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► **To cite this version:**

| Elliot Aurissergues. Monetary Policy Puzzle and wealth targeting consumers. 2017. halshs-01625347

**HAL Id: halshs-01625347**

**<https://shs.hal.science/halshs-01625347>**

Preprint submitted on 27 Oct 2017

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# Monetary Policy Puzzle and wealth targeting consumers

Elliot Aurissergues

## Abstract

In this paper, I document that the three equation new keynesian model predicts a strong overreaction of real wages to monetary policy shocks, whereas the response is close to zero in datas. This puzzle may be solved by sticky wages but I show that this overreaction is created by the intertemporal choice of households, on which the recent forward guidance literature have cast doubts. Then, I build a simple new keynesian model with bounded rationality. At each period, households do not form a consistent plan for their lifetime but choose between leisure, consumption and future wealth. It transposes joy of giving model to the business cycles analysis. I show that this simple model generates more realistic response to monetary policy shocks than the three equation new keynesian model. The model also highlights the importance of perceived future wealth and asset supply. Their response to changes in real interest rates deeply affects consumption and leisure decision. This point could be useful for more complicated model like Heterogenous agents or Behavioral new keynesian model.

**JEL Classification:** D83,D84

**Keyword:** Forward guidance, nonseparable preferences, bounded rationality, euler equation, monetary policy shocks

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## Introduction

The low volatility of real wages at business cycles frequency is a well established stylized fact for most advanced economies. These small fluctuations suggest that the response of real wages to various economic shocks is weak. The empirical literature have confirmed this intuition for monetary policy shocks. Real wages underreact to unexpected and exogenous changes in interest rates. The response is positive for the United States but close to zero. Surprisingly, the standard three equation new keynesian model is extremely bad at replicating this small response of real wages. Indeed, it implies that real wages should overreact to monetary policy shocks. This overreaction is created by the supply side effect of monetary policy. When consumption and leisure are chosen by a representative agent maximizing its utility over its infinite lifetime, real interest rates have an important impact on leisure and thus desired hours worked. This supply side effect is not a well known feature of the new keynesian model. Indeed, the labor supply equation is usually written as the equality between the marginal utility of leisure and the marginal utility of consumption times the real wages. But, you can also write an euler equation for leisure which will relate the growth of the marginal utility of leisure with real wage growth over the interest factor. This supply side effect is not very intuitive. But a more important issue is that it has counterfactual implications for the real wage response to change in real interest rate. A lower real interest rate not only pushes upward the labor demand curve by stimulating aggregate demand, but it will also pushes downward the labor supply curve and this last effect does not depend from frisch labor supply elasticity. The combined movement of both labor demand and labor supply curve implies a large increase of real wages.

This supply effect of real interest rate comes from the intertemporal optimization of the representative household. Interestingly, the literature have recently cast doubts on the standard model of intertemporal choice. Indeed, it has also counterfactual implications regarding the effects of forward guidance. Following the financial crisis, Central banks hit the zero lower bound and had to rely on unconventional tools to further stimulate the economy. Among those tools, forward guidance is especially interesting. Forward guidance means that central banks announce a monetary policy shock in advance. If the current nominal rate is hold constant, the standard new keynesian model predicts very large effects on inflation and output. The practice of forward guidance by central banks allow to test this forecast. Evidences by Del Negro et al. (2013) show that forward guidance has some effects but much smaller than predicted. This discrepancy between the theory and the datas

leads to reconsider some aspects of the standard model. The consumption euler equation is among obvious suspects. It implies that consumption is quite sensitive to change in future real interest rates. In turn, a change in consumption affects current and future inflation, further lowering future and current real interest rates. This puzzle have led some authors, like Gabaix(2016) or Fahri and Werning (2017) to propose new models of intertemporal choice based on some form of "bounded rationality".

Another strand of the literature have not challenged the principle of the intertemporal choice but have questioned its quantitative significance. Auclert (2016) suggests that real interest rate affects consumption mostly through indirect channels. A lower interest rate reduces interest repayments by indebted households, increasing their income and their consumption. Using an heterogenous agent framework, Kaplan, Moll and Vioante (2016) are able to replicate the response of consumption to permanent and transitory income shocks. They also document the sensitivity of consumption to real interest rate in the model. They find that the direct effect through intertemporal substitution is four times lower than in the representative agent model and than most of the consumption response comes from indirect effects.

To solve the puzzle of real wage response to monetary policy shocks, I study an alternative to the standard theory of intertemporal choice. To keep tractability and better understand the underlying mechanisms, I propose a very simple model. I label it the wealth taregting consumer (WTC) model. In a nutshell, instead of planning their consumption path along their lifetime at each period, households choose between current consumption and leisure, and future assets. Motivations for holding assets, like precautionary savings, retirement provision or inheritance, are not explicitly modelled but are implicitly integrated into the utility function. It is an application of the joy of giving model to the analysis of business cycles. Joy of giving models have been introduced by Yaari (1964) to explain bequests. When the future wealth of households is kept constant, the WTC model leads to consumption equation in levels and not in growth. It dampens the response to current and future shocks on interest rate. It allows to introduce intertemporal nonseparable preferences. It becomes possible to parametrize the supply side effect of interest rate with more freedom. I also examine the implications of the model for the forward guidance puzzle.

A useful extension of this simple model is to consider that wealth is varying and in particular react to real interest rate changes. I show that results may critically depends on the wealth response. I construct several examples in which the response of wealth dampens or enhance the response of consumption and labor supply. More

complicated models often makes "simplifying" assumption about asset distribution (see for example, McKay, Nakamura and Steinsson 2015). My results suggest that modellers should be careful before making such assumption which can deeply affect their results.

I detail the real wage puzzle in the first section. I also provide a formal characterization of the forward guidance puzzle. The second section is dedicated to the model itself and how it modifies responses to monetary policy shocks. The third section analyze the differences between the WTC model and the standard model. The fourth section discusses how the WTC model performs with respect to other piece of evidences. The fifth section discuss the relation between the WTC model and other proposed solution to the two puzzles.

