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Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

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2017.33

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

Multivariate Reflection Symmetry of Copula Functions

[1], Monica Billio[1], Lorenzo Frattarolo *[1], and Dominique Guégan[2]

[1] University Ca' Foscari of Venice, Department of Economics, Fondamenta San Giobbe, Cannareggio 873, 30121 Venice, Italy
[2]University Paris-1 Panthéon-Sorbonne, MSE-CES, 106, avenue de l' Hopital, 75013 Paris, France

Abstract

We propose a multivariate nonparametric copula test of reflection symmetry. The test is valid in any number of dimensions, extending previous results that cover the bivariate case. Furthermore, the asymptotic theory for the test relies on recent results on the dependent multiplier bootstrap, valid for sub-exponentially strongly mixing data. Consequently to the introduction of those two features, the procedure is suitable for financial time series whose asymmetric dependence, in distressed periods, has already been documented elsewhere. We conduct an extensive simulation study of empirical size and power and provide several examples of applications. In particular, we investigate the use of the statistic as a financial stress indicator by comparing it with the CISS , the leading ECB indicator.

Keywords: Dependence Asymmetry, Copula, Reflection Symmetry, Radial Symmetry, Empirical Process, Dependent Multiplier Bootstrap, Financial Stress

1 Introduction

Statistical description of multivariate data has suffered from serious limitations, due to the mathematical difficulties associated multivariate probability distributions. Accordingly, multivariate Gaussian distribution was the building block of the vast majority of models used. The situation has slowly changed when Sklar, Sklar [1959] introduced copula functions, allowing to model, separately, the dependence structure and the marginals. Recently, copula functions have become essential ingredients, in many applied fields concerning the modeling of multivariate data, such as actuarial sciences, finance, hydrology and survival analysis, Genest and Favre [2007], Cherubini et al. [2004],Joe [2014]. Nevertheless, in most high-dimensional applications of related methods, in particular in presence of temporal dependence, the Gaussian and t-copulas remain the

*lorenzo.frattarolo@unive.it

standard copula models.This is also true in a semiparametrc framework, for example, the popular dynamic quantile regression proposed Bouyé and Salmon [2009], implicitly assumes reflection symmetry. A test of the symmetry proposed in this paper allows to justify or reject the use of those models. In addition, testing this property could be evidence of asymmetric dependence that was empirically documented by different statistical devices in Longin and Solnik [2001], Ang and Chen [2002] and Hong et al. [2007] for financial time series. Moreover, consequences of this stylized fact are highly relevant for portfolio risk management,(Patton [2001])and multivariate option pricing,(Cherubini and Romagnoli [2010],Cherubini et al. [2011]).

Our multivariate non-parametric copula framework delivers a measure that includes the effect of more than two series and investigates, directly, a system-wide change in the dependence structure. This joint change in dependence properties is different from a simultaneous change in the characteristics of marginal distributions. Those are important advantages with respect to correlation and conditional correlation-based measures but also with respect to already proposed bivariate nonparametric copula tests. Motivated by this financial application, we also include a proper treatment of weakly dependent data. This is important because unconditional dependence asymmetry could be dynamically generated and the usual strategy of using i.i.d test on the residuals could be misleading. Conversely, our asymptotic test, being based on empirical copula, survival empirical copula, and dependent multiplier bootstrap, is completely nonparametric and is valid under the vast majority of hypothesis underlying most common strictly stationary parametric models, Segers et al. [2012], Bücher et al. [2016]. This paper proposes a multivariate extension beyond dimension 2 of the Cramér–von Mises statistic introduced in Bouzebda and Cherfi [2012] and studied in Dehgani et al. [2013], Genest and Nešlehová [2013]. In particular, according to Genest and Nešlehová [2013] the chosen Cramér–von Mises was the most powerful statistic in the bivariate case, allowing us to restrict our analysis to this functional form. Different investigations of the same problem by other bivariate non-parametric approaches are in Rosco and Joe [2013], Li and Genton [2013], Quessy [2016].The extension beyond dimension 2 is difficult due to the complex multivariate generalization of the relation between the copula and the survival copula (see for example Georges et al. [2001]). Conversely, we avoid this issue by the direct use of the survival empirical copula process, in the spirit pioneered for the nonparametric modelization of survival tail copula in the bivariate context by Schmidt and Stadtmüller [2006]. The use of empirical (tail) survival copula, computed directly from the observations, allow those authors to bypass the complex functional expression of the survival copula as a function of the copula, but to derive, in any case, the asymptotics of the empirical survival copula process in complete analogy to the usual empirical copula process. Even if this is the case Schmidt and Stadtmüller [2006] still remain in a bivariate framework and to our knowledge, no extension of their work to dimension 3 or beyond has been done. The multivariate extension in Schmid and Schmidt [2007] is explicit only for lower tail dependence i.e. for the usual empirical copula process. In addition with respect to them, due to the recent advances in the study of empirical copula process Segers et al. [2012], Bücher et al. [2016], our asymptotic results are less restrictive in terms of copula derivatives and dependence assumptions. To our knowledge, the only paper that proposes a multivariate nonparametric reflection test and conducts a simulation study

with dimensionality greater or equal to three is Krupskii [2016]. The approach of this paper is aggregation by a, cleverly chosen weighted statistic, and has, with respect to our work, the advantage of deriving a closed form asymptotics of a signed statistic. This feature implies less computational burden, even if it requires bootstrap or jackknife estimation of the asymptotic variance, but does not cover the case of weak dependence in the data sample. In addition, a closer inspection to the statistic proposed for higher (≥ 10) dimensions reveal that it is a sum of all possible equivalent trivariate statistics.
The paper is structured as follows: In section 2 we introduce the rank reflection symmetry and motivate its importance for the right modelization of extreme events and asymmetric dependence. In section 3 we discuss the asymptotics of the test and introduce the dependent multiplier bootstrap. In section 4 we provide an extensive simulation study where we discuss the empirical size and power of the test. Section 5 reports a simple application of the methodology to the Cook Jonhson database and financial data. At the end of the section, we compare the behavior of our statistic, computed rolling on European stock indexes, with the CISS Holló et al. [2012], the main financial stress indicator of the BCE. In section 6 we summarize our findings discussing their implications and extensions.

2 Copulas and Rank Reflection

The aim of this paper is the introduction of a new non parametric test for the study of the null hypothesis

$$H_0: \quad C(u_1, \ldots, u_D) = \bar{C}(u_1, \ldots, u_D) \tag{1}$$

In the previous display C is a copula and $\bar{C}$ is the corresponding survival copula, to be properly introduced in the following. This relationship was first introduced, in the bivariate copula context by Nelsen [1993] and Nelsen [2007] under the name of radial symmetry of the copula function. Moreover, the reason, motivating the use of this name, was that this invariance property is linked to the so-called radial symmetry of the bivariate distribution. Instead, we follow Rosco and Joe [2013] and Joe [2014], in calling it reflection symmetry. In this section, after introducing copulas and survival copulas, we discuss reflection symmetry, its probabilistic implication and which copulas and time series models manifest the symmetry.
We begin introducing the well known relationship among the cumulative marginal distribution function F_i of the i-th variable X_i and the marginal survival function $\bar{F}_i$ of the same variable:

$$\mathbb{P}(X_i > x_i) = \bar{F}_i(x_i) = 1 - \mathbb{P}(X_i \leq x_i) = 1 - F_i(x_i) \tag{2}$$

Moreover, applying the probability integral transforms $F_i(X_i) = U_i$ and $\bar{F}_i(X_i) = \bar{U}_i$ we can translate it to a relationship among uniform random variables :

$$\bar{U}_i = 1 - U_i \tag{3}$$

.
Consequently, the sample version of U_i represents the univariate ranks of the

sample from X_i divided by the number of observations in the sample and, using (3), $\bar{U}_i$ can be interpreted as the one dimensional reflection around the center of the unitary interval of U_i. In this way the usual concept of symmetry of the distribution :

$$\begin{cases} F_i\left(c_i - x_i\right) & = & 1 - F_i\left(c_i + x_i\right) = \bar{F}_i\left(c_i + x_i\right) \\ F_i\left(c_i\right) & = & 1 - F_i\left(c_i\right) = \bar{F}_i\left(c_i\right) = \dfrac{1}{2} \end{cases} \tag{4}$$

can be interpreted (see the appendix) as a rank reflection symmetry in probability:

$$\mathbb{P}\left(U_i \leq u_i\right) = \mathbb{P}\left(\bar{U}_i \leq u_i\right). \tag{5}$$

The extension to the multivariate setting can be translated in a relationship between the copula and its corresponding survival.
In fact, according to the Sklar Theorem, Sklar [1959], a multivariate cumulative distribution could be expressed using the univariate marginal cumulative distributions and a copula function

$$\mathbb{P}\left(X_1 \leq x_1, \ldots, X_D \leq x_D\right) = F\left(x_1, \ldots, x_D\right) = C\left(F_1\left(x_1\right), \ldots, F_D\left(x_D\right)\right) \tag{6}$$

