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Income Distribution by Age Group and Productive Bubbles

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Income distribution by age group and productive bubbles*

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Abstract

The aim of this paper is to study the role of the distribution of income by age group on the existence of speculative bubbles. A crucial question is whether this distribution may promote a bubble associated to a larger level of capital, i.e. a productive bubble. We address these issues in a three period overlapping generations (OG) model, where productive investment done in the first period of life is a long term investment whose return occurs in the following two periods. A bubble is a short term speculative investment that facilitates intertemporal consumption smoothing. We show that the distribution of income by age group determines both the existence and the effect of bubbles on aggregate production. We also show that fiscal policy, by changing the distribution of income, may facilitate or prevent the existence of bubbles and may also modify the effect that bubbles have on aggregate production.

JEL classification: E22; E44.

Keywords: Bubble; efficiency; income distribution; overlapping generations.

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1 Introduction

Individuals have heterogeneous savings behaviors over the life cycle. This suggests that the population size of each generation may affect the asset market and is a determinant of the asset price. This has been studied by Abel (2001) and Geanakoplos *et al.* (2004), among others, who have shown that the relative size of the different age groups affects the price of the assets.

We adopt a complementary view taking into account that the distribution of income by age group is an important determinant of the aggregate savings. Accordingly, we examine whether the distribution of income by age group also affects the asset market. Interestingly, cross-country differences in this distribution are very large. Table 1 shows a cross-country comparison of the distribution of income by age group when we consider three age groups: young, middle age and old. This table shows that middle age individuals generally obtain the largest fraction of total income, whereas the old individuals obtain the smallest fraction. However, beyond this common feature, there are large cross-country differences in the distribution of income by age group. For example, the minimum value of the fraction of total income obtained by the young individuals is 33%, whereas the maximum value is 30% larger. These cross-country differences are even larger if we consider the fraction of total income obtained by the old individuals. The maximum value of this fraction is 47% larger than its minimum value.

We are interested in the interplay between income distribution by age group and the value of assets without fundamental value, i.e. bubbles. Indeed, the literature has already shown that the existence of bubbles depends on the savings decisions over the life cycle. In particular, Tirole (1985) shows that bubbles arise when the equilibrium of an overlapping generations model is dynamically inefficient.¹ This form of inefficiency is explained by imperfections that force individuals to use productive capital to postpone consumption. In this case, they overaccumulate capital and, hence, the equilibrium is dynamically inefficient. Tirole (1985) shows that, in this situation, individuals may use an asset without fundamental value to postpone consumption. Therefore, when the equilibrium without bubbles is dynamically inefficient, an equilibrium with bubbles may also exist.² These bubbles reduce the stock of productive capital and also gross domestic product (GDP). However, more recently, Caballero *et al.* (2006) and Martin and Ventura (2012) provide convincing evidence showing that bubbles arise during economic booms. Obviously, this evidence suggests that GDP should be larger in the equilibrium with bubbles. To explain this evidence, we refer to the concept of productive bubbles, defined as bubbles that facilitate a larger accumulation of productive capital. Therefore, we can distinguish between unproductive bubbles, that arise when the equilibrium without bubbles is dynamically inefficient, and productive bubbles, that may arise when the equilibrium without bubbles is dynamically efficient. Martin and

¹See Abel, et al. (1989) for an analysis of dynamic efficiency in OG models.

²The existence of bubbles has been studied in OG models by Samuelson (1958), Tirole (1985) and Weil (1987), and more recently, by Bosi and Seegmuller (2010), Caballero *et al.* (2006), Fahri and Tirole (2012) or Martin and Ventura (2012, 2016). There is a large literature that also studies the possibility of bubbles in infinite horizon models. Some relevant references of this literature are Hirano and Yanagawa (2013), Kamihigashi (2008), Kocherlakota (1992, 2009), Miao and Wang (2011).

Ventura (2012) and Raurich and Seegmuller (2015) show that productive bubbles may exist when there are heterogenous agents that are differentiated by their productivity of investment.

The purpose of this paper is to contribute to the aforementioned literature by showing how the distribution of income by age group affects dynamic efficiency of the bubbleless equilibrium and the existence of productive bubbles. To this end, we extend the three period OG model studied in Raurich and Seegmuller (2015) by assuming that individuals work in the first two periods of life. As a consequence, labor income is distributed between young and middle age individuals. We will highlight that this assumption plays a crucial role for our results. In this model, the distribution of labor income between young and middle age individuals and the distribution of capital income between middle age and old individuals determine the distribution of total income by age group. We show that the model can generate the income distributions displayed in Table 1.

In the model, productive investment is done by young individuals and it is a long term investment whose return occurs in the following two periods of life. The bubble is a short term investment that facilitates intertemporal consumption smoothing. Note that this model introduces an important distinction between young and middle age individuals. The former invest in productive capital, whereas the later only invest in financial assets to smooth consumption. This distinction introduces heterogeneity across individuals that, as mentioned, is necessary to have productive bubbles. Therefore, bubbles can be either productive or unproductive.

We first show that if a large part of the labor income is earned by middle age individuals and a large part of the capital income is earned by old individuals then neither the young, nor the middle age individuals are interested in holding the speculative asset in order to postpone consumption. In this case, an equilibrium with bubbles does not exist.

In addition to its existence, we also study how the distribution of income by age group affects whether a bubble is productive or not. On the one hand, we show that if a large fraction of the labor income is earned by the young individuals and a large fraction of the capital income is earned by the middle age individuals, households overaccumulate capital to postpone consumption. In this case, the equilibrium without bubbles is dynamically inefficient. As in Tirole (1985), an equilibrium with bubbles exists, but these bubbles are unproductive because they are aimed to postpone consumption.

On the other hand, we show that bubbles can be productive in two different cases: when the income obtained by the middle age individuals is sufficiently large and when it is sufficiently small.³ In the first case, bubbles are used to transfer consumption from the middle age period to the other two periods of life. Young agents are short sellers of the bubble. The larger wealth obtained by the young individuals is invested in productive capital and, hence, the bubble is productive. In the second case, the bubble is used to transfer consumption from the young and the old periods of life to the middle

³Note that middle age individuals obtain a large (small) fraction of total income when the fraction of labor income obtained by the young is small (large) and when the fraction of capital income obtained by the middle age is large (small). Thus, the two situations in which bubbles can be productive correspond to polar cases of the distribution of income by age group.

age period. In this case, middle age households are short sellers of the bubble. Young individuals increase savings to compensate the reduction that the bubble causes on the level of consumption in the last period of life. Part of the increase in savings is invested in productive capital, which explains that the bubble is productive.

The distribution of income by age group is largely modified by fiscal policy. As this distribution determines the existence of productive bubbles, fiscal policies may facilitate or prevent the existence of productive bubbles. To study the effect of fiscal policy, we introduce capital and labor income taxes. We differentiate labor income taxes by the age group of the tax payers, in order to introduce progressive taxes. Regarding the effect that taxes have on the existence of bubbles, we first show that, depending on the age group of the tax payers, an increase in the labor income taxes may either hinder or facilitate the existence of an equilibrium with bubbles. We also show that an increase in the capital income tax reduces the income of both middle age and old individuals. As a consequence, the introduction of this tax facilitates the existence of an equilibrium with bubbles that will be used to postpone consumption.

We also study the effect of fiscal policy on the stock of productive capital and, hence, on aggregate production. On the one hand, in the absence of bubbles, productive capital is used to smooth consumption and is mainly determined by the discounted incomes received at each age. This explains that an increase in the labor income tax paid by the young individuals reduces productive investment and that the labor income tax paid by the middle age individuals has the opposite effect. On the other hand, in the presence of bubbles, productive capital is determined by a non-arbitrage condition between the returns from capital and the returns from bubbles. This implies that labor income taxes do not affect the stock of productive capital in the equilibrium with bubbles. Since the bubbly and bubbleless steady states may coexist, we conclude that the effect of the labor income tax on capital depends on who is paying the tax and also on which steady state individuals coordinate.

We illustrate numerically the effects of actual fiscal policies by comparing the tax rates in the US and in several European economies. We show that capital income taxes are clearly larger in the US, whereas labor income taxes are larger in European economies. These differences in fiscal policy have two clear implications. First, the tax burden is more concentrated on the young individuals in European countries, which limits capital accumulation in these economies. Second, the larger capital income taxes in the US facilitate the existence of a bubble. We illustrate these two implications by simulating the model using the actual values of the tax rates in the US and in European economies. We first show that only in the US economy the value of the parameters is consistent with the existence of a bubble. We also show that if European economies change their fiscal policy and set tax rates at the level of the US economy, then (i) the stock of productive capital will substantially increase, (ii) several European economies could exhibit a bubble, and (iii) this bubble will be productive in those economies where the distribution of income is such that middle age individuals obtain a large fraction of total income.

The paper is organized as follows. Section 2 presents the model. Section 3 studies the equilibrium without bubbles and characterizes dynamic efficiency. Section 4 studies the equilibrium with bubbles and obtains the distribution of income by age group for which bubbles exist and are productive. Section 5 discusses the effect of fiscal policy

on the existence of productive bubbles. Section 6 concludes the paper. Some technical details are relegated to an Appendix.

2 Model

Consider a three period OG economy that in period t is populated by N_t young individuals. Let $n = N_t/N_{t-1} > 0$ be the constant ratio between the number of young and middle age individuals in period t . The utility of an individual born in period t is

$$\ln c_{1,t} + \beta \ln c_{2,t+1} + \beta^2 \ln c_{3,t+2}, \quad (1)$$

where $c_{1,t}$ is the consumption when young, $c_{2,t+1}$ is the consumption in the middle age, $c_{3,t+2}$ is the consumption when old and $\beta \in (0, 1)$ is the subjective discount rate.