## 1 Puzzles

In this section, I outline two important puzzles for monetary policy: The real wage response to monetary policy shocks and the forward guidance puzzle.

### 1.1 The real wage response to monetary policy shocks

Analysis of monetary policy shocks have mostly focused on the response of output and prices. This approach is not without merit. Monetary policy shocks have to be properly identified and a parsimonious VAR model makes this identification easier. But, this appraoch has also some danger. The risk is to developed models which performs well at matching output and price responses but poorly when confronted to a broader set of datas. Real wage is one of these datas. The simple new keynesian model imply that real wage growth overreact to real interest rate change, which implies a particularly large response of real wages levels to monetary policy shocks. Empirical evidences do not really support the theory. Wages change over the business cycles are notoriously small. VAR evidences from Christiano, Eichenbaum and Evans (1997,2005) only shows a small decrease of real wages when real interest rate rises.

#### 1.1.1 The supply side effect of monetary policy

The overreaction of real wages is not an obvious feature of the new keynesian model. It is a consequence of the supply side effect of monetary policy. Intutively and in good new keynesian logic, a lower real interest rate increases consumption. Firms serve the demand and thus increase their labor demand. Real wages should increase

to clear the labor market. This is the demand effect of monetary policy. The intuition is correct but incomplete. The New Keynesian model hides another powerful supply effect of monetary policy. A lower real interest rate does not only increase consumption, it also increases leisure for the same reason, pushing the labor supply curve downward and the real equilibrium wage upward.

To see this effect, reconsider the choice of the representative household. The agent maximizes

$$E_0 \sum_{t=0}^{+\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} + \frac{L_t^{1-\theta}}{1-\theta}$$

Without loss of generality, I assume perfect foresights and drop the expectation. I derive the usual first order condition

$$\begin{aligned} \frac{C_{t+1}^\sigma}{C_t} &= \beta R_{t+1} \\ L_t^{-\theta} &= W_t C_t^{-\sigma} \end{aligned}$$

where  $R$  is the real interest factor and  $W$  the real wage I now rewrite the labor supply equation. I first divide the labor supply equation of period  $t+1$  by the labor supply equation of period  $t$

$$\frac{L_{t+1}^{-\theta}}{L_t} = \frac{W_{t+1}}{W_t} \frac{C_{t+1}^{-\sigma}}{C_t}$$

I combine with the consumption euler equation. It gives another euler equation for leisure

$$\frac{L_{t+1}^{-\theta}}{L_t} = \frac{W_{t+1}}{W_t} \frac{1}{\beta R_{t+1}} \quad (1)$$

This euler equation relates changes in leisure with changes in wages and real interest rate. A very interesting feature of this equation is that the inverse of the elasticity of intertemporal substitution (EIS thereafter) for consumption  $\sigma$  does not appear. Even if the real interest rate has zero effects on consumption because the EIS is equal to zero, it has still an impact on an economy by modifying the leisure decision of households.

A linearized form of equation (1) is more convenient.

$$\theta(l_{t+1} - l_t) = rr_{t+1} - (w_{t+1} - w_t) \quad (2)$$

Where lowercases denote linearized variables and  $rr$  is the difference between the real interest rate and its steady state value.

I denote by  $N$  the steady state working time. Hours evolve in the opposite direction of leisure  $n_t = \frac{N}{1-N} l_t = \eta l_t$ .

Equation (2) becomes

$$\theta \eta (n_{t+1} - n_t) = w_{t+1} - w_t - rr_{t+1} \quad (3)$$

### 1.1.2 The overreaction of real wages

The euler equation for hours predicts a positive correlation between hours growth and real wage growth and a negative correlation between hours growth and real interest rate. Whereas the first is not a surprise, the second is more intriguing and seems worth investigating. Real interest rate is not often seen as an important determinant of labor supply. However, computing the observed correlation between hours and real interest rates in the spirit of Canzoneri et al. (2007) would not be useful. Indeed, the demand side of the model is given by the consumption euler equation, the good market clearing condition and the production function

$$\begin{aligned}\sigma(c_{t+1} - c_t) &= rr_{t+1} \\ y_t &= c_t \\ y_t &= \alpha n_t\end{aligned}$$

When combining the three, another equation relating hours and interest rate appears

$$\sigma\alpha(n_{t+1} - n_t) = rr_{t+1} \quad (4)$$

Correlation between hours and real interest rate would actually estimate the parameter  $\sigma\alpha$ . This is true for both the unconditional correlation and the correlation conditional to monetary policy shocks.

However, the existence of the supply side effect of monetary policy has more useful implications for the correlation between real wages and real interest rate conditional to a monetary policy shocks. To deduce this prediction, I assume the real interest rate is exogenous and follows the sequence  $rr_{t+1} = \rho rr_t + \epsilon_t$ . Then, I compute multipliers relative to this shock for hours growth and real wage growth (denoted respectively  $\delta_n$  and  $\delta_w$ ) and for hours and wages levels (denoted respectively  $\Psi_n$  and  $\Psi_w$ ). Multipliers refers to the contemporaneous response of one of the model variable to an exogenous shock. I find

$$\begin{aligned}\delta_n &= \frac{1}{\sigma\alpha} \\ \delta_w &= \theta\eta\delta_n + 1 = \frac{\theta\eta}{\sigma\alpha} + 1\end{aligned}$$

The multiplier of wage growth is superior to one. Wage growth overreact to real interest rate. The multiplier for wages level can be easily deduced