Then, with the application of the probability integral transform to the original random variable we produce an uniform random vector $\mathbf{U} = \left(U_1, \ldots, U_D\right)^T$ with $U_i = F_i\left(X_i\right)$. In addition, given the joint distribution and the marginals we can define a copula by:

$$\mathbb{P}\left(U_1 \leq u_1, \ldots, U_D \leq x_D\right) = C\left(u_1, \ldots, u_D\right) = F\left(F_1^{-1}\left(u_1\right), \ldots, F_D^{-1}\left(u_D\right)\right) \tag{7}$$

Analogously, a multivariate survival function could be expressed using the univariate survival functions and the survival copula

$$\mathbb{P}\left(X_1 > x_1, \ldots, X_D > x_D\right) = \bar{F}\left(x_1, \ldots, x_D\right) = \bar{C}\left(\bar{F}_1\left(x_1\right), \ldots, \bar{F}_D\left(x_D\right)\right) \tag{8}$$

The survival copula is in this way a distribution function on the hypercube (not a survival function) of the random vector $\bar{\mathbf{U}} = \left(\bar{U}_1, \ldots, \bar{U}_D\right)^T$ with $\bar{U}_i = \bar{F}_i\left(X_i\right)$

$$\mathbb{P}\left(\bar{U}_1 \leq u_1, \ldots, \bar{U}_D \leq x_D\right) = \bar{C}\left(u_1, \ldots, u_D\right) = \bar{F}\left(\bar{F}_1^{-1}\left(u_1\right), \ldots, \bar{F}_D^{-1}\left(u_D\right)\right). \tag{9}$$

In this paper, we are concerned with the study of the null hypothesis of rank reflection symmetry, that is represented by the following generalization of (5):

$$\mathbb{P}\left(U_1 \leq u_1, \ldots, U_D \leq u_D\right) = \mathbb{P}\left(\bar{U}_1 \leq u_1, \ldots, \bar{U}_D \leq u_D\right) \tag{10}$$

$$\Leftrightarrow C\left(u_1, \ldots, u_d\right) = \bar{C}\left(u_1, \ldots, u_d\right) \tag{11}$$

Since the sample version of $\mathbf{U}$ represents the vector of univariate ranks of the sample divided by the number of observations, and $\bar{\mathbf{U}}$ is $\mathbf{U}$ reflected in the center of the unit hypercube, we follow Joe [2014] in choosing the name reflection symmetry for this property. Moreover we remark that this is only a necessary condition for reflection symmetry of the multivariate distribution around a point. In fact in order to have reflection symmetry of the multivariate distribution, copula symmetry must be supplemented by the symmetry of all the

marginals (c.f. Patton [2001]).
In terms of probabilities, rank reflection symmetry correspond to:

$$\begin{aligned} &\mathbb{P}\left(U_1 \leq u_1, \ldots, U_D \leq u_D\right) = \mathbb{P}\left(\bar{U}_1 \leq u_1, \ldots, \bar{U}_D \leq u_D\right) \\ \Leftrightarrow\ &\mathbb{P}\left(X_1 \leq x_1, \ldots, X_D \leq x_D\right) = \mathbb{P}\left(X_1 > F_1^{-1}\left(1 - F_1\left(x_1\right)\right), \ldots, X_D > F_D^{-1}\left(1 - F_D\left(x_D\right)\right)\right) \end{aligned} \tag{12}$$

To have a better understanding of (12), we compute the previous relation in the vector of marginal u-th quantiles $\left\{F_d^{-1}\left(u\right)\right\}_{d=1}^{D}$.

$$\mathbb{P}\left(X_1 \leq F_1^{-1}\left(q\right), \ldots, X_D \leq F_D^{-1}\left(u\right)\right) = \mathbb{P}\left(X_1 \geq F_1^{-1}\left(1-u\right), \ldots, X_D \geq F_D^{-1}\left(1-u\right)\right).$$

Accordingly, one consequence of copula reflection symmetry is that the probability of having all variables less than their respective u-th quantile is the same of the probability of having all the variables greater than the complementary marginal quantile. In this way, in particular, upper and lower joint extreme events are equiprobable. Moreover introducing the multivariate generalization of upper and lower tail dependence coefficients Sibuya [1959], Joe [1997] allow us to express in another way the same concept :

$$\lambda_U = \lim_{v \to 1} \frac{\bar{C}\left(1-v, \cdots, 1-v\right)}{1-v} = \lim_{u \to 0} \frac{\bar{C}\left(u, \cdots, u\right)}{u} \tag{13}$$

$$\lambda_L = \lim_{u \to 0} \frac{C\left(u, \cdots, u\right)}{u} \tag{14}$$

From (13) and (14), we have that rank reflection symmetry implies the equality $\lambda_U = \lambda_L$ and this is, also, true for more refined measure of tail dependence as the one introduced in Schmid and Schmidt [2007]. Consequently, given the above interpretation of reflection symmetry, testing this property could be evidence of asymmetric dependence and financial stress as already discussed in the introduction. For a complete discussion of which parametric models are rank reflection symmetric in the two-dimensional case, we refer to the seminal paper of Nelsen, Nelsen [1993] and more the recent works of Rosco and Joe [2013] and Dehgani et al. [2013]. In particular, all elliptical copula are rank reflection symmetric and that Frank,Frank [1979] explicitly constructed the only rank reflection symmetric bivariate Archimedean copula. Moreover, for higher dimensions, to our knowledge, only elliptical copulas are known to exhibit this property. In time series context,it is well known, linear models with elliptic innovation are elliptically distributed and, a fortiori, exhibit rank reflection symmetry, because the class of elliptical distributions is closed under affine transformations and reflection is an affine transformation. In general, assuming, as is usually the case for common processes, that the marginal are symmetric with support $[-\infty, \infty]$, and, for simplicity, that the process has zero median and that the innovations are i.i.d., reflection is simply a joint sign change of the process components. For linear processes, this can be translated to a joint sign change of the innovations path. Accordingly, the process is reflection symmetric if and only if the innovations are symmetric. For a multivariate GARCH process in order to relate symmetry properties to the symmetry properties of the innovations, we should evaluate the effect of a joint sign change of the innovation path on the implied covariance process. For an ordinary BEKK process, the covariance process is a quadratic form in the innovations and it is invariant with respect to a joint sign change, so the symmetry properties are inherited from the innovation distribution. In the case of the asymmetric BEKK instead, the situation is more involved since additional asymmetry is also dynamically generated in the covariance process.

3 Asymptotics with Empirical Processes

In this section, after introducing the empirical copula, the survival empirical copula and the related empirical processes, we discuss the dependent multiplier bootstrap as a way to approximate the asymptotic distribution and introduce our test statistic.
The bivariate survival copula can be expressed in terms of the corresponding copula through the relationship:

$$\bar{C}(u,v) = u + v - 1 + C(1-u, 1-v) \tag{15}$$

Accordingly, this relationship was the main device in testing this invariance property Bouzebda and Cherfi [2012], Rosco and Joe [2013], Dehgani et al. [2013], Genest and Nešlehová [2013] and Li and Genton [2013]. However, due to the intricacy of the multivariate analogue, reported inGeorges et al. [2001] and Cherubini et al. [2011], and omitted here, this approach is difficult to generalize for a number of variables greater than 2. Conversely, by making explicit use of empirical survival copula process, in this paper we propose an alternative way of testing the null hypothesis (1) of multivariate reflection simmetry.

3.1 Empirical Copula Processes

Let $\{\{X_{id}\}_{i=1}^{n}\}_{d=1}^{D} \equiv \{\mathbf{X}_i\}_{i=1}^{n}$ a D dimensional multivariate sample of size n of strongly-mixing random variables. The empirical distribution function $\hat{F}(\mathbf{x})$, the empirical survival function $\hat{\bar{F}}(\mathbf{x})$ and their marginals are

$$\hat{F}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\prod_{d=1}^{D}\mathbb{I}(X_{id} \leq x_d)) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}(\mathbf{X_i} \leq \mathbf{x}) \tag{16}$$

$$\hat{F}_d(x_d) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}(X_{id} \leq x_d) \tag{17}$$

$$\hat{\bar{F}}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\prod_{d=1}^{D}\mathbb{I}(X_{id} > x_d) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}(\mathbf{X_i} > \mathbf{x}) \tag{18}$$

$$\hat{\bar{F}}_d(x_d) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}(X_{id} > x_d) \tag{19}$$

$$\tag{20}$$

Moreover, let us define the pseudo observations $\hat{U}_{id} = \hat{F}_d(X_{id})$ and $\hat{\bar{U}}_{id} = \hat{\bar{F}}_d(X_{id}) = 1 - \hat{U}_{id}$. Then, the empirical copula and the empirical survival copula are

$$\hat{C}_n(\mathbf{u}) = \hat{F}\left(\hat{F}_1^{-1}(u_1), \ldots, \hat{F}_D^{-1}(u_D)\right) = \frac{1}{n}\sum_{i=1}^{n}\prod_{d=1}^{D}\mathbb{I}\left(X_{id} \leq \hat{F}_d^{-1}(u_d)\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(\hat{\mathbf{U}}_i \leq \mathbf{u}\right) \quad (21)$$
$$\hat{\bar{C}}_n(\mathbf{u}) = \hat{\bar{F}}\left(\hat{\bar{F}}_1^{-1}(u_1), \ldots, \hat{\bar{F}}_D^{-1}(u_D)\right) = \frac{1}{n}\sum_{i=1}^{n}\prod_{d=1}^{D}\mathbb{I}\left(X_{id} > \hat{\bar{F}}_d^{-1}(u_d)\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(\hat{\bar{\mathbf{U}}}_i \leq \mathbf{u}\right) \quad (22)$$

where we used the fact that empirical survivals are non increasing.
Accordingly, the empirical processes are:

$$\hat{\mathbb{C}}_n = \sqrt{n}\left(\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\right) \quad (23)$$
$$\hat{\bar{\mathbb{C}}}_n = \sqrt{n}\left(\hat{\bar{C}}_n(\mathbf{u}) - \bar{C}(\mathbf{u})\right). \quad (24)$$

