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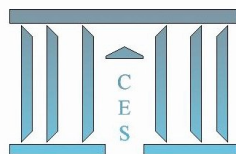
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**Rank-discounting as a resolution to a dilemma in
population ethics**

Geir B. ASHEIM, Stéphane ZUBER

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Rank-discounting as a resolution to a dilemma in population ethics*

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Abstract

When evaluating well-being distributions in an anonymous (and replication invariant) manner, one faces a dilemma between (i) assigning dictatorship to a single worst-off person, thus succumbing to a tyranny of non-aggregation and (ii) assigning dictatorship to (unboundedly) many better-off persons, thus succumbing to a tyranny of aggregation. We show how this corresponds to a population-ethical dilemma in the variable population setting between, on the one hand, a reversed repugnant conclusion (preferring a very small population with high well-being) and, on the other hand, a repugnant conclusion (preferring a sufficiently large population with lives barely worth living to a population with good lives) or very sadistic conclusion (not preferring a large population with lives worth living to a population with terrible lives).

The dilemma can be resolved by relaxing replication invariance and thus allowing that evaluation in the fixed population setting might change with population size even though the relative distributions of well-being remain unchanged. Rank-dependent criteria are evaluation criteria that resolve this dilemma but fails replication invariance. We provide conditions under which rank-dependent criteria are the only way out of the dilemma. Furthermore, we discuss the following perspective of the relaxation of replication invariance: It becomes important to associate with the notion of total population not the people that are alive at any one time, but the people that will ever live.

Keywords: Social evaluation, population ethics, rank-dependent welfare.

JEL Classification numbers: D63, D71.

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1 Introduction

One prominent population ethics view is that there exists a critical level of lifetime well-being which, if experienced by an added individual without changing the well-being levels of the existing population, leads to an alternative which is as good as the original. Combined with a prioritarian criterion where, for fixed population size, the undiscounted sum of the individuals' transformed utilities is maximized, this leads to critical-level prioritarianism (also known in the economics literature as critical-level generalized utilitarianism; Broome, 2004; Blackorby, Bossert and Donaldson, 2005).

The standard objection to critical-level generalized utilitarianism is that

- if we set the critical level at or below the individually neutral well-being level (above which life is worth living, and below which it is not), then we get the repugnant conclusion (Parfit, 1976, 1982, 1984) where, for any population with excellent lives, there is a population with lives barely worth living that is better, provided that the latter includes sufficiently many people,
- if we set the critical level strictly above the individually neutral well-being level, then we get the very sadistic conclusion (Arrhenius, 2000, 2012) where, for any population with terrible lives not worth living, there is a population with good lives that is worse, provided that the latter includes sufficiently many people.

There are other criteria that avoid the repugnant and very sadistic conclusions, but they all have their own serious shortcomings. According to average utilitarianism, the contributive value of a life can vary in all respects: a life worth living can have negative contributive value and a life not worth living can have positive contributive value. According to critical-level leximin, as defined by Blackorby, Bossert and Donaldson (1996), any population with excellent lives is worse than a population with one added individual even when the well-being of all the individuals in the latter population is barely above the critical level. According to leximin, as suggested by Arrhenius (2012, Sect. 6.8), any population is worse than a population consisting of one individual, provided that the worst-off individual of the former has lower well-being than the single

individual of the latter.

Even in a fixed population framework, where there is no difference between the various prioritarian criteria discussed above, and also no difference between the two kinds of leximin criteria, both the prioritarian and the egalitarian (viz. leximin) approaches have shortcomings if there are many present and future people. If one considers a completely egalitarian well-being stream in an intergenerational setting with many future generations, and seeks to evaluate whether the present generation should make a sacrifice leading to a uniform benefit of all future generations, then prioritarian and leximin approaches reach opposite and extreme conclusions: According to prioritarianism, the sacrifice should always be made provided that there are sufficiently many future generations, while according to leximin, it should never be made. This has been labelled by Fleurbaey and Tungodden (2010) respectively as the “tyranny of aggregation” and the “tyranny of non-aggregation”.

In their paper, Fleurbaey and Tungodden (2010) suggest a class of rank-dependent welfare orderings that do not suffer from the tyranny of aggregation and the tyranny of non-aggregation. Rank-dependent criteria generalize prioritarian ones by adding a weight on transformed utilities that depend on the rank of the individual in the distribution of well-being. In the context of population ethics, Asheim and Zuber (2014) have proposed and axiomatized a family of such rank-dependent criteria, namely the rank-discounted critical-level generalized utilitarianism criteria. They showed that they can avoid shortcomings such as the repugnant and very sadistic conclusions. This chapter studies to what extent other rank-dependent criteria solve dilemmas in population ethics.

In Sections 2 and 3 we introduce the framework, present some general desirable properties of social welfare orderings in the variable population framework and discuss three families of criteria: prioritarian ones, egalitarian ones, and proper rank-dependent criteria.

In Sections 4 and 5, we discuss the tyranny of aggregation and the tyranny of non-aggregation. We show that rank-dependent criteria may escape both tyrannies, at the

cost of not satisfying a replication-invariance property. We argue that this property may not be so appealing in the variable population context.

In Section 6 we consider a related dilemma in population ethics. We show that rank-dependent criteria may simultaneously avoid the repugnant conclusion, the very sadistic conclusion and the reversed repugnant conclusions. A key condition is that the cumulative effect of adding individuals at a given level of lifetime well-being is bounded. In Section 7 we show how negative results extend to a larger class of prioritarian and egalitarian criteria.

Finally, in Section 8 we conclude, while all proofs are contained in an appendix.

2 Framework and basic properties

Let \mathbb{N} denote the set of positive integers (natural numbers), let \mathbb{R} denote the real numbers, and let respectively $\mathbb{R}_+/\mathbb{R}_{++}/\mathbb{R}_-/\mathbb{R}_{--}$ denote the non-negative/positive/non-positive/negative real numbers.

We consider distributions of (lifetime) well-being within finite populations of variable sizes. We favor the interpretation where the population consists of people that either are alive now or will exist in the future. The set of such distributions is $\mathbf{X} = \cup_{k \in \mathbb{N}} \mathbb{R}^k$, with typical element $\mathbf{x} = (x_1, \dots, x_i, \dots, x_k)$, and where x_i is the utility of individual i in this population. Well-being is supposed to be at least level comparable. For every $k \in \mathbb{N}$, and each distribution $\mathbf{x} \in \mathbb{R}^k$, the finite population size in \mathbf{x} is $n(\mathbf{x}) = k$. Following the usual convention in population ethics, a well-being level equal to 0 represents *neutrality*. Hence, well-being is normalized so that above neutrality, a life, as a whole, is worth living; below neutrality, it is not.

We assume that distributions of well-being can be normatively ranked by a *social welfare ordering* (SWO) \succsim . Such a social welfare ordering is a binary relation, which is complete and transitive. For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ means that the distribution \mathbf{x} is deemed socially at least as good as \mathbf{y} . The fact that \succsim is complete means that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, either $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$ (or both). The fact that \succsim is transitive means that, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$, if $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{z}$ then $\mathbf{x} \succsim \mathbf{z}$. We let \sim and \succ denote the symmetric

and asymmetric parts of \succsim . We say that a *social welfare function* (SWF) $W : \mathbf{X} \rightarrow \mathbb{R}$ represents an SWO \succsim on \mathbf{X} if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if and only if $W(\mathbf{x}) \geq W(\mathbf{y})$.

For each $\mathbf{x} \in \mathbf{X}$, we let $\mathbf{x}_{[]} = (x_{[1]}, \dots, x_{[r]}, \dots, x_{[n(\mathbf{x})]})$ denote the non-decreasing distribution that is obtained by reordering \mathbf{x} ; i.e., for each rank $r \in \{1, \dots, n(\mathbf{x}) - 1\}$, $x_{[r]} \leq x_{[r+1]}$. For every $k \in \mathbb{N}$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, we write $\mathbf{x}_{[]} \gg \mathbf{y}_{[]}$ whenever $x_{[r]} > y_{[r]}$ for all $r \in \{1, \dots, n(\mathbf{x})\}$, and $\mathbf{x}_{[]} > \mathbf{y}_{[]}$ whenever $\mathbf{x}_{[]} \geq \mathbf{y}_{[]}$ and $\mathbf{x}_{[]} \neq \mathbf{y}_{[]}$. For any real number $z \in \mathbb{R}$ and any positive integer $k \in \mathbb{N}$, we let $(z)_k \in \mathbb{R}^k$ denote the *egalitarian* allocation where all k individual have a well-being level equal to z . For any $\mathbf{x} \in \mathbf{X}$ and $z \in \mathbb{R}$, $(\mathbf{x}, (z)_k)$ denotes the distribution \mathbf{x} with k added individuals, all with lifetime well-being equal to $z \in \mathbb{R}$.

We now introduce through the following axioms some basic requirements that seem appealing for an SWO.

Anonymity. An SWO \succsim on \mathbf{X} satisfies *anonymity* if, for all $k \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{R}^k$, $\mathbf{x} \sim \mathbf{y}$ if \mathbf{y} is obtained by reordering the elements of \mathbf{x} .

ExtCont. An SWO \succsim on \mathbf{X} satisfies *extended continuity* if, for all $k, \ell \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{R}^k$, the sets $\{\mathbf{y} \in \mathbb{R}^\ell \mid \mathbf{y} \succsim \mathbf{x}\}$ and $\{\mathbf{y} \in \mathbb{R}^\ell \mid \mathbf{x} \succsim \mathbf{y}\}$ are closed in \mathbb{R}^ℓ .