Young individuals work and obtain a labor income $\xi_1 w_t$ that they use to consume $c_{1,t}$ and invest in both a speculative asset, $b_{1,t}$, and a non-speculative asset, a_{t+1} . The wage per efficiency unit is w_t and $\xi_1 > 0$ measures the efficiency units of a young worker. We assume that only the young individuals can invest in the non-speculative asset, which is a long term investment that provides returns in the following two periods of life. In the second period of life, agents also work and obtain a labor income $\xi_2 w_{t+1}$, where $\xi_2 > 0$ measures the efficiency units of a middle age worker. These workers also obtain capital income from the return on the non-speculative asset, $\phi_1 q_{t+1}$. The return of one unit of productive capital is q_{t+1} and ϕ_1 are the units of productive capital that middle age individuals obtain from one unit of investment. Finally, they sell the speculative asset, $R_{t+1} b_{1,t}$. The return from selling the bubble, R_{t+1} , is the growth rate of the price of the bubble. The income obtained by middle age individuals is used to consume, $c_{2,t+1}$, and invest in speculative assets, $b_{2,t+1}$. In the last period of life, individuals are retired and, hence, they do not obtain labor income. They sell the speculative asset, $R_{t+2} b_{2,t+1}$, and they obtain $\phi_2 q_{t+2}$ from the return on the non-speculative asset, where ϕ_2 are the units of productive capital that old individuals obtain from one unit of investment done in the first period of life. Old individuals consume $c_{3,t+2}$. It follows that the budget constraints of the young, middle age and old individuals are, respectively,

$$c_{1,t} + a_{t+1} + b_{1,t} = \xi_1 w_t, \quad (2)$$

$$c_{2,t+1} + b_{2,t+1} = \xi_2 w_{t+1} + q_{t+1} \phi_1 a_{t+1} + R_{t+1} b_{1,t}, \quad (3)$$

$$c_{3,t+2} = R_{t+2} b_{2,t+1} + q_{t+2} \phi_2 a_{t+1}. \quad (4)$$

We note first that the investment in the non-speculative asset only when young is a simplifying assumption aimed to introduce a relevant difference in the productivity of the investment decisions of the different age groups. In fact, it is a reasonable assumption once this productive investment is considered as investment in education or investment in new companies. These forms of productive investment clearly decline as individuals get older. We also note that the return on productive investment depends on whether the investment has been done one or two periods before. This is a consequence of assuming that the productivity of capital depends on the period in which investment has been done. This is formalized through a simple form of vintage capital. This second assumption is introduced to generate the distribution of capital income between middle

age and old individuals. Similarly, the difference in the efficiency units of labor between young and middle age individuals is introduced to generate the distribution of labor income between these two groups of individuals. The joint distribution of labor and capital income will be used in our analysis to determine the distribution of total income by age group.

Technology is characterized by the following aggregate production function:

$$Y_t = AK_t^\alpha L_t^{1-\alpha}, \text{ with } A > 0 \text{ and } \alpha \in (0, 1),$$

where Y_t is aggregate production, L_t the total amount of efficiency units of labor and K_t the stock of productive capital in the economy. Using $k_t \equiv K_t/L_t$, $Y_t/L_t = Ak_t^\alpha$ and competitive factor prices satisfy:

$$w_t = (1 - \alpha) Ak_t^\alpha, \quad (5)$$

and

$$q_t = \alpha Ak_t^{\alpha-1}. \quad (6)$$

We complete the characterization of the model with the market clearing conditions for capital, labor and the speculative asset. The market clearing condition for capital is:

$$K_t = N_{t-1}\phi_1 a_t + N_{t-2}\phi_2 a_{t-1},$$

where $\phi_1 a_t$ and $\phi_2 a_{t-1}$ measure, respectively, the units of productive capital owned by middle age and old individuals. The market clearing condition for efficiency units of labor is:

$$L_t = N_t \xi_1 + N_{t-1} \xi_2,$$

where ξ_1 and ξ_2 measure, respectively, the efficiency units of labor provided by young and middle age workers. We use these two market clearing conditions to define the fraction of productive capital owned by the middle age individuals:

$$\Omega_t = \frac{n\phi_1 a_t}{n\phi_1 a_t + \phi_2 a_{t-1}}, \quad (7)$$

and the fraction of efficiency units of employment provided by the young individuals:

$$\Sigma = \frac{n\xi_1}{n\xi_1 + \xi_2}. \quad (8)$$

Note that at a steady state with $a_t = a_{t-1}$, the fraction of productive capital simplifies to the following parameter:

$$\Omega = \frac{n\phi_1}{n\phi_1 + \phi_2}.$$

The fractions Σ and Ω measure the distribution of labor and capital income by age group. In Appendix E, we use the distribution of total income by age group displayed in Table 1 and two plausible assumptions of the model, the old do not obtain labor income and the young do not obtain capital income, to obtain the values of Σ and Ω displayed in Table 2. This table shows huge differences across-countries in the value of Σ and Ω . As an example, the largest value of Σ is 58% larger than its minimum value

and the largest value of Ω is 91% larger than its minimum value. Note that these very large differences are the consequence of both differences in the relative size of the age groups and also differences in the average income of each age group.

From the previous two market clearing conditions, we also obtain that capital per efficiency unit of labor is:

$$k_t = \frac{N_{t-1}\phi_1 a_t + N_{t-2}\phi_2 a_{t-1}}{N_t \xi_1 + N_{t-1} \xi_2},$$

which can be rewritten as:

$$k_t = \frac{\phi_1}{n\xi_1 + \xi_2} a_t + \frac{\phi_2}{n^2\xi_1 + n\xi_2} a_{t-1}. \quad (9)$$

We assume that the speculative asset is supplied in one unit at a price p_t in period t . New investments in this asset by young and middle age individuals are in quantities ϵ_t and $1 - \epsilon_t$, respectively. Therefore, the values of this asset bought or sold by these agents are $B_{1,t} = b_{1,t}N_t = p_t\epsilon_t$ and $B_{2,t} = b_{2,t}N_{t-1} = p_t(1 - \epsilon_t)$. Since this asset has no fundamental value, it is a bubble if $p_t = B_{1,t} + B_{2,t} > 0$, which happens when $nb_{1,t} + b_{2,t} > 0$. Finally, the market clearing condition for the speculative asset at period $t + 1$ is:

$$N_{t+1}b_{1,t+1} + N_t b_{2,t+1} = R_{t+1} (N_t b_{1,t} + N_{t-1} b_{2,t}).$$

The left-hand side of the previous equation is the value of the speculative asset bought by young and middle age individuals, whereas the right-hand side is the value of the speculative asset sold by middle age and old individuals. The speculative asset sold in period $t + 1$ is multiplied by the growth rate of the price, R_{t+1} , as it was purchased in period t . This equation can be rewritten as:

$$nb_{1,t+1} + b_{2,t+1} = \frac{R_{t+1}}{n} (nb_{1,t} + b_{2,t}). \quad (10)$$

From the previous arguments, it follows that there is a bubble when $nb_{1,t} + b_{2,t} > 0$, while a bubbleless equilibrium is given by $b_{1,t} = b_{2,t} = 0$.

3 Equilibria without bubble

We start by analyzing the model when there is no bubble, i.e. $b_{1,t} = b_{2,t} = 0$. In this case, the household's budget constraint rewrites:

$$c_{1,t} = \xi_1 w_t - a_{t+1}, \quad (11)$$

$$c_{2,t+1} = \xi_2 w_{t+1} + q_{t+1} \phi_1 a_{t+1}, \quad (12)$$

$$c_{3,t+2} = q_{t+2} \phi_2 a_{t+1}. \quad (13)$$

Maximizing the utility under the budget constraints (11)-(13), we get:

$$w_{t+1} = \frac{q_{t+1} a_{t+1} \phi_1}{\xi_2} \left[\frac{(\beta + \beta^2) \xi_1 w_t - (1 + \beta + \beta^2) a_{t+1}}{(1 + \beta^2) a_{t+1} - \beta^2 \xi_1 w_t} \right]. \quad (14)$$

From using (5) and (6), the previous equation can be rewritten as

$$k_{t+1} = \frac{\alpha a_{t+1} \phi_1}{(1-\alpha) \xi_2} \left[\frac{(\beta + \beta^2) \xi_1 (1-\alpha) A k_t^\alpha - (1 + \beta + \beta^2) a_{t+1}}{(1 + \beta^2) a_{t+1} - \beta^2 \xi_1 (1-\alpha) A k_t^\alpha} \right]. \quad (15)$$

Note that using (15), we can implicitly define a_{t+1} as a function of k_{t+1} and k_t . Substituting it into (9), we deduce that k_{t+1} implicitly depends on k_t and k_{t-1} . This explains that two initial conditions, $k_{-1} > 0$ and $k_0 > 0$, are required.

Definition 1 *Given $k_{-1} \geq 0$ and $k_0 \geq 0$, an equilibrium without bubble is a path $\{k_t, a_t\}_{t=1}^\infty$ that solves the system of equations (9) and (15).*

In the following, we restrict our attention to steady states, because our main aim is to compare stationary equilibria with and without bubbles, and understand the role of the distribution of income by age group.

3.1 Steady State

We use (9) and (15) to show that there is a unique steady state and, using (7) and (8), it can be shown that the steady state values of productive investment, a^* , and capital, k^* , are:

$$a^* = \frac{\Omega n \xi_1}{\phi_1 \Sigma} k^*, \quad (16)$$

$$k^* = \left(\frac{(1-\alpha)(1-\Sigma) + \alpha \Omega}{(1-\alpha)\beta^2(1-\Sigma) + \alpha \Omega(\beta + \beta^2)} + 1 \right)^{\frac{1}{\alpha-1}} \left(\frac{\Omega n}{(1-\alpha)\phi_1 A \Sigma} \right)^{\frac{1}{\alpha-1}}. \quad (17)$$

Note that the capital stock at the steady state increases with the fraction of labor income obtained by the young individuals, Σ , and it also increases with the fraction of capital income obtained by the middle age individuals, Ω . On the one hand, an increase in Σ rises the income obtained by the young individuals, who then increase investment in productive capital. On the other hand, an increase in Ω reduces the income obtained by old individuals. Young individuals then compensate this reduction by increasing the investment in the productive asset.

The previous arguments show that the willingness to postpone consumption is large when Σ and Ω are large, which suggests that in this case the equilibrium will be dynamically inefficient. This is analyzed in the following subsection.