Consequently, an application of the functional Delta method van der vaart and Wellner [1996] on results for central limit theorem (CLT) of multivariate empirical process for strongly mixing data with mixing coefficient $\alpha_n = o(n^{-a})$ for some $a > 0$, that can be found in Rio [1999], allow Bücher and Volgushev [2013] (see also Bücher et al. [2016] for the sequential version) to obtain the following weak convergence result for the empirical copula process

$$\hat{\mathbb{C}}_n \rightsquigarrow \mathbb{C} = \mathbb{B}_C(\mathbf{u}) - \sum_{d=1}^{D}\frac{\partial C(\mathbf{u})}{\partial u_d}\mathbb{B}_{d,C}(u_d), \quad (25)$$

where $\mathbb{B}_C$ is a D-dimensional Brownian sheet with covariance function

$$\mathrm{Cov}\left(\mathbb{B}_C(\mathbf{u}), \mathbb{B}_C(\mathbf{v})\right) = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v}) \quad (26)$$

where $\wedge$ is the component-wise minimum.
In particular, we derive in the next proposition the weak convergence results for the empirical survival copula process, for strongly mixing data, under an assumption analogous to the one introduced in Segers et al. [2012] (see the appendix for the proof):

A 1 *For each $j \in \{1, 2, 3\}$, the jth first-order partial derivative $\frac{\partial \bar{C}}{\partial u_j}$ exists and is continuous on the set $V_{D,j} := \left\{u \in [0,1]^D : 0 < u_j < 1\right\}$.*

proposition 1 *Suppose that conditions **A1** hold and the strongly mixing coefficients α_n of the sample are such that $\alpha_n = o(n^{-a})$ for some $a > 0$, then the empirical survival copula process $\hat{\bar{\mathbb{C}}}_n = \sqrt{n}\left(\hat{\bar{C}}_n(\mathbf{u}) - \bar{C}(\mathbf{u})\right)$ weakly converges towards a Gaussian field $\bar{\mathbb{C}}$*

$$\hat{\bar{\mathbb{C}}}_n \rightsquigarrow \bar{\mathbb{C}} = \mathbb{B}_{\bar{C}}(\mathbf{u}) - \textstyle\sum_{d=1}^{D}\frac{\partial \bar{C}(\mathbf{u})}{\partial u_d}\mathbb{B}_{d,\bar{C}}(u_d) \qquad in \qquad \ell^\infty [0,1]^D$$

To our knowledge, the weak convergence of the empirical survival copula process, under the stated assumptions, is new in literature. The result closer to ours is obtained for upper tail copula processes in Schmidt and Stadtmüller [2006] under more restrictive assumptions on copula derivatives. Moreover we stress that the use of delta method could make the result valid for a variety of weakly dependent conditions for which the CTL on the multivariate empirical processes are known, as argued in Bücher and Kojadinovic [2013]. Specifically, the assumption of strongly mixing processes, assures the validity of the dependent multiplier bootstrap, to be introduced in the next section, known, to our knowledge only for this kind of mixing condition.

3.2 Dependent Multiplier Bootstrap

Unfortunately, the dependence of the limiting processes from the unknown copula and related survival copula, through the covariance of $\mathbb{B}_C$ and $\mathbb{B}_{\bar{C}}$ and derivatives, forbids a closed form inference, based on the their distribution. Notwithstanding this difficulty, the multiplier central limit theorem allows to obtain the distribution of the limiting process trough simulations. Consquently, here, we will use a recently introduced version of the multiplier central limit theorem valid for strictly stationary strongly mixing data, Bücher and Kojadinovic [2013]. We consider now a dependent multiplier sequence $\{\xi_{i,n}\}_{i\in\mathbb{Z}}$ i.e. a sequence that satisfies

1. The sequence $\{\xi_{i,n}\}_{i\in\mathbb{Z}}$ is strictly stationary with $\mathbb{E}\left(\xi_{0,n}\right) = 0$, $\mathbb{E}\left(\xi_{0,n}^2\right) = 1$ and $\mathbb{E}\left(|\xi_{0,n}|^{\nu}\right) 2$ and independent from the available sample.
2. There exists a sequence $\ell_n \to \infty$ of strictly positive constants such that $\ell_n = o\left(n\right)$ and the sequence $\{\xi_{i,n}\}_{i\in\mathbb{Z}}$ is ℓ_n-dependent i.e $\xi_{i,n}$ is independent from $\xi_{i+h,n}$ for all $h > \ell_n$ and $i \in \mathbb{N}$.
3. There exists a function $\phi : \mathbb{R} \to [0, 1]$, symmetric around 0, continous at 0, satisfying $\phi\left(0\right) = 1$ and $\phi\left(x\right) = 0$ for all $|x| > 1$ such that $\mathbb{E}\left(\xi_{0,n}\xi_{h,n}\right) = \phi\left(h/\ell_n\right)$ for all $h \in \mathbb{Z}$.

Given M independent copies of the dependent multiplier sequence $\left\{\xi_{i,n}^{[1]}\right\}_{i\in\mathbb{Z}}, \ldots, \left\{\xi_{i,n}^{[M]}\right\}_{i\in\mathbb{Z}}$ we can define the new processes following:

$$\tilde{\mathbb{B}}_n^{[m]}\left(\mathbf{u}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_{i,n}^{(m)}\left(\mathbb{I}\left(\hat{\mathbf{U}}_i \leq \mathbf{u}\right) - C\left(\mathbf{u}\right)\right) \tag{27}$$

$$\tilde{\bar{\mathbb{B}}}_n^{[m]}\left(\mathbf{u}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_{i,n}^{(m)}\left(\mathbb{I}\left(\hat{\bar{\mathbf{U}}}_i \leq \mathbf{u}\right) - \bar{C}\left(\mathbf{u}\right)\right) \tag{28}$$

$$\tilde{\mathbb{C}}_n^{[m]}\left(\mathbf{u}\right) = \tilde{\mathbb{B}}_n^{[m]}\left(\mathbf{u}\right) - \sum_{d=1}^{D}\mathcal{D}_{u_d}^{FD}\hat{C}_n\left(\mathbf{u}\right)\tilde{\mathbb{B}}_{d,n}^{[m]}\left(u_d\right) \tag{29}$$

$$\tilde{\bar{C}}_n^{[m]}\left(\mathbf{u}\right) = \tilde{\bar{\mathbb{B}}}_n^{[m]}\left(\mathbf{u}\right) - \sum_{d=1}^{D}\mathcal{D}_{u_d}^{FD}\hat{\bar{C}}_n\left(\mathbf{u}\right)\tilde{\bar{\mathbb{B}}}_{d,n}^{[m]}\left(u_d\right) \tag{30}$$

In the previous display, $\mathcal{D}_{u_d}^{FD}$ is the partial finite difference derivative operator, that was introduced for derivative estimation in Rémillard and Scaillet [2009], we require the estimated derivatives to satisfy condition 4.1 of Bücher and Kojadinovic [2013], originally in Segers et al. [2012]. In addition, Proposition 4.2 in Bücher and Kojadinovic [2013] implies that if $\ell_n = O\left(n^{\frac{1}{2}-\epsilon}\right)$ with $0 < \epsilon < \frac{1}{2}$ and the sample is drawn by a strictly stationary sequence with strongly mixing coefficients $\alpha(r) = O(r^{-a})$, $a = 3 + 3D/2$, we have

$$\left(\hat{\mathbb{C}}_n, \tilde{\mathbb{C}}_n^{[1]}, \ldots, \tilde{\mathbb{C}}_n^{[M]}\right) \rightsquigarrow \left(\mathbb{C}, \mathbb{C}^{[1]}, \ldots, \mathbb{C}^{[M]}\right) \tag{31}$$

$$\left(\hat{\bar{\mathbb{C}}}_n, \tilde{\bar{\mathbb{C}}}_n^{[1]}, \ldots, \tilde{\bar{\mathbb{C}}}_n^{[M]}\right) \rightsquigarrow \left(\bar{\mathbb{C}}, \bar{\mathbb{C}}^{[1]}, \ldots, \bar{\mathbb{C}}^{[M]}\right). \tag{32}$$

Here $\mathbb{C}^{[1]}, \ldots, \mathbb{C}^{[M]}$ are M independent copies of $\mathbb{C}$ and $\bar{\mathbb{C}}^{[1]}, \ldots, \bar{\mathbb{C}}^{[M]}$ of $\bar{\mathbb{C}}$.

