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► **To cite this version:**

| Antonin Macé, Rafael Treibich. On the Weights of Sovereign Nations. 2018. halshs-01630090v3

HAL Id: halshs-01630090

<https://shs.hal.science/halshs-01630090v3>

Preprint submitted on 31 Jan 2018 (v3), last revised 19 Jun 2019 (v4)

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On the Weights of Sovereign Nations*

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January 31, 2018

Abstract

We study the design of voting rules for committees representing heterogeneous groups (countries, states, districts) when cooperation among groups is voluntary. While efficiency recommends weighting groups proportionally to their stakes, we show that accounting for participation constraints entails overweighting some groups, those for which the incentive to cooperate is the lowest. When collective decisions are not enforceable, cooperation induces more stringent constraints that may require granting veto power to certain groups. In the benchmark case where groups differ only in their population size (i.e, the apportionment problem), the model provides a rationale for setting a minimum representation for smaller groups.

JEL: F53, D02, C61, C73.

“You may if you wish go home from this Conference and say that you have defeated the veto. But what will be your answer when you are asked: Where is the Charter?”

U.S. Senator Tom Connally at the 1945 San Francisco Conference.

1 Introduction

In 1787, when the founding fathers met in Philadelphia to discuss the creation of a new constitution, the most contentious issue revolved around the composition of the future legislature.

*We would like to thank Alessandra Casella, Ernesto Dal Bó, Andreas Kleiner, Jean-François Laslier, Michel Lebreton, Aniol Llorente-Saguer, and Sharun Mukand for useful comments, as well as audiences from Barcelona, Berkeley, Copenhagen, Glasgow, Louvain-la-Neuve, Odense, Paris, Saint-Étienne, and Toulouse.

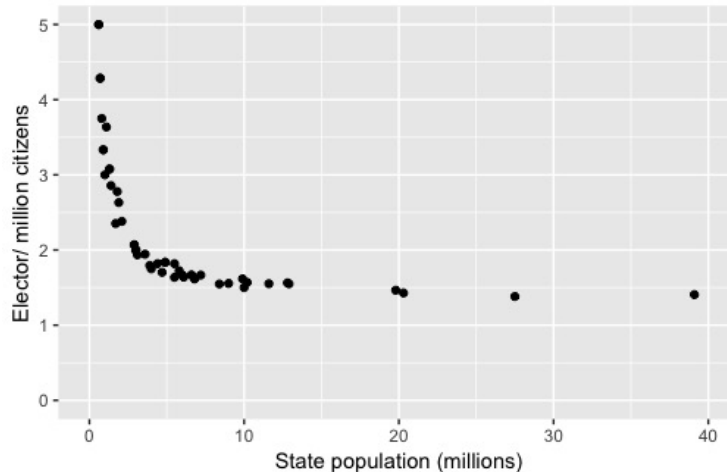
Larger states, led by Virginia, argued in favor of a bicameral legislature, under which states would receive a number of seats proportional to their population in both houses. Smaller states rejected the Virginia Plan, proposing instead the creation of a single house, under which states would receive an equal number of seats, independently of their population. The conflict was so serious that smaller states threatened to leave the union if larger states insisted on the idea of a purely proportional representation:

“The small ones would find some foreign ally of more honor and good faith, who will take them by the hand and do them justice.”

Gunning Bedford Jr., representative for Delaware, 1787.

The issue was resolved by the so-called *Connecticut Compromise*, and the creation of a bicameral legislature, under which states received a weight proportional to their population in the lower house (House of Representatives), but an equal weight in the upper house (Senate). The resulting distribution of seats in the Electoral College¹ appears as a compromise between the principle of “one man, one vote”, leading to efficient and democratic decision-making, and the principle of “one state, one vote”, ensuring the voluntary participation of all states. As illustrated in [Figure 1](#), this distribution is such that the number of electors per capita in each state decreases with the state’s population. In this article, we show why it may actually be optimal to assign weights in such a distorted manner when accounting for participation constraints.

Figure 1: Elector per million citizens in the U.S. Electoral College, by state population (2017).



¹The Electoral College is a representative committee designed to elect the U.S. president. The number of electors obtained by each state corresponds to the sum of its number of senators (2) and of its number of representatives in the House. In most states, electors are appointed through a winner-takes-all system.

The tension between the efficiency of a multi-party institution and its acceptability by all parties is not limited to the episode of the Constitutional Convention. In fact, such a tension is also inherently present for many intergovernmental organizations and confederations, when a group of *sovereign* states voluntarily commits to collectively decide on one or several policy areas. One prominent example is the Council of the European Union (EU), one of the EU's main decision-making bodies, whose decision rule has often been changed after fierce debates. The Treaty of Lisbon established the latest rule: a reform is adopted if it is approved by at least 55% of the Member States representing at least 65% of the EU population.² This rule again exhibits a compromise between proportionality and equality, which ensures its acceptability by the smallest Member States. Another important example is the UN Security Council, in which the five permanent members can veto any resolution and consequently benefit from a disproportionate power. When the Charter of the UN was ratified in San Francisco in 1945, “the issue was made crystal clear by the leaders of the Big Five: it was either the Charter with the veto or no Charter at all” (Wilcox, 1945). In that case, the acceptability of the rule entailed not only overweighting some countries, but ensuring them veto power.

The goal of this article is to study the design of voting rules for committees representing heterogeneous groups (countries, states, districts) when cooperation among groups is voluntary. Should groups be weighted according to their populations, or should small groups be overweighted? Should some important groups benefit from veto power? To address these questions, we take a second-best approach to institutional design by looking for normatively appealing rules among those that are politically feasible.

Our model features a fixed set of countries choosing whether to delegate some of their competences to a supranational entity.³ The choice to transfer a competence is made unanimously *ex ante*, before countries learn about their preferences over future decisions.⁴ If cooperation is agreed upon, decisions are made collectively according to a predetermined voting rule. If cooperation is rejected, countries remain sovereign and make their own decisions.

Our core assumption is that the choice to delegate reflects a trade-off between the efficiency gains from cooperation and a reduced control over decisions. Making collective decisions is profitable for many reasons: it generates coordination externalities (Loeper, 2011), allows for economies of scale (Alesina et al., 2005), increases bargaining power (Moravcsik, 1998),

²The rule applies for most policy areas. Additionally, a proposal cannot be blocked by less than four Member States.

³For instance, the European Union has exclusive competence over customs unions, competition policy, monetary policy (for countries in the Eurozone), common fisheries policy, and common commercial policy. The EU also holds shared competence (Member States cannot exercise competence in areas where the EU has done so) over various other domains, such as the internal market, agricultural policy, environmental policy, and consumer protection. See Treaty of Lisbon (2007b).

⁴The fact that all EU competences must be voluntarily transferred by its Member States is known as the principle of *conferral* (Treaty of Lisbon, 2007a).

strengthens commitment (Bown, 2004), etc. However, by forfeiting the right to make their own decisions, countries also lose some decision power. As a result, countries may reject cooperation if they expect to disagree too frequently with the collective decision. The voting rule, which determines how much influence each country exerts on the collective decision, thus plays a critical role in generating cooperation.

We consider in turn the cases of enforceable and non-enforceable collective decisions. When decisions are enforceable, countries commit to accepting the outcome of these decisions even if they end up disagreeing. In the model, countries enter a decision game in which they first choose whether to cooperate or to remain sovereign, and then, in case of cooperation, make a collective binary decision (i.e., to approve or reject a proposed reform). Crucially, countries must decide whether to cooperate before learning their preferences about the reform. We show that under enforceable decisions cooperation can be established (at equilibrium) if the voting rule satisfies a set of participation constraints. This leads us to study a constrained optimization problem: determining which rule delivers the highest (ex ante, utilitarian) welfare, subject to the participation constraint of all countries. Our first result asserts that the optimal rule is a weighted majority rule, whereby each country is assigned a fixed voting weight and a reform is adopted if the total weight of favorable countries exceeds a certain threshold. Furthermore, we show that the weight of a country is equal to its stake in the collective decision unless its participation constraint is binding, in which case it is larger. This result thus offers a justification for overweighting countries that have the lowest (endogenous) incentive to participate.

If collective decisions cannot be enforced, as is often the case in intergovernmental organizations (Maggi and Morelli, 2006), countries may decide not to comply with collective decisions ex post. In that case, looking at the previous one-shot game is not sufficient, since countries have no incentive to abide by collective decisions if the game ends right away. We thus consider the infinitely repeated version of the previous game where countries decide at each stage whether to cooperate and, in case of cooperation, whether to implement the collective decision. We say that self-enforcing cooperation is achieved if countries choose to cooperate at each stage and always comply with collective decisions on the equilibrium path. We show that a rule is self-enforcing if it satisfies a set of endogenous constraints: countries with veto power must satisfy their participation constraint, while countries without veto power must satisfy a more stringent “compliance” constraint. Building on this result, we show that optimal self-enforcing rules take the form of weighted majority rules. Countries for which the compliance constraint is not satisfied are strictly overweighted (relative to their stakes) and have veto power. Countries for which the compliance constraint is binding are weakly overweighted. Finally, countries for which the compliance constraint is satisfied but

not binding are not overweighted and do not have veto power. The result thus provides a rationale for the use of veto power. Compliance can sometimes be best achieved by giving some “negative power” to a country (i.e., veto power) rather than by compensating it with too much additional “positive power” (i.e., overly large weight).

