(L \leq l) = \Pr(g(X) \leq l)$.

Assumption A5: There is only one factor X, with a cdf $F_X(.)$ and $p_i(X)$ is a decreasing function of X.

Under assumption A5, the α-VaR of the portfolio loss L is defined as $VaR_L(\alpha) = F_L^{-1}(\alpha) = g\left(F_X^{-1}(1-\alpha)\right)$ or equivalently by

$$VaR_L(\alpha) = \sum_{i=1}^{n} \text{EAD}_i \times \mathbb{E}(\text{LGD}_i) \times p_i\left(F_X^{-1}(1-\alpha)\right) = \sum_{i=1}^{n} RC_i$$

where RC_i denotes the risk contribution of the credit i. The VaR of an infinitely fine-grained portfolio can be decomposed as a sum of independent risk contributions, since RC_i only depends on the characteristics of the i^{th} credit (exposure at default, loss given default and probability of default). Similarly, the marginal loss expectation is defined as

$$\mathbb{E}(L) = \sum_{i=1}^{n} \text{EAD}_i \times \mathbb{E}(\text{LGD}_i) \times p_i$$

where $p_i = \Pr(D_i = 1)$ corresponds to the unconditional probability of failure.

Assumption 6: *Let Z_i be the normalized asset value of the entity i. The default occurs when Z_i is below a given barrier B_i (level of debt).*

$$D_i = 1 \quad \text{if} \quad Z_i \leq B_i$$

Assumption 7: *The asset value Z_i depends on a common risk factor X and an idiosyncratic risk factor ε_i.*

$$Z_i = \sqrt{\rho} X + \sqrt{1-\rho}\varepsilon_i$$

where X and ε_i are two independent standard normal random variables, and ρ is the asset's correlation (or with the factor).

Under assumptions A6-A7, the conditional probability of default is

$$p_i(x) = \Phi\left(\frac{B_i - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)$$

where $\Phi(.)$ is the cdf of the standard normal distribution and the barrier B_i corresponds to the quantile associated to the unconditional probability of default, $B_i = \Phi^{-1}(p_i)$. Since $\Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha)$, we get

$$VaR_L(\alpha) = \sum_{i=1}^{n} \text{EAD}_i \times \mathbb{E}(\text{LGD}_i) \times \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

In order to determine the regulatory capital (RC), the BCBS considers the unexpected loss as the credit risk measure

$$\text{RC} = UL(\alpha) = VaR_L(\alpha) - \mathbb{E}(L)$$

Then, we get

$$\text{RC} = \sum_{i=1}^{n} \text{EAD}_i \times \mathbb{E}(\text{LGD}_i) \times \left(\Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) - p_i\right)$$

By considering a risk level $\alpha = 99.9\%$ and by denoting PD the unconditional probability of default, we get the IRB formula (without maturity adjustment).

B Maturity adjustment and correlation functions

The maturity adjustment suggested by the BCBS depends on the type of exposure. For the corporate, sovereign, and bank exposures, it is defined as

$$\gamma\left(\mathrm{M}\right)=\frac{1+\left(\mathrm{M}-2.5\right)\times b\left(\mathrm{PD}\right)}{1-1.5\times b\left(\mathrm{PD}\right)}$$

with the smoothed maturity adjustment equal to

$$b\left(\mathrm{PD}\right)=\left(0.11852-0.05478\log\left(\mathrm{PD}\right)\right)^{2}$$

For the retail exposures, there is no maturity adjustment, i.e. $\gamma\left(\mathrm{M}\right)=1$. The correlation function $\rho\left(\mathrm{PD}\right)$ describes the dependence of the asset value of a borrower on the general state of the economy. Different asset classes show different degrees of dependency on the overall economy, so it's necessary to adapt the correlation coefficient to these classes. The correlation function $\rho\left(\mathrm{PD}\right)$ for corporate, sovereign and bank exposures is defined as

$$\rho\left(\mathrm{PD}\right)=0.12\times\left(\frac{1-e^{-50\ \mathrm{PD}}}{1-e^{-50}}\right)+0.24\times\left(1-\left(\frac{1-e^{-50\ \mathrm{PD}}}{1-e^{-50}}\right)\right)$$