3.3 Test Statistic

Following what previously said, we want to test the null hypothesis

$$H_0: \quad C(u_1, \ldots, u_D) = \bar{C}(u_1, \ldots, u_D) \tag{33}$$

against the alternative

$$H_1: \quad C(u_1, \ldots, u_D) \neq \bar{C}(u_1, \ldots, u_D) \tag{34}$$

Among the possible alternatives, we choose a Cramér–von Mises test statistic under the random measure generated by the empirical copula:

$$\mathbb{T}_n = \int_{(0,1]^D} \left(\hat{C}_n - \hat{\bar{C}}_n\right)^2 d\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{C}_n\left(\hat{\mathbf{U}}_i\right) - \hat{\bar{C}}_n\left(\hat{\mathbf{U}}_i\right)\right)^2. \tag{35}$$

Specifically, the main motivation for using the random measure associated with the empirical copula, instead of the uniform measure is that a Cramér–von Mises statistic based on the empirical copula measure leads to a more powerful test than competitors in the bivariate case, Genest and Nešlehová [2013]. Moreover, this functional form leads to a simple average over pseudo-observations of the difference among empirical copulae. In Dehgani et al. [2013] and Rosco and Joe [2013], there is a study of the extremes of reflection asymmetry, in the bivariate case. Those results, allow for a proper normalizzation of measures of asymmetries proposed there. Contrary to the exchangiability case Harder and Stadtmüller [2014], the equivalent study of multivariate extremes are not available and the derivation of the bounds for our measure, although of great importance, are outside the scope of this paper. The normalizzation proposed in Krupskii [2016] beyond the bivariate framework is based on a limiting argument in the number of dimensions and they admit that it is possible that the absolute value of their measure could be greater than one. This makes the range of our test statistic unknown, but does not alter the conclusion of our testing procedure.
Under the null we have:

$$n\mathbb{T}_n = \int_{(0,1]^D} \left(\hat{\mathbb{C}}_n - \hat{\bar{\mathbb{C}}}_n\right)^2 d\hat{C}_n \tag{36}$$

and we can construct multiplier copies

$$n\tilde{\mathbb{T}}_n^{[m]} = \int_{(0,1]^D} \left(\tilde{\mathbb{C}}_n^{[m]} - \tilde{\bar{\mathbb{C}}}_n^{[m]}\right)^2 d\hat{C}_n. \tag{37}$$

In the following proposition we obtain the weak limits under the null

proposition 2 *If C is a normalized rank reflection symmetric copula i.e. $C = \bar{C}$ we have, as $n \to \infty$*

$$\left(n\mathbb{T}_n, n\tilde{\mathbb{T}}_n^{[1]}, \ldots, n\tilde{\mathbb{T}}_n^{[M]}\right) \rightsquigarrow \left(\mathbb{T}, \mathbb{T}^{[1]}, \ldots, \mathbb{T}^{[M]}\right) \tag{38}$$

$$\mathbb{T} = \int_{(0,1]^D} \left(\mathbb{C}_n - \bar{\mathbb{C}}_n\right)^2 dC \tag{39}$$

where $\mathbb{T}^{[1]}, \ldots, \mathbb{T}^{[M]}$ are independent copies of $\mathbb{T}$

It follows from proposition 2, whose proof is postponed to the appendix, that approximate P values for the tests of H_0 based on $\mathbb{T}_n$ are given by

$$\frac{1}{M}\sum_{m=1}^{M} \mathbb{I}\left(\tilde{\mathbb{T}}_n^{[m]} > \mathbb{T}_n\right). \tag{40}$$

Consequently, our inference procedure is robust, both under the null and under the alternative, for a data generating process whose dynamics can be modeled by strictly stationary strongly mixing processes, thanks to a CTL for strongly mixing empirical processes and the dependent multiplier bootstrap. Even if we are not aware of a data driven procedure for investigating the strongly mixing condition on real data we remark that the most used linear and non linear stationary time series models, usually satisfy this assumption Carrasco and Chen [2002]. The residual dependence, in addition, can be described through, the vast majority of copula parametric models, being the Segers condition on derivatives valid for them. For a clear exposition of this point and several examples we refer to the original paper Segers et al. [2012]. In this way, the proposed, non parametric, statistical methodology can be applied jointly to any number of random variables that satisfy the assumption of most common parametric time series models. For this reason is suited for multivariate financial and economic datasets. Theoretical results for the power of the test against local alternatives and for the size when of the null is satisfied are not available and difficult to obtain, for this reason in the next section we conduct an extensive simulation study to investigate the empirical power and size of the test.

4 Simulation Study

In this section, we report the results for the empirical power and size of our non-parametric test for different numbers of observations, for 2 and 10 dimensions. Even if a study of the power under local alternative is conceivable using skewed elliptical distributions, Genton [2004],Ma and Genton [2004] is beyond the scope of this paper. In addition, as shown in the framework of bivariate goodness of fit tests by Berg and Quessy [2009] Cramér–von Mises type of statistic distributions under local alternatives cannot, in any case,be computed without the use of

the bootstrap. All the test use the dependent multiplier bootstrap of Bücher et al. [2016], in their moving average approach with the Bartlett kernel and are conducted at the 5% nominal level. We, slightly, modify the bandwidth selection procedure, integrating with respect to the empirical copula measure and not with respect to the uniform one, as in Bücher et al. [2016]. This choice is, once again, equivalent to averaging on pseudo-observations. We postpone the description of the algorithm used to the appendix C and refer to Bücher et al. [2016], that introduced this methodology, for additional details. All the copula simulations are obtained by the use of the copula R package Kojadinovic et al. [2010].Different experiments are conducted with the same number $M = 2500$ of multiplier replicates varying the dimensionality $D = 2, 10$, different number of observations $N = 250, 500, 1000$, Kendall's τ equal to $0.1, 0.3, 0.5, 0.7, 0.9$ and five different copula families, two elliptic: the Gaussian and the Student t with 1 degree of freedom, and three Archimedean: Frank Clayton and Gumbel. For each of the previous cases the experiment is repeated $N_s = 1000$ times. The elliptic copulas are reflection symmetric, Gumbel and Clayton are not, the Frank Copula is symmetric only in 2 dimensions. Under Gaussian and Student t-copula, which satisfy the null we compute the percentage of times the test is rejecting the symmetry hypothesis when it is true, i.e. the size of the test, and this should be close to the 5% nominal level. When simulating under Clayton or Gumbel copulae, that are not symmetric, we compute the number of times the test is rejecting the null under the alternative, i.e. the power of the test. In the case of Frank copula, we compute the size in 2 dimensions and the power in 10 dimensions.
We use three different data generating processes, i.i.d., autoregressive process and a conditionally heteroscedastic process.

4.1 i.i.d. Data Generating Process

The first one is the i.i.d. case, where we draw the $U_{i,d}$ with $i \in \{1, \dots, n\}$ and $d \in \{1, \dots, D\}$ from the choosen copula model C and put

$$X_{id} = U_{id} \tag{41}$$

In table 1 we show the results in case the simulated data are i.i.d. The size of the test is close to the nominal value of 5%, also for a moderate number of observations, but is lower for the Frank copula than for elliptical copulas and in general for high values of τ, and it is also lower in 10 dimensions. The empirical power increases, not only, with the number of observations, but also with the number of dimensions. In particular in the case D=10 the test is quite restrictive rejecting the alternatives almost always and delivering a size moderately but consistently less than the nominal level. This increase of power with dimension is consistent with the findings in Krupskii [2016].

4.2 AR(1) Data Generating Process

For the second DGP again we draw $U_{i,d}$ from a copula C , but now with $i \in \{-100, \dots, n\}$ and $d \in \{1, \dots, D\}$. Then we impose Gaussian marginal

Table 1: Power/Size of the test based on the statistic $\mathbb{T}_n$ (35) in i.i.d. case, Ns=1000 M=2500

D	2						10					
N		Size			Power			Size			Power	
250	τ	Gaussian	t	Frank	Clayton	Gumbel	τ	Gaussian	t	Frank	Clayton	Gumbel
	0.1	0.055	0.077	0.042	0.237	0.055	0.100	0.011	0.018	0.992	0.503	0.949
	0.3	0.048	0.065	0.052	0.894	0.245	0.300	0.019	0.036	1.000	1.000	1.000
	0.5	0.050	0.061	0.048	0.998	0.431	0.500	0.028	0.019	1.000	1.000	1.000
	0.7	0.038	0.049	0.029	1.000	0.404	0.700	0.009	0.002	0.697	1.000	1.000
	0.9	0.015	0.016	0.016	1.000	0.137	0.900	0.000	0.000	1.000	0.000	0.000
500	τ	Gaussian	t	Frank	Clayton	Gumbel	τ	Gaussian	t	Frank	Clayton	Gumbel
	0.1	0.046	0.050	0.042	0.396	0.097	0.1	0.023	0.033	1.000	0.987	1.000
	0.3	0.046	0.042	0.034	0.993	0.557	0.3	0.031	0.040	1.000	1.000	1.000
	0.5	0.049	0.042	0.034	1.000	0.781	0.5	0.032	0.028	1.000	1.000	1.000
	0.7	0.050	0.045	0.031	1.000	0.810	0.7	0.015	0.002	1.000	1.000	1.000
	0.9	0.014	0.028	0.011	1.000	0.498	0.9	0.000	0.000	1.000	0.435	0.000
1000	τ	Gaussian	t	Frank	Clayton	Gumbel	τ	Gaussian	t	Frank	Clayton	Gumbel
	0.1	0.047	0.048	0.051	0.609	0.199	0.1	0.035	0.045	1.000	1.000	1.000
	0.3	0.048	0.052	0.048	1.000	0.877	0.3	0.039	0.047	1.000	1.000	1.000
	0.5	0.037	0.049	0.043	1.000	0.989	0.5	0.028	0.036	1.000	1.000	1.000
	0.7	0.043	0.053	0.040	1.000	0.993	0.7	0.028	0.020	1.000	1.000	1.000
	0.9	0.020	0.033	0.024	1.000	0.952	0.9	0.000	0.000	1.000	1.000	1.000

innovations, applying Φ^{-1} the inverse of standard normal cumulative distribution, and an AR(1) dynamic on the marginal processes:

$$\begin{aligned} \epsilon_{i,d} &= \Phi^{-1}\left(U_{i,d}\right) \\ X_{i,d} &= 0.5X_{i-1,d} + \epsilon_{i,d} \\ X_{-100,d} &= \epsilon_{-100,d} \end{aligned}$$

and then, we discard the first observations from the sample.
In table 2 we show the results for the AR(1) copula DGP of Bücher and Kojadinovic [2013]. All the remarks done for the independent case remain valid, but as can be seen the test is less powerful for dependent data requiring a greater number of observations to be reliable, between $N = 500$ and $N = 1000$ according to the specific copula dependence.