3.2 Dynamic efficiency

The steady state equilibrium is dynamically efficient when aggregate consumption increases with investment. This is a direct implication of the results obtained by Abel *et al.* (1989) and de la Croix and Michel (2002). As it is well known, this occurs when the return on investment is larger than population growth. In this model, this condition implies that $(\phi_1 + \phi_2/n)q > n$. Using (6) and the steady state value of capital, we obtain that the steady state is dynamically efficient when the following condition holds:

$$\left(\frac{(1-\alpha)(1-\Sigma)+\alpha\Omega}{(1-\alpha)\beta^2(1-\Sigma)+\alpha\Omega(\beta+\beta^2)} + 1 \right) \left(\frac{\alpha}{1-\alpha} \right) > \Sigma. \quad (18)$$

Using this condition, we get the following result:

Proposition 1 *The equilibrium is dynamically efficient if either (i) $\Sigma < \Sigma_1$ or (ii) $\Sigma \in (\Sigma_1, \Sigma_2)$ and $\Omega < \bar{\Omega}$, where $\Sigma_1 = \frac{\alpha}{1-\alpha} \frac{1+\beta+\beta^2}{\beta+\beta^2}$, $\Sigma_2 = \frac{\alpha}{1-\alpha} \frac{1+\beta^2}{\beta^2}$ and*

$$\bar{\Omega} = \left(\frac{\Sigma_2 - \Sigma}{\Sigma - \Sigma_1} \right) \left(\frac{1 + \beta^2}{\beta + \beta^2} \right) \left(\frac{1 - \Sigma}{\Sigma_2} \right).$$

Proof. See Appendix A. ■

The result in Proposition 1 implies that the equilibrium is dynamically inefficient when either Σ or Ω are sufficiently large. This result is obtained because there is a positive relationship between the savings rate and the values of both Σ and Ω . In order to illustrate this mechanism that relates dynamic efficiency with the distribution of income by age group and that it is based on savings, we next show the relation between the savings rate and condition (18). We first use (5) and (6) to obtain $w/q = (1 - \alpha)k/\alpha$. We use this equation, the expression of k^* and (14) to obtain:

$$\frac{\xi_1 w}{a} = \frac{(1 - \alpha)(1 - \Sigma) + \alpha\Omega}{(1 - \alpha)(1 - \Sigma)\beta^2 + \alpha\Omega(\beta + \beta^2)} + 1, \quad (19)$$

where $a/\xi_1 w$ is the savings rate defined as the ratio between savings and the labor income of the young. Using (19), condition (18) can then be written as

$$\frac{1}{\Sigma} \left(\frac{\alpha}{1-\alpha} \right) > \frac{a}{\xi_1 w}.$$

Therefore, the steady state equilibrium is dynamically efficient when the savings rate is smaller than $\alpha/(1 - \alpha)$. This is exactly the same condition that the literature has obtained for dynamic efficiency. In fact, if $\Sigma = 1$, condition (18) simplifies to $\alpha/(1 - \alpha) > (\beta + \beta^2)/(1 + \beta + \beta^2)$, which is the condition obtained in Raurich and Seegmuller (2015). However, in this case, the savings rate and the condition for dynamic efficiency are independent from the distribution of income by age group. In contrast, as follows from (19), the savings rate increases with both Σ and Ω when $\Sigma < 1$. Note that this is a crucial difference that explains that dynamic efficiency depends on the income distribution by age group and it will also explain some of the main results in the following section.

4 Equilibria with a bubble

We introduce in this section the portfolio decision of the consumer between a short term speculative asset, $b_{1,t}$ and $b_{2,t+1}$, and a long term productive asset, a_{t+1} . Hence, the consumer decides a_{t+1} , $b_{1,t}$ and $b_{2,t+1}$ to maximize the utility (1) subject to the budget

constraints (2)-(4). The solution to this maximization problem is characterized by the first order conditions with respect to $b_{1,t}$, $b_{2,t+1}$, and a_{t+1} , which are, respectively,

$$\frac{1}{c_{1,t}} = \beta \frac{R_{t+1}}{c_{2,t+1}}, \quad (20)$$

$$\frac{1}{c_{2,t+1}} = \beta \frac{R_{t+2}}{c_{3,t+2}}, \quad (21)$$

$$\frac{1}{c_{1,t}} = \beta \phi_1 \frac{q_{t+1}}{c_{2,t+1}} + \beta^2 \phi_2 \frac{q_{t+2}}{c_{3,t+2}}. \quad (22)$$

From combining (20)-(22) and using (6), we obtain the following non-arbitrage condition between the returns from investing one unit in the speculative asset and the returns from investing the same unit in productive capital:

$$R_{t+1} = \phi_1 \alpha A k_{t+1}^{\alpha-1} + \frac{\phi_2 \alpha A k_{t+2}^{\alpha-1}}{R_{t+2}}. \quad (23)$$

In Appendix B, we combine (2)-(6), (20), (21) and (23) to obtain the following two equations:

$$b_{1,t} = \frac{(\beta + \beta^2) \xi_1 (1 - \alpha) A k_t^\alpha - \frac{\xi_2 (1 - \alpha) A k_{t+1}^\alpha}{R_{t+1}}}{1 + \beta + \beta^2} - a_{t+1}, \quad (24)$$

$$b_{2,t+1} = \frac{\beta^2 \xi_2 (1 - \alpha) A k_{t+1}^\alpha + \beta^2 \xi_1 (1 - \alpha) A k_t^\alpha R_{t+1}}{1 + \beta + \beta^2} + a_{t+1} (\phi_1 \alpha A k_{t+1}^{\alpha-1} - R_{t+1}). \quad (25)$$

Definition 2 Given $k_{-1} \geq 0$ and $k_0 \geq 0$, an equilibrium is a path of $\{a_t, k_t, b_{1,t}, b_{2,t}, R_t\}_{t=1}^\infty$ that solves the system of difference equations (23), (24) and (25) and the market clearing conditions (9) and (10).

We proceed to obtain the steady state and then we characterize the distributions of income for which an equilibrium with bubbles exists and also the distributions for which these bubbles are productive, i.e. are associated with a larger level of capital per unit of labor.

4.1 Steady state

We first use (10) to obtain $R = n$. Next, from (23), we obtain that the steady state value of capital in the equilibrium with bubbles, k , is:

$$k = \left(\frac{\phi_1 \alpha A}{\Omega n} \right)^{\frac{1}{1-\alpha}}.$$

We use (16) to deduce the steady state value of productive investment, a . From (24), we obtain the steady state value of the bubbles owned by the young individuals:

$$b_1 = \frac{(n\xi_1 + \xi_2)(1 - \alpha) A k^\alpha}{n} (\Sigma - \Sigma_{b_1}), \quad (26)$$

where $\Sigma_{b_1} = \frac{1}{1+\beta+\beta^2} + \frac{\alpha}{1-\alpha}$. From (25), we obtain the steady state value of the bubbles owned by the middle age individuals:

$$b_2 = (n\xi_1 + \xi_2) \alpha A k^\alpha (\Omega - \Omega_{b_2}), \quad (27)$$

where $\Omega_{b_2} = 1 - \left(\frac{1-\alpha}{\alpha}\right) \left(\frac{\beta^2}{1+\beta+\beta^2}\right)$. Finally, as explained in Section 2, the price of the bubble is $N_{t-1} (nb_1 + b_2)$, where

$$nb_1 + b_2 = (n\xi_1 + \xi_2) (1 - \alpha) A k^\alpha \left(\Sigma - \Sigma_{b_1} + \frac{\alpha}{1-\alpha} (\Omega - \Omega_{b_2}) \right).$$

Recall that b_1 is used to smooth consumption between young and middle age individuals, whereas b_2 is used to smooth consumption between middle age and old individuals. This explains that the sign of b_1 depends on Σ , whereas the sign of b_2 depends on Ω . If $\Sigma > \Sigma_{b_1}$ then a large fraction of labor income is obtained by the young individuals. The bubble is then used to transfer wealth to the second period of life, i.e. $b_1 > 0$. In contrast, if $\Sigma < \Sigma_{b_1}$ then a large part of labor income is obtained by middle age individuals. The bubble is then used to transfer wealth to the first period of life, $b_1 < 0$. Similarly, if $\Omega > \Omega_{b_2}$ then a large fraction of capital income is obtained by the middle age individuals. These individuals use the bubble to transfer wealth to the last period of life, i.e. $b_2 > 0$. Obviously, the opposite occurs when $\Omega < \Omega_{b_2}$.

We next obtain conditions for which an equilibrium with bubbles exists.

Proposition 2 *A steady state with a bubble exists if $\Omega > \tilde{\Omega}$ where*

$$\tilde{\Omega} = \left(\frac{1-\alpha}{\alpha} \right) (\Sigma_3 - \Sigma),$$

and $\Sigma_3 = \frac{1-\beta^2}{1+\beta+\beta^2} + \frac{2\alpha}{1-\alpha}$.

Proof. A bubble exists when its price is positive, which occurs when $nb_1 + b_2 > 0$. Using (26) and (27), the previous inequality implies that $\Omega > \tilde{\Omega}$.

From Proposition 2, it follows that a bubble does not exist when either Σ or Ω are sufficiently small. A bubble may only exist if either the young individuals buy the speculative asset ($b_1 > 0$), or the middle age individuals buy this asset ($b_2 > 0$). As already explained, the young individuals buy the speculative asset if they obtain a sufficiently large income, which requires large Σ . Similarly, middle age individuals buy this asset when they obtain a sufficiently large amount of income, which requires a sufficiently large value of Ω .

4.2 Productive bubbles

Bubbles are a financial instrument that facilitates consumption smoothing and, hence, individuals do not need to use productive capital to smooth consumption. As a consequence, the introduction of bubbles modifies the stock of productive capital, which may either increase or decrease. More specifically, bubbles are productive when $k > k^*$. From the comparisons between these two stocks of capital, it is easy to show

that the bubble is unproductive if and only if the equilibrium without bubbles is dynamically inefficient. In this case, as in Tirole (1985), the bubble is used to postpone consumption and, as a consequence, productive investment declines. The bubbly steady state corresponds to the golden rule.

We have shown that a bubble may exist when the young generation obtains a large fraction of the labor income and when the middle age generation obtains a large fraction of the capital income. We have also shown that if these two fractions are not too large then the steady state without bubbles is dynamically efficient and, hence, the bubble is productive. The following proposition summarizes these findings and provides a complete characterization of the conditions implying the existence of productive bubbles.

Proposition 3 *The steady state equilibrium satisfies the following properties:*

1. If $\Sigma < \Sigma_1$, then (i) the bubble exists and is productive when $\Omega > \tilde{\Omega}$ and (ii) the bubble does not exist when $\Omega < \tilde{\Omega}$.
2. If $\Sigma > \Sigma_1$, then (i) the bubble exists and is not productive when $\Omega > \max\{\tilde{\Omega}, \bar{\Omega}\}$, (ii) the bubble exists and is productive when $\Omega \in (\tilde{\Omega}, \bar{\Omega})$ and (iii) the bubble does not exist when $\Omega < \tilde{\Omega}$.