Monotonicity. An SWO \succsim on \mathbf{X} satisfies *monotonicity* if, for all $k \in \mathbb{N}$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $\mathbf{x} \geq \mathbf{y}$ implies $\mathbf{x} \succsim \mathbf{y}$.

RestrDom. An SWO \succsim on \mathbf{X} satisfies *restricted dominance* if, for all $k \in \mathbb{N}$ and all $z, z' \in \mathbb{R}$ with $z > z'$, $(z)_k \succ (z', (z)_{k-1})$.

PigouDalton. An SWO \succsim on \mathbf{X} satisfies *Pigou-Dalton* if, for all $k \in \mathbb{N}$, all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, all $\delta > 0$ and all $i, j \in \{1, \dots, k\}$, if (i) $x_h = y_h$ for all $h \in \{1, \dots, k\}$, $h \neq i, j$, and (ii) $y_i - \delta = x_i \geq x_j = y_j + \delta$, then $\mathbf{x} \succsim \mathbf{y}$.

For all $\ell \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{R}^k$, let $\ell * \mathbf{x}$ denote the ℓ -replica of \mathbf{x} , i.e. an allocation giving \mathbf{x} to ℓ disjoint population of size k .

ReplicInv. An SWO \succsim on \mathbf{X} satisfies *replication invariance* if, for all $k \in \mathbb{N}$, all $\ell \in \mathbb{N}$,

all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $\mathbf{x} \sim \mathbf{y}$ implies $\ell * \mathbf{x} \sim \ell * \mathbf{y}$

Our last axiom is an independence or consistency axiom that holds on ordered (“comonotonic”) allocations. It was proposed by Köbberling and Wakker (2003) to represent the idea that we have a consistent way to measure transformed well-being. To introduce this axiom, we need further notation. Let \mathbf{x} be any allocation, and i be an integer in $\{1, \dots, n(\mathbf{x})\}$ and $z \in \mathbb{R}$ a real number. The allocation $\mathbf{x}_{[i]}z$, if it exists, is the allocation $\tilde{\mathbf{x}}$ such that $n(\mathbf{x}) = n(\tilde{\mathbf{x}})$, $\tilde{x}_r = x_{[r]}$ for all $r \in \{1, \dots, n(\mathbf{x})\}$ such that $r \neq i$ and $\tilde{x}_i = z$. Hence $\mathbf{x}_{[i]}z$ is an ordered vector equivalent to \mathbf{x} (up to a permutation) except that the well-being of individual with rank i is now z instead of $x_{[i]}$. Such a vector is *well-defined* only if $x_{[i-1]} \leq z \leq x_{[i+1]}$ (or $i = 1$ and $z \leq x_{[2]}$, or $i = n(\mathbf{x})$ and $z \geq x_{[n(\mathbf{x})-1]}$). Trade-off consistency for ordered vectors is the following axiom:

TradeOffOrdVec. An SWO \succsim on \mathbf{X} satisfies *trade-off consistency for ordered vectors* if, for all $k \in \mathbb{N}$, all $\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbb{R}^k$, all $i, j \in \{1, \dots, k\}$ and all $z_1, z_2, z_3, z_4 \in \mathbb{R}$ and $z_3' > z_3$, if $\mathbf{x}_{[i]}z_1, \mathbf{y}_{[i]}z_2, \mathbf{x}_{[i]}z_3, \mathbf{y}_{[i]}z_4, \tilde{\mathbf{x}}_{[j]}z_1, \tilde{\mathbf{y}}_{[j]}z_2, \tilde{\mathbf{x}}_{[j]}z_3'$ and $\tilde{\mathbf{y}}_{[j]}z_4$ are all well-defined and $\mathbf{x}_{[i]}z_1 \sim \mathbf{y}_{[i]}z_2, \mathbf{x}_{[i]}z_3 \sim \mathbf{y}_{[i]}z_4, \tilde{\mathbf{x}}_{[j]}z_1 \sim \tilde{\mathbf{y}}_{[j]}z_2$, then $\tilde{\mathbf{x}}_{[j]}z_3' \succeq \tilde{\mathbf{y}}_{[j]}z_4$.¹

This axiom is rather complicated to formulate but it represents the following idea. The fact that $\mathbf{x}_{[i]}z_1 \sim \mathbf{y}_{[i]}z_2$ and $\mathbf{x}_{[i]}z_3 \sim \mathbf{y}_{[i]}z_4$ means that the difference in well-being brought by z_1 and z_2 has the same impact on social welfare than the difference in well-being brought by z_3 and z_4 . Trade-off consistency means that this cardinal measurement of the effect of changes in well-being on social welfare should be consistent in the sense that they do not depend on the rank of the individuals (provided we compare people at the same rank). This is exactly what the axiom above means (that we can cardinally measure the effect of some individual level of well-being on social welfare). It also add that social judgements can be independent of the well-being at ranks where well-being is fixed. Indeed, Wakker (1989) and Köbberling and Wakker (2003) have

¹With $\tilde{\mathbf{x}}_{[j]}z_3' \succ \tilde{\mathbf{y}}_{[j]}z_4$ for any essential rank j such that there exists an ordered vector for which increasing the well-being of the individual with j th level of well-being yields a strict social welfare improvement.

shown that, under additional conditions (implied by **Anonymity**, **ExtCont** and **Monotonicity**), **TradeOffOrdVec** implies the following independence condition, applied by Ebert (1988).

IndOrdVec. An SWO \succsim on \mathbf{X} satisfies *independence wrt. ordered vectors* if, for all $k \in \mathbb{N}$, all $M \subseteq \{1, \dots, k\}$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $\mathbf{x} \sim \mathbf{y}$ implies $\mathbf{x}' \sim \mathbf{y}'$ whenever $x_{[r]} = y_{[r]}$ and $x'_{[r]} = y'_{[r]}$ for all $r \in M$ and $x'_{[r]} = x_{[r]}$ and $y'_{[r]} = y_{[r]}$ for all $r \in \{1, \dots, k\} \setminus M$.

3 Basic representation results

With the basic properties introduced in Section 2, we are able to present three representation results. The proofs of these and other propositions are contained in an appendix.

Proposition 1 *Assume that an SWO \succsim on \mathbf{X} satisfies **Anonymity**, **ExtCont**, **Monotonicity** and **RestrDom**. Then there exist,*

- *for each $k \in \mathbb{N}$, a continuous, symmetric and non-decreasing function $e_k : \mathbb{R}^k \rightarrow \mathbb{R}$ satisfying $e_k((z)_k) = z$ for all $z \in \mathbb{R}$, and*
- *a continuous function $V : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$, which is strictly increasing in its second variable,*

such that $W(\mathbf{x}) = V(n(\mathbf{x}), e_{n(\mathbf{x})}(\mathbf{x}))$ represents \succsim .

The SWOs obtained in Proposition 1 decompose the evaluation of well-being distributions in two steps. The functions e_k embody the principles of justice within populations of the same size. The function V embody the trade-off between population size and equally-distributed equivalent well-being in the population.

In the proposition, for each k , the function $e_k : \mathbb{R}^k \rightarrow \mathbb{R}$ is an SWF that represents the SWO \succsim on \mathbb{R}^k — the restricted domain of populations of size k — such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $\mathbf{x} \succsim \mathbf{y}$ if and only if $e_k(\mathbf{x}) \geq e_k(\mathbf{y})$. Furthermore, the normalization $e_k((z)_k) = z$ for all $z \in \mathbb{R}$ ensures that these welfare functions have the property

that, for all $\mathbf{x} \in \mathbb{R}^k$, $e_k(\mathbf{x})$ is the equally-distributed equivalent of \mathbf{x} . This means that the well-being level $z \in \mathbb{R}$ determined by $e_k(\mathbf{x}) = z$ has the property that an egalitarian distribution where all individuals have well-being z is socially as good as \mathbf{x} . The axioms of **ExtCont**, **Monotonicity** and **RestrDom** imply that such a level exists and is unique.

The following are three examples of SWOs that are consistent with this decomposition:

- *Critical-level generalized utilitarianism* (CLU) has $V(k, e) = k(\phi(e) - \phi(c))$ for some *critical level* $c \geq 0$ and

$$e_k(\mathbf{x}) = \phi^{-1} \left(\frac{1}{k} \sum_{r=1}^k \phi(x_{[r]}) \right) \quad (1)$$

for some increasing and continuous function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ that transforms well-being into transformed well-being, so that the CLU SWO \succsim is represented by

$$W(\mathbf{x}) = \sum_{r=1}^k (\phi(x_{[r]}) - \phi(c)) .$$

For consistency with our discussion below we let the index sum according to rank r , but this is clearly immaterial for the CLU SWO. The CLU SWO has a prominent place in the literature on population ethics (Blackorby, Bossert and Donaldson, 2005; Broome, 2004).

- *Average generalized utilitarianism* (AU) has $V(k, e) = \phi(e)$ with $e_k(\mathbf{x})$ given by (1), so that the AU SWO \succsim is represented by

$$W(\mathbf{x}) = \frac{1}{k} \sum_{r=1}^k \phi(x_{[r]}) .$$

- *Maximin* has $V(k, e) = \phi(e)$ and $e_k(\mathbf{x}) = x_{[1]}$, so that the maximin SWO is represented by

$$W(\mathbf{x}) = \phi(x_{[1]}) . \quad (2)$$

In the presentation of maximin we have chosen (2) instead of the equivalent representation $W(\mathbf{x}) = x_{[1]}$ in order to be consistent with our discussion below.