4.3 GARCH(1,1) Data Generating Process

The third choice of DGP is the same as the second one but now we impose a GARCH dynamics on the marginal processes.

$$\begin{aligned} \epsilon_{i,d} &= \Phi^{-1}\left(U_{i,d}\right) \\ X_{i,d} &= h_{i,d}^{-\frac{1}{2}}\epsilon_{i,d} \\ h_{i,d} &= \omega + \alpha\epsilon_{i-1,d}^2 + \beta h_{i-1} \\ h_{i-100} &= \frac{\omega}{1-\alpha-\beta} \end{aligned}$$

and we discard the first 100 observations as before. The value of parameters used are $\alpha = 0.919$, $\beta = 0.072$, $\omega = 0.012$ as estimated from Jondeau et al. [2007] for the S&P500 and already used in the simulations in Bücher and Ruppert [2013] and Bücher and Kojadinovic [2013].

Table 2: Power/Size of the test based on the statistic $\mathbb{T}_n$ (35) with marginal AR(1) dependence Ns=1000 M=2500

D		2					10					
N		Size			Power			Size		Power		
250	τ	Gaussian	t	Frank	Clayton	Gumbel	τ	Gaussian	t	Frank	Clayton	Gumbel
	0.1	0.050	0.072	0.049	0.118	0.032	0.1	0.000	0.010	0.486	0.026	0.529
	0.3	0.047	0.053	0.042	0.396	0.080	0.3	0.013	0.021	0.986	0.793	0.992
	0.5	0.043	0.058	0.045	0.552	0.087	0.5	0.014	0.018	0.889	0.940	0.977
	0.7	0.025	0.034	0.018	0.640	0.066	0.7	0.004	0.001	0.104	0.686	0.444
	0.9	0.010	0.010	0.006	0.263	0.012	0.9	0.000	0.000	1.000	0.000	0.000
500	τ	Gaussian	t	Frank	Clayton	Gumbel	τ	Gaussian	t	Frank	Clayton	Gumbel
	0.1	0.037	0.067	0.060	0.160	0.053	0.1	0.007	0.016	0.976	0.151	0.992
	0.3	0.040	0.071	0.035	0.604	0.125	0.3	0.011	0.026	1.000	1.000	1.000
	0.5	0.051	0.070	0.037	0.861	0.167	0.5	0.020	0.027	1.000	1.000	1.000
	0.7	0.032	0.032	0.023	0.934	0.144	0.7	0.006	0.004	0.921	1.000	0.998
	0.9	0.006	0.013	0.005	0.690	0.019	0.9	0.000	0.000	1.000	0.000	0.000
1000	τ	Gaussian	t	Frank	Clayton	Gumbel	τ	Gaussian	t	Frank	Clayton	Gumbel
	0.1	0.047	0.048	0.043	0.645	0.210	0.1	0.044	0.036	1.000	0.776	1.000
	0.3	0.056	0.055	0.047	1.000	0.891	0.3	0.034	0.038	1.000	1.000	1.000
	0.5	0.047	0.040	0.048	1.000	0.981	0.5	0.038	0.041	1.000	1.000	1.000
	0.7	0.047	0.051	0.043	1.000	0.993	0.7	0.018	0.028	1.000	1.000	1.000
	0.9	0.019	0.029	0.016	1.000	0.966	0.9	0.000	0.000	1.000	0.194	0.003

Table 3 reports results for marginal GARCH(1,1) processes. The greater persistence in those models leads to an additional power reduction and we need, $N \geq 1000$ for proper inference, in particular for Gumbel distributed innovations. Summarizing we conclude that the testing procedure can lead to correct

Table 3: Power/Size of the test based on the statistic $\mathbb{T}_n$ (35) with GARCH(1,1) marginal dependence Ns=1000 NB=2500

D		2					10					
N		Size			Power			Size		Power		
250	τ	Gaussian	t	Frank	Clayton	Gumbel	τ	Gaussian	t	Frank	Clayton	Gumbel
	0.1	0.046	0.070	0.052	0.090	0.037	0.1	0.010	0.025	0.781	0.042	0.754
	0.3	0.056	0.066	0.063	0.252	0.075	0.3	0.019	0.034	0.997	0.653	1.000
	0.5	0.056	0.056	0.053	0.368	0.074	0.5	0.025	0.021	0.948	0.804	1.000
	0.7	0.052	0.040	0.061	0.255	0.060	0.7	0.017	0.003	0.187	0.492	0.719
	0.9	0.024	0.043	0.025	0.096	0.036	0.9	0.000	0.000	1.000	0.000	0.000
500	τ	Gaussian	t	Frank	Clayton	Gumbel	τ	Gaussian	t	Frank	Clayton	Gumbel
	0.1	0.054	0.066	0.061	0.108	0.057	0.1	0.013	0.032	0.997	0.174	0.999
	0.3	0.048	0.046	0.059	0.401	0.100	0.3	0.028	0.033	1.000	0.973	1.000
	0.5	0.046	0.048	0.063	0.537	0.142	0.5	0.030	0.030	1.000	0.989	1.000
	0.7	0.049	0.045	0.047	0.479	0.091	0.7	0.029	0.009	0.815	0.905	0.995
	0.9	0.031	0.036	0.016	0.215	0.042	0.9	0.000	0.000	1.000	0.001	0.000
1000	τ	Gaussian	t	Frank	Clayton	Gumbel	τ	Gaussian	t	Frank	Clayton	Gumbel
	0.1	0.056	0.056	0.056	0.179	0.066	0.1	0.035	0.035	1.000	0.599	1.000
	0.3	0.050	0.071	0.050	0.691	0.201	0.3	0.034	0.043	1.000	0.999	1.000
	0.5	0.056	0.043	0.042	0.836	0.227	0.5	0.030	0.041	1.000	1.000	1.000
	0.7	0.052	0.049	0.057	0.808	0.177	0.7	0.036	0.021	1.000	1.000	1.000
	0.9	0.037	0.042	0.037	0.418	0.072	0.9	0.002	0.000	1.000	0.140	0.260

inference for most common models if we have more than 1000 observations and the reliability increase with the number of variables D considered.

5 Data Applications

In this section, we report the application of our testing procedure to real data. Datasets have been chosen in order to have more than 1000 observation, a high number of variables, and with exception of the first one, temporal dependence. In this way, we hope to highlight the advantages of using a non-parametric high dimensional and dependence robust tests of reflection symmetry. Bandwidth selection is done as in the previous section and P-values are approximated using the same number of multipliers $M = 2500$. As a first, simple illustration of the methodologies described, we apply the reflection test, based on the statistic $\mathbb{T}_n$ (35), to the dataset introduced in Cook and Johnson [1981], Cook and Johnson [1986], as an example of non-elliptically distributed multivariate data. This seven-dimensional data set contains the log-concentration of uranium (U), lithium (Li), cobalt (Co), potassium (K), cesium (Cs), scandium (Sc) and Titanium (Ti) measured in n = 655 data samples taken near Grand Junction, Colorado, US. Recently, the pairs of this data set have been tested for a possible modeling by certain selected Archimedean copula models (Ali–Mikhail–Haq, Clayton, Frank, Gumbel–Hougaard) in Genest et al. [2006]. Ben Ghorbal et al. [2009] and Quessy [2012] carried out various tests on the same data set for the hypothesis that the pairs can be modeled by an arbitrary extreme-value copula. In table 4, we report the P-values for all pairwise rank reflection test and the

Table 4: Pairwise and all variables test Pvalues for the Cook & Johnson Database

	Li	Co	K	Cs	Sc	Ti
U	0.709	0.178	0.064	0.286	0.469	0.168
Li		0.109	0.176	0.189	0.120	0.214
Co			0.008	0.010	0.572	0.241
K				0.076	0.000	0.034
Cs					0.001	0.000
Sc						0.010
All	0.238					

P-value for the test of rank reflection on the joint distribution of all variables. Specifically, as can be seen, we are not able to reject the reflection symmetry for the majority of the pairs with the exception of pairs with titanium and scandium. Moreover, If we compare to the goodness of fit test reported in Genest et al. [2006], where the only reflection symmetric copula considered is the Frank copula, we accept the null of reflection symmetry for the couples (U,Li), (U,SC), (Li,Ti) and (Co,Ti) where the best model chosen by their test is the Frank copula. Consequently, for two out of four cases when the best model chosen by their procedure is not reflection symmetric i.e. (Co,Cs)and (Cs,Sc) we reject the null at 5% level and for the other two cases i.e. (U,Co) and (Li,Sc), even if we are not able to reject the null at 5% we have moderately low p-value. Additionally, we cannot reject the rank reflection symmetry for the distribution of all the variables.