Finally, we consider a simpler model in which utilities are binary and countries differ only in their population size, but are otherwise (ex ante) identical. This model allows us to address the classic problem of *apportionment*: how should countries’ populations be translated into voting weights of representatives in an international committee? We obtain sharper results in that model. Countries must receive weights proportional to their populations, except for the smallest ones, which must all be weighted equally. The result thus offers a rationale for a minimum representation of smaller countries, as required explicitly in the Treaty of Lisbon ([Treaty of Lisbon, 2007a](#)). It also echoes the distribution of weights in the U.S. Electoral College, where each state is allocated a baseline of two seats plus a number of seats proportional to its population. Then, we investigate the implication of self-enforceability in this simplified model and show that it never leads to recommending veto power for a subset of countries: either the rule must be a weighted majority rule with no veto or it must be the unanimity rule. We further show that the former case prevails when the efficiency gain is high and/or the discount factor is high. We also establish that, in that case, the minimum representation threshold decreases with the efficiency gain and the discount factor.

1.1 Related Literature

Our article combines both a normative and a positive approach to voting rules in representative committees. On the normative side, we follow the literature on *apportionment*, which studies the allocation of weights to nations (states) of different sizes in international unions (federations). A first branch of the literature focuses on how to best approximate proportionality when weights are constrained to be integers, such as for the allocation of seats in a parliament ([Balinski and Young, 1982](#)). A second branch of the literature questions the desirability of proportionality, arguing instead in favor of a principle of *degressive proportionality*, which requires weights to increase less than proportionally to states’ populations.⁵ Our

⁵The literature on degressive proportionality has focused in particular on the *square-root law*, which recommends weights that are proportional to the square-root of each state’s population. Arguments in favor of the *square-root law* are developed by [Penrose \(1946\)](#), [Felsenthal and Machover \(1999\)](#), and [Barberà and Jackson \(2006\)](#), on the grounds of (respectively) equalizing each citizen’s influence, minimizing the mean *majority deficit* (extent of disagreement with the federation-wise majority rule), and following the utilitarian principle. These works are extended by [Beisbart and Bovens \(2007\)](#) and [Kurz et al. \(2017\)](#), who show the fragility of the law to the introduction of a small degree of correlation in citizen’s preferences. Finally, [Koriyama et al. \(2013\)](#) offer a different rationale for degressive proportionality based on the utilitarian principle when citizens exhibit decreasing marginal utility. See [Laslier \(2012\)](#) for a survey.

article follows this second strand, building in particular on the utilitarian approach⁶ proposed by Barberà and Jackson (2006) to study voting rules in two-tier democracies, where citizens elect representatives that vote on their behalf. They show in a general framework that an efficient voting rule must weight each state proportionally to its *stake* in the collective decisions.⁷ Depending on the assumption made on the correlation of preferences within states, the stake of a state coincides either with its population or with the square root of its population.

We depart from this literature by adding political feasibility constraints. The premise is that countries' decision to cooperate is voluntary. Starting with the same assumption, but inspired by the formation of monetary unions, Casella (1992) shows that a two-country partnership may require overweighting (in the welfare function of the partnership's decision-maker) the country most tempted to remain sovereign. Our first result on enforceable decisions generalizes her argument to committees with more than two countries by analyzing this trade-off in a voting game. Barberà and Jackson (2004) also follow a positive approach to the design of voting rules, but their focus is on the stability of the voting rule with respect to a constitutional change: a voting rule is *self-stable* if it cannot be overthrown by another voting rule. In contrast, we study the stability of a rule with respect to the composition of the union and require that a rule induces the cooperation of all of its members. Note that the optimal rules and optimal self-enforcing rules that we identify are self-stable⁸ among those satisfying the same feasibility constraints, since they are obtained from a welfare-maximization program. The assumption of enforceable decisions is relaxed in a pioneering article by Maggi and Morelli (2006), who consider a union of homogeneous countries engaging in repeated collective decisions. They prove that the optimal self-enforcing rule is either the (efficient) qualified majority rule or the unanimity rule. Our section on self-enforcing voting rules extends their analysis to the case of a heterogeneous union. In particular, we show that the optimal self-enforcing rule may give veto power to a strict subset of countries. In the apportionment model, veto power is given to all countries or none, but the optimal self-enforcing rule may be neither the (efficient) qualified majority rule nor the unanimity rule, for intermediate values of the discount factor.

Finally, a central assumption in our article is that a country's decision to cooperate results from a trade-off between the efficiency of collective decisions and the loss of power in the union. Following the seminal article of Alesina and Spolaore (1997) on the (endogenous) size of nations, several articles have explored this rationale for cooperation between countries.

⁶The ex-ante utilitarian approach to binary voting rules was initiated by Rae (1969) to provide an argument for the majority rule.

⁷A similar result is provided by Azrieli and Kim (2014) in a mechanism design context. See also Brighthouse and Fleurbaey (2010) for a discussion of this idea at the level of political philosophy.

⁸With respect to the unanimity rule, taken as the benchmark constitutional rule.

Alesina et al. (2005) study the composition and size of international unions when efficiency gains stem from externalities in public good provisions. Renou (2011) studies the effect of the stringency of the supermajority rule on the endogenous composition of the union. Similar to Renou (2011), our article emphasizes the importance of the voting rule on the stability of the union, but differs in that we take into account the heterogeneity of countries. Finally, let us note that some authors have provided other rationales for international cooperation, such as information aggregation (Penn, 2016), or even pure preference aggregation (Crémer and Palfrey, 1996).

1.2 Example

Consider a union of five countries which must decide, repeatedly, whether to impose embargoes on tax havens. A sanction is only effective if implemented by all countries. Countries are uncertain about whether to support the embargoes. Country 1 is generally unfavorable and has a probability $1/3$ of supporting a sanction, while countries 2 to 5 are generally favorable and have a probability $2/3$ of supporting the sanctions. Preferences are independent across countries and across decisions.⁹ Ex post, if the embargo is effective, country 1 gets a utility of 1 if it is favorable and a disutility of 2 if it is unfavorable. In contrast, countries 2 to 5 get a utility of 2 if they are favorable and a disutility of 1 if they are unfavorable. If the embargo is not effective, all countries get a utility of 0. Countries' preferences are summarized in the table below.

	Probability of support	Utility if Favorable + Embargo Effective	Utility if Unfavorable + Embargo Effective	Utility if Embargo not Effective
Country 1	$1/3$	1	-2	0
Countries 2, 3, 4, 5	$2/3$	2	-1	0

Table 1: Countries' Preferences.

Before preferences over future decisions are realized, countries must decide whether to sign a cooperation treaty (i.e., agree to take all embargo decisions collectively with all other countries in the union) or remain sovereign (i.e., take all embargo decisions independently of other countries). The treaty is only effective if signed by all countries in the union and is assumed enforceable.

⁹Which means we can focus, without loss of generality, on a single (representative) collective decision.

Under sovereignty, the embargo is implemented effectively only when all countries are favorable, which happens with a small probability of $16/3^5 \approx 0.07$. Ex ante, country 1 gets a utility of $U_1^\emptyset = 16/3^5$, while all other countries get a utility of $U_{2,3,4,5}^\emptyset = 32/3^5$ (from any single decision). Social welfare is equal to $144/3^5$.

In contrast, under cooperation, the embargo may be implemented effectively even if some countries are unfavorable since they must all accept the collective decision. Ex ante, the utility of each country depends both on the preferences of other countries and on the decision rule used to make the collective decision after preferences have realized.

Because all countries have the same stake in the collective decision, the efficient voting rule consists in adopting the embargo by simple majority ([Theorem 1](#)).¹⁰ Ex ante, countries 2 to 5 get a utility of:

$$U_{2,3,4,5}^e = \frac{2}{3} \times 2 \times \mathbb{P}(\text{emb. adopted} \mid \text{fav.}) - \frac{1}{3} \times 1 \times \mathbb{P}(\text{emb. adopted} \mid \text{unfav.}) = \frac{228}{3^5} > \frac{32}{3^5},$$

and are thus much better off than under sovereignty. In contrast, country 1, which tends to disagree with the four other countries, is now much worse off:

$$U_1^e = \frac{1}{3} \times 1 \times \mathbb{P}(\text{emb. adopted} \mid \text{fav.}) - \frac{2}{3} \times 2 \times \mathbb{P}(\text{emb. adopted} \mid \text{unfav.}) = -\frac{120}{3^5} < \frac{16}{3^5},$$

which means it would not agree to cooperate ex ante. The only way to ensure cooperation is to give some additional voting power to country 1. The optimal decision rule¹¹ ([Theorem 2](#)) consists in *overweighting* country 1 just enough, so that its participation constraint becomes binding: the embargo is adopted either if country 1 and at least one other country are in favor or if all but country 1 are in favor. This voting rule can be represented as a weighted majority rule with weights $\mathbf{w}^* = (3, 1, 1, 1, 1)$ and threshold $t^* = 1/2$.¹² Country 1 gets exactly its stand-alone utility, $U_1^* = 16/3^5 = U_1^\emptyset$, while countries 2 to 5 now get a reduced utility of $U_{2,3,4,5}^* = 146/3^5$. Social welfare is reduced from $792/3^5$ (under the efficient decision rule) to $600/3^5$, but still much larger than under sovereignty.