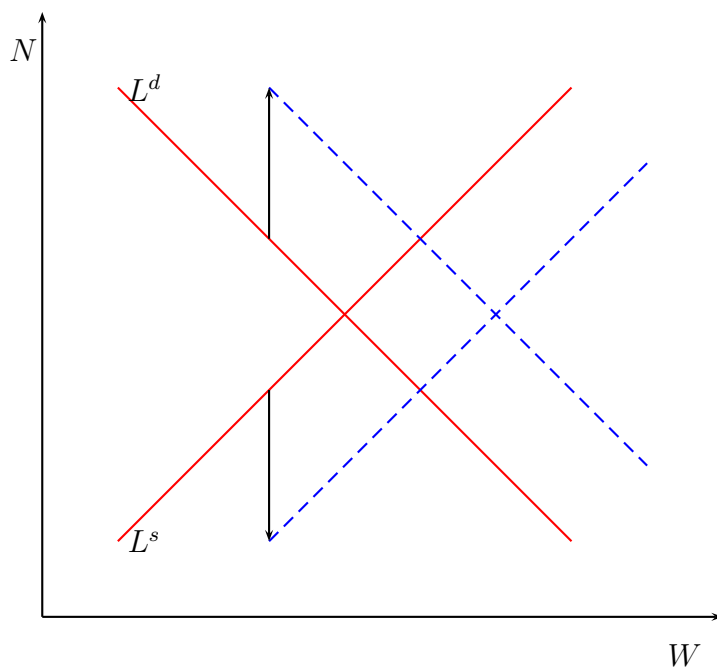
$$\Psi_w = (\rho - 1)\delta_w \quad (5)$$

A quick computation clearly shows the problem. Suppose that  $\sigma$  and  $\theta$  are equal to one whereas  $\eta$  and  $\alpha$  equal to 0.7, values which are consistent from a theoretical

viewpoint. Moreover, I assume a persistence parameter  $\rho$  equal to two thirds in line with the persistence of monetary policy shocks observed in VAR models. A one point increase in real rate should generate a two percent wage growth and thus a decline of wage levels by six percent ! If  $\sigma$  is infinite which implies that aggregate demand do not respond to monetary stimulus, real wage levels should still fall by three percent.

### 1.1.3 Intuition

To better understand the intuition behind this overreaction, it is useful to consider the partial equilibrium on labor markets. In the short run, the labor demand is given by the production function combined to the market clearing condition. In first approximation, it is flat and does not depend on real wages. Labor supply increases with real wages for a given level of real interest rate. The effect of the shock on labor market equilibrium is represented in figure (1).



A rise of real rate has two effects on the figure. It depresses aggregate demand, shifting downward the labor demand curve and the equilibrium real wages. This is the standard Keynesian channel. Mathematically, it is represented by the term  $\frac{\theta\eta}{\sigma\alpha}$  in the expression of the real wage multiplier.

A second effect is the supply side effect. The rise in interest rate reduces the discounted sum of future real wages and thus the wealth of households. It creates an income effect which pushes the labor supply curve upward, further lowering the



equilibrium real wages.

#### 1.1.4 The real wage response in reality

Real wages response to monetary policy shocks have attracted less attention than output or inflation. Nevertheless, a respectable body of literature have dealt with the issue. Unlike other topics, results are not really controversial. The response from real wages to monetary policy shocks is consistently small. Real wages underreact instead of overreacting. Sims and Zha (1998) finds that a very persistent increase of nominal rate by 0.4 percent have not a significant impact on average real wages when looking at US postwar datas. A discrepancy between nominal and real interest rate cannot explain thsi result. The shock on the nominal rate is actually a close approximation of a shock on real rate, the response of inflation being small and delayed. The theoretical response according to the new keynesian model whould have been a substantial rise of real wages, between one and two percent on impact. Christiano, Eichenbaum and Evans (1997) results are only slightly better for the standard model. A less persistent 0.7 percent increase in nominal interest rate rises real wages by 0.1 percent. Like for Sims and Zha, response of inflation is quite small. The theoretical response should have been around one percent. Both papers focus on US datas, but estimation across countries are consistent. Peersman and Smets (2001) finds that response of real wages is small in most countries of the euro area. Normandin (2006) finds a similar result for United Kingdom and Canada.

## 1.2 The forward guidance puzzle

The second puzzle which motivates this paper is the forward guidance puzzle. Central banks perfom forward guidance when they announce a departure from their usual policy. It becomes a standard tool for monetary policy when the zero lower bound is binding.

The popularity of the forward guidance comes from its large theoretical effects in the new keynesian model as shown by Del Negro, Giannoni and Patterson (2013) or McKay, Nakamura and Steinsson (2016). These papers show that expected shock on interest rate generates large response of output and inflation, larger than the response to current monetary policy shocks.

The intuition is straightforward. Suppose that an expansionnary monetary policy is announced for the period  $t+n$ . The output  $y_{t+n}$  will increase stimulating inflation  $\pi_{t+n}$ . In period  $t+n-1$ , with standard notations, euler equation for output and

philips curve will give

$$y_{t+n-1} = y_{t+n} + \frac{1}{\sigma}\pi_{t+n}$$

$$\pi_{t+n-1} = \xi y_{t+n} + \beta\pi_{t+n}$$

Because the forward looking component of both output and inflation equation are equal or close to one, output and inflation in  $t + n - 1$  will overreact to their expected increase in  $t + n$ . The euler equation implies that current output increases as much as the future output. Moreover, as future inflation is lower, real interest rate falls, amplifying the rise of  $y_{t+n-1}$ . The inflation follows a similar path. The same reasoning can be done for  $t+n-2$  until period  $t$ . At each peirod, teh effect on output is amplified.

### 1.2.1 Forward Guidance and reality

Such large effects occurs if the central bank do not react to the increase of current inflation before the realization of the shock. The condition is not fulfilled in normal times but may be when the zero lower bound is binding. The long period of zero nominal interest rate in the US, in japan, and in the euro area following and the use of forward guidance by central banks to loose moneatry policy further, provides an opportunity to test the overreaction predicted by the new keynesian model. Using High frequency identification, Del Negro, Giannoni and Patterson (2013) finds that forward guidance announcements matters and affects asset prices. But, they also find that effects are much smaller than those implied by the model. Expected shock on interest rate generates smaller response than contemporary shock, not larger ones.