For small and medium-sized enterprises (SME), a firm-size adjustment is introduced that depends on the sales. In the sequel, we neglect this adjustment for simplicity. For retail exposures, the correlation function $\rho\left(\mathrm{PD}\right)$ depends on the exposures. For residential mortgage exposures the BCBS recommends to fix the correlation at 0.15, for revolving retail exposures at 0.04 and for other retail exposures, to use the following formula:

$$\rho\left(\mathrm{PD}\right)=0.03\times\left(\frac{1-e^{-35\ \mathrm{PD}}}{1-e^{-35}}\right)+0.16\times\left(1-\left(\frac{1-e^{-35\ \mathrm{PD}}}{1-e^{-35}}\right)\right)$$

C Proof of proposition 2

Proof. Under assumptions A1-A2, the capital charge expected loss can be expressed as

$$\begin{aligned}\mathcal{L}_{CC,m} &= \mathbb{E}\left(g\left(\eta_{i,m}\right)\right)\\ &= \mathbb{E}\left(g\left(\mathrm{EAD}_i\times\delta\left(\mathrm{PD}\right)\times\gamma\left(\mathrm{M}\right)\times\varepsilon_{i,m}\right)\right)\\ &= g\left(\delta\left(\mathrm{PD}\right)\right)\times g\left(\gamma\left(\mathrm{M}\right)\right)\times\mathbb{E}\left(g\left(\mathrm{EAD}_i\times\varepsilon_{i,m}\right)\right)\end{aligned}$$

since $\delta\left(\mathrm{PD}\right)$ and $\gamma\left(\mathrm{M}\right)$ are positive constant terms. Rewrite $\mathcal{L}_{CC,m}$ as

$$\mathcal{L}_{CC,m}=\Delta\times cov\left(g\left(\mathrm{EAD}_i\right),g\left(\varepsilon_{i,m}\right)\right)+\Delta\times\mathbb{E}\left(g\left(\mathrm{EAD}_i\right)\right)\times\mathcal{L}_m$$

with $\Delta = g(\delta(\text{PD})) \times g(\gamma(\text{M}))$ and $\mathcal{L}_m = \mathbb{E}(g(\varepsilon_{i,m}))$. Consider two LGD models m and $m+1$. The rankings of the two models are consistent as soon as $\mathcal{L}_m < \mathcal{L}_{m+1}$ and $\mathcal{L}_{CC,m} < \mathcal{L}_{CC,m+1}$. Since $\Delta > 0$, these conditions can be expressed as

$$cov(g(\text{EAD}_i), g(\varepsilon_{i,m})) + \mathbb{E}(g(\text{EAD}_i)) \times \mathcal{L}_m < cov(g(\text{EAD}_i), g(\varepsilon_{i,m+1})) + \mathbb{E}(g(\text{EAD}_i)) \times \mathcal{L}_{m+1}$$

Or equivalently as

$$cov(g(\text{EAD}_i), g(\varepsilon_{i,m})) - cov(g(\text{EAD}_i), g(\varepsilon_{i,m+1})) < \mathbb{E}(g(\text{EAD}_i))(\mathcal{L}_{m+1} - \mathcal{L}_m)$$

with $\mathcal{L}_{m+1} - \mathcal{L}_m < 0$ and $\mathbb{E}(g(\text{EAD}_i)) > 0$. ■

D Proof of corollary 3

Proof. If the variables EAD_i and $\varepsilon_{i,m}$ are independent, the variables $g(\text{EAD}_i)$ and $g(\varepsilon_{i,m})$ are also independent. Then, the capital charge expected loss becomes

$$\mathcal{L}_{CC,m} = \mathbb{E}\left(g\left(\eta_{i,m}\right)\right) = g(\delta(\text{PD})) \times g(\gamma(\text{M})) \times \mathbb{E}(g(\text{EAD}_i)) \times \mathbb{E}(g(\varepsilon_{i,m}))$$

Consider two LGD models m and $m+1$, $\forall m = 1, ..., \mathcal{M} - 1$, for which $\mathcal{L}_m < \mathcal{L}_{m+1}$, then we have

$$\Delta_i \times \mathbb{E}(g(\varepsilon_{i,m})) < \Delta_i \times \mathbb{E}(g(\varepsilon_{i,m+1}))$$

with $\Delta_i = g(\delta(\text{PD})) \times g(\gamma(\text{M})) \times \mathbb{E}(g(\text{EAD}_i)) > 0$. The ranking of LGD models are necessarily consistent, i.e. $\mathcal{L}_{CC,m} < \mathcal{L}_{CC,m+1}$. ■