If collective decisions cannot be enforced, countries may choose not to implement the embargo even if it has been approved collectively. In that case, looking at the previous one-shot game is not sufficient since countries have no incentive to abide by collective decisions if the

¹⁰[Theorem 1](#) asserts that any efficient rule is represented by $[\mathbf{w}^e; t^e]$, where $\mathbf{w}^e = (1, 1, 1, 1, 1)$ and $t^e = 6/15$. When two countries are favorable to the embargo, there is a tie, and it is easy to see that the tie-breaking rule can be chosen arbitrarily (see, for instance, the proof of [Theorem 1](#) in [Barberà and Jackson \(2006\)](#)). Therefore, the simple majority rule is efficient (ties are resolved by blocking a proposal supported by only two countries).

¹¹All optimal rules and optimal self-enforcing rules mentioned in the article have been checked by an algorithm. When the union is divided in two groups of identical countries, the algorithm finds an optimal (resp. optimal self-enforcing) rule among rules that are symmetric within groups. This is without loss of generality, by linearity of the problem. The algorithm is available upon request.

¹²According to [Theorem 2](#), the optimal rule is represented by $[\mathbf{w}^*; t^*]$ with $t^* = 10/21$. We note that the rule is equally well represented by $[\mathbf{w}^*; 1/2]$.

game ends right away. We thus consider the infinitely repeated version of that game, where countries decide at each stage whether to cooperate and, in case of cooperation, whether to implement the collective decision. Let $\delta = 5/6$ be the common discount factor. In order for the voting rule to be self-enforcing (i.e., induce cooperation and compliance on the equilibrium path), the benefit of not implementing the embargo for unfavorable countries must be outweighed by the long-term cost of not sustaining cooperation. The associated compliance constraints turn out to be more stringent than the participation constraints ([Proposition 2](#)). As a result, the previous optimal rule cannot be self-enforcing since country 1's participation constraint is already binding. Here, self-enforcement can only be achieved by granting veto power to country 1 ([Theorem 3](#)). The optimal self-enforcing voting rule is such that the embargo is adopted if and only if country 1 and at least two other countries are in favor. This voting rule can be represented as a weighted majority rule with weights $\mathbf{w}^{**} = (3, 1, 1, 1, 1)$ and threshold $t^{**} = 2/3$. Country 1 gets a utility of $U_1^{**} = 72/3^5 > U_1^\emptyset$, while countries 2 to 5 now get an even more reduced utility of $U_{2,3,4,5}^{**} = 84/3^5$. Social welfare is reduced from $600/3^5$ (under the optimal rule) to $408/3^5$. Note that even though collective decisions cannot be enforced, social welfare is still much larger under the optimal self-enforcing rule than under sovereignty. The following table summarizes the rules and utilities obtained in each of the four considered benchmarks (to simplify the expressions, utilities are multiplied by a factor 3^5).

Benchmark	Sovereignty	Efficient	Optimal	Optimal Self-Enforcing	
Symbol	\emptyset	e	*	**	
Voting rule		Simple Majority	1 overweighted	1 overweighted + veto	
w_1		1	3	3	
$w_{2,3,4,5}$		1	1	1	
t		1/2	1/2	2/3	
U_1	\propto	16	-120	16	72
$U_{2,3,4,5}$	\propto	32	228	146	84
Welfare	\propto	144	792	600	408

Table 2: Summary of example.

1.3 Outline

Section 2 introduces the model and the decision game. **Section 3** derives the optimal voting rule when collective decisions are enforceable. **Section 4** introduces an infinitely repeated version of the decision game and derives the optimal self-enforcing rule. Then the model is applied in **Section 5** to a simple environment in which utilities are binary and countries differ only in their populations. **Section 6** concludes. All the proofs are gathered in **Section 7**.

2 Model

An international union N is made of n countries. Each country has a representative who takes decisions on behalf of its citizens. Representatives must decide whether to remain sovereign or to cooperate, and in the latter case, whether to implement a reform or to stick with the status quo. This is modeled as a game with four stages.

2.1 The Decision Game

In the first stage, each country $i \in N$ decides to remain sovereign, $d_i = 0$, or to cooperate, $d_i = 1$. If at least one country wants to remain sovereign, cooperation is aborted (the game ends), and each country i derives a stand-alone utility $U_i^\emptyset \in \mathbb{R}$. If all countries decide to cooperate, the game continues, and countries have to make a collective decision on the adoption of a proposed reform.¹³

In the second stage, countries learn the realization of their preferences for the proposed reform. A vector of utilities $u = (u_i)_{i \in N}$ is drawn from a distribution μ . The number u_i measures country i 's aggregate utility if the reform is adopted by all countries. The utilities are drawn independently across countries,¹⁴ and such that for all $i \in N$, $\mathbb{P}_\mu[u_i > 0] > 0$, $\mathbb{P}_\mu[u_i = 0] = 0$ and $\mathbb{P}_\mu[u_i < 0] > 0$. Each country i privately observes its own utility u_i , and the prior μ is common knowledge. If the reform is not adopted by all countries, each country derives a utility of 0.

The third stage is a voting stage. Each country reports a message $m_i \in \{0, 1\}$, where $m_i = 1$ is interpreted as a vote in favor of the reform, and $m_i = 0$ is interpreted as a vote against the reform. The collective decision to adopt the reform is made according to a

¹³Equivalently, one could assume that countries have to make repeated *independent* collective decisions. We assume a single decision for ease of exposition.

¹⁴The independence assumption is ubiquitous in the literature. It emphasizes the conflict of preferences across countries that is central to the model, and it allows for a tractable framework. Note that, if arbitrary patterns of correlation are allowed, the efficient rule may not be weighted.

predetermined voting rule v . To keep the model flexible, we define a voting rule as a non-decreasing function $v : \{0, 1\}^N \rightarrow [0, 1]$, where $v(\mathbf{m})$ denotes the probability of accepting the reform, given the vector of messages \mathbf{m} .¹⁵ We denote by \mathcal{V} the set of all such voting rules and by $\hat{v}(\mathbf{m}) \in \{0, 1\}$ the realized collective decision; i.e., a random variable $\hat{v}(\mathbf{m})$ such that $\mathbb{P}[\hat{v}(\mathbf{m}) = 1] = v(\mathbf{m})$. For a given profile of votes \mathbf{m} , $\hat{v}(\mathbf{m}) = 0$ indicates that countries must keep the status quo and $\hat{v}(\mathbf{m}) = 1$ means that countries must implement the reform.

In the fourth stage, each country i takes an action $a_i \in \{0, 1\}$, taking the value 1 if country i implements the reform, and 0 otherwise. If collective decisions are enforceable, each country must abide by the collective decision, $a_i = \hat{v}(\mathbf{m})$ for all $i \in N$. If collective decisions are not enforceable, then countries may choose to go against the collective decision.

The game thus defined is denoted by $\Gamma_E(v)$ if decisions are enforceable and by $\Gamma_{NE}(v)$ if decisions are not enforceable. In the game $\Gamma_E(v)$, a strategy for $i \in N$ is a vector $s_i = (d_i, m_i)$, with $d_i \in \{0, 1\}$ and $m_i : \mathbb{R} \rightarrow \{0, 1\}; u_i \mapsto m_i(u_i)$. In the game $\Gamma_{NE}(v)$, a strategy for $i \in N$ is a vector $s_i = (d_i, m_i, a_i)$, with $d_i \in \{0, 1\}$, $m_i : \mathbb{R} \rightarrow \{0, 1\}; u_i \mapsto m_i(u_i)$ and $a_i : \mathbb{R} \times \{0, 1\}^N \times \{0, 1\} \rightarrow \{0, 1\}; (u_i, \mathbf{m}, \hat{v}(\mathbf{m})) \mapsto a_i(u_i, \mathbf{m}, \hat{v}(\mathbf{m}))$.

In this article, we particularly focus on the *cooperative profile* of the game; i.e., the profile of strategies such that, for all $i \in N$, $d_i = 1$, $m_i = \mathbf{1}_{u_i > 0}$ and $a_i = \hat{v}(\mathbf{m})$. The expected aggregate utility of country i associated with this profile is given by:

$$U_i(v) = \mathbb{E}_\mu \left[v((\mathbf{1}_{u_j > 0})_{j \in N}) u_i \right].$$

The main objective of the article is to identify conditions for which this cooperative profile can be implemented as an equilibrium. [Section 3](#) tackles this question when decisions are enforceable, and [Section 4](#) studies the non-enforceable case. Before incorporating such strategic constraints, we introduce the notions of weighted rules, vetoes, welfare, and (first-best) efficient voting rules.