### 1.2.2 A formal characterization

I now try to give a more formal characterization of forward guidance. The response to an expected shock on interest rate will depends on the duration between the announcement and the realization of the shock but also depends on the contemporary reaction of monetary policy with respect to inflation and output gap. I choose the lower computationnal burden. I compute output and inflation multipliers with respect to an expected shock on nominal interest rate with two additionnal assumption. First, the expected shock occurs in period  $t + 1$ . Second, the nominal interest rate in period  $t$  is kept constant by the central bank and do not react either to inflation or output gap. The underlying idea is that multipliers for other forward guidance shock are linked to multipliers for this simple case.

I consider a standard linear new keynesian model.

$$\begin{aligned}
\sigma(c_{t+1} - c_t) &= r_t - \pi_{t+1} \\
-\theta l_t &= w_t - \sigma c_t \\
l_t &= -\eta n_t \\
\mu_t &= w_t + n_t - y_t \\
y_t &= \alpha n_t \\
\pi_t &= \kappa \mu_t + \beta \pi_{t+1} \\
r_t &= \phi_\pi \pi_t + \epsilon_t \\
y_t &= c_t
\end{aligned}$$

Where  $c$  is consumption,  $r$  is nominal interest rate,  $\pi$  is inflation,  $l$  is leisure,  $n$  is hours worked,  $y$  is output,  $w$  is real wage,  $\mu$  is the marginal cost (the inverse of the markup), and  $\epsilon$  is a monetary policy shock.

First, I compute multipliers for a contemporaneous monetary policy shock. They are denoted  $\psi_{(\cdot)}$ . the subscript denotes the variable of interest.

$$\begin{aligned}
\psi_w &= \frac{\theta\eta + \sigma\alpha}{\alpha} \psi_y \\
\psi_\mu &= \frac{\theta\eta + \sigma\alpha + 1 - \alpha}{\alpha} \psi_y \\
\psi_\pi &= \frac{\kappa}{1 - \beta\rho} \frac{\theta\eta + \sigma\alpha + 1 - \alpha}{\alpha} \psi_y \\
\psi_y &= \frac{(1 - \beta\rho)\alpha}{(1 - \beta\rho)\alpha\sigma(\rho - 1) - (\phi_\pi - \rho)\kappa(\theta\eta + \sigma\alpha + 1 - \alpha)}
\end{aligned}$$

I now compute Multiplier for a shock occurring in  $t + 1$  under the assumption that nominal interest rate in  $t$  is fixed. I denote these multipliers  $\mathcal{M}_{(\cdot)}$ . I derived first the system relating current multiplier for output and inflation to future ones.

$$\begin{aligned}
\sigma \mathcal{M}_y &= \sigma \psi_y + \psi_\pi \\
\mathcal{M}_\pi &= \kappa \frac{\theta\eta + \sigma\alpha + 1 - \alpha}{\alpha} \mathcal{M}_y + \beta \psi_\pi
\end{aligned}$$

Solving the system leads to the expressions

$$\mathcal{M}_y = \left( 1 + \frac{\kappa(\theta\eta + \sigma\alpha + 1 - \alpha)}{\sigma(1 - \beta\rho)\alpha} \right) \psi_y \tag{6a}$$

$$\mathcal{M}_\pi = \psi_\pi \left( \beta + \frac{\kappa}{\alpha\sigma} (\theta\eta + \sigma\alpha + 1 - \alpha) \right) \tag{6b}$$

Current output overreacts. Its response to the future shock is always superior to the response to a current shock. Inflation is very likely to overreact as well. The response of inflation depends on  $\beta$  which is usually very close to 1.

**Forward guidance and the supply side effect of monetary policy** It is worth noting that the supply side effect of monetary policy enhance the effects of a forward guidance shock. Indeed, the term  $\sigma\alpha$  at the numerator of the expression of  $\mathcal{M}_y$  is a consequence of the supply side effect of interest rate. In a imaginary world in which labor supply would only depends on wages without modifying the euler equation, the expression for  $\mathcal{M}_y$  would be  $\left(1 + \frac{\kappa(\theta\eta+1-\alpha)}{\sigma(1-\beta\rho)\alpha}\right)\psi_y$ . There is still a substantial overreaction but lower than in the standard model.

## 2 The model with constant wealth

The standard model of intertemporal choice seems not compatible with observed response to unexpected and expected monetary policy shocks. In this section, I propose an alternative model which may help reconciling theory with facts whereas keeping tractability.

### 2.1 Baseline model

#### 2.1.1 The household program

Consumers wants to solve the following utility function

$$U(C_t, L_t, \Omega + A_{t+1}) \tag{7}$$

under the budget constraint

$$Q_t A_{t+1} + W_t L_t + C_t = A_t + W_t + \Pi_t \tag{8}$$

where  $C$  is the consumption,  $L$  is leisure time,  $W$  is real wage, and  $\Pi$  are profits distributed by firms.  $A$  is an asset which give the right to receive one unit of consumption good at the next period. To buy one unit of this asset, the consumer should pay a price  $Q$ . This price is the inverse of the interest factor

$$Q_t = \frac{1}{1 + rr_{t+1}} \tag{9}$$

where  $rr$  is the real interest rate.

It is important to note that households only maximize  $U$  in period  $t$ , and not the discounted sum of current and future utility streams. The presence of  $A_{t+1}$  in the

utility function is indeed enough to generate an intertemporal behavior. Households make a choice between current consumption and future consumption goods.

### 2.1.2 First order conditions

Obviously, this program is easy to solve. First order conditions are

$$U_C(C_t, L_t, A_{t+1}) = \Lambda_t \quad (10a)$$

$$U_L(C_t, L_t, A_{t+1}) = W_t \Lambda_t \quad (10b)$$

$$U_A(C_t, L_t, A_{t+1}) = Q_t \Lambda_t \quad (10c)$$

Where  $U_c$  is the first derivative of the utility function with respect to consumption.

In order to make comparisons with the new keynesian model, I keep these first order conditions, and do not derive demand function.