Next, we consider financial data that usually exhibit relevant dependence properties. First of all, we use the dataset of international financial indexes, introduced in Chen [2007], where he proposes a moment-based test in a dynamic copula framework to discriminate between different dynamic copula models. The Dataset includes log returns from six stock indexes taken from Yahoo! fi-

nance: CAC40(France), The FTSE1000(UK),), The Hang Seng Index (Ch), Nikkei (Japan), S&P500(US), Russel 2000(US) and TWSE(Taiwan) from July 7 1997 to December 30,2003. In the original article Chen [2007] the sample was from January 1, 1995, to December 31, 2003, but we were not able to find on Yahoo! finance the time series of TWSE preceding July 7, 1997. This sample reduction cannot qualitatively change the results, in our opinion.
In table 5 we show the P-values for the pairwise test of reflection. The results show three rejections of reflection symmetry at 5% level. The first one, Russel2000-TWSE is one of the four couples for which a Gaussian copula (rank reflection symmetric) is rejected in Chen [2007]. Moreover, the second one S&P 500-TWSE is one of the three couples for which in the same paper a Gumbel (not reflection symmetric) copula is accepted. Finally, the third case, HangSeng-S&P500, they accept the Survival Gumbel (not rank reflection symmetric). Accordingly, at least for the rejected couples, we are in agreement with the conclusion of Chen [2007]. Moreover, as an additional check we conducted the test also for the selected terns of indexes they choose. Results are in table 6 together with the test on all the variables. The table shows three rejection at 5% level, S&P500-Russel2000-Nikkei, Russel2000-FTSE100-TWSE and CAC40-Nikkei-HangSeng. Of those three only for S&P500-Russel2000-Nikkei some of the test for trivariate Gaussian copula reject the hypothesis in Chen [2007], so we have a partial agreement. We must stress that all their test with Student t copula accept the null and that they do not test in the trivariate

Table 5: Pairwise variables test Pvalues for the international Stock indexes returns used in Chen [2007]

		UK	China	Japan	US Broad	US Top	Taiwan
		FTSE	Hang Seng	Nikkei	Russel2000	S&P500	TWSE
France	CAC	0.723	0.053	0.055	0.846	0.554	0.207
UK	FTSE		0.561	0.252	0.962	0.783	0.668
China	HangSeng			0.297	0.058	0.009	0.690
Japan	Nikkei				0.255	0.111	0.452
US Broad	Russel2000					0.078	0.000
US Top	S&P500						0.001

Table 6: Selected Terns and all variables test Pvalues for the Stock index returns used in Chen [2007]

S&P500	Russel2000	FTSE100	0.657
S&P500	Russel2000	Nikkei	0.044
Russel2000	FTSE100	CAC40	0.866
Russel2000	FTSE100	Nikkei	0.110
Russel2000	FTSE100	TWSE	0.023
FTSE100	CAC40	Nikkei	0.075
FTSE100	CAC40	TWSE	0.294
CAC40	Nikkei	Hang Seng	0.020
CAC40	Nikkei	TWSE	0.092
		All	0.016

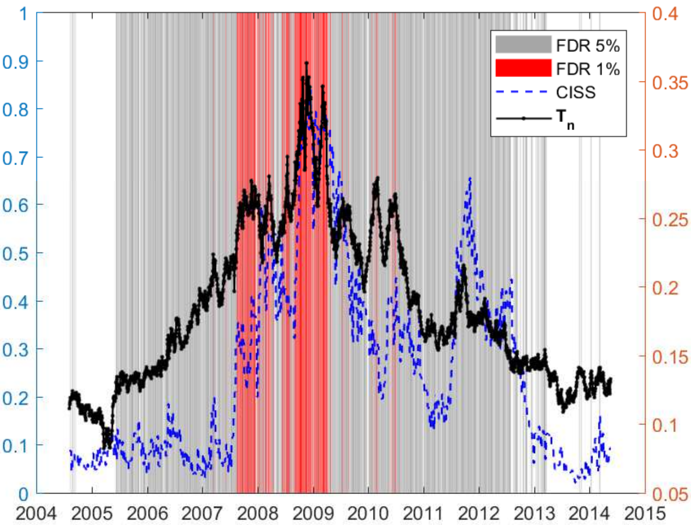
setting for non elliptical copulas. In addition the test for all the variables reject the rank reflection symmetry so that overall our conclusion is the opposite of that reported in Chen [2007] and is in favor of asymmetric dependence in stock returns as reported by older studies Longin and Solnik [2001], Ang and Chen [2002] and Hong et al. [2007].
To further explore this issue, we apply the proposed test to log returns of stocks indexes of Belgium, France, Germany, Greece, Ireland, Italy, Netherlands, Portugal, Spain and United Kingdom from January 2,2007 to May 20, 2014 during the past subprime and sovereign debt crisis. The prices are total return index net of dividends, taken from Bloomberg. As can be seen in Table 7, at 5%

Table 7: Pairwise and all variables test Pvalues for European Stock index returns

		Germany	UK	Netherlands	Belgium	Ireland	Greece	Spain	Italy	Portugal
		DAX	FSTE	AEX	BEL20	ISEQ	ASE	IBEX	FTSEMIB	PSI20
France	CAC	0.913	0.496	0.451	0.608	0.814	0.013	0.692	0.453	0.006
Germany	DAX		0.281	0.384	0.489	0.791	0.090	0.632	0.174	0.022
UK	FSTE			0.598	0.755	0.094	0.004	0.835	0.559	0.029
Netherlands	AEX				0.415	0.494	0.024	0.983	0.882	0.035
Belgium	BEL20					0.260	0.019	0.985	0.686	0.001
Ireland	ISEQ						0.063	0.413	0.496	0.448
Greece	ASE							0.019	0.011	0.083
Spain	IBEX								0.512	0.001
Italy	FTSEMIB									0.055
All	0.018									

level, the majority of the couples with Greece and Portugal are not reflection symmetric and the test for all the variables is rejected. This is again in favor of dependence asymmetry and contagion in the euro area during the last period. As a final way to investigate this finding, we computed our statistic from all the european index on a rolling window of 1500 observations on a day basis, starting from 1/5/1998 to 5/20/2014. Moreover, we control for familywise error rate using a false discovery rate approach (FDR) Benjamini and Hochberg [1995]. As Figure 1 shows for most of the sample data asymmetry at the 5% level is exhibited and during the crisis period the test is significant 1% level. In addition, the statistic has a smooth behavior, increasing the first part of the sample peaking at the end of 2008 after the AIG bailout and other FED intervention measure and then decreases. It, then, reincreases, again, during the sovereign debt crisis, in 2010 and, to a lesser extent in the second part of 2011. We also report in the same figure for visual comparison the CISS Holló et al. [2012] the main financial stress index indicator of the ECB in dashed blue line. Specifically, we stress that our statistic is computed only using stock market data, while the CISS uses all the most important asset classes. In particular, including sovereign bond data, the CISS reacted more sensibly to the sovereign bond crisis. Even with those limitations, the correlation between the two indicators is 0.7782. Moreover, we remark that the determination of stress periods is the direct outcome of the test procedure. In this regards a nice feature of our application is that it shows how before and after the crisis the joint distribution of the European stock returns appear reflection symmetric. Those preliminary results seem to suggest that a properly normalized asymmetry measure could be used as a stress indicator for contagion in financial markets. Further investigation of this possibility, on different time periods and datasets, is left for future research.

Figure 1: Daily Rolling Statistic



6 Conclusions

In this article, we propose a test for multivariate reflection symmetry. Our framework allows an easier extension to more than two dimensions with respect previously proposed tests of this type. In addition, the use of probabilistic result for the CLT of multivariate empirical processes for strongly mixing data and of a new dependent multiplier bootstrap procedure allows the application of the test to dependent data, covering most of the known stationary parametric models. This is particularly important in light of the application of the test, we have done, to financial stress detection. Our extensive simulation study showed that the test exhibit sufficient power already with a moderate number of observations and that increasing the dimension make the test more powerful, but also more restrictive. Moreover, dependence in the data lower the power, but with more than 1000 observations the test appears reliable for most of the DGP used. Notwithstanding those results, we must remark that the use of the dependent bootstrap in case of i.i.d. data comes at the cost of a loss of power with respect to the simple multiplier bootstrapBücher and Kojadinovic [2013]. Even with this cost in mind, in our opinion this approach is preferable to the usual one that apply the test to the residuals of some parametric model, the vast majority of the times non taking properly into account the estimated nature of the residuals. Moreover, in our application to financial time series, we showed that our test could be used to detect dependence asymmetries. Our application to European Stock indexes hints, also, to a link to financial stress being able to isolate Greece and Portugal as two of the biggest sources of asymmetry during the sovereign debt crisis. This connection with financial stress and the consequent possibility of using the test statistic as stress indicator is, partly, explored computing our statistic on a rolling window and and comparing it to the CISS, the ECB financial stress index in Figure 1.The concordance with the ECB indicator and the significativity of the statistic during well known crisis episodes enlight, the potential of the proposed methodology for building a financial stress index, but further investigation on different time periods and datasets, is needed and left for future research.