Proof. From Proposition 2, it is immediate to show that the bubble exists if $\Omega > \tilde{\Omega}$. From Propositions 1, it is easy to show that the equilibrium without bubbles is dynamically efficient and the bubble is productive if either $\Sigma < \Sigma_1$ or $\Sigma > \Sigma_1$ and $\Omega < \bar{\Omega}$, where the expressions of $\bar{\Omega}$ and Σ_1 are defined in Proposition 1. ■

Proposition 3 provides the main result of the paper. It shows that the distribution of income by age group crucially determines the existence of productive bubbles. It extends the analysis provided in Raurich and Seegmuller (2015), where it is already shown that bubbles can increase the stock of productive capital when productive investment is a long term investment. However, that paper restricts its attention to the case where $\Sigma = 1$ and, hence, productive bubbles only arise if $\Sigma_1 > 1$. Therefore, the existence of productive bubbles does not depend on the distribution of income by age group. Here, this distribution plays a crucial role, not only on the existence of productive bubbles but also their features, i.e. whether they are characterized by $b_i < 0$ or $b_i > 0$. This is studied in the following proposition.

Proposition 4 *We distinguish among the following parametric regions:*

1. If $\frac{\alpha}{1-\alpha} \in \left(\frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)}, \frac{\beta^2}{1+\beta+\beta^2} \right) \cup \left(\frac{\beta/2+\beta^2}{1+\beta+\beta^2}, \frac{\beta+2\beta^2}{1+\beta+\beta^2} \right)$ then productive bubbles satisfy $b_1 < 0$ and $b_2 > 0$. It requires $\Sigma < \Sigma_{b_1}$.
2. If $\frac{\alpha}{1-\alpha} \in \left(\max \left\{ \frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)}, \frac{\beta^2}{1+\beta+\beta^2} \right\}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2} \right)$ then productive bubbles satisfy $b_1 < 0$ and $b_2 > 0$ when $\Sigma < \Sigma_{b_1}$ and $b_1 > 0$ and $b_2 < 0$ otherwise.

3. If $\frac{\alpha}{1-\alpha} \in \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)} \right)$ then productive bubbles satisfy $b_1 > 0$ and $b_2 < 0$. It requires $\Sigma > \Sigma_{b_1}$.
4. If $\frac{\alpha}{1-\alpha} < \min \left\{ \frac{\beta^2}{1+\beta+\beta^2}, \frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)} \right\}$ or $\frac{\alpha}{1-\alpha} > \frac{\beta+2\beta^2}{1+\beta+\beta^2}$ then the equilibrium does not exhibit productive bubbles.

Proof. See Appendix C.

This proposition implies that, depending on the values of α and β , we can distinguish among four possible regions. In the first region, bubbles are productive only when $b_1 < 0$ and $b_2 > 0$. Panel a of Figure 1 shows this case by displaying the relationship between Ω and Σ implied by the functions $\tilde{\Omega}$ and $\tilde{\Omega}$ when $b_1 < 0$ and $b_2 > 0$ is the only possible productive bubble.⁴ Observe from Panel a that productive bubbles emerge when Σ is small and Ω is large. This implies that productive bubbles arise when $b_1 < 0$ and $b_2 > 0$ if the middle age individuals obtain a sufficiently large fraction of total income. In this case, individuals use the bubble to transfer wealth from the middle age to the other two periods of life. On the one hand, middle age individuals postpone consumption, which implies that $b_2 > 0$. On the other hand, middle age individuals transfer wealth to the young individuals, which implies that $b_1 < 0$.

In the second region, bubbles can be productive when either $b_1 < 0$ and $b_2 > 0$ or when $b_1 > 0$ and $b_2 < 0$. This case is displayed in Panel b of Figure 1. This figure shows that, as in the previous case, bubbles are productive when $b_1 < 0$ and $b_2 > 0$ if the middle age obtains a sufficiently large fraction of income (Σ small and Ω large). The figure also shows that bubbles are productive when $b_1 > 0$ and $b_2 < 0$ if the middle age individuals obtain a small fraction of income (Σ large and Ω small). In this case, consumption smoothing implies that wealth is transferred from the young and old individuals to the middle age individuals. In the third region of the previous proposition, bubbles can be productive only when $b_1 > 0$ and $b_2 < 0$. This case is displayed in Panel c of Figure 1. As in the second region, this productive bubble arises when the middle age individuals obtain a small fraction of total income. Finally, the last region is displayed in Panel d of Figure 1. In this region, productive bubbles do not exist for any income distribution.

From inspection of Figure 1, we obtain clear insights about the effects of non-marginal increases in Σ and Ω that change the characteristics of the equilibrium. On the one hand, an increase in Σ facilitates the existence of an equilibrium with bubbles. These bubbles can be productive or unproductive, depending on the value of Ω . A large value of Σ implies that the fraction of income obtained by young individuals is large and, hence, young individuals are willing to hold the bubble to postpone consumption. On the other hand, an increase in Ω also facilitates the existence of an equilibrium with bubbles. A larger value of Ω increases the income obtained by middle age individuals. These individuals are then willing to hold the bubble to postpone consumption.

Proposition 4 shows that bubbles can be productive in two very different situations: (i) when $b_1 < 0$ and $b_2 > 0$, and (ii) when $b_1 > 0$ and $b_2 < 0$. In the first situation, the

⁴The productive bubbles obtained in Raurich and Seegmuller (2015) are a particular example of the bubbles obtained in Case 1.

bubble is productive as it is used to transfer wealth to the young, who then increase productive investment. In the second situation, the bubble reduces the consumption of the old individuals and, as a consequence, young individuals increase savings in order to keep their consumption when old. If the increase in the savings of the young is large enough, productive investment increases even though part of the savings are used to transfer wealth to the middle age individuals ($b_1 > 0$). In order to show more explicitly this argument, we compare the savings rate in the economy with bubbles with the savings rate in the economy without bubbles. The savings rate is defined as the ratio between assets accumulated when young and the income of the young individuals. We first use (5) and (24) to obtain the savings rate in the economy with bubbles,

$$\frac{a + b_1}{\xi_1 w} = \frac{\beta + \beta^2 - \frac{1-\Sigma}{\Sigma}}{1 + \beta + \beta^2}$$

Note that in the economy with bubbles young individuals accumulate both productive assets and speculative assets. Using (19), we obtain the savings rate in the economy without bubbles, where the young individuals only accumulate productive assets, i.e.:

$$\frac{a}{\xi_1 w} = \frac{(1 - \alpha)(1 - \Sigma)\beta^2 + \alpha\Omega(\beta + \beta^2)}{(1 - \alpha)(1 - \Sigma)(1 + \beta^2) + \alpha\Omega(1 + \beta + \beta^2)}.$$

Note that both expressions of the savings rate are different when $\Sigma < 1$, whereas they coincide when $\Sigma = 1$. As a consequence, when $\Sigma = 1$, productive capital is larger with bubbles if and only if $b_1 < 0$. On the contrary, when $\Sigma < 1$, capital can be larger with bubbles even if $b_1 > 0$, as the savings rates can be larger in the economy with bubbles. From the comparison between the two savings rates, it follows that the savings rate of the economy with bubbles is larger when the following condition on the distribution of income by age group holds:

$$\Omega < \left(\Sigma - \frac{1 + \beta^2}{1 + \beta + \beta^2} \right) \frac{1 - \alpha}{\alpha}$$

This condition implies that the savings rate is larger in the economy with bubbles when either Σ is sufficiently large or when Ω is sufficiently small. Therefore, these two conditions show that the savings rate is larger when the middle age individuals are poor, which is precisely the condition that makes bubbles be productive when $b_1 > 0$ and $b_2 < 0$.

5 Fiscal Policy

Fiscal policies cause large changes in the distribution of income by age group. The analysis in the previous sections suggests that these changes may affect both the existence of bubbles and also the effect of bubbles on production. The purpose of this section is to study a fiscal policy that consists of taxes on capital income, τ_k , taxes on labor income paid by the young individuals, τ_w^1 , and taxes on labor income paid by the middle age individuals, τ_w^2 .⁵ We assume that tax rates on labor income depend on

⁵Taxes on bubble returns could have been introduced. If they were introduced, the after tax return from the bubbles would be $\tilde{R} = 1 + (R - 1)(1 - \tau_b)$, where τ_b is the tax rate on bubble returns. As

the age group of the tax payers to introduce progressive taxes. Finally, we assume that government revenues are used to finance a useless government spending, G_t . Thus, an increase in the tax rates will cause a variation in this government spending that will not affect individuals decisions, as government spending is assumed to be useless. The government budget constraint is:

$$\tau_w^1 \xi_1 w_t N_t + \tau_w^2 \xi_2 w_t N_{t-1} + \tau_k (q_t \phi_1 a_t N_{t-1} + q_t \phi_2 a_{t-1} N_{t-2}) = G_t.$$

The budget constraints of the individuals, equations (2)-(4), are modified as follows:

$$c_{1,t} + a_{t+1} + b_{1,t} = (1 - \tau_w^1) \xi_1 w_t, \quad (28)$$

$$c_{2,t+1} + b_{2,t+1} = (1 - \tau_w^2) \xi_2 w_{t+1} + (1 - \tau_k) q_{t+1} \phi_1 a_{t+1} + R_{t+1} b_{1,t}, \quad (29)$$

$$c_{3,t+2} = R_{t+2} b_{2,t+1} + (1 - \tau_k) q_{t+2} \phi_2 a_{t+1}. \quad (30)$$

5.1 Steady state

We start by analyzing the equilibrium of the model when there are no bubbles. In this case, the household maximizes the utility under the budget constraints (28)-(30) when $b_{1,t} = b_{2,t} = 0$. We get:

$$\frac{1}{(1 - \tau_w^1) \xi_1 w_t - a_{t+1}} = \frac{(1 - \tau_k) \phi_1 \beta}{(1 - \tau_w^2) \xi_2 \frac{w_{t+1}}{q_{t+1}} + (1 - \tau_k) \phi_1 a_{t+1}} + \frac{\beta^2}{a_{t+1}}.$$

In order to obtain the steady state without bubbles, we substitute in the previous equation (5), (6), and (9), and we use the expressions of Σ and Ω to obtain the steady state value of capital:

$$k^* = \left(\frac{n\Omega}{A\phi_1(1-\alpha)(1-\tau_w^1)\Sigma} \right)^{\frac{1}{\alpha-1}} \left(\frac{(1+\beta+\beta^2)\alpha(1-\tau_k)\Omega + (1+\beta^2)(1-\alpha)(1-\tau_w^2)(1-\Sigma)}{(\beta+\beta^2)\alpha(1-\tau_k)\Omega + \beta^2(1-\alpha)(1-\tau_w^2)(1-\Sigma)} \right)^{\frac{1}{\alpha-1}}. \quad (31)$$

In this section, Ω and Σ still measure the distribution by age group of before taxes labor and capital income.