2.2 Weighted Majority Rules and Vetoes

In practice, decision rules used by international committees often take the form of a weighted majority whereby each country is assigned a fixed voting weight and a reform is approved if the total weight of countries in favor exceeds a given threshold (e.g., IMF or Council of the EU before 2014). Formally, a rule v is a *weighted majority rule* if there exist a vector of weights $\mathbf{w} = (w_i)_{i \in N} \in \mathbb{R}^N$, and a threshold $t \in [0, 1]$ such that, for any profile of votes

¹⁵This expression allows for probabilistic decisions, in order to break possible ties. See [Koriyama et al. \(2013\)](#) for an introduction of this class of voting rules, labeled *probabilistic simple games*.

$$\mathbf{m} = (m_i)_{i \in N} \in \{0, 1\}^N,$$

$$\left\{ \begin{array}{l} \sum_{i|m_i=1} w_i > t \sum_{i \in N} w_i \Rightarrow v(\mathbf{m}) = 1 \\ \sum_{i|m_i=1} w_i < t \sum_{i \in N} w_i \Rightarrow v(\mathbf{m}) = 0. \end{array} \right.$$

We say that rule v is weighted and can be represented by the system $[\mathbf{w}; t]$.¹⁶ Whether weighted or not, some rules mechanically grant veto power to some countries (e.g., UN Security Council). Formally, we say that a country $i \in N$ has veto power under rule v if $v(\mathbf{m}) = 0$ whenever $m_i = 0$. We denote by $VE(v) \subseteq N$ the set of countries having veto power under the rule v :

$$VE(v) = \left\{ i \in N \mid m_i = 0 \Rightarrow v(\mathbf{m}) = 0 \right\}.$$

2.3 Welfare and Efficient Voting Rule

For any voting rule v , we define the welfare associated with the cooperative profile under v as:

$$W(v) = \mathbb{E}_\mu \left[v((\mathbf{1}_{u_j > 0})_{j \in N}) \sum_{i \in N} u_i \right] = \sum_{i \in N} U_i(v).$$

We say that a rule is *efficient* if it achieves the maximum welfare at the cooperative profile; that is, absent any incentive constraint. Following the analysis of [Barberà and Jackson \(2006\)](#), it is useful to define country i 's expected utility from a favorable reform, $w_i^+ = \mathbb{E}_\mu[u_i | u_i > 0]$, and its expected disutility from an unfavorable reform, $w_i^- = -\mathbb{E}_\mu[u_i | u_i < 0]$. From these two numbers, we define country i 's *stake* in the decision as $w_i^e = w_i^+ + w_i^-$, and its *efficient threshold* as $t_i^e = w_i^- / w_i^e$.

Theorem 1. ([Barberà and Jackson, 2006](#); [Azrieli and Kim, 2014](#)) *Any efficient voting rule v^e is a weighted majority rule. It is represented by $[\mathbf{w}^e; t^e]$, where the threshold t^e is defined by:*

$$t^e = \frac{\sum_{i \in N} w_i^e t_i^e}{\sum_{i \in N} w_i^e}.$$

The result asserts that the efficient rule is essentially unique, in the sense that any efficient rule is represented by the same system of weights $[\mathbf{w}^e; t^e]$, although the tie-breaking rule may differ between two efficient rules.¹⁷ Therefore, we will refer to w_i^e as country i 's *efficient*

¹⁶Note that the definition is agnostic with respect to the tie-breaking rule. Note also that the representation of v may not be unique, even after re-scaling the weights \mathbf{w} by a common factor.

¹⁷Note some subtleties associated with the weight representation. First, there may be other systems of weights

weight, and the threshold t_i^e is efficient in the sense that it is the threshold of an efficient rule if all countries have the same “efficient threshold”. Note that the result focuses on first-best efficiency and that the associated cooperative profile may not be an equilibrium of the decision game. Incorporating such constraints is the main goal of our article and is the object of the following two sections. In what follows, we assume, without substantial loss of generality, that no country has veto power under the efficient rule (*Assumption NEV*, for no efficient veto).¹⁸

3 Enforceable Decisions

We start the analysis by considering the case where decisions prescribed by the voting rule v are enforceable: each country $i \in N$ commits to follow the action plan $a_i = \hat{v}(\mathbf{m})$ for any realization of the messages \mathbf{m} , whenever they cooperate. We are interested in voting rules that induce cooperation (in the first stage) at equilibrium.

Proposition 1. *The cooperative profile is a perfect Bayesian equilibrium of the game $\Gamma_E(v)$ if and only if each country satisfies the participation constraint: $U_i(v) \geq U_i^0$ for all $i \in N$.*

We denote by $\mathcal{PC} \subseteq \mathcal{V}$ the set of voting rules satisfying the participation constraints.

3.1 Optimal Voting Rules

We look for voting rules maximizing social welfare when participation is voluntary. We say that a voting rule is *optimal* if it is a solution of the maximization problem $\max_{v \in \mathcal{PC}} W(v)$.¹⁹ The following theorem describes optimal voting rules.

Theorem 2. *There exists a system of weights $[\mathbf{w}^*; t^*]$ such that any optimal voting rule v^* is weighted and represented by $[\mathbf{w}^*; t^*]$. Countries for which the participation constraint is binding are overweighted relative to their efficient weight, while countries for which the participation constraint is not binding receive their efficient weight.*

Similar to Theorem 1, the results asserts that the optimal rule v^* is essentially unique, in the sense that any optimal rule is represented by the same system of weights $[\mathbf{w}^*; t^*]$. However, contrary to the efficient rule, the optimal rule is such that countries that do not strictly benefit from cooperation may receive more than their efficient weight. We say that these countries

$[\mathbf{w}'; t']$ such that efficient rules are represented by $[\mathbf{w}'; t']$. Second, there may be weights $[\mathbf{w}'; t']$ such that some efficient rules are represented by these weights and some other are not.

¹⁸Formally, we require that $\forall i \in N, w_i^- < \sum_{j \neq i} w_j^+$ (*Assumption NEV*).

¹⁹Note that the existence of a solution is guaranteed, as the objective function is linear and the set of voting rules \mathcal{PC} is a closed subset of $[0, 1]^{2^N}$.

are *overweighted*. In contrast, countries that get strictly more than their stand-alone utility receive their efficient weight. Formally, there exists a system $[\mathbf{w}^*; t^*]$, such that any optimal rule v^* is represented by $[\mathbf{w}^*; t^*]$, with, for each country $i \in N$:

$$\begin{cases} U_i(v^*) = U_i^\emptyset & \Rightarrow w_i^* \geq w_i^e \\ U_i(v^*) > U_i^\emptyset & \Rightarrow w_i^* = w_i^e, \end{cases}$$

and where the threshold t^* is the associated weighted average of countries' efficient thresholds:

$$t^* = \frac{\sum_{i \in N} w_i^* t_i^e}{\sum_{i \in N} w_i^*}.$$

At one extremity, if stand-alone utilities are low enough, all countries are willing to cooperate under the efficient voting rule. In that case, the constraints are inoperative, and the efficient rule coincides with the optimal rule. However, as stand-alone utilities become larger, the constraint starts to bind for some countries. The result asserts that, in comparison to the efficient benchmark, these countries should be overweighted, and that the threshold t^* should be closer to their efficient thresholds. This is illustrated in the example of section [Section 1.2](#), where the optimal voting rule, represented by $[(3, 1, 1, 1, 1); 1/2]$, is such that country 1 is overweighted, while countries 2 to 5 get their efficient weight. Country 1's utility $16/3^5$ is equal to its stand-alone utility, while countries 2 to 5's utility $146/3^5$ is larger than their stand-alone utility $32/3^5$.

In contrast with the efficient voting weights, which can be computed independently for each country, the optimal voting weights cannot be obtained separately since they each depend on the complete probability distribution μ and on the vector of stand-alone utilities $(U_i^\emptyset)_{i \in N}$. A country may be overweighted at the optimum if it gains relatively little from cooperation or if it often disagrees with the (efficient) collective decision (as in the example of [Section 1.2](#)). The level of heterogeneity across countries, both in stakes and preferences, thus plays a crucial role in determining the optimal rule.

Inducing all countries to cooperate may turn out costly if some countries do not benefit enough from cooperation or if they disagree too often with the (endogenous) collective decision. Mechanically, the *cost of participation*, the loss of welfare from having to satisfy the participation constraints,²⁰ increases with each country's stand-alone utility: decreasing a country's stand-alone utility means relaxing its participation constraint, and thus improving the welfare reached at the optimal rule. However, understanding the effect of other aspects of the model (such as the probability distribution μ) on the cost of participation is more difficult

²⁰That is, the difference in welfare between the efficient and the optimal rules.

due to the simultaneous effect on the participation constraints and on the efficient decision rule. This ambiguous interplay may lead to counter-intuitive effects. For example, an increase in the efficiency of cooperation may actually increase the cost of cooperation. Consider, for instance, a situation where the efficient decision rule is optimal, and assume that the stake of one country increases (thus increasing the overall efficiency of cooperation). As the new efficient rule weights this country more, other countries whose (ex ante) preferences are opposite to the first country's may end up with a reduced utility. Such countries may then require some additional voting power to cooperate, thus leading to an increase in the cost of participation (from zero to positive). Similarly, an increase in the degree of preference homogeneity may actually increase the cost of participation. Again, starting from a situation where the efficient rule satisfies the participation constraints, raising the homogeneity of preferences may change the efficient voting rule, leading one country's participation constraint to be violated.²¹ A more homogeneous union may thus induce a larger cost of participation.