### 2.1.3 Supply block

The rest of the model do not differ from the standard new keynesian model.

Asset supply is null

$$A_{t+1} = 0 \quad (11)$$

The production function relates output with hours

$$Y_t = N_t^\alpha \quad (12)$$

Working and leisure time are equal to the unity

$$L_t + N_t = 1 \quad (13)$$

The only component of aggregate demand is consumption and is equal to output at equilibrium

$$Y_t = C_t \quad (14)$$

Firms maximize the discounted sum of their profits under Calvo price rigidity. The discount factor is the real interest rate. Thus, the supply block is unchanged compare to the new keynesian model.

### 2.1.4 Linearized model

The linear model is easy to derive. The supply block and equilibrium conditions are

$$a_{t+1} = 0 \quad (15a)$$

$$l_t = -\eta n_t \quad (15b)$$

$$\mu_t = w_t + n_t - y_t \quad (15c)$$

$$y_t = \alpha n_t \quad (15d)$$

$$\pi_t = \kappa \mu_t + \beta \pi_{t+1} \quad (15e)$$

$$r_t = \phi_\pi \pi_t + \epsilon_t \quad (15f)$$

$$y_t = c_t \quad (15g)$$

$$q_t = -r r_{t+1} \quad (15h)$$

$$r_t - \pi_{t+1} = r r_{t+1} \quad (15i)$$

These equations are not controversial. Thus, I focus on behavioral equations for households which are given by

$$\frac{U_{CCC}}{U_C} c_t + \frac{U_{CAA}}{U_C} a_{t+1} + \frac{U_{CLL}}{U_C} l_t = \lambda_t \quad (16a)$$

$$\frac{U_{CLC}}{U_L} c_t + \frac{U_{LAA}}{U_L} a_{t+1} + \frac{U_{LLL}}{U_L} l_t = w_t + \lambda_t \quad (16b)$$

$$\frac{U_{CAC}}{U_A} c_t + \frac{U_{AAA}}{U_A} a_{t+1} + \frac{U_{LAL}}{U_A} l_t = q_t + \lambda_t \quad (16c)$$

### 2.1.5 Separable preferences

I first assume that preferences are separable. It implies that cross derivative of the utility function are equal to zero :  $U_{CL} = 0$ ,  $U_{AL} = 0$ ,  $U_{CA} = 0$ . To make notation easier, I introduce parameters  $\sigma = \frac{U_{CCC}}{U_C}$  and  $\omega = \frac{U_{LLL}}{U_L}$ . By combining first order conditions of household program with asset supply equation, I get a consumption equation and a leisure equation

$$-\theta l_t = w_t - \sigma c_t \quad (17)$$

$$\sigma c_t = q_t \quad (18)$$

The leisure equation is not different from the standard new keynesian model. All the difference lies in the equation. Intead of having an equation for consumption growth, I have an equation for consumption **levels** with respect to interest rate.

It is easy to see that a level equation for consumption may prevent overreaction to expected interest rate shock and makes the response of real wages to current real rate shock much more reasonable.

**Forward guidance** Indeed, multiplier for forward guidance shocks are now given by the system

$$\begin{aligned}\sigma\mathcal{M}_y &= \psi_\pi \\ \mathcal{M}_\pi &= \kappa \frac{\theta\eta + \sigma\alpha + 1 - \alpha}{\alpha} \mathcal{M}_y + \beta\psi_\pi\end{aligned}$$

The response of current output to expected interest rate shock only depends of the response of current real interest rate and thus the response of future inflation. The forward looking component of the output equation have disappeared. The output multiplier becomes

$$\mathcal{M}_y = \frac{\kappa(\theta\eta + \sigma\alpha + 1 - \alpha)}{\sigma(1 - \beta\rho)\alpha} \psi_y$$

The multiplier with respect to expected shocks is not always superior to the multiplier with respect to current monetary policy shock. The overreaction of current output is no longer systematic. The multiplier for inflation remains unchanged with respect to  $\psi_\pi$ .

**Real wages** The system (19)-(20) may be rearranged to get equations for supply and demand of hours worked (conditionnal to asset market equilibrium)

$$\begin{aligned}\theta\eta n_t &= w_t + rr_{t+1} \\ \sigma\alpha n_t &= -rr_{t+1}\end{aligned}$$

Multiplier for real wages and hours level may be computed directly in our new model.

$$\begin{aligned}\Psi_W &= -1 - \frac{\theta\eta}{\sigma\alpha} \\ \Psi_N &= -\frac{1}{\theta\alpha}\end{aligned}$$

Multiplier for wage levels is equal to the inverse of the multiplier for wage growth in the new keynesian model. Wages still overreacts to changes in real interest rate, but the rise (resp. the fall) is no longer magnified by the persistence of shock. In my view, it is a step in the good direction. Shocks in macroeconomics are quite persistent, implying implausibly large variations for control variables.

## 2.2 The model with nonseparable intertemporal preferences

The case of separable preferences is a step in the good direction but is still quite insufficient. I now show that nonseparable preferences may prevent the overreaction of real wages and lower again the output response to forward guidance shock.

I allow the cross derivative between asset and consumption to be different from zero  $U_{CA} \neq 0$ . A positive cross derivative implies that assets and consumption are complement in the sense of Edgeworth, whereas a negative cross derivative means the two are substitutes.

I keep the separability assumption for leisure.  $U_{AL} = 0, U_{CL} = 0$ . This is a strong assumption but the goal is to keep a tractable model and to focus on intertemporal choice.

Nonseparability between consumption and leisure have been extensively analyzed by Bilbiie (2009). It may also help to lower the response of real wages to monetary policy shocks and is also a candidate to explain the response of consumption to fiscal policy shocks. My analysis heavily borrows from Bilbiie(2009) and transposes many of his concepts to the intertemporal choice.