We proceed to obtain the steady state of the equilibrium with bubbles. To this end, we assume that the consumer can smooth consumption using bubbles. Hence, the consumer maximizes the utility function subject to (28)-(30). The first order conditions of the household problem are:

$$c_{2,t+1} = \beta R_{t+1} c_{1,t} \quad (32)$$

$$c_{3,t+2} = \beta^2 R_{t+2} R_{t+1} c_{1,t} \quad (33)$$

$$\frac{1}{c_{1,t}} = \beta \phi_1 \frac{(1 - \tau_k) q_{t+1}}{c_{2,t+1}} + \phi_2 \beta^2 \frac{(1 - \tau_k) q_{t+2}}{c_{3,t+2}}. \quad (34)$$

From combining (32)-(34) and using (6), we obtain the following non-arbitrage condition between the returns from the speculative asset and the returns from

$R = n$ then $\tilde{R} = 1 + (n - 1)(1 - \tau_b) = n - (n - 1)\tau_b$. However, these taxes will reduce the increase of the price of the bubble if $n > 1$ and, hence, they would be a subsidy when the household is a short-seller of the bubble. To avoid this problem, we do not introduce these taxes.

productive capital:

$$R_{t+1} = (1 - \tau_k) \phi_1 \alpha A k_{t+1}^{\alpha-1} + \frac{(1 - \tau_k) \phi_2 \alpha A k_{t+2}^{\alpha-1}}{R_{t+2}}. \quad (35)$$

We use (9), (28)-(30), (32), (33) and (35) to obtain the following steady state values (see Appendix D):

$$k = \left(\frac{(1 - \tau_k) \phi_1 A \alpha}{\Omega n} \right)^{\frac{1}{1-\alpha}}, \quad (36)$$

$$b_1 = \frac{(n\xi_1 + \xi_2) A k^\alpha}{n} \left[(1 - \alpha) \frac{\beta(1+\beta)(1-\tau_w^1)\Sigma - (1-\tau_w^2)(1-\Sigma)}{1+\beta+\beta^2} - \alpha(1 - \tau_k) \right], \quad (37)$$

and

$$b_2 = (n\xi_1 + \xi_2) A k^\alpha \left[(1 - \alpha) \frac{\beta^2(1-\tau_w^2)(1-\Sigma) + \beta^2(1-\tau_w^1)\Sigma}{1+\beta+\beta^2} - \alpha(1 - \tau_k)(1 - \Omega) \right], \quad (38)$$

The steady state price of the bubble is positive if $nb_1 + b_2 > 0$, where

$$nb_1 + b_2 = A k^\alpha (n\xi_1 + \xi_2) F,$$

and

$$F = (1 - \alpha) \frac{\beta(1+2\beta)(1-\tau_w^1)\Sigma - (1-\beta^2)(1-\tau_w^2)(1-\Sigma)}{1+\beta+\beta^2} - \alpha(1 - \tau_k)(2 - \Omega).$$

A steady state with a bubble exists if the price of the bubble is positive, which requires that $F > 0$. Therefore, the effects of fiscal policy on the existence of bubbles are obtained from a simple comparative static analysis on the function F . The results obtained from this analysis are summarized in the following proposition:

Proposition 5 *The following fiscal policies facilitate the existence of an equilibrium with bubbles: (i) a reduction in the labor income taxes paid by the young individuals; (ii) an increase in the labor income taxes paid by the middle age individuals; (iii) an increase in the capital income taxes.*

Proof. The results follow directly from using the function F . ■

Labor income taxes paid by young individuals reduce their income after taxes. This limits the possibility to postpone consumption using bubbles. Labor income taxes paid by middle age individuals reduce their income after taxes. As a consequence, individuals use bubbles to postpone consumption towards the middle age individuals. Thus, an increase in this tax facilitates the existence of an equilibrium with bubbles. Finally, capital income taxes reduce the after tax income of both middle age and old individuals. Since capital has a lower return, traders have more incentive to invest in the speculative asset. Therefore, an increase in these taxes facilitates the existence of bubbles that will be used to postpone consumption.

We distinguish between the labor income taxes paid by young and middle age individuals to introduce progressive taxes in the analysis. The average labor income of middle age individuals is generally larger than the average labor income of young

individuals and, thus, it is reasonable to assume that $\tau_w^2 > \tau_w^1$. However, we could have assumed the same tax rate. In this case, the effect of an increase in the labor income tax on the existence of bubbles depends on the value of Σ . If Σ is large, the labor income tax is mainly a tax on the income of the young and, hence, an increase in this tax hinders the possibility of bubbles. The opposite occurs when Σ is small.

5.2 Analysis of the effects of fiscal policy

We proceed to study the effect of fiscal policies on production both in the economy without bubbles and in the economy with bubbles. This will allow us to characterize those fiscal policies that promote productive bubbles. At this point, it is important to clarify that the effect of fiscal policy on production follows directly from the effect that fiscal policy has on the stock of productive capital.

Using equation (31), it can be shown that the steady state stock of productive capital of the economy without bubbles, k^* , decreases when (i) the tax rate on the labor income of the young individuals increases; (ii) the tax rate on the labor income of the middle age individuals decreases; and (iii) the tax rates on capital income increases. The effects of labor income taxes are explained because, in the absence of bubbles, productive capital is used to smooth consumption. Therefore, an increase in the labor income tax paid by the young individuals reduces their income net of taxes, which causes a reduction in productive investment. An increase in the labor income tax paid by the middle age individuals reduces their after tax income. Young individuals then increase investment in productive capital to postpone consumption. Finally, taxes on capital income reduce the return from productive capital, which implies a raise of the discounted income. Therefore, young households consume more, which causes the reduction in productive investment.

Using (36), we can easily see that the steady state stock of productive capital of the economy with bubbles, k , decreases following an increase of the tax on capital income. This result follows from the fact that this tax reduces the return from productive investment and there is a no-arbitrage condition between holding capital and the bubble. As a direct implication, this stock of productive capital does not depend on the different taxes on labor income.

The previous results imply that the effect on the stock of capital of taxes on labor income depend on the existence of bubbles. As a consequence, fiscal policy may make bubbles productive or unproductive. To study the effect of fiscal policy, we compare the stocks of capital k and k^* and we show that $k^* < k$ when $\Psi > 0$, where

$$\Psi = 1 + \frac{(1 - \alpha)(1 - \tau_w^2)(1 - \Sigma) + \alpha(1 - \tau_k)\Omega}{(1 + \beta)\beta\alpha(1 - \tau_k)\Omega + \beta^2(1 - \alpha)(1 - \tau_w^2)(1 - \Sigma)} - \frac{(1 - \alpha)(1 - \tau_w^1)\Sigma}{\alpha(1 - \tau_k)}.$$

Straightforward comparative statics on the function Ψ show that bubbles may become productive as a consequence of the following fiscal policies: (i) an increase in the labor income taxes paid by the young individuals and (ii) a reduction in the labor income taxes paid by the middle age individuals. As explained before, an increase in the labor income taxes paid by the young makes individuals use bubbles to transfer wealth to the first period of life. As a consequence, bubbles either disappear or become productive. Obviously, the effect is the opposite when the fiscal policy consists of increasing the

taxes paid by the middle age individuals, either existence of bubbles is facilitated or bubbles become unproductive. Finally, an increase in the taxes on capital income has an ambiguous effect on the existence of productive bubbles. This is explained by the fact that these taxes reduce the stock of capital both when the equilibrium exhibits bubbles and when it does not exhibit bubbles.

These results on the effect of fiscal policy on the stock of capital are summarized in Figure 2, that shows how both stocks of productive capital depend on the taxes on labor income. Panel a shows that if the tax rate on the labor income of the young is sufficiently small, then the bubble will be used to postpone consumption and, hence, it will be unproductive. To see this, note that $k < k^*$ for low values of this tax rate. As the tax rate increases, the bubble becomes productive and, eventually, the bubble disappears. Panel b shows the effects of the tax rate on the income of the middle age individuals. These effects are the opposite from the ones displayed in Panel a. When this tax rate is sufficiently small, the bubble may not exist. When the tax rate increases, a productive bubble exists. Finally, for sufficiently large values of the tax rate, $k^* > k$ and, hence, the bubble becomes unproductive.

Figure 2 introduces an important implication for fiscal policy. It shows that marginal increases in the labor income taxes that do not affect the existence of bubbles have no effect on the stock of productive capital in the economy with bubbles. However, a non-marginal increase in the tax rate on the labor income of the young that makes the bubble disappear will cause a dramatic reduction in the stock of capital since the only long run equilibrium is the steady state without bubble. A large decline in the stock of productive capital would also occur if we instead consider a non-marginal reduction in the tax rate on the labor income of the middle age individuals that also eliminates bubbles. These results point out an important discontinuity in the effects that fiscal policy has on production.

The effects illustrated in the first two panels of Figure 2 are obtained when fiscal policy makes a productive bubble disappear. However, for a different distribution of income by age group, the same fiscal policy may cause the disappearance of an unproductive bubble. This possibility is illustrated in Panels c and d of Figure 2, that display, respectively, the effects of taxes on the labor income of the young individuals and the effects of taxes on the labor income of the middle age individuals. These two panels show that in this case a non-marginal change in the tax rates that eliminates the bubble will cause an increase, instead of a decrease in the stock of capital once the economy reaches the bubbleless steady state.

We conclude that the effects of fiscal policies crucially depend on the distribution of income by age group. It follows that in order to obtain clear insights on the effects of fiscal policy we need to perform a numerical analysis based on a plausible parametrization. This is the purpose of the following section.