We hope to have clarified the reliability of our test procedure for dependent high dimensional data, with a vast simulation study and several empirical applications, focusing on the importace of testing reflection symmetry, in particular, for financial datasets.

Acknowledgements

This project has received funding from the European Union's Seventh Framework Programme (FP7-SSH/2007-2013) for research, technological development and demonstration under grant agreement 320270 - SYRTO. The first version of this paper was part of the PhD thesis of the second author. We wish to thank Umberto Cherubini and Olivier Scaillet for useful discussion and comments.

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A Inverse of survival

To find the inverse of the survival function in terms of the distribution we must solve the equation

$$G\left(\bar{F}(x)\right) = G\left(1 - F(x)\right) = x. \tag{42}$$

This is accomplished by computing (42) in $F^{-1}(q)$

$$\begin{aligned} G\left(1-F\left(F^{-1}(q)\right)\right) &= F^{-1}(q) \\ \Leftrightarrow G(1-q) &= F^{-1}(q) \\ \Leftrightarrow G(p) &= F^{-1}(1-p) \quad p=1-q \end{aligned}$$

$$F^{-1}\left(1-\bar{F}(x)\right) = F^{-1}(1-(1-F(x))) = F^{-1}(F(x)) = x. \tag{43}$$

B Proofs

B.1 Proof of proposition 1

Following Tsukahara [2005] we can express the empirical survival copula in a more convenient way. Consider a Sample from the uniform random variables $\bar{\mathbf{U}}$ that are distributed according to $\bar{C}$.
We define

$$\begin{aligned} \hat{\bar{G}}_n(\mathbf{u}) &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(\bar{\mathbf{U}}_i \le \mathbf{u}\right) & (44) \\ &= \frac{1}{n}\sum_{i=1}^{n}\prod_{d=1}^{D}\mathbb{I}\left(\hat{X}_{id} > \bar{F}_d^{-1}(u_d)\right) & (45) \\ &= \hat{\bar{F}}\left(\bar{F}_1^{-1}(u_1),\ldots,\bar{F}_D^{-1}(u_D)\right). & (46) \end{aligned}$$

The last equality follows from the fact that, by the probability integral transform, $\hat{X}_{id} = \bar{F}_d^{-1}\left(\bar{U}_{id}\right)$ has the same distribution as X_{id}. In this way we have:

$$\hat{\bar{G}}_n\left(\bar{F}_1(x_1)\ldots\bar{F}_1(x_1)\right) = \hat{\bar{F}}(x_1,\ldots,x_D) \tag{47}$$

$$\hat{\bar{G}}_{nd}\left(\bar{F}_d(x_d)\right) = \hat{\bar{F}}_d(x_d) \tag{48}$$

$$\bar{F}_d^{-1}\left(\hat{\bar{G}}_{nd}^{-1}(u_d)\right) = \hat{\bar{F}}_d^{-1}(u_d). \tag{49}$$

We get for the empirical survival copula

$$\bar{C}(u_1,\ldots,u_d) = \hat{\bar{G}}_n\left(\hat{\bar{G}}_{n1}^{-1}(u_1),\ldots,\hat{\bar{G}}_{nd}^{-1}(u_d)\right). \tag{50}$$

We use the map introduced in Bücher and Volgushev [2013] :

$$\Phi : \begin{cases} \mathbb{D}_\Phi \mapsto \ell^\infty [0,1]^D \\ H \mapsto H\left(H_1^{-1},\ldots,H_D^{-1}\right) \end{cases} \tag{51}$$

where $\mathbb{D}_\Phi$ denotes the set of all distribution functions H on $[0,1]^D$ whose marginal cdfs H_d satisfy $H_d(0) = 0$. Using this map, the empirical survival copula process can be expressed as maps from the multivariate empirical processes on $[0,1]^D$, $\bar{\mathbb{G}}_n = \sqrt{n}\left(\hat{\bar{G}}_n - \bar{C}\right)$:

$$\hat{\bar{\mathbb{C}}}_n = \sqrt{n}\left(\Phi\left(\hat{\bar{G}}_n(\mathbf{u})\right) - \Phi\left(\bar{C}(\mathbf{u})\right)\right) \tag{52}$$

since $\bar{\mathbb{G}}_n$ is a multivariate empirical process (not a multivariate survival empirical process). The already cited results in Rio [1999] for strongly mixing data lead directly to the following weak convergence limit:

$$\bar{\mathbb{G}}_n(\mathbf{u}) \rightsquigarrow \mathbb{B}_{\bar{C}}(\mathbf{u}) \tag{53}$$

$$\mathbb{C}\mathrm{ov}\left(\mathbb{B}_{\bar{C}}(\mathbf{u}), \mathbb{B}_{\bar{C}}(\mathbf{v})\right) = \bar{C}(\mathbf{u} \wedge \mathbf{v}) - \bar{C}(\mathbf{u})\bar{C}(\mathbf{v}). \tag{54}$$

Theorem 2.4 in Bücher and Volgushev [2013] implies it that, under **A 1**, Φ is Hadamard differentiable at $\bar{C}$ and the application of the functional delta method to (52) yields the result.

B.2 Proof of proposition 2

Let $C[0,1]^D$ the space of function $f:[0,1]^D \to \mathbb{R}$ that are continuous $D[0,1]^D$ the space of cadlag function on $[0,1]^D$ and $BV1[0,1]^D$ as the subspace of $D[0,1]^D$ consisting of the functions with total variation bounded by one. For notational convenience we consider only one multiplier replicate, the generalization to M replicates being straightforward. From continuous mapping theorem we get

$$\left(\left(\hat{\mathbb{C}}_n - \hat{\bar{\mathbb{C}}}_n\right)^2, \left(\tilde{\mathbb{C}}_n^{[1]} - \tilde{\bar{\mathbb{C}}}_n^{[1]}\right)^2, \hat{\mathbb{C}}_n\right) \rightsquigarrow \left(\left(\mathbb{C} - \bar{\mathbb{C}}\right)^2, \left(\mathbb{C}^{[1]} - \bar{\mathbb{C}}^{[1]}\right)^2, \mathbb{C}\right) \tag{55}$$

on $\left[\ell^\infty[0,1]^D\right]^4$ Because we can write

$$\left(\left(\hat{\mathbb{C}}_n - \hat{\bar{\mathbb{C}}}_n\right)^2, \left(\tilde{\mathbb{C}}_n^{[1]} - \tilde{\bar{C}}_n^{[1]}\right)^2, \hat{C}_n\right) = \sqrt{n}\left(\left(\hat{\mathbb{A}}_n, \hat{\mathbb{A}}_n^{[1]}, \hat{C}_n\right) - \left(\mathbb{A}, \mathbb{A}^{[1]}, C\right)\right) \tag{56}$$

where $\hat{\mathbb{A}}_n = \sqrt{n}\left(\hat{C}_n - \hat{\bar{C}}_n\right)^2$, $\hat{\mathbb{A}}_n^{[1]} = \frac{1}{\sqrt{n}}\left(\left(\tilde{\mathbb{C}}_n^{[1]} - \tilde{\bar{\mathbb{C}}}_n^{[1]}\right)^2\right)$ and $\mathbb{A} = \mathbb{A}^{[1]} = 0$. Let us introduce the map $\Psi : \ell^\infty[0,1]^D \times \ell^\infty[0,1]^D \times BV1[0,1]^D \longrightarrow \mathbb{R}^2$ defined by

$$\Psi(\alpha, \tilde{\alpha}, \beta) = \left(\int_{(0,1]^D} \alpha d\beta, \int_{(0,1]^D} \tilde{\alpha} d\beta\right) \tag{57}$$

we have then

$$\left(n\hat{\mathbb{T}}_n, n\tilde{\mathbb{T}}_n^{[1]}\right) = \sqrt{n}\left(\Psi\left(\hat{\mathbb{A}}_n, \hat{\mathbb{A}}_n^{[1]}, \hat{C}_n\right) - \Psi\left(\mathbb{A}, \mathbb{A}^{[1]}, C\right)\right). \tag{58}$$