4 Non-Enforceable Decisions

We have assumed so far that collective decisions were fully enforceable under cooperation. In fact, enforceability is a major concern for most international organizations, as countries always retain some form of sovereignty and full enforceability is never really achieved. Following Maggi and Morelli (2006), we thus relax the assumption of enforceability and consider an infinitely repeated version of our decision game where countries must repeatedly decide whether to cooperate and, if so, whether to respect the collective decision. In that framework, we show that inducing self-enforcing cooperation is more difficult than inducing cooperation under enforceability. Then, we characterize the optimal self-enforcing rule, which occasionally entails giving veto power to some countries, but not necessarily all.

4.1 Repeated Game

When decisions are not enforceable, considering the one-shot game $\Gamma_{NE}(v)$ is not sufficient, since countries have no incentive to abide by collective decisions in the fourth stage of the game if the game ends right away. A notion of self-enforcing cooperation can instead be introduced if we repeat the decision game. We thus consider the δ -discounted infinitely repeated game $\Gamma_{NE}^\delta(v)$. At each stage $T \in \mathbb{N}$, each country $i \in N$ decides whether to

²¹Consider, for example, a union of three countries, and assume that the simple majority rule is both efficient and optimal. The probability of favoring the reform is $1/2$ for country 1, $q \in (1/2, 1)$ for country 2, and 1 for country 3. As q increases, the union is more homogeneous, as the probability of any two (or three) countries agreeing is either constant or increasing. However, as U_1 decreases with q (the efficient rule is independent of q , and q only affects the probability of approving the reform when 1 is unfavorable), country 1 may require to be overweighted for high q , and this leads to a positive cost of participation.

participate or not, $d_i^T \in \{0, 1\}$. Preferences for the reform proposed at stage T , \mathbf{u}^T , are drawn from μ , independently of the previous stages. Each country $i \in N$ reports a message $m_i^T \in \{0, 1\}$, observes the action plan $\hat{v}^T(\mathbf{m}^T)$, and takes an action $a_i^T \in \{0, 1\}$, which can differ from $\hat{v}(\mathbf{m}^T)$. At each stage, \mathbf{d}^T , \mathbf{m}^T , $\hat{v}^T(\mathbf{m}^T)$, and \mathbf{a}^T are publicly observed. All countries are characterized by the same discount factor $\delta \in (0, 1]$.

For a given value of the discount factor δ , we say that a voting rule v is *self-enforcing* if there exists a perfect public equilibrium²² of $\Gamma_{NE}^\delta(v)$ such that the cooperating profile is played at each stage of the game on the equilibrium path. To construct such an equilibrium, we consider the profile of strategies for which each country follows the cooperative strategy absent any deviation and ceases to cooperate forever after any (publicly observed) deviation by a single country i , of the form $d_i^T = 0$ or $a_i^T \neq \hat{v}^T(\mathbf{m}^T)$ for some T .

We observe that, under such a profile, a deviation is profitable for a country when it is unfavorable to a reform approved by the committee. In that case, a deviation yields a short-term benefit for not complying at the current stage in addition to the stand-alone utility at the subsequent stages. Compared to the one-shot game, the repeated game thus creates an extra incentive to leave the union, which can only be mitigated by giving veto power to the country tempted to exit. To measure this new temptation to deviate, we define the maximal disutility that country i may suffer from a collective decision:

$$w_i^D = - \min \left\{ w \in \mathbb{R} \mid \mathbb{P}_\mu(u_i = w) > 0 \right\}.$$

Note that $w_i^D \geq w_i^- > 0$. We say that a country $i \in N$ satisfies the *compliance constraint* if

$$U_i(v) \geq U_i^\emptyset + \frac{1 - \delta}{\delta} w_i^D.$$

Proposition 2. *A voting rule v is self-enforcing if and only if each country either has veto power and satisfies the participation constraint, or does not have veto power and satisfies the compliance constraint.*

We denote by \mathcal{SE} the set of self-enforcing rules. The result establishes the equivalence between the notion of self-enforceability and a set of *endogenous* constraints. Indeed, the constraint that country i should satisfy under rule v is contingent on i having veto power under v . Moreover, we observe that the compliance constraints are more stringent than the participation constraints. As a result, if a voting rule is self-enforcing then it also satisfies the participation constraints. Note that the extreme case $\delta = 1$ coincides with the model of enforceable decisions.

²²The notion of *public perfect equilibrium* is a generalization of subgame perfection for games of incomplete information, commonly employed to analyze games of the type of $\Gamma_{NE}^\delta(v)$ as, for instance, in [Athey and Bagwell \(2001\)](#) or [Maggi and Morelli \(2006\)](#).

4.2 Optimal Self-Enforcing Rules

We say that the voting rule v is *optimal self-enforcing* if it maximizes social welfare among self-enforcing rules; i.e., if it is a solution of $\max_{v \in \mathcal{SE}} W(v)$. From Proposition 2, we immediately get that social welfare is lower under an optimal self-enforcing rule than under an optimal voting rule since $\mathcal{SE} \subseteq \mathcal{PC}$. The following theorem describes optimal self-enforcing rules.

Theorem 3. *For any optimal self-enforcing voting rule v^{**} there exists a system of weights $[\mathbf{w}^{**}; t^{**}]$ such that v^{**} is weighted and represented by $[\mathbf{w}^{**}; t^{**}]$. Countries for which the compliance constraint is not satisfied are strictly overweighted and have veto power. Countries for which the compliance constraint is binding are weakly overweighted. Countries for which the compliance constraint is satisfied but not binding receive their efficient weight and do not have veto power.*

Formally, for any optimal self-enforcing rule v^{**} , there exists a system $[\mathbf{w}^{**}; t^{**}]$ representing v^{**} , such that for all $i \in N$:

$$\begin{cases} U_i(v^{**}) < U_i^\emptyset + \frac{1-\delta}{\delta} w_i^D & \Rightarrow w_i^{**} > w_i^e \text{ and } i \in VE(v^{**}), \\ U_i(v^{**}) = U_i^\emptyset + \frac{1-\delta}{\delta} w_i^D & \Rightarrow w_i^{**} \geq w_i^e, \\ U_i(v^{**}) > U_i^\emptyset + \frac{1-\delta}{\delta} w_i^D & \Rightarrow w_i^{**} = w_i^e \text{ and } i \notin VE(v^{**}), \end{cases}$$

and where the threshold t^{**} satisfies:

$$t^{**} \geq \frac{\sum_{i \in N} w_i^{**} t_i^e}{\sum_{i \in N} w_i^{**}},$$

with an equality if no country has veto power, and a strict inequality otherwise.

Theorem 3 differs from **Theorem 2** in two main respects. First, the benchmark level of utility U_i^\emptyset that separates overweighted countries from non-overweighted countries is increased by an additional $(1-\delta)w_i^D/\delta$.²³ Countries that fall strictly below this augmented utility threshold are strictly overweighted, while countries that fall strictly above receive their efficient weight. Second, in contrast to **Theorem 2**, the benchmark utility also separates countries that benefit from veto power from countries that do not. This is illustrated in the example of **Section 1.2**, where the optimal self-enforcing rule grants veto power to country 1, but not to countries 2 to 5. Country 1's utility, $72/3^5 \approx 0.30$, falls below its augmented utility threshold of $16/3^5 + 2/5 \approx 0.47$, while countries 2 to 5's utility, $84/3^5 \approx 0.35$, falls above their augmented utility threshold of $32/3^5 + 1/5 \approx 0.33$. The fact that optimal self-enforcing rules

²³When $\delta = 1$, this additional term equals 0 so that $\mathcal{SE} = \mathcal{PC}$.

may grant veto power to only a strict subset of countries is a major difference to [Maggi and Morelli \(2006\)](#), in which either all countries have veto power or no country has veto power, and this stems from the generality of our model, which allows for heterogeneous countries.²⁴

5 A Model of Apportionment

In this section, we consider a more particular model of apportionment where utilities are binary and countries differ only in their population size, which allows for more tractable results. The specification of preferences in this simplified model follows [Koriyama et al. \(2013\)](#).