By imposing that the SWO \succsim on \mathbf{X} satisfies trade-off consistency for ordered vectors and Pigou-Dalton, we obtain a smaller class of SWOs, which contains the set of CLU, AU and maximin swos as special cases. To introduce the general class, we need the following index of non-concavity of a continuous and increasing function ϕ (see Chateauneuf, Cohen and Meilijson, 2005):

$$G_\phi = \sup_{z_1 < z_2 \leq z_3 < z_4} \left[\frac{\frac{\phi(z_4) - \phi(z_3)}{z_4 - z_3}}{\frac{\phi(z_2) - \phi(z_1)}{z_2 - z_1}} \right].$$

Proposition 2 *Assume that an SWO \succsim on \mathbf{X} , in addition to the axioms of Proposition 1, satisfies **TradeOffOrdVec** and **PigouDalton**. Then, for each $k \in \mathbb{N}$, the equally distributed equivalent of $\mathbf{x} \in \mathbb{R}^k$ is given by*

$$e_k(\mathbf{x}) = \phi_k^{-1} \left(\sum_{r=1}^k w_r^k \phi_k(x_{[r]}) \right), \quad (3)$$

where the weights w_r^k , for $r \in \{1, \dots, k\}$, are non-negative, non-increasing and sum up to 1, and the function ϕ_k is continuous, increasing and such that

$$G_{\phi_k} \leq \min_{r \in \{1, \dots, k-1\}} \frac{w_r^k}{w_{r+1}^k}.$$

An SWO satisfying the axioms of Propositions 1 and 2 is called a *rank-dependent generalized utilitarian* SWO. We can consider different subclasses of rank-dependent generalized utilitarian SWOs depending on the principles of justice applied for populations with the same size. We will pay particular attention to two subclasses, specified in Definitions 1 and 2.

Definition 1 An SWO \succsim on \mathbf{X} is a *prioritarian* (or *generalized utilitarian*) SWO if it satisfies the axioms of Propositions 1 and 2 with the weights in (3) given by $w_r^k = \frac{1}{k}$ for all $k \in \mathbb{N}$ and $r \in \{1, \dots, k\}$ and a function $\phi_k = \phi$ for all $k \in \mathbb{N}$.

The prioritarian class includes the CLU and AU SWOs. However, this class is much richer than these few examples, and can encompass many different ways of trading-off population size and average transformed well-being.

For the prioritarian class, Proposition 2 implies that $G_\phi \leq 1$ so that ϕ is weakly concave. If ϕ is affine, then we obtain (non-generalized) utilitarianism. The utilitarian

class is the one that has been most studied in population ethics. A famous example is the *total utilitarian* SWO, which is the member of the utilitarian class such that $V(k, z) = kz$ for all $k \in \mathbb{N}$ and $e_k(\mathbf{x}) = \frac{1}{k} \sum_{r=1}^k x_{[r]}$. This is a CLU SWO with $\phi(z) = z$ and $c = 0$; thus, it is also prioritarian.

The term “prioritarian” suggests that ϕ is a strictly concave transformation of well-being, thereby giving preference to redistribution of well-being from a better-off to a worse-off person. An extreme case of such preference for redistribution is given by equalitarian SWOs.

Definition 2 An SWO \succsim on \mathbf{X} is an *egalitarian* SWO if it satisfies the axioms of Propositions 1 and 2 with the weights in (3) given by $w_1^k = 1$ for all $k \in \mathbb{N}$.

To the best of our knowledge, only a few papers have considered this version of egalitarianism in the population ethics literature.² There have been discussions of other versions of egalitarianism that use a leximin rather than a maximin criterion to define e_k .³ For instance, Blackorby, Bossert and Donaldson (1996) have argued in favor of Critical-Level Leximin, while Arrhenius (2012, Sect. 6.8) discusses a version of leximin related to the Positional-Extension Leximin of Blackorby, Bossert and Donaldson (1996). One issue though is that leximin is not continuous so that we cannot apply Proposition 1 to define a family of leximin criteria. Given that leximin gives priority to the worst-off, like maximin, the conclusions that we would obtain would not be very different from the ones for maximin in the context of the present chapter.

Proposition 3 Assume that an SWO \succsim on \mathbf{X} , in addition to the axioms of Propositions 1 and 2, satisfies **RepInv**. Then $\phi_k = \phi$ for all $k \in \mathbb{N}$ and there exists a non-decreasing and concave weighting function $F : [0, 1] \rightarrow [0, 1]$, which is continuous on $(0, 1]$ and

²Only Bossert (1990), Blackorby, Bossert and Donaldson (2005, Chap. 5) and Arrhenius (2012) mention versions of egalitarianism.

³The leximin ordering \succsim_{lex}^k over \mathbb{R}^k is defined as follows. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$: $\mathbf{x} \sim_{\text{lex}}^k \mathbf{y}$ if and only if $\mathbf{x}_{[r]} = \mathbf{y}_{[r]}$; $\mathbf{x} \succ_{\text{lex}}^k \mathbf{y}$ if and only if there exists $r \in \{1, \dots, k\}$ such that $x_{[r]} > y_{[r]}$ and $x_{[\ell]} = y_{[\ell]}$ for all $\ell \in \{1, \dots, r-1\}$.

satisfies $F(0) = 0$ and $F(1) = 1$, such that the weights in (3) are given by $w_r^k = F\left(\frac{r}{k}\right) - F\left(\frac{r-1}{k}\right)$ for each $k \in \mathbb{N}$ and $r \in \{1, \dots, k\}$.

An SWO satisfying the axioms of Propositions 1, 2 and 3 is called a *relative rank-dependent generalized utilitarian* SWO. Both prioritarian and egalitarian SWOs are in this class. In the case of prioritarianism, we have that the weighting function F is given by $F(\rho) = \rho$ for all $\rho \in [0, 1]$. In the case of egalitarianism, we have that the weighting function F is given by $F(0) = 0$ and $F(\rho) = 1$ for all $\rho \in (0, 1]$.

4 Tyranny of aggregation, tyranny of non-aggregation and replication invariance

Fleurbaey and Tungodden (2010) have highlighted a dilemma in social ethics between what they call the “tyranny of aggregation” and the “tyranny of non-aggregation”. The tyranny of aggregation means that a tiny gain to sufficiently many well-off people might justify imposing a much larger sacrifice on the worst-off. On the other hand, the tyranny of non-aggregation means that a small gain by the worst-off might be sufficient to justify a sacrifice by a great number of people.

To avoid these two tyrannies, Fleurbaey and Tungodden (2010) have proposed the principles of minimal aggregation and minimal non-aggregation.

MinAgg. An SWO \succsim satisfies *minimal aggregation* if for some $k \in \mathbb{N}$ and for all $\mathbf{x} \in \mathbb{R}^k$, all $i \in \{1, \dots, k\}$, there exist $\delta > \varepsilon > 0$ such that for all $\mathbf{y} \in \mathbb{R}^k$, if (i) $0 \leq x_i - y_i \leq \varepsilon$; and (ii) for all $j \in \{1, \dots, k\}$, with $j \neq i$, $y_j - x_j \geq \delta$; then $\mathbf{y} \succsim \mathbf{x}$.

MinAgg means that for some population, if all individuals, except one, gain sufficiently, then it is tolerable to impose a loss on the one individual if the loss is sufficiently small. Note that the loss may be arbitrary low, so that the principle does not seem very demanding.

To introduce minimal non-aggregation, let us introduce for any distribution $\mathbf{x} \in \mathbf{X}$

the set of worst-off and best-off individuals in this distribution:

$$I_W(\mathbf{x}) = \left\{ i \in \{1, \dots, n(\mathbf{x})\} \mid x_i = \min_{j \in \{1, \dots, n(\mathbf{x})\}} x_j \right\}$$

and

$$I_B(\mathbf{x}) = \left\{ i \in \{1, \dots, n(\mathbf{x})\} \mid x_i = \max_{j \in \{1, \dots, n(\mathbf{x})\}} x_j \right\}.$$

MinNonAgg. An SWO \succsim satisfies *minimal non-aggregation* if there exist $0 < \underline{z} < \bar{z}$ and $\delta > \varepsilon > 0$ such that for all $k \in \mathbb{N}$, all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, all $i \in \{1, \dots, k\}$, if (i) $i \in I_W(\mathbf{y})$, $y_i \leq \underline{z}$ and $x_i - y_i \geq \delta$; and (ii) for all $j \in \{1, \dots, k\}$ with $j \neq i$ and $x_j \neq y_j$, it holds that $j \in I_B(\mathbf{x}) \cap I_B(\mathbf{y})$, $y_j \geq \bar{z}$ and $y_j - x_j \leq \varepsilon$; then $\mathbf{x} \succsim \mathbf{y}$.

MinNonAgg means that if a worst-off individual is sufficiently badly-off and gains enough, there is a small loss that is tolerable for all the best-off, no matter how numerous they are, provided they are sufficiently well-off. Again the principle does not seem very demanding given that the loss by the best-off may be arbitrary small.

The key remark by Fleurbaey and Tungodden (2010) is that prioritarian SWOs do not satisfy **MinNonAgg**, while egalitarian SWOs do not satisfy **MinAgg**. Furthermore, in their Proposition 2(i) they establish conditions under which **MinAgg** and **MinNonAgg** cannot be combined with **ReplicInv**. Our next proposition illustrates this result under conditions that are stronger than those imposed by Fleurbaey and Tungodden (2010, Proposition 2(i)).