E Dataset description

Table 6: List of the variables

Variables type	Variables name	Description
Contract	Duration	Duration of the contract
	Time to default	Number of months before default
	Relative duration	Time to default divided by duration
	Interest rate	Interest (or renting) rate
	Exposition type	Credit or leasing
	Customer type	Individual, professional (natural or legal)
	Brand of the car	Brand name of the car
	State of the car	New or second-hand
Macroeconomic	GDP Growth rate	Brazil, quaterly
	Unemployment rate	Brazil, monthly
	Interbank market interest rate	Brazil, monthly
Basel parameters	EAD	Exposure at default
	PD	Basel default probability estimated by the bank
	LGD	Loss Given Default

F Support vector machine

Suppose a set of training data $\{y_i, X_i\}_{i=1}^N$ in which y_i is the observed response value (i.e. LGD_i in our case) and X_i the associated k-vector of explanatory variables for the i^{th} individual. Let us assume that y_i can be approximated by a linear function such that

$$f(X_i) = X_i'\beta$$

where β is a k-vector of unknown parameters. In a SVM regression, β is determined by solving a risk minimization problem with respect to an ϵ-insensitive loss function ($\epsilon \geq 0$). This ϵ-insensitive loss function belongs to the so-called *robust regression* family and is known to provide reliable forecasts for many distributional hypothesis made on the regression noise (see Vapnik (2000) for more details). In the following, we consider the well-known *linear* ϵ-insensitive loss function defined as

$$L_\epsilon(y, f(X)) = \begin{cases} 0 & \text{if } |y - f(X)| \leq \epsilon \\ |y - f(X)| - \epsilon & \text{otherwise} \end{cases}$$

SVM regression hence consists in finding the value of β that solves a convex minimization problem subject to a L_ϵ-based constraint. One has to minimize

$$J(\beta) = \frac{1}{2}\beta'\beta$$

subject to

$$|y_i - Xi'\beta| \leq \epsilon, \qquad \forall i = 1, \ldots, N$$

The minimization of the objective $J(\beta)$ allows to control appropriately for overfitting, while the constraints impose $f(X_i)$ to deviate from y_i by a value no greater than ϵ for all the observations. As the constraints cannot be satisfied for some observations, slack variables $\{\xi_i, \xi_i^*\}_{i=1}^N$ are introduced in order to get a feasible problem. With these slack variables, the primal formula becomes

$$\Phi(\beta, \xi^*, \xi) = \frac{1}{2}\beta'\beta + C\left(\sum_{i=1}^N \xi_i^* + \sum_{i=1}^N \xi_i\right)$$

under the constraints

$$\begin{aligned} y_i - X_i'\beta &\leq \epsilon + \xi_i^*, \quad i = 1, \ldots, N \\ X_i'\beta - y_i &\leq \epsilon + \xi_i, \quad i = 1, \ldots, N \\ \xi_i^* &\geq 0, \qquad i = 1, \ldots, N \\ \xi_i &\geq 0, \qquad i = 1, \ldots, N \end{aligned}$$

The constant C is the box constraint, a positive regularization parameter that controls the penalty imposed on observations that lie outside the ϵ margin. Therefore, this parameter determines the trade-off between the model complexity (flatness) and the degree to which deviations larger than ϵ are tolerated. This optimization problem can be solved in a simpler way using its Lagrange dual formulation counterpart. The dual formula requires the introduction of nonnegative multipliers denoted $\{\alpha_i, \alpha_i^*\}_{i=1}^N$ in the optimization problem leading to the maximization of

$$W(\alpha, \alpha^*) = -\epsilon \sum_{i=1}^{N} (\alpha_i^* + \alpha_i) + \sum_{i=1}^{N} y_i (\alpha_i^* - \alpha_i) - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\alpha_i^* - \alpha_i) (\alpha_j^* - \alpha_j) X_i' X_j$$

subject to constraints

$$\sum_{i=1}^{N} \alpha_i^* = \sum_{i=1}^{N} \alpha_i,$$
$$0 \leq \alpha_i^* \leq C, \qquad i = 1, \ldots, N$$
$$0 \leq \alpha_i \leq C, \qquad i = 1, \ldots, N$$