We state the Hadamard differentiability of Ψ tangentially to $C[0,1]^D \times C[0,1]^D \times D[0,1]^D$ at each $(\alpha, \tilde{\alpha}, \beta)$ in $\ell^\infty[0,1]^D \times \ell^\infty[0,1]^D \times BV1[0,1]^D$ such that $\int |d\alpha| < \infty$ and $\int |d\tilde{\alpha}| < \infty$ in the lemma 1 below. Then an application of the functional delta method gives

$$\left(n\hat{\mathbb{T}}_n, n\tilde{\mathbb{T}}_n^{[1]}\right) \rightsquigarrow \Psi'_{A,A^{[1]},C}\left(\left(\mathbb{C} - \bar{\mathbb{C}}\right)^2, \left(\mathbb{C}^{[1]} - \bar{\mathbb{C}}^{[1]}\right)^2, \mathbb{C}\right) \tag{59}$$

with

$$\Psi'_{A,A^{[1]},C}\left(\left(\mathbb{C}-\bar{\mathbb{C}}\right)^2,\left(\mathbb{C}^{[1]}-\bar{\mathbb{C}}^{[1]}\right)^2,\mathbb{C}\right) \quad (60)$$

$$=\left(\int_{(0,1]^D}\mathbb{A}d\mathbb{C}+\int_{(0,1]^D}\left(\mathbb{C}-\bar{\mathbb{C}}\right)^2dC,\int_{(0,1]^D}\mathbb{A}^{[1]}d\mathbb{C}+\int_{(0,1]^D}\left(\mathbb{C}^{[1]}-\bar{\mathbb{C}}^{[1]}\right)^2dC\right) \quad (61)$$

$$=\left(\int_{(0,1]^D}\left(\mathbb{C}-\bar{\mathbb{C}}\right)^2dC,\int_{(0,1]^D}\left(\mathbb{C}^{[1]}-\bar{\mathbb{C}}^{[1]}\right)^2dC\right)=\left(\mathbb{T},\mathbb{T}^{[1]}\right). \quad (62)$$

Lemma 1 *The map Ψ defined in (57) is Hadamard Differentiable tangentially to $C[0,1]^D\times C[0,1]^D\times D[0,1]^D$ at each $(\alpha,\tilde{\alpha},\beta)$ in $\ell^\infty[0,1]^D\times\ell^\infty[0,1]^D\times BV1[0,1]^D$ such that $\int|d\alpha|<\infty$ and $\int|d\tilde{\alpha}|<\infty$ with derivative given by*

$$\Psi'_{A,\tilde{A},B}(\alpha,\tilde{\alpha},\beta)=\left(\int_{(0,1]^D}Ad\beta+\int_{(0,1]^D}\alpha dB,\int_{(0,1]^D}\tilde{A}d\beta+\int_{(0,1]^D}\tilde{\alpha}dB\right) \quad (63)$$

where $\int\alpha\beta,\int\tilde{\alpha}\beta$ are defined via the D-dimensional integration by parts formula exemplified for 2 dimension in Theorem 8.8 of Hildebrandt [1963] if β is not of bounded variation.

Lemma 1 is a vectorized D-dimensional version of lemma 3.9.17 in van der vaart and Wellner [1996] (see also lemma 4.3 of Carabarin-Aguirre and Ivanoff [2010]) and since the proof is similar, it will be omitted.

C Bandwidth Selection and generation of dependent multiplier sequences

C.1 Bandwidth ℓ_n

In this appendix we give more details on the concrete implementation of the dependent multiplier bootstrap by considering the bandwidth selection and the generation of the dependent multiplier sequence. We adapt the procedure introduced in Bücher and Kojadinovic [2013] for the estimation of the bandwidth parameter ℓ_n to be more coherent with our test statistic. In their paper, the optimal bandwidth is obtained by minimizing the integrated MSE of an estimator of

$$\sigma_C(\mathbf{u},\mathbf{v})=\mathbb{C}\mathrm{ov}\left(\mathbb{B}_C(\mathbf{u}),\mathbb{B}_C(\mathbf{v})\right) \quad (64)$$

given by

$$\hat{\sigma}_n(\mathbf{u},\mathbf{v}) = \sum_{k=-L}^{L}k_{F,0.5}(k/L)\,\hat{\gamma}_n(k,\mathbf{u},\mathbf{v}) \quad (65)$$

where L is an integer >1 to be chosen in the following,

$$\hat{\gamma}_n(k,\mathbf{u},\mathbf{v})=\begin{cases} n^{-1}\sum_{i=1}^{n-k}\left\{\mathbb{I}\left(\hat{U}_i<u\right)-\hat{C}_n(u)\right\}\left\{\mathbb{I}\left(\hat{U}_{i+k}<v\right)-\hat{C}_n(v)\right\} & k\geq 0\\ n^{-1}\sum_{i=1-k}^{n}\left\{\mathbb{I}\left(\hat{U}_i<u\right)-\hat{C}_n(u)\right\}\left\{\mathbb{I}\left(\hat{U}_{i+k}<v\right)-\hat{C}_n(v)\right\} & k<0\end{cases} \quad (66)$$

and $k_{F,c}$ is the flat top kernel

$$k_{F,c} = \{[(1-|x|)/(1-c)] \vee 0\} \wedge 1. \quad (67)$$

In order to obtain the optimal ℓ_n they minimize

$$IMSE_U(\hat{\sigma}_n(\mathbf{u},\mathbf{v})) = \int_{[0,1]^{2D}} MSE(\hat{\sigma}_n(\mathbf{u},\mathbf{v}))\, d\mathbf{u}d\mathbf{v} \quad (68)$$

approximating the integral with a finite grid. To avoid the arbitrary choice of the grid and to be coherent with our test statistic we choose to minimize

$$IMSE_{\hat{C}_n}(\hat{\sigma}_n(\mathbf{u},\mathbf{v})) = \int_{[0,1]^{2D}} MSE(\hat{\sigma}_n(\mathbf{u},\mathbf{v}))\, d\hat{C}_n(\mathbf{u})\, d\hat{C}_n(\mathbf{v}) \quad (69)$$

$$= n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} MSE\left(\hat{\sigma}_n\left(\hat{\mathbf{U}}_i,\hat{\mathbf{U}}_j\right)\right). \quad (70)$$

In complete analogy with their computations our optimal bandwidth is

$$\ell_{opt} = \left(\frac{4\hat{\hat{\Gamma}}_{n,\hat{C}_n}}{\hat{\hat{\Delta}}_{n,\hat{C}_n}}\right) n^{1/5} \quad (71)$$

$$\hat{\hat{\Gamma}}_{n,\hat{C}_n} = \frac{1}{4}\left.\frac{d^2\phi(x)}{dx^2}\right|_{x=0} n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=-L}^{L} k_{F,0.5}k^2\hat{\gamma}_n\left(k,\hat{\mathbf{U}}_i,\hat{\mathbf{U}}_j\right) \quad (72)$$

$$\hat{\hat{\Delta}}_{n,\hat{C}_n} = \left\{\int_{-1}^{1}\phi(x)^2\,dx\right\}\left[\left(n^{-1}\sum_{i=1}^{n}\hat{\sigma}_n\left(\hat{\mathbf{U}}_i,\hat{\mathbf{U}}_i\right)\right)^2 - n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\hat{\sigma}_n\left(\hat{\mathbf{U}}_i,\hat{\mathbf{U}}_j\right)\right]. \quad (73)$$

Then we choose L as the minimum lag for which autocorrelation of all series becomes negligible using the automatic procedure proposed in Politis and White [2004] in the matlab implementation that can be found on Andrew Patton website.

C.2 Dependent Multiplier sequence $\xi_{i,n}$

Once we have chosen the bandwidth we are ready to generate a dependent multiplier sequence according to the moving average method discussed in detail in Bücher and Kojadinovic [2013]. Let k be some positive bounded real function such that $k(x) > 0$ for all $|x| < 1$. let b_n be a sequence of integers such that $b_n \to 0$, $b_n = o(n)$ and $b_n \geq 1$ for all $n \in \mathbb{N}$. Let $Z_1,\ldots,Z_{n+2b_n-2}$ be i.e. random variables independent of the sample such that $\mathbb{E}(Z_1) = 0$,$\mathbb{E}\left(Z_1^2\right) = 1$ and $\mathbb{E}\left(|Z_1|^\nu\right) < \infty$ for all $\nu > 2$. Then let $\ell_n = 2b_n - 1$, for any $j \in \{1,\ldots,\ell_n\}$, let $w_{j,n} = k((j-b_n)/b_n)$ and $\tilde{w_{j,n}} = w_{j,n}\left(\sum_{j=1}^{\ell_n} w_{j,n}^2\right)^{-\frac{1}{2}}$. For each $i \in \{1,\ldots,n\}$ they show that the sequence

$$\xi_{i,n} = \sum_{j=1}^{\ell_n}\tilde{w}_{j,n}Z_{j+i-1} \quad (74)$$

is a dependent multiplier sequence as defined in subsection 3.2 with function ϕ given by

$$\phi(x) = \frac{k\star k(2x)}{k\star k(2x)}. \quad (75)$$

C.3 Additional Details

In all the simulations and data application performed, we draw Z_j from a standard normal distribution and we choose $k(x)$ to be the Bartlett kernel

$$k(x) = k_B(x) = (1 - |x|) \vee 0 \tag{76}$$

with the previous choice it follows that ϕ is the Parzen kernel

$$\begin{aligned} \phi(x) = k_P(x) &= \left(1 - 6x^2 - 6|x|^3\right) \mathbb{I}(|x| \leq 1/2) \\ &+ 2\left(1 - |x|^3\right) \mathbb{I}(1/2 < |x| \leq 1) \end{aligned} \tag{77}$$

and that the quantities needed for the bandwidth estimation are

$$\left.\frac{d^2\phi(x)}{dx^2}\right|_{x=0} = -12 \tag{78}$$

$$\int_{-1}^{1} \phi(x)^2 \, dx = 151/280. \tag{79}$$