Proposition 4 *Assume that an SWO \succsim on \mathbf{X} satisfies the axioms of Propositions 1, 2 and 3. Then \succsim fails either **MinAgg** or **MinNonAgg**.*

The proof of Proposition 4 shows that the failure of minimal aggregation or minimal non-aggregation depends on the properties of the weighting function F which determines the weights in (3).

- (a) The SWO \succsim fails minimal aggregation if $\lim_{\rho \rightarrow 0^+} F(\rho) = 1$ (so that \succsim is egalitarian).

- (b) The SWO \succsim fails minimal non-aggregation if $\lim_{\rho \rightarrow 0^+} F(\rho) < 1$ (this includes any prioritarian SWO).

The upshot of this result is that if we want to satisfy minimal aggregation and minimal non-aggregation, and we want to keep the axioms of Propositions 1 and 2, thus keeping the rank-dependent generalized utilitarian form, then replication invariance must go.

5 Escaping tyrannies through absolute rank-dependence

A particular class of SWO satisfying the axioms of Propositions 1 and 2, but not necessarily satisfying replication invariance, is obtained by letting weights depend on absolute rather than relative ranks.

Definition 3 An SWO \succsim on \mathbf{X} satisfying the axioms of Propositions 1 and 2 is an *absolute rank-dependent generalized utilitarian* SWO if there exist

- (i) non-negative and non-increasing *absolute weights* $(a_r)_{r \in \mathbb{N}}$ with $a_1 = 1$ such that, for all $k \in \mathbb{N}$, the weights in (3) are obtained by letting

$$w_r^k = \frac{a_r}{\sum_{r=1}^k a_r}$$

for all $r \in \{1, \dots, k\}$, and

- (ii) a function ϕ such that $\phi_k = \phi$ for all $k \in \mathbb{N}$.

Prioritarian and egalitarian SWOs are also absolute rank-dependent. In the case of prioritarianism, this is obtained by choosing $a_r = 1$ for all $r \in \mathbb{N}$ as absolute weights. This includes CLU and AU. In the case of egalitarianism, this is obtained by choosing $a_r = 0$ for all $r > 1$. This includes maximin. Hence, prioritarian and egalitarian SWOs are both relative and absolute rank-dependent generalized utilitarian SWOs.

Any absolute rank-dependent generalized utilitarian SWO satisfies the axioms of Propositions 1 and 2, but is not characterized by these axioms. To obtain absolute rank-dependent SWOs, we could add one additional independence property:

A social welfare ordering \succsim satisfies the *same population independence of the existence of the best-off* if, for any $k \in \mathbb{N}$, and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ and $z > \max\{\mathbf{x}_{[k]}, \mathbf{y}_{[k]}\}$, $\mathbf{x} \succsim \mathbf{y}$ if and only if $(\mathbf{x}, (z)_1) \succsim (\mathbf{y}, (z)_1)$.

Definition 4 An absolute rank-dependent generalized utilitarian SWO \succsim on \mathbf{X} is a *proper* rank-dependent generalized utilitarian SWO if the absolute weights $(a_r)_{r \in \mathbb{N}}$ satisfy $a_2 > 0$ and $\sum_{r \in \mathbb{N}} a_r < \infty$.

The restriction that a_2 be strictly positive means that no egalitarian SWOs is proper rank-dependent, and the restriction that the absolute weights be summable means that no prioritarian SWO is proper rank-dependent. If there exists a real number $\beta \in (0, 1)$ such that the absolute weights are $a_r = \beta^{r-1}$ for all $r \in \mathbb{N}$, then the SWO is proper rank-dependent and called a *rank-discounted generalized utilitarian* (RDU) SWO. We will return to this particular subclass in the next section.

Definition 5 An absolute rank-dependent generalized utilitarian SWO \succsim on \mathbf{X} is a *regular* rank-dependent generalized utilitarian SWO if there exists a positive number $\kappa \in \mathbb{R}_{++}$ such that, for all $k \in \mathbb{N}$, $a_k \geq \kappa \cdot (\sum_{r=k+1}^{+\infty} a_r)$.

The restriction imposed by regularity implies in particular that the absolute weights are summable (i.e. $\sum_{r \in \mathbb{N}} a_r < \infty$), but it is stronger than that. Hence no prioritarian SWO is regular rank-dependent. On the other hand, the restriction allows weights to be equal to zero from a certain rank r on. Hence egalitarian SWOs are regular rank-dependent.

As shown by the following result, the absolute rank-dependent generalized utilitarian SWO being *proper and regular* is the key to solving the dilemma posed by the requirements of minimal aggregation and minimal non-aggregation.

Proposition 5 Assume that an SWO \succsim on \mathbf{X} is an absolute rank-dependent generalized utilitarian SWO. Then \succsim satisfies both **MinAgg** and **MinNonAgg** if and only if \succsim is proper and regular rank-dependent.

The proof of Proposition 5 shows that:

- (a) If the SWO is egalitarian, then \succsim fails minimal aggregation.
- (b) If the SWO is not regular rank-dependent, then \succsim fails minimal non-aggregation.

The proposition implies that, within the class of absolute rank-dependent generalized utilitarian SWOs, there must be positive weights on more than the worst-off person, and the sum of the weights must be converging sufficiently fast. This gives a route out of the dilemma posed by Fleurbaey and Tungodden (2010), but at the cost of failing the requirement of replication invariance. Indeed, Fleurbaey and Tungodden (2010, p. 404) show existence of an SWO satisfying both minimal aggregation and minimal non-aggregation by providing an example of an RDU SWO in a form that is actually in the subclass considered in the subsequent section.

6 Simple rank-dependence and population ethics

Up to now, the general classes of SWOs that we have considered in our propositions and definitions have not specified the trade-off between population size and equally-distributed equivalent well-being in the population, as embodied by the function V of Proposition 1. We turn now to one particularly natural manner to make this trade-off.

Definition 6 An absolute rank-dependent generalized utilitarian SWO \succsim on \mathbf{X} with absolute weights $(a_r)_{r \in \mathbb{N}}$ is a *simple rank-dependent generalized utilitarian* SWO if there exists a critical level $c \in \mathbb{R}_+$ such that the function V of Proposition 1 is given by:

$$V(k, e) = \sum_{r=1}^k a_r (\phi(e) - \phi(c)) .$$

The term ‘simple’ is appropriate since, any rank-dependent generalized utilitarian SWO \succsim is represented by an SWF $W : \mathbf{X} \rightarrow \mathbb{R}$ of the form:

$$W(\mathbf{x}) = \sum_{r=1}^k a_r (\phi(x_{[r]}) - \phi(c)) . \quad (4)$$

The class of simple prioritarian SWO contains only CLU. In particular, the AU SWO is prioritarian, but not simple. The class of simple egalitarian SWO contains only

maximin. To the best of our knowledge, only the following two papers have considered SWFs similar to the one defined by (4) which are neither simple prioritarian or simple egalitarian.

Sider (1991) have proposed principle GV, which is more complicated than the form specified by (4). It first divides a population into two ordered sets: one set with the distribution of the people with positive well-being, in order of descending well-being; and another set with the distribution of the people with negative well-being, in order of ascending well-being. For people with negative well-being, principle GV is similar to (4) with decreasing weights a_r , an affine function ϕ and $c = 0$. For people with positive well-being, principle GV applies something like (4) with decreasing weights a_r , an affine function ϕ and $c = 0$, but it applies it to the distribution of people ordered in decreasing (and not increasing) order of well-being. An implication of principle GV is that, if everyone has positive well-being, it is always socially better to make a (non-leaky) well-being transfer from a poor to a rich. Hence, principle GV does not satisfy **Pigou-Dalton**.

Asheim and Zuber (2014) have proposed and axiomatized the class of *rank-discounted critical-level generalized utilitarian* (RDCLU) SWOs. They are a special case of the form specified by (4), where there exists a real number $\beta \in (0, 1)$ such that the weights are $a_r = \beta^{r-1}$, for all $r \in \mathbb{N}$. The fact that the absolute weights a_r are strictly decreasing implies that a RDCLU SWO may satisfy **Pigou-Dalton** even if ϕ is not concave: A RDCLU SWO satisfies **Pigou-Dalton** if and only if $G_\phi \leq \frac{a_r}{a_{r+1}} = \beta^{-1}$.

Any simple rank-dependent generalized utilitarian SWO satisfies the axioms of Propositions 1 and 2, but is not characterized by these axioms. To obtain simple rank-dependent SWOs from absolute rank-dependent SWOs, we could add yet another independence property:

An SWO \succsim satisfies the *independence of the utility of the worst-off* if for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and any $z', z'' \in \mathbb{R}$ such that $\max\{z', z''\} < \min\{x_{[1]}, y_{[1]}\}$, $(\mathbf{x}, (z')_1) \succsim (\mathbf{y}, (z')_1)$ if and only if $(\mathbf{x}, (z'')_1) \succsim (\mathbf{y}, (z'')_1)$.

According to Parfit (1984), a social welfare ordering leads to the *repugnant conclusion* if:

“For any possible population of at least ten billion people, all with a very high quality of life, there must be some much larger imaginable population whose existence, if other things are equal, would be better even though its members have lives that are barely worth living.”

The repugnant conclusion has caught much attention in the literature on population ethics as most of the literature have discussed ways to avoid such a conclusion. Formally, one can formulate the avoidance of the repugnant conclusion as follows:⁴

AvoidRepugConcl. An SWO \succsim on \mathbf{X} *avoids the repugnant conclusion* if there exist $k \in \mathbb{N}$, $y \in \mathbb{R}_{++}$ and $z \in (0, y)$ such that, for all $m \geq \ell \geq k$, $(y)_\ell \succsim (z)_m$.