5.3 Numerical Analysis

We fix the value of the parameters as follows. First, without loss of generality A and ϕ_1 are normalized to one. Second, $\alpha = 0.3$, which implies a labor income share equal

to 70%.⁶ Third, $\beta = 0.93$, which implies a savings rate of 7%.⁷ Apart from these parameters that are assumed to be common across countries, we consider two sets of country specific parameters. First, the values of Σ and Ω are displayed in Table 2 and they are computed as it is explained in Appendix E. Second, tax rates and the population growth rate are obtained from the OECD data set and they are displayed in Table 3.⁸

The results from this calibration are displayed in Table 4.⁹ This table shows the value of the capital stock in both economies (bubble and no bubble) and the value of F . The sign of F determines the existence of the bubble, with a negative sign implying that the economy does not exhibit a bubble. As it is clear from this table, only the US economy may exhibit a bubble. This bubble is productive, as follows from the comparison between the two capital stocks. In contrast, none of the European economies may exhibit a bubble according to this analysis. From the comparison between the fundamentals of the European economies and those of the US economy, displayed in Tables 2 and 3, it follows that the main difference is fiscal policy. In fact, there are no relevant differences between US and European economies in the population growth rate, nor in the distribution of income by age group. The only clear difference with respect to European economies are the larger taxes on capital income and the smaller taxes on labor income. This different fiscal policy implies that the tax burden in European economies is more concentrated on the young individuals, which limits investment in productive capital and prevents the existence of an equilibrium with bubbles.

The results in Table 4 suggest that if European economies change their fiscal policy then productive capital should increase. This is studied in Table 5, where we set the value of the tax rates in the European economies at the level of the US. In this table, we distinguish between three groups of European economies. The first group, formed by 6 countries (Belgium, Denmark, Finland, France, Germany and Italy), does not exhibit bubbles with this new fiscal policy. These economies are characterized by very low values of Ω , which, as follows from the analysis of the previous section, hinders the existence of bubbles. The second group, formed by 4 economies (Czech Republic, Greece, Norway and Poland) may exhibit unproductive bubbles. These are economies characterized by large values of both Σ and Ω and, hence, individuals in these economies use the bubble to postpone consumption. Finally, the last group of countries, formed by 7 countries (Austria, Hungary, Netherlands, Portugal, Spain, Sweden and United Kingdom) may exhibit productive bubbles. These seven economies are characterized

⁶There is not a consensus in the literature on the value of the labor income share. In a recent paper, Koh, Santaaulalia-Llopis and Zheng (2016) show that in the US the labor income share is stable and close to 70% if intellectual property capital is not considered as a form of capital income. We choose this stable value of the labor income share for our steady state analysis.

⁷Using the OECD savings rate, we obtain that the average savings rate in the period 1970-2015 in the countries displayed in Table 1 is equal to 7%. The average savings rate for these countries obtained from our simulation is also 7% when $\beta = 0.93$.

⁸The population growth rate n is obtained from the OECD data set as the ratio between the size of the young population and the size of the middle age population. In Table 3, we provide a unique tax rate on the labor income.

⁹The results in Table 4 cannot be used for cross country comparisons in the level of the GDP per capita, as countries may differ in both the efficiency units of labor and in the technology.

by intermediate values of both Σ and Ω . This distribution of income together with the fiscal policy facilitates that individuals use the bubble to smooth consumption by placing resources from the middle age towards the young and the old. The increase in the disposable income of the young makes the bubble productive.

An interesting remark is obtained from the comparison of the stocks of capital in Tables 4 and 5. From this comparison, it follows that the proposed change in fiscal policy will cause a substantial increase in the stock of capital of the European economies.¹⁰ The results in Table 5 show that the average increase in the stock of capital of these European economies would be 25% if the economy remains in an equilibrium without bubbles. At this point, it is important to introduce some words of caution on the large effects of fiscal policy obtained in the previous analysis. First, the changes in the stock of capital are obtained by comparing two different steady states. Thus, these effects of fiscal policy may only occur in the long run. Second, the effects of fiscal policy crucially depend on the value of Σ and Ω . To obtain these values, we have introduced assumptions on the distribution of labor and capital income by age group that, as explained in Appendix E, may introduce biases on the actual values of both Ω and Σ . Therefore, the results in Table 5 should only be considered as illustrative of the large effects that fiscal policies may have when they modify the distribution of income by age group.

6 Concluding remarks

We are interested in the interplay between distribution of income by age group and productive bubbles. We have studied a three period OG model where productive investment done in the first period of life is a long term investment whose return occurs in the following two periods. The bubble is a short term speculative investment that facilitates intertemporal consumption smoothing. Our main result shows that the distribution of labor and capital income by age group determines both the existence of bubbles and their effect on production. We first show that if a large part of the labor income is obtained by middle age individuals and a large part of the capital income is obtained by old individuals then the equilibrium does not exhibit bubbles. We also show that if the fraction of labor income obtained by the young individuals is large and the fraction of capital income obtained by the middle age individuals is also large then an equilibrium with unproductive bubbles exists. These bubbles are used to postpone consumption. Finally, we show that the equilibrium exhibits productive bubbles in two different situations: when the middle age individuals obtain a large fraction of total income and when these individuals obtain a small fraction of total income. In the first case, bubbles are productive because they are used to transfer wealth to the young individuals, who then increase investment in the productive asset. In the second case, bubbles are productive because the savings rate is larger in the equilibrium with bubbles.

Fiscal policies cause large changes in the distribution of income by age group and, as a consequence, they modify the effect that bubbles have on production and they can

¹⁰United Kingdom is an exception. Taxes are substantially lower in this country and, hence, the change in fiscal policy will increase taxes and reduce productive capital.

either facilitate or hinder the existence of bubbles. In particular, we show that large capital income taxes facilitate the existence of an equilibrium with bubbles. We also show that the effect of an increase in the labor income taxes depends on the age group of the tax payers. We conclude that the same fiscal policy may have very different effects on production depending on the distribution of income by age group. This conclusion is illustrated numerically by showing the effect that a fiscal policy has on several European economies.

Our analysis can be used to study the effects of shocks that modify the distribution of income by age group. An interesting example is population aging that will increase the size of the oldest age group. As a consequence, it will reduce the value of Ω in the following years, which will reduce the stock of productive capital. Our results suggest that population aging can be particularly harmful in those economies where productive bubbles finance a large stock of productive capital, as these bubbles, due to the reduction in Ω , may not be sustainable.

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Appendix

A Proof of Proposition 1

We rewrite condition (18) as

$$(1 - \alpha)(1 - \Sigma)\beta^2(\Sigma_2 - \Sigma) > \alpha\Omega(\beta + \beta^2)(\Sigma - \Sigma_1),$$

where Σ_2 and Σ_1 are defined in the main text.

As $(1 + \beta^2)/\beta^2 > (1 + \beta + \beta^2)/(\beta + \beta^2)$, there are only three possibilities: (i) $\Sigma > \Sigma_2$ and condition (18) is not satisfied; (ii) $\Sigma_1 < \Sigma < \Sigma_2$ and condition (18) is satisfied when $\Omega < \bar{\Omega}$, where $\bar{\Omega}$ is obtained from the above equation; and (iii) $\Sigma < \Sigma_1$ and condition (18) is always satisfied.

B Equilibrium with bubbles

We first use (2), (3) and (4), to rewrite equations (20) and (21) as

$$b_{1,t} = \frac{(\beta + \beta^2)(\xi_1 w_t - a_{t+1}) - \frac{\xi_2 w_{t+1}}{R_{t+1}} - \left[q_{t+1} \phi_1 + \frac{q_{t+2}}{R_{t+2}} \phi_2 \right] \frac{a_{t+1}}{R_{t+1}}}{1 + \beta + \beta^2}, \quad (39)$$

$$b_{2,t+1} = \frac{\beta^2 \xi_2 w_{t+1} + \beta^2 q_{t+1} \phi_1 a_{t+1} + \beta^2 (\xi_1 w_t - a_{t+1}) R_{t+1} - (1 + \beta) \frac{q_{t+2}}{R_{t+2}} \phi_2 a_{t+1}}{1 + \beta + \beta^2}. \quad (40)$$

From using (5) and (6), equations (39) and (40) can be rewritten as

$$b_{1,t} = \frac{(\beta + \beta^2)(\xi_1(1 - \alpha) A k_t^\alpha - a_{t+1}) - \frac{\xi_2(1 - \alpha) A k_{t+1}^\alpha}{R_{t+1}} - \left[\alpha A k_{t+1}^{\alpha-1} \phi_1 + \frac{\alpha A k_{t+2}^{\alpha-1}}{R_{t+2}} \phi_2 \right] \frac{a_{t+1}}{R_{t+1}}}{1 + \beta + \beta^2}, \quad (41)$$

$$b_{2,t+1} = \frac{\beta^2 \xi_2 (1 - \alpha) A k_{t+1}^\alpha + \beta^2 \alpha A k_{t+1}^{\alpha-1} \phi_1 a_{t+1} + \beta^2 [\xi_1(1 - \alpha) A k_t^\alpha - a_{t+1}] R_{t+1} - \frac{(1 + \beta) \alpha A k_{t+2}^{\alpha-1} \phi_2 a_{t+1}}{R_{t+2}}}{1 + \beta + \beta^2} \quad (42)$$

We use (23) to rewrite (41) and (42) as (24) and (25) in the main text.

C Proof of Proposition 4

We recall that a bubble exists iff:

$$\Omega > \tilde{\Omega}(\Sigma) = \left(\frac{1 - \alpha}{\alpha} \right) (\Sigma_3 - \Sigma)$$

Note that $\tilde{\Omega}(\Sigma)$ is a strictly decreasing line ($\tilde{\Omega}'(\Sigma) < 0$), $\tilde{\Omega}(\Sigma_3) = 0$ and $\tilde{\Omega}(\Sigma') = 1$, with $\Sigma' = \frac{1 - \beta^2}{1 + \beta + \beta^2} + \frac{\alpha}{1 - \alpha}$.