How does nonseparability affects equation. It allows to disentangle the consumption elasticity to interest rate from the leisure to interest rate. indeed, the system (16) becomes

$$\frac{U_{CCC}}{U_C} c_t = \lambda_t \quad (19a)$$

$$\frac{U_{LLL}}{U_L} l_t = w_t + \lambda_t \quad (19b)$$

$$\frac{U_{CAC}}{U_A} c_t = q_t + \lambda_t \quad (19c)$$

I now introduce the parameter  $\varphi \equiv \frac{U_{CAC}}{U_A} - \frac{U_{CCC}}{U_C}$

I get slightly modified consumption and leisure equation

$$-\theta l_t = w_t - \sigma c_t \quad (20)$$

$$\varphi c_t = q_t \quad (21)$$

Rewriting the leisure equation with respect to wages and interest rate leads to

$$-\theta l_t = w_t - \frac{\sigma}{\varphi} q_t \quad (22)$$

A low value of  $\sigma$  relative to  $\varphi$  will reduce the supply side effect of interest rate further lowering the response of leisure to monetary policy shocks and thus the response of real wages.

As a consequence, the response of real wages becomes more ambiguous

$$\Psi_W = -\frac{\sigma}{\varphi} - \frac{\theta\eta}{\varphi\alpha} \quad (23)$$



A low value of  $\sigma$  implies that the downward shift of the labor supply curve following a monetary expansion is small. The response of real wages is mostly the consequence of the upward shift of the labor demand, in line with keyensian intuition.

Reducing the supply side effect of interest rate may also lower the response to forward guidance shocks. The output multiplier for forward guidance becomes

$$\mathcal{M}_y = \frac{\kappa(\theta\eta + \sigma\alpha + 1 - \alpha)}{\varphi(1 - \beta\rho)\alpha} \psi_y$$

A low  $\sigma$  relative to  $\varphi$  lower the response of output to a  $t + 1$  forward guidance shock.

**The role of complementarity** To give some intuition for this result, I introduce the parameter  $\nu \equiv \frac{U_{CAC}}{U_A}$ . It is easy to see that  $\varphi = \sigma + \nu$ . The sensitivity of leisure with respect to interest rate is equal to  $\frac{\sigma}{\sigma + \nu}$ . A large and positive  $\nu$  implies  $U_{CA} > 0$  and thus complementarity between present consumption and future assets. A rise in  $q_t$  (i.e a fall in real rates) implies that the marginal utility of assets should rise relative to the marginal utility of consumption and relative to the marginal utility of leisure. With separable preferences, consumption increases and thus reduces the marginal utility of consumption. If consumption and assets are complement, the rise in consumption will decrease the marginal utility of consumption and increases the marginal utility of assets. Thus, a much lower rise in consumption may achieve the equality between marginal rate of substitution and the relative price. The increase in the marginal utility of assets also imply a lower fall in the marginal utility of leisure and thus a lower rise in leisure.

It should be understood that  $\sigma$  and  $\nu$  cannot be parameterize freely. They should respect concavity requirements for the utility function:

$$\begin{aligned} U_{CC} &\leq 0 \\ U_{LL} &\leq 0 \\ U_{AA} &\leq 0 \\ U_{AA}U_{CC} - U_{CA}^2 &\geq 0 \end{aligned}$$

The last condition implies that  $U_{CA}$  cannot be too large with respect to  $U_{CC}$  and thus  $\nu$  should not be too large with respect to  $\sigma$  in theory. However, with our particular asset supply function,  $U_{AA}$  can be parametrize freely allowing a low value for  $\frac{\sigma}{\sigma + \nu}$

Whereas not compulsory, it also seems reasonable to impose that assets and consumption are not inferior goods whose demand decreases when income rises. A

positive value of  $\nu$  (i.e. assets and consumption are complements) is however a sufficient condition for noninferiority.

### 3 The model with varying wealth

Until then, I have supposed that "wealth" which enters into the utility function is not varying. I now show the response of perceived wealth to monetary policy shocks may alter significantly the response of leisure and consumption.

Indeed, recall the system of first order conditions (16) may be rewritten under separable preferences

$$-\sigma c_t = -\gamma a_{t+1} - q_t \quad (24a)$$

$$-\theta l_t = w_t - q_t - \gamma a_{t+1} \quad (24b)$$

$$(24c)$$

where parameters  $\sigma, \theta$  and  $\gamma$  are defined by

$$\sigma = -\frac{U_{CC}C}{U_C}$$

$$\gamma = -\frac{U_{AA}A}{U_A}$$

$$\theta = -\frac{U_{LL}L}{U_L}$$

How wealth reacts to changes in interest rate will deeply modifies the response of leisure and consumption. If wealth is an increasing function of  $q_t$ , the two response will be enhanced whereas they will be dampened if wealth is a decreasing function of  $q_t$ .

I now give three examples of wealth which gives different outcomes for monetary policy shocks

#### 3.1 A broad definition of wealth

The first example is the closest to the standard model. The wealth is defined as the sum of the financial wealth and the labor wealth

$$A_t = F_t + \Omega_t$$

Where,  $F_t$  is the financial wealth and  $\Omega_t$  is the labor wealth defined by the recursive equation

$$\Omega_t = W_t + Q_t \Omega_{t+1}$$

If the previous equation is iterated forward, the labor wealth is the discounted sum of future real wages

$$\Omega_t = \sum_{T=0}^{+\infty} [\prod_{k=0}^T Q_{t+k}] W_{t+T}$$

It is easy to see this model is very close to the standard model in many respect. Indeed, the budget constraint is nothing else than the usual intertemporal budget constraint.  $\Omega_{t+1} + F_{t+1}$  is the income at period  $t + 1$  and thus is equal to the discounted sum of consumption and leisure spending. The intertemporal first order condition relates current consumption  $C_t$  with future wealth  $\Omega_{t+1} + A_{t+1}$ . In period  $t + 1$ , consumption  $C_{t+1}$  will be a function of this wealth. Thus, you recover an equation linking current consumption with future ones and real interest rate.