Although the CLU SWO does not avoid the repugnant conclusion if the critical level c equals 0, other generalized utilitarian SWO, e.g. CLU with positive c and AU do so. However, they are subject to other difficulties.

On the one hand, CLU with positive c imply the sadistic conclusion is that, for any allocation with negative well-being, there is an egalitarian allocation with low, but positive well-being that is worse. This is what Arrhenius (2000, 2012) refers to as the *very sadistic conclusion*. Formally, one can formulate the avoidance of the very sadistic conclusion as follows:

AvoidVerySadConcl. An SWO \succsim on \mathbf{X} *avoids the very sadistic conclusion* if there exists $\mathbf{x} \in \mathbf{X}$ such that $(z)_k \succsim \mathbf{x}$ for all $k \in \mathbb{N}$ and $z \in \mathbb{R}_{++}$.

On the other hand, AU and also maximin leads to the problematic conclusion that for any egalitarian allocation with very high positive well-being, there is a better one-individual allocation with slightly higher well-being. This might be referred to as

⁴Other formalization have been proposed, for instance by Blackorby, Bossert and Donaldson (2005). The formulation is slightly stronger than the one they have, but we think it is in the spirit of Parfit’s initial formulation.

the *reversed repugnant conclusion* (Arrhenius, 2012). Hence, one can formulate the avoidance of the reversed repugnant conclusion as follows:

AvoidRevRepugConcl. An SWO \succsim on \mathbf{X} *avoids the reversed repugnant conclusion* if there exists $k \in \mathbb{N}$, $y \in \mathbb{R}_{++}$ and $z \in (0, y)$ such that $(z)_k \succsim (y)_1$.

Our last result shows that in the class of simple rank-dependent generalized utilitarian SWOs, being proper is the key to avoiding all these three problematic conclusion.

Proposition 6 *Assume that an SWO \succsim on \mathbf{X} is a simple rank-dependent generalized utilitarian SWO. Then \succsim satisfies **AvoidRepugConcl**, **AvoidVerySadConcl** and **AvoidRevRepugConcl** if and only if \succsim is proper rank-dependent.*

The proof of Proposition 6 shows that:

- (a) If the SWO is egalitarian, then \succsim entails the reversed repugnant conclusion.
- (b) If the SWO has non-summable absolute weights, then \succsim entails either the repugnant conclusion (when $c = 0$) or the very sadistic conclusion (when $c > 0$).

Proposition 6 implies that, within the class of simple rank-dependent generalized utilitarian SWOs, there must be positive weights on more than the worst-off person, and the sum of the weights must be converging, in order to avoid all of the repugnant, very sadistic and reversed repugnant conclusions.

7 Additional population ethics results

Proposition 6 singles out proper rank-dependent SWOs in the class of simple rank-dependent generalized utilitarian SWOs. We can more generally compare proper and regular rank-dependent SWOs to a large class of prioritarian and egalitarian criteria, including those that are not simple. A first principle generalizing **AvoidVerySadConcl** can be proposed to do so (see Arrhenius, 2012, for a discussion of the principle):

WeakNonSadCond. An SWO \succsim on \mathbf{X} satisfies the weak non-sadism condition if there exist $k \in \mathbb{N}$ and $z \in \mathbb{R}_{--}$ such that for all $\mathbf{x} \in X$, $y \in \mathbb{R}_{++}$ and $\ell \in \mathbb{N}$ $(\mathbf{x}, (y)_\ell) \succsim (\mathbf{x}, (z)_k)$.

Any prioritarian social welfare ordering will be subject to one of the two problems: either it implies the repugnant conclusion or it cannot satisfy the weak non-sadism condition.

Proposition 7 *A prioritarian SWO \succsim on \mathbf{X} cannot satisfy both **AvoidRepugConcl** and **WeakNonSadCond**.*

Proposition 7 seems to reject prioritarian criteria for population ethics. One could therefore consider egalitarian criteria. However, they are also subject to a population ethics dilemma. They cannot satisfy the weak non-sadism condition if we want to insist on the intuitive principle that we should not create lives with a negative level of well-being.

NegMereAdd. An SWO \succsim on \mathbf{X} satisfies the negative mere addition principle if for all $\mathbf{x} \in \mathbf{X}$ and all $z \in \mathbb{R}_{--}$, $\mathbf{x} \succ (\mathbf{x}, (z)_1)$.

Proposition 8 *An egalitarian SWO \succsim on \mathbf{X} cannot satisfy both **WeakNonSadCond** and **NegMereAdd**.*

In contrast with these negative results, proper and regular simple rank-dependent SWOs may escape all the population dilemma we have discussed up to now provided function ϕ is concave and bounded above.

Proposition 9 *Assume that an SWO \succsim on \mathbf{X} is a simple rank-dependent generalized utilitarian SWO. Then \succsim satisfies **WeakNonSadCond** and **NegMereAdd** if \succsim is proper and regular rank-dependent and the function ϕ is concave and bounded above.*

Note that, by Proposition 6, we already know that the simple rank-dependent generalized utilitarian SWO described in Proposition 9 satisfies **AvoidRepugConcl**,

AvoidVerySadConcl and **AvoidRevRepugConcl**. So we obtain a family of SWOs that comply with many population ethics requirements. This family deserves receiving further attention.

8 Concluding remark

In this chapter we have argued that the escape from the tyrannies of aggregation and non-aggregation and the resolution of an important population-ethical dilemma require that we relax the requirement of replication invariance. Thus the normative evaluation of populations and their well-beings may change as populations and their well-beings are replicated. The RDCLU SWO (Asheim and Zuber, 2014) is one example of a population-ethical criterion that offers a solution by relaxing replication invariance. However, as we have shown, a larger class of simple rank-dependent generalized utilitarian SWOs satisfy all the requirements that we have imposed, save replication invariance.

The relaxation of replication invariance yields the following interesting perspective on population ethics: It becomes important to associate with the notion of total population not only the people that are alive at any one time, but also the people that will ever live (and perhaps those that have ever lived). This perspective has important implications for the conclusions that follow from population ethics. In particular, we must consider today the population policies that should be and will be implemented in the far future. Thus, we are forced to depart from the the separability of people promoted by Broome (2004, chap. 13) and Blackorby, Bossert and Donaldson (2005, pp. 131–133). We must also depart from the generation-relative population ethics that Dasgupta (2005, pp. 430–431) supports, because the reproductive choices made by future generations will affect the number and well-being of future people, therefore modifying the goodness of changing the population today.

This change of perspective might have important implications for the conclusions that follow from population ethics.

Appendix: Proofs

Proof of Proposition 1. If the SWO \succsim on \mathbf{X} satisfies **Monotonicity** and **RestrDom**, then it satisfies Minimal Increasingness in Blackorby, Bossert and Donaldson (2005, chap. 5). To see this, consider the distributions $(z)_k$ and $(z')_k$ with $z, z' \in \mathbb{R}$ satisfying $z > z'$. Construct $\mathbf{x} \in \mathbb{R}^k$ by letting $x_i = z'$ for some $i \in \{1, \dots, k\}$ and $x_j = z$ for all $j \in \{1, \dots, k\} \setminus \{i\}$. Then $(z)_k \succ \mathbf{x}$ by **RestrDom** and $\mathbf{x} \succsim (z')_k$ by **Monotonicity**, so $(z)_k \succ (z')_k$ by transitivity. Thus, the result follows from Blackorby, Bossert and Donaldson (2005, Theorem 5.2). ■

Proof of Proposition 2. From Köbberling and Wakker (2003, Theorem 8), for each population size $k \in \mathbb{N}$, there must exist non-negative weights w_r^k , for $r \in \{1, \dots, k\}$ and a continuous increasing function ϕ_k such that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ with $n(\mathbf{x}) = n(\mathbf{y}) = k$,

$$\mathbf{x} \succsim \mathbf{y} \iff \sum_{r=1}^k w_r^k \phi_k(x_{[r]}) \geq \sum_{r=1}^k w_r^k \phi_k(y_{[r]}).$$

Using the definition of equally distributed equivalent, we can take the weights w_r^k summing to one. By the uniqueness result in Köbberling and Wakker (2003, Theorem 8), both the weights and the function ϕ_k are unique after normalization, provided that $w_r^k > 0$ for at least two ranks r .

By Chateauneuf, Cohen and Meilijson (2005, Theorem 1) (or a similar result in Asheim and Zuber, 2014), for each $k \in \mathbb{N}$ such social orderings satisfy **PigouDalton** if and only if

$$G_{\phi_k} \leq \min_{r \in \{1, \dots, k-1\}} \frac{w_r^k}{w_{r+1}^k}.$$

Note that, by Chateauneuf, Cohen and Meilijson (2005, Proposition 3), $G_{\phi_k} \geq 1$, so that this implies $w_r^k \geq w_{r+1}^k$ for all $r \in \{1, \dots, k-1\}$: the weights are non-increasing. ■

Proof of Proposition 3. By **RepInv**, and by the uniqueness result of functions ϕ_k in Proposition 2, it is easy to show that there must exist a continuous and increasing function ϕ such that $\phi_k = \phi$ for all $k \in \mathbb{N}$, provided that there exist at least two ranks r such that $w_r^k > 0$, for $k \geq 2$ (otherwise, we have $w_1^k = 1$ for all $k \in \mathbb{N}$, and function

ϕ_k does not matter: we can take it identical without loss of generality).