We also recall that a bubble is productive iff $\Sigma < \Sigma_1$ or $\Sigma \in (\Sigma_1, \Sigma_2)$ and:

$$\Omega < \bar{\Omega}(\Sigma) = \left(\frac{\Sigma_2 - \Sigma}{\Sigma - \Sigma_1} \right) \left(\frac{1 + \beta^2}{\beta + \beta^2} \right) \left(\frac{1 - \Sigma}{\Sigma_2} \right)$$

It can be shown that $\Sigma_1 < \Sigma_2$, $\bar{\Omega}(\Sigma_1) = +\infty$, $\bar{\Omega}(\Sigma_2) = 0$ and $\bar{\Omega}(1) = 0$. Moreover,

$$\bar{\Omega}'(\Sigma) = \frac{1}{\Sigma_2(\Sigma - \Sigma_1)^2} \frac{1 + \beta^2}{\beta + \beta^2} [(\Sigma_1 - \Sigma_2)(1 - \Sigma) - (\Sigma_2 - \Sigma)(\Sigma - \Sigma_1)] < 0$$

for all $\Sigma \in (\Sigma_1, \min\{\Sigma_2; 1\})$. In addition,

$$\bar{\Omega}''(\Sigma) = \frac{1 + \beta^2}{(\beta + \beta^2)\Sigma_2} \left[\frac{2(\Sigma_2 - \Sigma_1)(1 - \Sigma)}{(\Sigma - \Sigma_1)^3} + \frac{2(\Sigma_2 - \Sigma_1)}{(\Sigma - \Sigma_1)^2} \right] > 0$$

for all $\Sigma \in (\Sigma_1, 1)$. Hence, $\bar{\Omega}(\Sigma)$ is a convex function, decreasing for all Σ such that $\bar{\Omega}(\Sigma) \geq 0$.

We further note that $\tilde{\Omega}(\Sigma_{b_1}) = \Omega_{b_2}$ and $\bar{\Omega}(\Sigma_{b_1}) = \Omega_{b_2}$. This means that $\tilde{\Omega}(\Sigma)$ and $\bar{\Omega}(\Sigma)$ crosses once at the point $(\Sigma, \Omega) = (\Sigma_{b_1}, \Omega_{b_2})$. Since $\tilde{\Omega}(\Sigma)$ is a line and $\bar{\Omega}(\Sigma)$ is convex, they cross at most twice.

We know that, on the one hand, $b_1 > 0$ if $\Sigma > \Sigma_{b_1}$ and $b_2 > 0$ if $\Omega > \Omega_{b_2}$ and, on the other hand, a bubble is productive if $\Omega < \bar{\Omega}(\Sigma)$. Since $\Omega = \bar{\Omega}(\Sigma)$ goes through $(\Sigma_{b_1}, \Omega_{b_2})$ and is a convex function, decreasing for all Σ such that $\bar{\Omega}(\Sigma) \geq 0$, a bubble cannot be productive if $b_1 > 0$ and $b_2 > 0$, whatever the values of Σ_{b_1} and Ω_{b_2} . Hence, a bubble is productive if either $b_1 < 0$ and $b_2 > 0$ or $b_1 > 0$ and $b_2 < 0$.

The existence of productive bubbles with $b_1 < 0$ and $b_2 > 0$ requires either $\Sigma_1 \geq \Sigma'$, which is equivalent to:

$$\frac{\alpha}{1 - \alpha} \geq \frac{(1 - \beta^2)(\beta + \beta^2)}{1 + \beta + \beta^2}$$

or $\Sigma_1 < \Sigma'$ and $\bar{\Omega}(\Sigma') > 1$, i.e.

$$\frac{(1 - \beta^2)(\beta + \beta^2)}{1 + \beta + \beta^2} > \frac{\alpha}{1 - \alpha} > \frac{(1 - \beta^2)(\beta + 2\beta^2)}{(1 + \beta + \beta^2)(2 + \beta)}$$

Moreover, $\Sigma' < 1$ if and only if:

$$\frac{\alpha}{1 - \alpha} < \frac{\beta + 2\beta^2}{1 + \beta + \beta^2}$$

When these inequalities are satisfied, there is a non-empty set of Σ such that there exists a productive bubble for $\Omega = 1$. By continuity, this result holds for $\Omega < 1$ but sufficiently close to 1. We deduce the existence of productive bubbles with $b_1 < 0$ and $b_2 > 0$ for $\frac{\alpha}{1 - \alpha} \in \left(\frac{(1 - \beta^2)(\beta + 2\beta^2)}{(1 + \beta + \beta^2)(2 + \beta)}, \frac{\beta + 2\beta^2}{1 + \beta + \beta^2} \right)$. This occurs if $\Sigma < \Sigma_{b_1}$.

To show the existence of productive bubbles with $b_1 > 0$ and $b_2 < 0$, we first prove that $\Sigma_3 < 1$ is equivalent to:

$$\frac{\alpha}{1 - \alpha} < \frac{\beta/2 + \beta^2}{1 + \beta + \beta^2}$$

and $\Sigma_3 < \Sigma_2$ iff:

$$\frac{\alpha}{1 - \alpha} > \frac{\beta^2}{1 + \beta + \beta^2}$$

If these two inequalities are satisfied, there is a non-empty interval for Σ such that there exists a productive bubble for $\Omega = 0$. By continuity, this result holds for $\Omega > 0$ but sufficiently close to 0. We deduce the existence of productive bubbles with $b_1 > 0$ and $b_2 < 0$ for $\frac{\alpha}{1-\alpha} \in \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$. This occurs if $\Sigma > \Sigma_{b_1}$.

We deduce the different cases of Proposition 4 comparing the two intervals $\left(\frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)}, \frac{\beta+2\beta^2}{1+\beta+\beta^2}\right)$ and $\left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$ and taking into account that $\max\left\{\frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)}, \frac{\beta^2}{1+\beta+\beta^2}\right\} < \frac{\beta/2+\beta^2}{1+\beta+\beta^2} < \frac{\beta+2\beta^2}{1+\beta+\beta^2}$.

D Equilibrium with bubbles and taxes

We use (28), (29) and (30) to rewrite equations (32) and (33) as

$$b_{2,t+1} = \frac{\beta^2(1-\tau_w^2)\xi_2 w_{t+1} + \beta^2(1-\tau_k)a_{t+1}q_{t+1}\phi_1 + \beta^2 R_{t+1}[(1-\tau_w^1)\xi_1 w_t - a_{t+1}] - (1+\beta)(1-\tau_k)a_{t+1}\frac{q_{t+2}\phi_2}{R_{t+2}}}{1+\beta+\beta^2}, \quad (43)$$

$$b_{1,t} = \frac{(\beta+\beta^2)[(1-\tau_w^1)\xi_1 w_t - a_{t+1}] - \frac{(1-\tau_w^2)\xi_2 w_{t+1}}{R_{t+1}} - \frac{(1-\tau_k)a_{t+1}}{R_{t+1}}\left(\frac{q_{t+2}\phi_2}{R_{t+2}} + q_{t+1}\phi_1\right)}{(1+\beta+\beta^2)}. \quad (44)$$

From using (5), (6) and (35), equations (43) and (44) can be rewritten as

$$b_{1,t} = \frac{\beta(1+\beta)(1-\tau_w^1)\xi_1(1-\alpha)Ak_t^\alpha - \frac{(1-\tau_w^2)\xi_2(1-\alpha)Ak_{t+1}^\alpha}{R_{t+1}}}{1+\beta+\beta^2} - a_{t+1},$$

$$b_{2,t+1} = \frac{\beta^2(1-\tau_w^2)\xi_2(1-\alpha)Ak_{t+1}^\alpha + \beta^2 R_{t+1}(1-\tau_w^1)\xi_1(1-\alpha)Ak_t^\alpha}{1+\beta+\beta^2} + a_{t+1}[(1-\tau_k)\phi_1\alpha Ak_{t+1}^{\alpha-1} - R_{t+1}].$$

In a steady state, $R = n$ and $a = n[(n\xi_1 + \xi_2)/(n\phi_1 + \phi_2)]k$. We deduce:

$$b_1 = (1-\alpha)Ak^\alpha \left[\frac{\beta(1+\beta)(1-\tau_w^1)\xi_1 - (1-\tau_w^2)\frac{\xi_2}{n}}{1+\beta+\beta^2} - \left(\frac{n\xi_1 + \xi_2}{n\phi_1 + \phi_2}\right) \frac{n}{(1-\alpha)Ak^{\alpha-1}} \right],$$

and

$$b_2 = (1-\alpha)Ak^\alpha \left[\frac{\beta^2[(1-\tau_w^2)\xi_2 + (1-\tau_w^1)n\xi_1]}{1+\beta+\beta^2} + \left(\frac{n\xi_1 + \xi_2}{n\phi_1 + \phi_2}\right) \frac{n[(1-\tau_k)\phi_1\alpha Ak^{\alpha-1} - n]}{(1-\alpha)Ak^{\alpha-1}} \right].$$

From (35), we also obtain:

$$Ak^{\alpha-1} = \left(\frac{n^2}{n\phi_1 + \phi_2}\right) \frac{1}{(1-\tau_k)\alpha}.$$

We use the last three equations to obtain (37) and (38) in the main text.

E Empirical strategy to obtain Σ and Ω

In this appendix we describe how the data in Table 2 on the distribution of gross labor and capital income by age has been obtained. The data sources used are the US census

and the Eurostat. US government census provides average income and total population in 2015 for the following age groups: young (age 25-44), middle age (age 45-64) and old (65 and over). Eurostat provides the same data in 2015 for the different European economies shown in Table 1 and for the following age groups: young (age 25-49), middle age (age 50-64) and old (65 and over). As the number of years people belong to each age group is different with the Eurostat data, we divide total income of each age group by the number of years individuals belong to each age group.¹¹ This normalization makes the different age groups comparable. From using these data, we obtain the total income of each age group and the total income of the economy is obtained as the sum of the income of each age group. The fraction of total income obtained by each age group is displayed in Table 1.

We use the labor income share and total income to obtain for each country the labor income and the capital income.¹² Consistent with the assumptions in the model, we assume that (i) the young individuals do not obtain capital income and (ii) the old individuals do not obtain labor income. Based on these assumptions, we obtain Σ as the ratio between the income of the young and the total labor income in the economy and we obtain $\widehat{\Omega}$ as the difference between one and the ratio between the income of the old and the total capital income of the economy. The values of Σ and $\widehat{\Omega}$ are displayed in Table 2.

The value of Σ and $\widehat{\Omega}$ are obviously biased because of the two aforementioned assumptions. To measure how problematic are these two assumptions, we use the US census data to obtain that the fraction of labor income obtained by the old individuals is only 4% and the net worth owned by the young is only 9.4%. These small numbers imply that the two assumptions are not too strong and, hence, the bias in the measures of Σ and $\widehat{\Omega}$ should be small.