Predictions with respect to monetary policy effects are also close. Assuming that supply of financial assets is null and linearizing of labor wealth gives

$$\begin{aligned} a_{t+1} &= \omega_{t+1} \\ \omega_t &= (1 - \beta)w_t + \beta(\omega_{t+1} + q_t) \end{aligned}$$

A persistent fall in real interest rate will increase interest factors  $q$  and the future wealth  $\omega_{t+1}$ . It will cause a rise in consumption and in leisure<sup>1</sup> implying a strong response of real wage to a current shock and strong responses to forward guidance.

### 3.2 Fixed public debt

The second example leads to a very different conclusion. I assume that wealth is only financial and take the form of public debt. response to monetary policy are affected by fiscal policy.

I assume that the government has a very simple fiscal policy rule. Public debt, denoted  $B_t$  is fixed, equal to  $\bar{B}$ . Market clearing for public debt implies

$$Q_t A_{t+1} = \bar{B}$$

The linearized equation is simply

$$a_{t+1} = -q_t \tag{26}$$

Consumption and leisure may be expressed with respect to real wages and interest rate

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<sup>1</sup> $\gamma - \nu m$  is equal to  $-\frac{U_{\Omega\Omega}\Omega}{U_{\Omega}}$  and thus is always positive

$$\begin{aligned}\sigma c_t &= (1 - \gamma)q_t \\ \theta l_t &= (1 - \gamma)q_t - w_t\end{aligned}$$

If  $\gamma > 1$ , both leisure and consumption becomes an increasing function of real interest rate. Obviously, under such parameters, effect of current and future monetary policy shocks on output are reversed.

### 3.3 A two agent framework

The third example shows how preferences heterogeneity may affect response to monetary policy. I consider a model with two type of agents. Agents differ by the elasticity of their wealth to interest rate for a given marginal utility of consumption and by their frisch elasticity of leisure. The first type is called "debtors" and the second type "creditors". Behavioral equations are more complicated, but aggregation and linearization remain straightforward (see appendix). Combining the linearized equation leads to a three equation system for leisure, consumption and financial assets of creditors.

Aggregate consumption and labor supply are given

$$- \sigma c_t = -q_t + [\lambda_c \gamma_c - \gamma_d(1 - \lambda_c)] a_{t+1} \quad (27a)$$

$$- (\theta_d(1 - \lambda_l) + \theta_c \lambda_l) l_t = w_t - q_t + (\lambda_l \gamma_c - \gamma_d(1 - \lambda_l)) a_{t+1} \quad (27b)$$

The law of motion for debtor's asset is

$$\left(1 - m - v + \frac{v\gamma_d}{\theta_d}\right) a_{t+1} = w_t \left(\frac{m + v}{q} - v + \frac{v}{\theta_d}\right) + a_t \left(\frac{1 - m - v}{q}\right) - q_t \left(1 - m - v + \frac{v}{\theta_d} + \frac{m}{\sigma}\right) \quad (28)$$

Where  $\lambda_c$  and  $\lambda_l$  are the share of debtors for respectively consumption and leisure at steady state.  $m$  and  $v$  are respectively average propensity to consume for debtors at steady state and its equivalent for leisure.  $a$  denotes debtor's wealth.

The labor supply keeps a term  $w_t - q_t$  but the endogenous response of debtors asset allows to disentangle the effects of real wages and of real interest rates. Moreover, the sign of the real interest rate effect becomes ambiguous.

Interestingly, there is now an endogenous amplification mechanism in the model. The shift in asset distribution in period  $t$  will affect labor supply in period  $t + 1$ . Following an expansionary monetary policy shocks, creditors desaccumulate assets and increase their labor supply on impact. At the next period, they have less assets

and through income effect, further reduce their labor supply and their consumption. This amplification effect may dampen current and future inflation and helps to solve the forward guidance puzzle.

## 4 Comparison between WTC model and standard model

Asset targeting consumers may help to explain some puzzles of monetary policy. However, other pieces of evidence can favor the standard model. In this section, I argue it is not the case.

### 4.1 Empirical evidence on Euler equation

Asset targeting model may for example fail to replicate stylized facts related to the correlation between consumption growth and interest. Since the seminal paper of Hall, a large literature has dealt with the issue since the seminal paper of Hall. It should be first noted that the results from that literature are very mixed for the standard model. The seminal paper of Hall (1988) found no evidence of intertemporal substitution, a result confirmed for example by Yogo (2004). Another disappointing result was the negative correlation found between the FED funds rate and the real rate implied by consumption growth, found by Canzoneri et al. (2007) for several widely used consumption models. More positive results have come from several papers by Attanasio and Weber (1993, 1995, 2010) using microeconomic data on individual consumption and introducing controls for demographics and labor supply, they find a larger elasticity of intertemporal substitution. Results from these papers are conflicting but evidence accumulated by the literature does not endorse clearly the standard model.

More importantly for our purpose, such tests do not really allow to choose between the two models. Indeed, an equation giving consumption levels as a decreasing function of real interest rate also implies a positive correlation between consumption growth and real interest rate if changes in real interest rate are positively autocorrelated. Conversely, under the same condition, an equation giving consumption growth as an increasing linear function of real interest rate implies a negative correlation between real interest rate and consumption levels.

Indeed, consider a consumption equation in levels.  $c_t = -\varphi rr_{t+1}$ . Consumption growth becomes  $c_{t+1} - c_t = -\varphi(rr_{t+2} - rr_{t+1})$ , implying  $cov(c_{t+1} - c_t, rr_{t+1}) = \varphi(1 - \rho_r)$  where  $\rho_r$  is the autocorrelation coefficient of real interest rate.

Consider now a consumption equation in growth  $c_{t+1} - c_t = \sigma rr_{t+1}$ . Correlation between consumption levels and real interest rate is equal to  $cov(c_t, rr_{t+1}) = cov(c_t, \frac{c_{t+1} - c_t}{\sigma}) = \frac{\rho_c - 1}{\sigma}$  where  $\rho_c$  denotes consumption autocorrelation

Most macroeconomic shocks are positively autocorrelated and persistence coefficient are often close to one. As a consequence, both consumption and real interest rate are very persistent at business cycles frequencies. Asset targeting consumer model and standard model have similar predictions for the sign of the two correlations.