Then we can mimic Donaldson and Weymark (1980, Theorem 4) to show that there exists a non-decreasing and concave weighting function F satisfying all the required properties. ■

Proof of Proposition 4. *Case 1:* $\lim_{\rho \rightarrow 0^+} F(\rho) = 1$. In this case, the weighting function F is given by $F(0) = 0$ and $F(\rho) = 1$ for all $\rho \in (0, 1]$, so that the SWO \succsim on \mathbf{X} is egalitarian. Any egalitarian SWO clearly contradicts minimal aggregation since no weight is given any person but the worst-off.

Case 2: $\lim_{\rho \rightarrow 0^+} F(\rho) < 1$. To show failure of minimal non-aggregation, we must show that, for all $0 < \underline{z} < \bar{z}$ and $\delta > \varepsilon > 0$ there exist $k \in \mathbb{N}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ with $\mathbf{x} \prec \mathbf{y}$, even though $y_i \leq \underline{z}$ and $x_i - y_i \geq \delta$ for some $i \in I_W(\mathbf{y})$, and it holds that $j \in I_B(\mathbf{x}) \cap I_B(\mathbf{y})$, $y_j \geq \bar{z}$ and $y_j - x_j \leq \varepsilon$ for all $j \in \{1, \dots, k\}$ with $j \neq i$ and $x_j \neq y_j$.

Choose any $0 < \underline{z} < \bar{z}$ and $\delta > \varepsilon > 0$. Since $\lim_{\rho \rightarrow 0^+} F(\rho) < 1$ and $F(\rho)$ is continuous for $\rho \in (0, 1]$, we can choose $m \in \mathbb{N}$ sufficiently large so that $F\left(\frac{1}{m+1}\right) < 1$, and write $\Delta := 1 - F\left(\frac{1}{m+1}\right)$. Construct, for each $\ell \in \mathbb{N}$ with $\ell \geq 2$, \mathbf{x}^ℓ and \mathbf{y}^ℓ such that $n(\mathbf{x}^\ell) = n(\mathbf{y}^\ell) = \ell(m+1)$, $x_1^\ell = y_1^\ell = y_2^\ell = \underline{z}$, $x_2^\ell = \underline{z} + \delta$, $x_h^\ell = y_h^\ell = \underline{z} + \delta$ for $h = 3, \dots, \ell$ and $x_{h'}^\ell = \bar{z} - \varepsilon$, $y_{h'}^\ell = \bar{z}$ for $h' = \ell + 1, \dots, \ell(m+1)$. For each $\ell \geq 2$, let

$$\Gamma(\ell) := F\left(\frac{2}{\ell(m+1)}\right) - F\left(\frac{1}{\ell(m+1)}\right),$$

where it follows from our assumptions that $\Gamma(\ell) > 0$ for all $\ell \geq 2$ and $\lim_{\ell \rightarrow \infty} \Gamma(\ell) = 0$.

Hence, we can choose $\ell' \geq 2$ such that

$$\Gamma(\ell')(\phi(\underline{z} + \delta) - \phi(\underline{z})) < \Delta(\phi(\bar{z}) - \phi(\bar{z} - \varepsilon)).$$

By definition of the SWOs obtained in Propositions 1, 2 and 3, $\mathbf{y}^{\ell'} \succ \mathbf{x}^{\ell'}$, which contradicts minimal non-aggregation.

Hence, either minimal aggregation or minimal non-aggregation fails. ■

Proof of Proposition 5. *Case 1:* The SWO \succsim on \mathbf{X} is egalitarian. Clearly minimal aggregation is contradicted since no weight is given any person but the worst-off.

Case 2: The SWO \succsim on \mathbf{X} is not regular rank-dependent. Assume that there exist real numbers $0 < \underline{z} < \bar{z}$ and $\delta > \epsilon > 0$ as required in the statement of **MinNonAgg**. Consider any $k \in \mathbb{R}$, $m \in \mathbb{R}$, with $k < m$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^l$ be such that $x_i = y_i = \underline{z}$ for all $i < k$, $y_k = \underline{z}$, $x_k = \underline{z} + \delta$, $x_j = z$ and $y_j = z + \epsilon$ for all $j > k$, where $z > \max\{\underline{z} + \delta, \bar{z}\}$. By definition of an absolute rank-dependent SWO, we have:

$$\mathbf{x} \succsim \mathbf{y} \iff a_k \phi(\underline{z} + \delta) + \left(\sum_{r=k+1}^m a_r \right) \phi(z) \geq a_k \phi(\underline{z}) + \left(\sum_{r=k+1}^m a_r \right) \phi(z + \epsilon).$$

This can be written:

$$\mathbf{x} \succsim \mathbf{y} \iff a_k \geq \frac{\phi(z + \epsilon) - \phi(z)}{\phi(\underline{z} + \delta) - \phi(\underline{z})} \left(\sum_{r=k+1}^m a_r \right).$$

But denoting

$$\kappa = \frac{\phi(z + \epsilon) - \phi(z)}{\phi(\underline{z} + \delta) - \phi(\underline{z})},$$

and given that \succsim is not regular rank-dependent, there must exist $k' \in \mathbb{N}$ such that $a_{k'} < \kappa \cdot \left(\sum_{r=k'+1}^{+\infty} a_r \right)$. This means that there must also exist $m' > k'$ such that $a_{k'} < \kappa \cdot \left(\sum_{r=k'+1}^{m'} a_r \right)$. So, **MinNonAgg** must sometimes be violated.

Case 3: The SWO \succsim on \mathbf{X} is proper and regular rank-dependent. Let $\lambda \in (0, 1)$ be such that $\kappa > \lambda \cdot \frac{a_1}{a_2}$. By Chateauneuf, Cohen and Meilijson (2005, Lemma 1), for any $x, y \in \mathbb{R}$, with $x \geq y$:

$$\lambda \cdot G_\phi \geq \frac{\phi(x + \lambda) - \phi(x)}{\phi(y) - \phi(y - 1)}.$$

Because the SWO \succsim satisfies **PigouDalton**, we have

$$G_\phi \leq \inf_{k \in \mathbb{N}} \frac{a_r}{a_{r+1}} \leq \frac{a_1}{a_2}.$$

Therefore, for any $x, y \in \mathbb{R}$, with $x \geq y$:

$$\kappa > \lambda \cdot \frac{a_1}{a_2} \geq \frac{\phi(x + \lambda) - \phi(x)}{\phi(y) - \phi(y - 1)}.$$

If the SWO \succsim is proper and regular rank-dependent, than it clearly satisfies **MinAgg**. Let us show that it also satisfies **MinNonAgg**, where in the statement of this property we take $\underline{z} = 0$, $\bar{z} > 1$, $\delta = 1$ and $\epsilon = \lambda$. Consider any $k \in \mathbb{N}$, and any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ and $i \in \{1, \dots, k\}$, such that (i) $i \in I_W(\mathbf{y})$, $y_i \leq 0$ and $x_i - y_i \geq 1$; and

(ii) for all $j \in \{1, \dots, k\}$ with $j \neq i$ and $x_j \neq y_j$, it holds that $j \in I_B(\mathbf{x}) \cap I_B(\mathbf{y})$, $y_j \geq \bar{z}$ and $y_j - x_j \leq \lambda$. We need to show that $\mathbf{x} \succsim \mathbf{y}$. Let $R = |\{\ell \in \{1, \dots, k\} : x_\ell \leq x_i\}|$ the number of individuals with well-being lower than x_i in distribution \mathbf{x} ; $R' = |\{\ell \in \{1, \dots, k\} : y_\ell \leq y_i\}|$ the number of individuals with well-being lower than y_i in distribution \mathbf{y} ; $K = |\{j \in \{1, \dots, k\} : j \neq i, x_j \neq y_j\}|$ the number of individuals other than i with different well-being in \mathbf{x} and \mathbf{y} . By definition of a proper rank-dependent SWO:⁵

$$\begin{aligned} \mathbf{x} \succsim \mathbf{y} &\iff \sum_{r=1}^{R'-1} a_r \phi(x_{[r]}) + \sum_{r=R'}^R a_r \phi(x_{[r]}) + \sum_{r=R+1}^{k-K} a_r \phi(x_{[r]}) + \sum_{r=k-K+1}^k a_r \phi(x_{[r]}) \\ &\geq \sum_{r=1}^{R'-1} a_r \phi(y_{[r]}) + \sum_{r=R'}^R a_r \phi(y_{[r]}) + \sum_{r=R+1}^{k-K} a_r \phi(y_{[r]}) + \sum_{r=k-K+1}^k a_r \phi(y_{[r]}) \end{aligned}$$

By definition of \mathbf{x} and \mathbf{y} , this can be written:

$$\mathbf{x} \succsim \mathbf{y} \iff \sum_{r=R'}^R a_r \phi(x_{[r]}) + \sum_{r=k-K+1}^k a_r \phi(x_{[r]}) \geq \sum_{r=R'}^R a_r \phi(y_{[r]}) + \sum_{r=k-K+1}^k a_r \phi(y_{[r]})$$

Given that $y_j - x_j \leq \lambda$ for all $j \in \{1, \dots, k\}$ with $j \neq i$ and $x_j \neq y_j$ (and $x_j = x_{[k]}$ for all such j), a sufficient condition to have $\mathbf{x} \succsim \mathbf{y}$ is:

$$\sum_{r=R'}^R a_r [\phi(x_{[r]}) - \phi(y_{[r]})] \geq \left[\sum_{r=k-K+1}^k a_r \right] (\phi(x_{[k]} + \lambda) - \phi(x_{[k]}))$$

But $\sum_{r=R'}^R a_r [\phi(x_{[r]}) - \phi(y_{[r]})] \geq a_R [\phi(x_i) - \phi(y_i)] \geq a_R [\phi(x_i) - \phi(x_i - 1)]$, and

$$\frac{a_R}{\sum_{r=k-K+1}^k a_r} \geq \frac{a_R}{\sum_{r=R}^{+\infty} a_r} \geq \kappa > \frac{\phi(x_{[k]} + \lambda) - \phi(x_{[k]})}{\phi(x_i) - \phi(x_i - 1)}.$$

It is then the case that $\mathbf{x} \succsim \mathbf{y}$. ■

Proof of Proposition 6. *Case 1: The SWO \succsim on \mathbf{X} is egalitarian.* Clearly the reversed repugnant conclusion is not avoided.