5.1 Model

We start by describing a more specific version of the model introduced in [Section 2](#). Under sovereignty, each country now chooses independently which reforms to implement. In each country, for any given reform, a (randomly chosen) fraction $q > 1/2$ of citizens agrees with the reform. Citizens who are favorable get a utility of 1, while citizens who are unfavorable get a disutility of 1. Ex ante, the aggregate (stand-alone) utility of country i with population p_i is thus equal to:

$$U_i^0 = qp_i - (1 - q)p_i = (2q - 1)p_i.$$

Under cooperation, proposals are determined exogenously. Ex ante, each country's representative has a probability 1/2 of agreeing with any of the proposed reforms, independently of each other. In each country, for any given reform, a fixed fraction $q > 1/2$ of citizens agrees with the opinion of its country's representative. If the reform ends up being implemented effectively (by all countries), favorable citizens get a utility of e , while unfavorable citizens get a disutility of e . The parameter e can be interpreted as a per-capita efficiency gain from cooperation, and we assume that $e > 1$.²⁵ If the reform is not adopted effectively, all citizens

²⁴Note that the possibility of having only a strict subset of veto countries at an optimal self-enforcing rule does not hinge on countries having biased preferences (as assumed in the example of [Section 1.2](#)). For example consider $N = \{1, 2, 3, 4, 5\}$, μ such that $\mathbb{P}_\mu(u_1 = 2) = \mathbb{P}_\mu(u_1 = -2) = \mathbb{P}_\mu(u_{2-5} = 1) = \mathbb{P}_\mu(u_{2-5} = -1) = 1/2$, $U_1^0 = 0.8$, and $U_{2-5}^0 = 0$. Then for $\delta = 0.95$, an optimal self-enforcing rule is such that a proposal is accepted (i) with probability 1 whenever country 1 and at least two of the remaining countries are in favor and (ii) with probability 0.45 whenever country 1 and one of the remaining countries are in favor. That voting rule gives country 1 veto power.

²⁵Note that cooperation is assumed to increase the utility from a favorable reform *and* the disutility from an unfavorable one, by the same factor e , consistent with the view that the collective action goes further in the desired/undesired direction. In a previous version of the article, it was assumed that the disutility of an unfavorable reform was multiplied by a factor e^- , below or above 1. With that alternative (and more general) assumption, the subsequent [Theorem 4](#) remains valid, with a suitable adaptation of the threshold of the optimal rule.

get a utility of 0. The probability distribution μ associated with this model is such that:

$$\forall i \in N, \quad \mathbb{P}_\mu(u_i = (2q - 1)ep_i) = \mathbb{P}_\mu(u_i = -(2q - 1)ep_i) = \frac{1}{2}.$$

The efficient weight of country $i \in N$ is thus given by $w_i^+ = w_i^- = w_i^e/2 = (2q - 1)ep_i$, and its efficient threshold is $t_i^e = 1/2$.

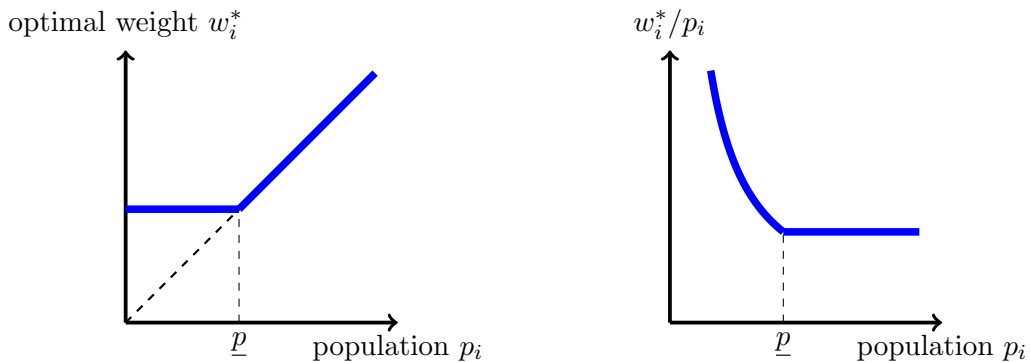
In that more specific model, countries thus vary in their stakes w_i^e , which are proportional to their population p_i , but are otherwise identical.²⁶ In particular, they have the same ex-ante probability of agreeing with a given reform, equal to 1/2. Note that Assumption NEV boils down to $p_i < (\sum_{j \in N} p_j)/2$ for all $i \in N$, which means that any country accounts for less than half of the total population.

5.2 Optimal Voting Rules

We now obtain sharper predictions for the optimal voting rule: first, overweighted countries are those with the smallest populations; and second, these countries must be given the same voting weight.

Theorem 4. *In the model of apportionment, there exists $\underline{p} \in \mathbb{R}$ such that any optimal voting rule is a weighted majority rule represented by $[\mathbf{w}^*; 1/2]$, with $w_i^* = \max(p_i, \underline{p})$ for all $i \in N$.*

Figure 2: Optimal weights (absolute and per capita) in the model of apportionment



The optimal apportionment rule is illustrated in Figure 2, in absolute and per-capita terms. We first note that the distribution of weights is *degressively proportional*: weights increase with countries' populations (left panel's curve is increasing), but less than proportionally (right

²⁶Note that we have assumed that the per-capita efficiency gain from cooperation, e , is the same for all countries. This assumption is maintained throughout the section to keep the interpretation of the results simple. However, with varying gains e_i , all the results remain valid by replacing p_i by $e_i p_i$ in the subsequent [Theorem 4](#) and [Theorem 5](#).

panel’s curve is decreasing). A sizable literature on apportionment has already argued in favor of this property,²⁷ but on different grounds than the one we put forth here. In particular, our argument focuses on the bottom of the distribution and supports overweighting small countries that may otherwise have almost no say in the collective decisions. By contrast, previous models recommend degressively proportional rules that have noticeable implications for medium to larger states, often with weights in the order of p^α with $1/2 \leq \alpha \leq 1$.²⁸

The requirement that smaller countries shall be given a minimal and equal representation is actually found explicitly in the Treaty of Lisbon, which specifies a set of constraints for the composition of the European Parliament.²⁹ Indeed, article 14.2 states that “representation of citizens [in the European Parliament] shall be degressively proportional, with a minimum threshold of six members per Member State” (Treaty of Lisbon, 2007a). Our article thus offers a theoretical rationale for such a minimal representation threshold.

Finally, the apportionment formula proposed here combines in a simple manner the notions of proportionality and equality, which is reminiscent of several prominent examples. In particular, the overweighting of smaller states echoes the distribution of seats in the U.S. Electoral College,³⁰ and the optimal weights per capita observed in the right panel of Figure 2 mirror the actual ones exhibited in Figure 1. The eight smaller states are allocated the same number of three seats,³¹ representing 4.5% of the seats for only 1.9% of the total population. The same type of apportionment formula has also been proposed for the allocation of seats in the European Parliament, under the name of the *Cambridge Compromise*.³²

5.3 Optimal Self-Enforcing Rules

We also obtain sharper predictions for the optimal self-enforcing rule: either no country has veto power or all countries have it, and we can map these two cases on a graph parametrized by the per-capita efficiency gain e and the discount factor δ .

Theorem 5. *In the model of apportionment, any optimal self-enforcing rule is either the unanimity rule or a weighted majority rule for which no country has veto power. There exist a threshold $\underline{e} > 0$, and two non-increasing functions $\delta^c, \delta^{eff} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that*

²⁷Laslier (2012) offers a review of the different arguments in favor of such rules.

²⁸For instance, in the model of Barberà and Jackson (2006), the optimal α is approximately equal to 1/2 in the *fixed-size block model*, and equal to 1 in the *fixed-number-of-blocks model*. See also Beisbart and Bovens (2007).

²⁹For a discussion of the application of the model to the allocation of seats in a federal parliament, rather than voting weights in a federal council, see Koriyama et al. (2013).

³⁰Each state is allocated a baseline of two electors plus a number of electors proportional to its population.

³¹Alaska, Delaware, District of Columbia, Montana, North Dakota, South Dakota, Vermont, and Wyoming.

³²The Cambridge Compromise was the result of an academic initiative by the European Parliament, which aimed at formulating a transparent and fair allocation of the seats in the European Parliament. The proposed allocation is based on a similar base + prop formula as in the U.S. Electoral College, whereby each country is allocated a base of six seats plus a number of seats proportional to its population. See Grimmert (2012).