However, the WTC model seems more appropriate to fit the relatively small response of consumption levels to changes in nominal interest rate in VAR models, as shown by Bernanke and Gertler (1995). Thus, direct evidences about consumption and interest rate do not eliminate the WTC model.

## 4.2 Microeconomic evidences on household behavior

Last, we can invoke direct evidences of households behavior to discriminate between the two models. The WTC model implies that agents are rational enough to choose between current consumption and future assets but are not rational enough to compute how macroeconomic shocks and in particular changes in real interest rates affect their labor wealth. Does evidences about household behavior provide support for this irrationality? Actually, this is rather the rational part of the WTC model which could be challenged. Observations of households intertemporal behavior repeatedly show significant violations of rationality. Camerer et al (1997) shows no intertemporal substitution between high wage and low wage days from experienced taxi drivers and "reversed" intertemporal substitution from inexperienced ones ( they work less during high wages days). A survey by Lusardi(2008) shows very small level of financial literacy among US workers, a limited ability to plan for the future including for predictable events like retirement and difficulty to compound interest rates.

These surveys make hard to believe that a significant fraction of consumers adjust their consumption and their leisure by computing accurately how hardly observable macroeconomic shocks will modify future real interest rates and real wages and thus their labor wealth.

## 4.3 Real wage response, consumption leisure substitutability and wage rigidity

In this paragraph, I compare the WTC model with other potential solutions for the real wages puzzle. To recover more realistic real wages response, the literature have

mostly relied on wage stickiness. Whereas they have been put forward to solve fiscal policy puzzles (see Bilbiie 2009), nonseparable preferences between leisure and consumption may also reduce the supply side effect of real interest rate if consumption and leisure are gross substitutes.

I see the WTC model as a complement to these approaches and not as a substitute. The WTC model is a step in the good direction for two important puzzles. But, it remains insufficient to fully recover the response of real wages. Sticky wages may further improve the fit of the model.

Sticky wage may also help the standard model to improve its empirical performances. For any response of real wages smaller than the standard response, there is always an appropriate level of wage stickiness which is able to explain it. But, you will still need much more wage stickiness in the standard model than in the WTC model.

A similar reasoning can be done for nonseparable preferences between leisure and consumption. For some calibration, they allow to completely eliminate the supply side effect of monetary policy. However, that implies a large substitutability between consumption and leisure. Combined to the WTC model, a very low supply side effect of monetary policy can be achieved for more reasonable levels of consumption leisure substitutability.

## 5 Conclusion

In this paper, I have shown that wealth targeting consumer may help the new keynesian model to match observed response to monetary policy shocks. In particular, it allows to parametrize with more freedom the supply side effect of monetary policy, which have not received in the literature the attention it deserves. It may also be a useful tool for the forward guidance puzzle. The advantage of the WTC is that it remains extremely simple and tractable, whereas capturing many intuitions of bounded rationality and heterogenous agents model. Another one is that it is relatively easy to introduce more sophisticated behavior like intertemporal non separable preferences. Obviously, it is not a panacea and more realistic models for consumption and labor supply are still needed. But, it may provide very useful lessons for the two literature, especially the importance of the asset supply and the asset distribution. For example, McKay, Nakamura and Steinsson, (2016) have developed a tractable version of heterogenous agents model in which the asset distribution is degenerate. My results suggest that this assumption is not only a simplifying one but may deeply affect the outcome of the model.

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## A Model with two agents

Compare to the standard model, the labor supply and the consumption equation are replaced by first order conditions for both agents, and aggregation equation. In their linear version, it gives

$$\begin{aligned}
-\sigma c_t^d &= -\gamma^d a_{t+1}^d - q_t \\
-\theta_d l_t^d &= w_t - q_t - \gamma a_{t+1}^d \\
-\sigma c_t^c &= -\gamma^c a_{t+1}^c - q_t \\
-\theta_c l_t^c &= w_t - q_t - \gamma a_{t+1}^c \\
\lambda_c c_t^c + (1 - \lambda_c) c_t^d &= c_t \\
\lambda_l l_t^c + (1 - \lambda_l) l_t^d &= l_t \\
a_{t+1}^d &= -a_{t+1}^c
\end{aligned}$$

To solve the problem, I need the budget constraint of one of the two agents. The budget constraint of debtors is

$$W_t + A_t^d = W_t L_t + C_t + Q_t A_{t+1}^d$$

I denote  $m = \frac{C}{W+A+\Pi}$  the average propensity to consume at steady state and  $v = \frac{WL}{W+A}$  its equivalent for leisure.

The linearized budget constraint gives

$$vw_t + vl_t + mc_t^d + (1 - m - v)(q_t + a_{t+1}^d) = \frac{1 - m - v}{q} a_t^d + \frac{m + v}{q} w_t \quad (29)$$

I can now derive an equation for debtor asset law of motion by combining the budget constraint with first order conditions for debtors

$$\left(1 - m - v + \frac{v\gamma_d}{\theta_d}\right) a_{t+1}^d = w_t \left(\frac{m + v}{q} - v + \frac{v}{\theta_d}\right) + a_t^d \left(\frac{1 - m - v}{q}\right) - q_t \left(1 - m - v + \frac{v}{\theta_d} + \frac{m}{\sigma}\right) \quad (30)$$

Equations for consumption and aggregate leisure are

$$\begin{aligned}
-\sigma c_t &= -q_t + [\lambda_c \gamma_c - \gamma_d(1 - \lambda_c)] a_{t+1}^d \\
-(\theta_d(1 - \lambda_l) + \theta_c \lambda_l) l_t &= w_t - q_t + (\lambda_l \gamma_c - \gamma_d(1 - \lambda_l)) a_{t+1}^d
\end{aligned}$$