Case 2: The SWO \succsim on \mathbf{X} has non-summable absolute weights and $c = 0$. Consider any $k \in \mathbb{N}$, $y \in \mathbb{R}_{++}$ and $z \in (0, y)$. By definition of a simple rank-dependent generalized

⁵We use the convention that $\sum_{r=k}^{k-1} u_r = 0$.

utilitarian SWO:

$$(y)_k \succsim (z)_m \iff \left(\sum_{r=1}^k a_r \right) \phi(y) \geq \left(\sum_{r=1}^m a_r \right) \phi(z).$$

But given that the absolute weights are non-summable, there must exist m' such that

$$\frac{\left(\sum_{r=1}^{m'} a_r \right)}{\left(\sum_{r=1}^k a_r \right)} > \frac{\phi(y)}{\phi(z)}$$

so that $(z)_{m'} \succ (y)_k$. This contradicts **AvoidRepugConcl.**

Case 3: The SWO \succsim on \mathbf{X} has non-summable absolute weights and $c > 0$. Assume that there exists $\mathbf{x} \in \mathbf{X}$ such that $(z)_k \succsim \mathbf{x}$ for all $k \in \mathbb{N}$ and $z \in \mathbb{R}_{++}$. Let $\ell = n(\mathbf{x})$. By definition of a simple rank-dependent generalized utilitarian SWO, $\mathbf{x} \succsim (x_{[1]})_n$. Take any $z' \in \mathbb{R}_{++}$ such that $z < c$. By definition of a simple rank-dependent generalized utilitarian SWO:

$$(x_{[1]})_n \succ (z')_k \iff \left(\sum_{r=1}^n a_r \right) (\phi(x_{[1]}) - \phi(c)) > \left(\sum_{r=1}^k a_r \right) (\phi(z') - \phi(c)).$$

If $x_{[1]} > z'$ then $(x_{[1]})_n \succ (z')_n$, so that by transitivity $\mathbf{x} \succ (z')_n$, which is a contradiction. If $x_{[1]} \leq z' < c$ then $(\phi(x_{[1]}) - \phi(c)) \leq (\phi(z) - \phi(c)) < 0$ and

$$(x_{[1]})_n \succ (z')_k \iff \frac{\left(\sum_{r=1}^k a_r \right)}{\left(\sum_{r=1}^n a_r \right)} > \frac{\phi(c) - \phi(x_{[1]})}{\phi(c) - \phi(z)}.$$

But given that the absolute weights are non-summable, there must exist k' such that

$$\frac{\left(\sum_{r=1}^{k'} a_r \right)}{\left(\sum_{r=1}^n a_r \right)} > \frac{\phi(c) - \phi(x_{[1]})}{\phi(c) - \phi(z)}$$

so that $(x_{[1]})_n \succ (z)_{k'}$ and therefore $\mathbf{x} \succ (z^{k'})_n$. This contradicts **AvoidVerySad-Concl.**

Case 4: The SWO \succsim on \mathbf{X} is proper rank-dependent. Let us check that \succsim satisfies all the required properties.

Because \succsim is proper rank-dependent $a_2 > 0$ and $(a_1 + a_2)/a_1 > 1$. By continuity of ϕ , for any $y \in \mathbb{R}_{++}$ such that $y > c$ there exists $z \in \mathbb{R}_{++}$ such that

$$1 < \frac{\phi(y) - \phi(c)}{\phi(z) - \phi(c)} < \frac{a_1 + a_2}{a_1}$$

so that $(a_1 + a_2)(\phi(z) - \phi(c)) > a_1(\phi(y) - \phi(c))$. By definition of a simple rank-dependent generalized utilitarian SWO, this implies that $(z)_2 \succ (y)_1$. **AvoidRevRepugConcl** is satisfied.

Because \succsim is proper rank-dependent so that $\sum_{r=1}^{+\infty} a_r < +\infty$, it is also the case that for any $\varepsilon \in \mathbb{R}_{++}$ there exists $\ell \in \mathbb{N}$ such that $(\sum_{r=1}^{+\infty} a_r) / (\sum_{r=1}^{\ell} a_r) < 1 + \varepsilon$.

Thus consider $\ell \in \mathbb{N}$, $y \in \mathbb{R}_{--}$ such that

$$\frac{\sum_{r=1}^{+\infty} a_r}{\sum_{r=1}^{\ell} a_r} < \frac{\phi(c) - \phi(y)}{\phi(c) - \phi(0)} = 1 + \frac{\phi(0) - \phi(y)}{\phi(c) - \phi(0)}.$$

Let us show that $(0)_k \succsim (y)_{\ell}$ for any $k \in \mathbb{N}$. For this to be the case, we need that $(\sum_{r=1}^k a_r)(\phi(0) - \phi(c)) \geq (\sum_{r=1}^{\ell} a_r)(\phi(y) - \phi(c))$ for all $k \in \mathbb{N}$. Given that $0 \leq c$, a sufficient condition is then that $(\sum_{r=1}^{+\infty} a_r)(\phi(c) - \phi(0)) \leq (\sum_{r=1}^{\ell} a_r)(\phi(c) - \phi(y))$, which is true by definition of ℓ and y . For any $z \in \mathbb{R}_{++}$ and any $k \in \mathbb{N}$, $(z)_k \succ (0)_k$ and therefore $(z)_k \succ (y)_{\ell}$. Hence **AvoidVerySadConcl** is satisfied.

If $c > 0$, for any $y > c > z > 0$ and for any $m \geq \ell$, by definition of a simple rank-dependent generalized utilitarian SWO, $(y)_{\ell} \succ (z)_m$ so that **AvoidRepugConcl** is satisfied. If $c = 0$ then for any $y > z > 0$, because \succsim is proper rank-dependent, there exists $k \in \mathbb{N}$ such that

$$\frac{\sum_{r=1}^{+\infty} a_r}{\sum_{r=1}^k a_r} < \frac{\phi(y) - \phi(0)}{\phi(z) - \phi(0)} = 1 + \frac{\phi(y) - \phi(z)}{\phi(z) - \phi(0)}.$$

Hence, for all $m \geq \ell \geq k$:

$$\begin{aligned} (\sum_{r=1}^{\ell} a_r)(\phi(y) - \phi(0)) &\geq (\sum_{r=1}^k a_r)(\phi(y) - \phi(0)) > (\sum_{r=1}^{+\infty} a_r)(\phi(z) - \phi(0)) \\ &\geq (\sum_{r=1}^m a_r)(\phi(z) - \phi(0)). \end{aligned}$$

By definition of a simple rank-dependent generalized utilitarian SWO, $(y)_{\ell} \succ (z)_m$. Hence **AvoidRepugConcl** is satisfied. ■

Proof of Proposition 7. Consider any $\hat{k} \in \mathbb{N}$, $\hat{y} \in \mathbb{R}_{++}$, $\hat{z} \in [0, y]$ and $\hat{w} \in (0, z)$.

Let $z < 0$ be the well-being level in the statement of **WeakNonSadCond**, and k the integer in this statement. For $\ell \geq k$, let $\theta_{\ell} = [(\ell + k)\phi(\hat{y}) - k\phi(z)]/\ell$. When ℓ is large enough, θ_{ℓ} can become arbitrary close to $\phi(\hat{y})$, so that $\phi^{-1}(\theta_{\ell})$ is well-defined (ϕ

is continuous and increasing and thus invertible). Let $\bar{\ell}$ be an integer such that $\phi^{-1}(\theta_{\bar{\ell}})$ is well-defined and denote $\zeta = \phi^{-1}(\theta_{\bar{\ell}})$.

By definition of a prioritarian SWO, $((\zeta)_{\bar{\ell}}, (z)_k) \sim (\hat{y})_{\bar{\ell}+k}$. But, by **WeakNonSadCond**, for any $m \in \mathbb{N}$, $((\zeta)_{\bar{\ell}}, (\hat{w})_m) \succsim ((\zeta)_{\bar{\ell}}, (z)_k)$. However, for m large enough we have:

$$\bar{\ell}\phi(\zeta) + m\phi(\hat{w}) < (\bar{\ell} + m)\phi(\hat{z}),$$

because $\hat{w} < \hat{z}$. Thus, by definition of a prioritarian SWO, there exists \bar{m} large enough (and in particular such that $\bar{m} > k$) such that $(\hat{z})_{\bar{\ell}+\bar{m}} \succ ((\zeta)_{\bar{\ell}}, (z)_{\bar{m}})$. By transitivity of social welfare orderings, $(\hat{z})_{\bar{\ell}+\bar{m}} \succ (\hat{y})_{\bar{\ell}+k}$.