A more serious problem with the data is that the income of the old also includes the pensions they receive, which should not be considered as capital income. Using the notation introduced in Section 5 and the definition of $\widehat{\Omega}$, we obtain

$$\widehat{\Omega}_t = 1 - \frac{q_{t+2}\phi_2 a_{t+1} + p_{t+2}}{q_{t+1}\phi_1 a_{t+1}n + q_{t+2}\phi_2 a_{t+1} + p_{t+2}},$$

where p_{t+2} are the pensions received by individuals when old. Note that $\widehat{\Omega}_t$ is the difference between one and the ratio between the income of the old and the total capital income. As follows from the data, pensions are included in the income of the old and also in the total income. $\widehat{\Omega}_t$ at the steady state simplifies as

$$\widehat{\Omega} = \frac{n\phi_1}{n\phi_1 + \phi_2 + \frac{p}{qa}},$$

where p is the steady state value of the pension. Let σ the replacement rate of pensions and, hence, $p = \sigma\xi_2 w$. Using the replacement rate, (5), (6), and (9), we obtain

$$\Omega = \widehat{\Omega} \left[1 + (1 - \Sigma) \frac{\sigma(1 - \alpha)}{\alpha n} \right],$$

¹¹We consider that 20 is the number of years individuals are old. This is approximately the value of the life expectancy at 65 in the economies considered.

¹²The labor income share in 2014 is obtained from the Penn World Table.

where $\Omega = n\phi_1 / (n\phi_1 + \phi_2)$ is the fraction of capital income obtained by the middle age individuals and that we have used in the main text of this paper. The previous equation clearly shows that $\widehat{\Omega}$ is a biased measure of the distribution of capital income by age group when pensions are introduced. In the last step of our empirical strategy, we use this equation to obtain the value of Ω . To this end, we must obtain the values of σ , α and n . The value of σ is obtained from OECD data set 2014, where the replacement rate is defined as the gross pension divided by the gross pre-retirement wage and, hence, it corresponds to our definition of σ . The value of α is obtained from the labor income share in the Penn World Table 2014 and the value of n is obtained from OECD data as the ratio between total population age 45-64 divided by total population age 65 and over. The value of Ω obtained from this analysis is displayed in the last column of Table 2.

F Figures and Tables

Table 1. Income distribution by age group¹³

	Young	Middle Age	Old
Austria	0,36	0,40	0,24
Belgium	0,38	0,40	0,22
Czech Republic	0,43	0,37	0,20
Denmark	0,36	0,42	0,22
Finland	0,35	0,42	0,23
France	0,33	0,39	0,28
Germany	0,36	0,40	0,24
Greece	0,37	0,36	0,27
Hungary	0,39	0,37	0,24
Italy	0,34	0,38	0,28
Netherlands	0,37	0,41	0,22
Norway	0,38	0,40	0,22
Poland	0,41	0,39	0,20
Portugal	0,37	0,37	0,26
Spain	0,39	0,37	0,24
Sweden	0,36	0,40	0,24
United Kingdom	0,40	0,38	0,22
United States	0,36	0,45	0,19

Source. US census data and Eurostat.

¹³The second column is the fraction of income obtained by young individuals, the third column is the fraction of income obtained by middle age individuals and the last column is the fraction of income obtained by old individuals.

Table 2. Income distribution by age group¹⁴

	Σ	$\hat{\Omega}$	Ω
Austria	0,62	0,44	0,55
Belgium	0,61	0,42	0,49
Czech Republic	0,84	0,60	0,63
Denmark	0,56	0,38	0,51
Finland	0,57	0,40	0,49
France	0,53	0,26	0,33
Germany	0,57	0,36	0,42
Greece	0,78	0,48	0,52
Hungary	0,65	0,41	0,48
Italy	0,63	0,39	0,47
Netherlands	0,61	0,44	0,56
Norway	0,72	0,54	0,59
Poland	0,74	0,55	0,58
Portugal	0,64	0,38	0,47
Spain	0,66	0,42	0,52
Sweden	0,63	0,45	0,55
United Kingdom	0,66	0,44	0,47
United States	0,60	0,52	0,58

Source. See Appendix E.

¹⁴ Σ is the fraction of labor income obtained by the young individuals. $\hat{\Omega}$ is the fraction of capital income obtained by the middle age individuals when pensions are considered part of the capital income of the old. Finally, Ω is the fraction of capital income obtained by middle age individuals when pensions are not considered as capital income of the old.

Table 3. Taxes and population growth¹⁵

	τ_w	τ_k	n
Austria	0,50	0,25	0,96
Belgium	0,55	0,34	0,98
Czech Republic	0,43	0,19	1,14
Denmark	0,36	0,23	0,94
Finland	0,44	0,20	0,89
France	0,48	0,38	0,97
Germany	0,49	0,30	0,84
Greece	0,39	0,26	1,13
Hungary	0,49	0,19	1,07
Italy	0,48	0,31	0,99
Netherlands	0,36	0,25	0,91
Norway	0,37	0,27	1,08
Poland	0,35	0,19	1,10
Portugal	0,42	0,29	1,02
Spain	0,40	0,28	1,16
Sweden	0,43	0,22	1,01
United Kingdom	0,31	0,20	1,05
United States	0,32	0,39	1,00

Source. OECD Data base.

¹⁵The population growth rate is obtained from the ratio between the population in the interval 25-44 years and the population in the interval 45 to 64. The population growth rate is obtained for all countries in the year 2013, except for Belgium, France, Greece, Netherlands and Poland that it is obtained in the year 2012.

Table 4. Results from the simulation

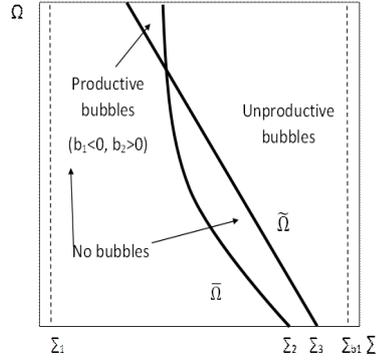
	k^*	k	F
Austria	0, 12	---	-0, 55
Belgium	0, 12	---	-0, 63
Czech Republic	0, 16	---	-0, 07
Denmark	0, 17	---	-0, 50
Finland	0, 16	---	-0, 65
France	0, 19	---	-0, 74
Germany	0, 19	---	-0, 70
Greece	0, 20	---	-0, 09
Hungary	0, 14	---	-0, 63
Italy	0, 16	---	-0, 50
Netherlands	0, 18	---	-0, 32
Norway	0, 16	---	-0, 05
Poland	0, 18	---	-0, 13
Portugal	0, 18	---	-0, 39
Spain	0, 15	---	-0, 27
Sweden	0, 14	---	-0, 46
United Kingdom	0, 23	---	-0, 30
United States	0, 15	0, 19	0, 01

Table 5. Results from the simulation

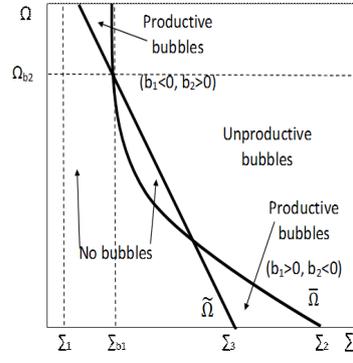
	k^*	k	F
Austria	0,18	0,22	0,05
Belgium	0,20	---	-0,05
Czech Republic	0,20	0,14	0,69
Denmark	0,18	---	-0,16
Finland	0,21	---	-0,14
France	0,28	---	-0,42
Germany	0,29	---	-0,21
Greece	0,23	0,19	0,42
Hungary	0,20	0,23	0,06
Italy	0,22	---	-0,01
Netherlands	0,19	0,23	0,04
Norway	0,18	0,17	0,34
Poland	0,19	0,17	0,38
Portugal	0,22	0,25	0,02
Spain	0,17	0,18	0,13
Sweden	0,18	0,21	0,07
United Kingdom	0,22	0,24	0,06
United States	0,15	0,19	0,01

We assume that $\tau_k = 0,39$ and $\tau_w = 0,32$

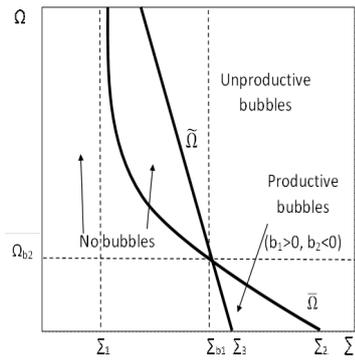
Figure 1. Bubbles and the distribution of income



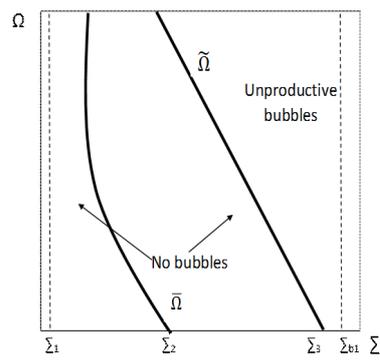
Panel a



Panel b

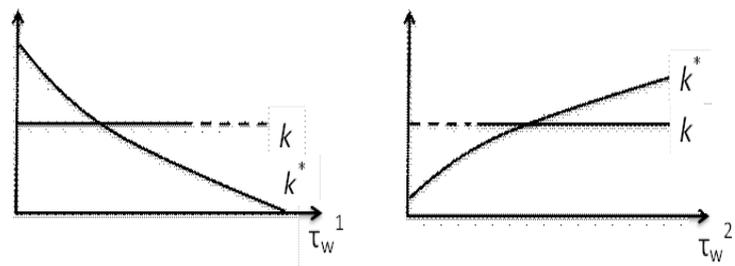


Panel c



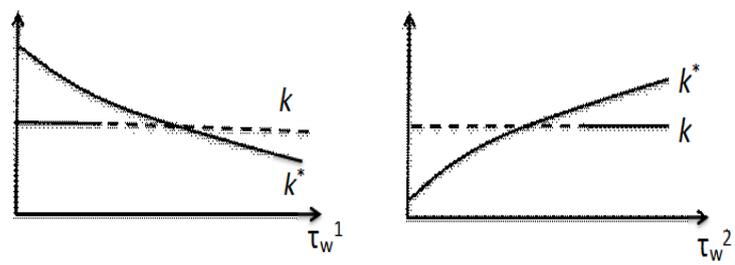
Panel d

Figure 2. The effect of fiscal policies on capital



Panel a

Panel b



Panel c

Panel d