$\delta^c(e) \leq \delta^{eff}(e)$ for all $e \in \mathbb{R}_+$, $\lim_{e \rightarrow \infty} \delta^{eff}(e) < 1$, and:

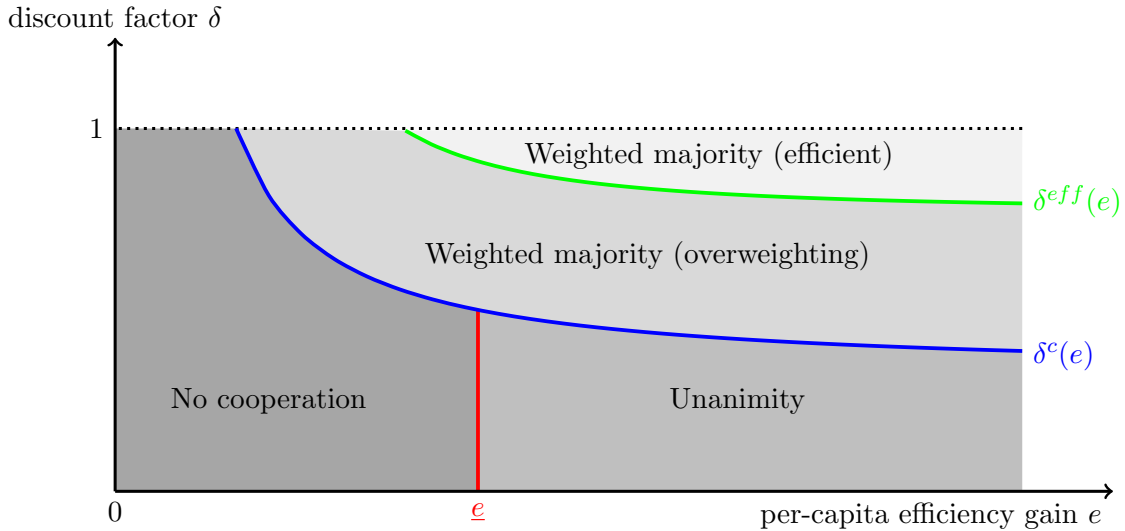
- (i) if $\delta \geq \delta^{eff}(e)$, any optimal self-enforcing rule is an efficient weighted majority rule,
- (ii) if $\delta^c(e) \leq \delta < \delta^{eff}(e)$, any optimal self-enforcing rule is a weighted majority rule, with overweighting of small countries,
- (iii) if $\delta < \delta^c(e)$ and $e \geq \underline{e}$, the optimal self-enforcing rule is the unanimity rule,
- (iv) if $\delta < \delta^c(e)$ and $e < \underline{e}$, there is no self-enforcing rule.

Moreover, for $\delta^c(e) \leq \delta < \delta^{eff}(e)$, there exists a minimal weight $\underline{p}(e, \delta)$, non-increasing in both e and δ , such that any optimal self-enforcing rule is represented by $[\mathbf{w}^{**}; 1/2]$, defined by:

$$w_i^{**} = \max \left(p_i, \underline{p}(e, \delta) \right) \quad \text{for all } i \in N.$$

Theorem 5 defines four regions in the space (e, δ) that yield different (or no) optimal self-enforcing rules, as represented in **Figure 3** below.

Figure 3: Optimal self-enforcing rule in the model of apportionment



The figure can be interpreted either horizontally or vertically. First, the line $\delta = 1$ depicts the results we obtain for enforceable decisions. If the per-capita efficiency gain e is too small, there is no rule inducing cooperation. If the per-capita efficiency gain e is large enough, the efficient voting rule induces cooperation and is therefore optimal. However, for intermediate values of e , the optimal rule involves overweighting small countries, and the extent to which small countries are overweighted decreases with e .

Reading [Figure 3](#) vertically reveals how [Theorem 5](#) extends the main result of [Maggi and Morelli \(2006\)](#). In that article, countries are homogeneous, and there exists a threshold $\bar{\delta}$, below which the optimal self-enforcing rule is the unanimity, and above which the optimal self-enforcing rule is the (efficient) majority rule. In our model, for $e \geq \underline{e}$, there are two thresholds: $\delta^{eff}(e)$ and $\delta^c(e)$. As in the homogeneous model, the efficient rule is the optimal self-enforcing rule when the discount factor is high ($\delta \geq \delta^{eff}(e)$), and the unanimity rule is the optimal self-enforcing rule when the discount factor is low ($\delta < \delta^c(e)$). What is new here is that we obtain a region of intermediate values of the discount factor ($\delta^c(e) \leq \delta < \delta^{eff}(e)$), for which the optimal self-enforcing rule is a weighted majority rule with overweighting of small countries. Moreover, the extent to which small countries are overweighted decreases with the per-capita efficiency gain e and with the discount factor δ .

6 Conclusion

We have studied the design of voting rules in representative committees when decisions are binary and cooperation is voluntary. In contrast to the efficient voting rule, which assigns to each country a voting weight proportional to its stake, the optimal voting rule sometimes involves overweighting certain countries, namely those that have the lowest *endogenous* incentive to cooperate. In the apportionment case, where the heterogeneity reduces to the population size and countries have identical ex-ante preferences, the optimal voting rule assigns an equal (and larger than proportional) weight to smaller countries. The theory thus provides a new rationale for the use of degressive proportional apportionment rules. When collective decisions are not enforceable, and cooperation must be agreed on repeatedly, voting rules must satisfy stronger compliance constraints, thus reducing social welfare. At the optimum, countries that do not satisfy compliance constraints benefit from veto power. In the apportionment case, the optimal self-enforcing rule is either unanimous (thus giving veto power to all countries), or such that no country enjoys veto power.

We have assumed throughout the article that cooperation was only beneficial if all countries participated, the “pure” collective action case ([Maggi and Morelli, 2006](#)). Note that all of our results on enforceable decisions remain valid for more general forms of cooperation gains. Indeed, as participation is decided at the unanimity in our model, either all countries cooperate or all remain sovereign. Nevertheless, inducing the cooperation of all countries may prove costly, as in our main example, in which the welfare drops by 24% from the efficient to the optimal rule. When the cooperation gains are “impure”, this cost may be alleviated by allowing a strict subset of countries to cooperate while the others remain sovereign. This flexible form of cooperation, known as *enhanced cooperation*, was first introduced by the Euro-

pean Union in the Treaty of Amsterdam (1997).³³ It allows a subset of nine or more countries to move forward with cooperation without the consent of the other countries.

Our model can be extended to study the design of voting rules under such flexible forms of cooperation. The first stage of the game is replaced by a simultaneous move game where countries decide whether to cooperate or not. Cooperation only follows with the subset of countries that have decided to cooperate in the first stage, while the other countries remain sovereign and make their own independent decisions. The stand-alone utility of sovereign countries is not necessarily fixed, as it may depend both on the coalition of cooperating countries and on the particular voting rule used under such coalition. The problem now consists in designing a *constitution*, i.e. a menu of voting rules (one for each possible coalition of participating countries) to be used in case of cooperation. The optimal constitution,³⁴ and the associated voting rule used by the participating coalition at equilibrium, may differ substantially from the optimal voting rule under rigid cooperation, for which only the grand coalition can form. Social welfare under such flexible cooperation is necessarily larger than under rigid cooperation and may even, in some cases, reach the same level as under the efficient voting rule.

Assume, for instance, in the example of [Section 1.2](#) that the embargo becomes effective if only four countries decide to implement it (instead of five). Everything else stays identical. Under sovereignty, country 1 gets a utility of $-16/3^5$, while countries 2 to 5 get a utility of $136/3^5$, yielding a social welfare of $528/3^5$. Country 1 remains unwilling to cooperate (with the grand coalition) under the efficient voting rule, since its ex-ante utility $U = -120/3^5$ is still inferior to its stand-alone utility.³⁵ Inducing the participation of all countries still requires overweighting country 1. Similar to the original example, the optimal voting rule can be represented as a weighted majority rule with weights $(3, 1, 1, 1, 1)$ and a threshold of $1/2$, but with a slightly different tie-breaking rule.³⁶ Country 1 gets exactly its stand-alone utility, $U = -16/3^5$, while countries 2 to 5 now get a reduced utility of $U = 166/3^5$ (instead of $228/3^5$ under the efficient voting rule). Social welfare is reduced from $792/3^5$ (under the efficient voting rule) to $648/3^5$.

If enhanced cooperation is allowed, then a strict subset of countries may cooperate while

³³Its use has been further extended by the treaties of Nice in 2001 and Lisbon in 2007. It has been used to establish a European regulation for divorce law (16 participating countries as of Dec. 2017), a European patent with unitary effect (25 countries as of Dec. 2017) and most recently a common legislation for the property regimes of international couples (18 countries as of Dec. 2017).

³⁴An optimal constitution could be defined as a constitution for which the welfare at a welfare-maximizing equilibrium is maximal.

³⁵Note that the efficient voting rule remains identical to the one in the original example since decisions are enforceable, which means either all countries or no country implements the embargo. The assumption that only 4 countries implementing the embargo is sufficient for the embargo to be effective does not make a difference.

³⁶Such that if only three of the countries 2 to 5 are in favor, the embargo is implemented with probability $1/4$ (instead of 0).

the others remain sovereign. A constitution specifies a voting rule for each possible coalition of cooperating countries. Here, the optimal constitution is such that collective decisions are made by simple majority with a threshold of $1/3$ among cooperating countries. At the associated equilibrium only countries 2 to 5 choose to participate, while country 1 remains sovereign. Countries 2 to 5 get a utility of $U = 252/3^5$, while country 1 gets a utility of $U = -216/3^5$, yielding the efficient level of welfare $W = 792/3^5$, which is substantially larger than the welfare under the optimal voting rule, $W = 648/3^5$. In contrast to the optimal voting rule under rigid cooperation, where countries 2 to 5 bear the cost of inducing country 1's cooperation, it is now country 1 that suffers from the cooperation of countries 2 to 5. This example shows how allowing for enhanced cooperation may improve social welfare (to the point of full efficiency) if cooperation gains are not of the "pure" collective action type.