In conclusion, for any $k \in \mathbb{N}$, $\hat{y} \in \mathbb{R}_{++}$ and $\hat{z} \in [0, \hat{y}]$, there exists $\bar{m} > \bar{\ell}$ such that $(\hat{z})_{\bar{\ell}+\bar{m}} \succ (\hat{z})_{\bar{\ell}+k}$. This is a violation of **AvoidRepugConcl.** ■

Proof of Proposition 8. By **WeakNonSadCond**, there exist $z < 0$ and $k \in \mathbb{N}$ such that, for any $z' < z$ and $y \in \mathbb{R}_{++}$: $((z')_1, (y)_{k+1}) \succsim ((z')_1, (z)_k)$. By definition of an egalitarian SWO and transitivity, this implies that $(z')_{k+2} \succsim (z')_{k+1}$. This is a violation of **NegMereAdd.** ■

Proof of Proposition 9. Let us (without loss of generality) normalize function ϕ so that $\phi(0) = 0$.

The simple rank-dependent SWO \succsim satisfies **WeakNonSadCond**. Assume that function ϕ is concave and bounded above and that there exists $\kappa \in \mathbb{R}_{++}$ such that, for all $k \in \mathbb{N}$, $a_k \geq \kappa \cdot (\sum_{r=k+1}^{+\infty} a_r)$. Let $\bar{u} := \sup\{\phi(z) : z \in \mathbb{R}\}$. Choose $\hat{z} \in \mathbb{R}_{--}$ such that⁶

$$-\phi(\hat{z}) > \frac{\bar{u}}{\kappa}. \quad (5)$$

Then, for any initial allocation $\mathbf{x} \in \mathbf{X}$, let r_1 be the highest rank of an individual with well-being strictly lower than \hat{z} and r_2 the highest rank of an individual with well-being strictly lower than 0. By definition of a simple rank-dependent, for any $\ell \in \mathbb{N}$, we

⁶There must exist such a $\hat{z} \in \mathbb{R}_{--}$: given that $\phi(0) = 0$ and ϕ is concave, so that it is not bounded below on \mathbb{R} .

have $(\mathbf{x}, (0)_\ell) \succsim (\mathbf{x}, (\hat{z})_1)$ if and only if:

$$\begin{aligned}
& \sum_{r=1}^{r_1} a_r (\phi(x_{[r]}) - \phi(c)) + a_{r_1+1} (\phi(\hat{z}) - \phi(c)) + \sum_{r=r_1+2}^{n(x)+1} a_r (\phi(x_{[r-1]}) - \phi(c)) \\
& \leq \sum_{r=1}^{r_1+r_2} a_r (\phi(x_{[r]}) - \phi(c)) + \left(\sum_{r=r_1+r_2+1}^{r_1+\ell+r_2} a_r \right) (\phi(0) - \phi(c)) \\
& \quad + \sum_{r=r_1+\ell+r_2+1}^{\ell+n(x)} a_r (\phi(x_{[r-\ell]}) - \phi(c))
\end{aligned} \tag{6}$$

Given that $0 = \phi(0) > \phi(x_{[r]}) \geq \phi(\hat{z})$ for all $r_2 \geq r > r_1$, we have

$$\begin{aligned}
& a_{r_1+1} (\phi(\hat{z}) - \phi(c)) + \sum_{r=r_1+2}^{r_1+r_2+1} a_r (\phi(x_{[r-k]}) - c) \\
& \leq \sum_{r=r_1+1}^{r_1+r_2} a_r (\phi(x_{[r]}) - c) + a_{r_1+r_2+1} (\phi(\hat{z}) - \phi(c)),
\end{aligned}$$

and,

$$\sum_{r=r_1+r_2+2}^{n(x)+1} a_r (\phi(x_{[r-k]}) - \phi(c)) \leq \sum_{r=r_1+r_2+2}^{n(x)+1} a_r (\bar{u} - \phi(c)) < \left(\sum_{r=r_1+r_2+2}^{+\infty} a_r \right) (\bar{u} - \phi(c)).$$

Hence,

$$\begin{aligned}
& \sum_{r=1}^{r_1} a_r (\phi(x_{[r]}) - \phi(c)) + a_{r_1+1} (\phi(\hat{z}) - \phi(c)) + \sum_{r=r_1+2}^{n(x)+1} a_r (\phi(x_{[r-1]}) - \phi(c)) \\
& < \sum_{r=r_1+1}^{r_1+r_2} a_r (\phi(x_{[r]}) - c) + a_{r_1+r_2+1} (\phi(\hat{z}) - \phi(c)) + \left(\sum_{r=r_1+r_2+2}^{+\infty} a_r \right) (\bar{u} - \phi(c)).
\end{aligned}$$

Also, given that $\phi(x_{[r]}) \geq \phi(0)$ for all $r \geq r_2 + 1$ and $0 \leq c$ (so that $\phi(0) - \phi(c) \leq 0$):

$$\begin{aligned}
& \left(\sum_{r=r_1+r_2+1}^{r_1+r_2+\ell} a_r \right) (\phi(0) - \phi(c)) + \sum_{r=r_1+\ell+r_2+1}^{\ell+n(x)} a_r (\phi(x_{[r-\ell]}) - \phi(c)) \\
& > \left(\sum_{r=r_1+r_2+1}^{+\infty} a_r \right) (\phi(0) - c).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \sum_{r=1}^{r_1+r_2} a_r (\phi(x_{[r]}) - \phi(c)) + \left(\sum_{r=r_1+r_2+1}^{+\infty} a_r \right) (\phi(0) - \phi(c)) \\
& \leq \sum_{r=1}^{r_1+r_2} a_r (\phi(x_{[r]}) - \phi(c)) + \left(\sum_{r=r_1+r_2+1}^{r_1+\ell+r_2} a_r \right) (\phi(0) - \phi(c)) \\
& \quad + \sum_{r=r_1+\ell+r_2+1}^{\ell+n(x)} a_r (\phi(x_{[r-\ell]}) - \phi(c))
\end{aligned}$$

A sufficient condition for inequality (6) to hold is therefore:

$$a_{r_1+r_2+1} (\phi(\hat{z}) - \phi(c)) + \left(\sum_{r=r_1+r_2+2}^{+\infty} a_r \right) (\bar{u} - \phi(c)) \leq \left(\sum_{r=r_1+r_2+1}^{+\infty} a_r \right) (\phi(0) - \phi(c)).$$

Given that $\phi(0) = 0$, this sufficient condition can be written

$$a_{r_1+r_2+1} \phi(\hat{z}) + \left(\sum_{r=r_1+r_2+2}^{+\infty} a_r \right) \bar{u} \leq 0.$$

But we know that:

$$\frac{1}{\kappa} \geq \frac{\sum_{r=k+1}^{+\infty} a_r}{a_k} \text{ for all } k \in \mathbb{N}, \text{ and } -\phi(z) > \frac{\bar{u}}{\kappa}.$$

This sufficient condition is thus satisfied.

Therefore we have shown that there exist $\hat{z} \in \mathbb{R}_{--}$ such that $(\mathbf{x}, (0)_\ell) \succsim (\mathbf{x}, (\hat{z})_1)$ for any $\ell \in \mathbb{N}$. The Suppes-Sen principle implies that, for any $y \in \mathbb{R}_{++}$ $(\mathbf{x}, (y)_\ell) \succ (\mathbf{x}, (0)_\ell)$, and by transitivity $(\mathbf{x}, (y)_\ell) \succ (\mathbf{x}, (\hat{z})_1)$.

*The simple rank-dependent SWO \succsim satisfies **NegMereAdd**.* Consider any $\mathbf{x} \in \mathbf{X}$ and any $z \in \mathbb{R}_{--}$. Let r_1 be the highest rank of an individual with well-being strictly lower than z and r_2 the highest rank of an individual with well-being strictly lower than c . By definition of a simple rank-dependent SWO $\mathbf{x} \succ (\mathbf{x}, (z)_1)$ if and only if:

$$\begin{aligned}
& \sum_{r=1}^{r_1} a_r (\phi(x_{[r]}) - \phi(c)) + a_{r_1+1} (\phi(z) - \phi(c)) + \sum_{r=r_1+2}^{r_2+1} a_r (\phi(x_{[r-1]}) - \phi(c)) \\
& \quad + \sum_{r=r_2+2}^{n(x)+1} a_r (\phi(x_{[r-1]}) - \phi(c)) \\
& < \sum_{r=1}^{r_1} a_r (\phi(x_{[r]}) - \phi(c)) + \sum_{r=r_1+1}^{r_2} a_r (\phi(x_{[r]}) - \phi(c)) + \sum_{r=r_2+1}^{n(x)} a_r (\phi(x_{[r]}) - \phi(c))
\end{aligned} \tag{7}$$

Inequality (7) can be written:

$$\begin{aligned}
& a_{r_1+1} (\phi(z) - \phi(x_{[r_1+1]})) + \sum_{r=r_1+2}^{r_2} a_r (\phi(x_{[r-1]}) - \phi(x_{[r]})) + a_{r_2+1} (\phi(x_{[r_2]}) - \phi(c)) \\
& < \sum_{r=r_2+1}^{n(x)} (a_r - a_{r+1}) (\phi(x_{[r]}) - \phi(c))
\end{aligned}$$

By definition of r_1 and r_2 the left-hand side of the above inequality is strictly negative, while the right-hand side is non-negative. Hence inequality (7) holds, and therefore $\mathbf{x} \succ (\mathbf{x}, (z)_1)$. ■

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