Using optimized $\{\alpha_i, \alpha_i^*\}_{i=1}^N$ multipliers and $\{X_i\}_{i=1}^N$ covariates allow to compute

$$\beta = \sum_{i=1}^{N} (\alpha_i^* - \alpha_i) X_i$$

We finally get

$$f(X_i) = X_i' \sum_{i=1}^{N} (\alpha_i^* - \alpha_i) X_i$$

G Empirical means of the losses

Table 7: Empirical means of the LGD and capital charge losses

			FRR	ANN	TREE	SVM	RF	GB
Common PD (0.40451)								
	Standard	LGD	0.1169	0.1160	0.1176	0,1428	0.1170	0.1160
MSE		CC	3,815,235	3,799,711	3,732,987	4,604,789	3,737,030	3,720,010
	Asymmetric	LGD	0.2010	0.2014	0.2018	0.2646	0.1982	0.2019
		CC	6,036,659	5,898,476	6,131,352	8,212,090	5,835,150	6,179,716
MAE	Standard	LGD	0.2906	0.2856	0.2908	0.2729	0.2856	0.2896
		CC	1,373.96	1,353.61	1,366.04	1,306.95	1,342.20	1,367.34
	Asymmetric	LGD	0.3815	0.3817	0.3817	0.4243	0.3752	0.3843
		CC	1,815.28	1,800.22	1,819.83	2,016.85	1,758.73	1,836.43
Basel PD								
MSE	Standard	LGD	0.1169	0.1160	0.1176	0.1428	0.1170	0.1160
		CC	1,041,302	1,034,879	1,009,601	1,173,002	1,008,480	1,005,158
	Asymmetric	LGD	0.2010	0.2014	0.2018	0.2646	0.1982	0.2019
		CC	1,450,604	1,435,094	1,504,186	2,008,958	1,414,640	1,502,921
MAE	Standard	LGD	0.2906	0.2856	0.2908	0.2729	0.2856	0.2896
		CC	693.70	682.04	686.89	651.49	677.57	688.98
	Asymmetric	LGD	0.3815	0.3817	0.3817	0.4243	0.3752	0.3843
		CC	869.97	861.95	878.73	975.01	844.15	884.70

Table 8: Empirical means of the LGD and capital charge losses (LGD models with macroeconomic variables)

			FRR	ANN	TREE	SVM	RF	GB
Common PD (0,40451)								
MSE	Standard	LGD	0.1125	0.1108	0.1117	0.1307	0.1092	0.1101
		CC	3,589,815	3,530,643	3,470,465	4,107,855	3,441,599	3,476,828
	Asymmetric	LGD	0.1994	0.1870	0.1904	0.2425	0.1843	0.1921
		CC	5,792,513	5,480,483	5,678,123	7,376,466	5,443,856	5,716,877
MAE	Standard	LGD	0.2805	0.2731	0.2768	0.2626	0.2705	0.2766
		CC	1,316.02	1,279.29	1,296.38	1,243.49	1,270.13	1,304.60
	Asymmetric	LGD	0.3807	0.3613	0.3683	0.4030	0.3583	0.3721
		CC	1,784.87	1,691.39	1,741.11	1,897.82	1,686.07	1,758.48
Basel PD								
MSE	Standard	LGD	0.1125	0.1108	0.1117	0.1307	0.1092	0.1101
		CC	991,435	971,479	952,024	1,066,160	942,159	945,860
	Asymmetric	LGD	0.1994	0.1870	0.1904	0.2425	0.1843	0.1921
		CC	1,442,541	1,389,007	1,462,395	1,851,081	1,367,273	1,412,801
MAE	Standard	LGD	0.2805	0.2731	0.2768	0.2626	0.2705	0.2766
		CC	666.01	646.16	651.15	620.44	641.51	658.36
	Asymmetric	LGD	0.3807	0.3613	0.3683	0.4030	0.3583	0.3721
		CC	861.93	814.91	850.51	921.58	813.93	849.72

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