We have taken a "constrained" normative approach where a benevolent social planner looks for the welfare-maximizing voting rule while accounting for the effect of the voting rule on countries' participation and compliance. A natural alternative would be to consider a fully positive approach where countries bargain over the voting rule. The first participation stage in our game (as described in [Section 2](#)) could then be replaced by a bargaining stage where the voting rule (to be used in the subsequent stages) is determined by the following Nash program:

$$\max_{v \in \mathcal{V}} \prod_{i \in N} (U_i(v) - U_i^\emptyset)$$

The rest of the game would follow in the same fashion. In contrast to the original model, the voting rule is not chosen exogenously so as to maximize social welfare but is obtained endogenously as the outcome of a Nash bargaining between countries in the initial stage. Using a similar technique as in the proof of [Theorem 1](#), one can show that under enforceable decisions the outcome is a weighted voting rule such that countries that are closer to their stand-alone utility under the bargained voting rule are relatively more overweighted.³⁷ The result captures a similar intuition as [Theorem 1](#). Countries that gain less from cooperation (relative to their outside option) under the bargained voting rule are more overweighted. For instance, in the example of [Section 1.2](#), the bargained rule can be represented as a weighted majority rule with weights $(4, 1, 1, 1, 1)$ and a threshold of $1/2$.³⁸ Country 1's utility increases from $16/3^5$ (under the optimal voting rule) to $38.8/3^5$, while the utility of countries 2 to 5 decreases from $146/3^5$ to $123.2/3^5$. Social welfare decreases from $600/3^5$ to $531.6/3^5$. More

³⁷Denoting by v^* the bargained voting rule, for any two countries $i, j \in N$: $U_i(v^*) - U_i^\emptyset > U_j(v^*) - U_j^\emptyset$ implies $w_i^*/w_i^\emptyset < w_j^*/w_j^\emptyset$.

³⁸And such that the embargo is approved with probability 0.65 if all but country 1 are in favor and with probability 0.4 if only country 1 is in favor.

generally, overweighted countries under the optimal voting rule are necessarily better off under the bargained voting rule since their utility under the optimal voting rule is equal to their stand-alone utility. However, social welfare is necessarily lower since the bargained voting rule satisfies all participation constraints. Finally, note that the bargained voting rule is more difficult to compute theoretically because it is defined as the solution of a non-linear maximization problem.

References

- Alesina, A., Angeloni, I., and Etro, F. (2005). International unions. *American Economic Review*, 95(3):602–615.
- Alesina, A. and Spolaore, E. (1997). On the number and size of nations. *The Quarterly Journal of Economics*, 112(4):1027–1056.
- Athey, S. and Bagwell, K. (2001). Optimal collusion with private information. *RAND Journal of Economics*, pages 428–465.
- Azrieli, Y. and Kim, S. (2014). Pareto efficiency and weighted majority rules. *International Economic Review*, 55(4):1067–1088.
- Balinski, M. L. and Young, H. P. (1982). *Fair representation: Meeting the ideal of one man, one vote*. Yale University Press.
- Barberà, S. and Jackson, M. O. (2004). Choosing how to choose: Self-stable majority rules and constitutions. *The Quarterly Journal of Economics*, 119(3):1011–1048.
- Barberà, S. and Jackson, M. O. (2006). On the weights of nations: Assigning voting power to heterogeneous voters. *Journal of Political Economy*, 114(2):317–339.
- Beisbart, C. and Bovens, L. (2007). Welfarist evaluations of decision rules for boards of representatives. *Social Choice and Welfare*, 29(4):581–608.
- Bown, C. (2004). On the economic success of gatt/wto dispute settlement. *The Review of Economics and Statistics*, 86:811–823.
- Brighouse, H. and Fleurbaey, M. (2010). Democracy and proportionality. *Journal of Political Philosophy*, 18(2):137–155.
- Casella, A. (1992). Participation in a currency union. *American Economic Review*, 82(4):847–63.

- Crémer, J. and Palfrey, T. R. (1996). In or out? centralization by majority vote. *European Economic Review*, 40(1):43–60.
- Felsenthal, D. S. and Machover, M. (1999). Minimizing the mean majority deficit: The second square-root rule. *Mathematical Social Sciences*, 37(1):25–37.
- Grimmett, G. R. (2012). European apportionment via the cambridge compromise. *Mathematical Social Sciences*, 63(2):68–73.
- Koriyama, Y., Laslier, J.-F., Macé, A., and Treibich, R. (2013). Optimal apportionment. *Journal of Political Economy*, 121(3):584–608.
- Kurz, S., Maaser, N., and Napel, S. (2017). On the democratic weights of nations. *Journal of Political Economy*, 125(5):1599–1634.
- Laslier, J.-F. (2012). Why not proportional? *Mathematical Social Sciences*, 63(2):90–93.
- Loeper, A. (2011). Coordination in heterogeneous federal systems. *Journal of Public Economics*, 95(7):900–912.
- Maggi, G. and Morelli, M. (2006). Self-enforcing voting in international organizations. *The American Economic Review*, 96(4):1137–1158.
- Moravcsik, A. (1998). *The choice for Europe: social purpose and state power from Messina to Maastricht*. Cornell University Press.
- Penn, E. M. (2016). Engagement, disengagement, or exit: A theory of equilibrium associations. *American Journal of Political Science*, 60(2):322–336.
- Penrose, L. S. (1946). The elementary statistics of majority voting. *Journal of the Royal Statistical Society*, 109(1):53–57.
- Rae, D. W. (1969). Decision-rules and individual values in constitutional choice. *American Political Science Review*, 63(1):40–56.
- Renou, L. (2011). Group formation and governance. *Journal of Public Economic Theory*, 13(4):595–630.
- Treaty of Lisbon (2007a). Consolidated version of the treaty on european union - title i: Common provisions.
- Treaty of Lisbon (2007b). Consolidated version of the treaty on the functioning of the european union - part one: Principles.

Wilcox, F. O. (1945). The yalta voting formula. *American Political Science Review*, 39(5):943–956.

7 Appendix: Proofs

7.1 Proof of **Proposition 1**

In the game $\Gamma_E(v)$, the only relevant beliefs are given by the prior μ at the first stage, and by $\mu(u_i, \cdot)$ for a player i at the third stage. Thus, the only condition to check to see if the cooperating profile is a perfect Bayesian equilibrium is that of sequential rationality.

First, it is clear that sending $m_i = \mathbb{1}_{u_i > 0}$ is always rational in the third stage, as v is non-decreasing. Second, playing $d_i = 1$ at the first stage is rational if and only if the expected outcome of the game under cooperation is no worse than under sovereignty (obtained if $d_i = 0$). The result follows.

7.2 Proof of **Theorem 2**

In this proof, and in the subsequent proofs, we abuse notation and write $v(M)$ for $v(\mathbf{m})$, where $M \subseteq N$ denotes the coalition of countries that vote in favor of the proposal: $M = \{i \in N | m_i = 1\}$.

Let $\mathcal{V}^* = \arg \max_{v \in \mathcal{PC}} W(v)$ be the set of optimal rules. The set \mathcal{V}^* is convex: for any $v, v' \in \mathcal{V}^*$ and $\alpha \in (0, 1)$, it is clear that $\alpha v + (1 - \alpha)v' \in \mathcal{V}^*$, by linearity of the objective function and of the constraints that define \mathcal{V}^* . Let v^* be an optimal rule such that:

$$v^* \in \arg \max \#\{M \subseteq N | v(M) \in (0, 1)\}.$$

In words, the rule v^* maximizes the recourse to probabilistic tie-breaking among optimal rules. We claim that, if v^* is represented by $[\mathbf{w}; t]$, then any optimal rule $v' \in \mathcal{V}^*$ is also represented by $[\mathbf{w}; t]$. To see that, assume that v^* is represented by $[\mathbf{w}; t]$, but that $v' \in \mathcal{V}^*$ is not. Then, without loss of generality, we can assume that there exists $M \subseteq N$, such that $\sum_{i \in M} w_i > t \sum_{i \in N} w_i$, but $v'(M) < 1$. We have by assumption $v^*(M) = 1$. We obtain a contradiction by considering the rule $v'' = (v^* + v')/2$, as $v'' \in \mathcal{V}^*$, but $\#\{M \subseteq N | v''(M) \in (0, 1)\} > \#\{M \subseteq N | v^*(M) \in (0, 1)\}$. In the remainder of the proof, we solely focus on v^* , as we know that any representation of v^* will be a representation of any optimal rule.

We claim that v^* is also a solution of the following maximization problem (note that v is not constrained to be non-decreasing in that problem) :

$$(\mathcal{P}) : \left| \begin{array}{l} \max \\ \{v(M)\}_{M \subseteq N} \in [0, 1]^{(2^N)} \sum_{i \in N} U_i(v) \\ \text{s.t. } \forall i \in N, U_i(v) \geq U_i^\emptyset. \end{array} \right.$$

It suffices to show that any solution of (\mathcal{P}) is non-decreasing. For that, let v be a solution of (\mathcal{P}) such that $v(M) > v(M')$ with $M \subset M'$. It is straightforward that the rule v' , obtained from v by permuting M and M' , will increase the expected utility of countries in $M' \setminus M$, while decreasing the expected utility of no country. Hence, v' improves the welfare and satisfies the constraints, a contradiction.

Let P be the probability distribution on the coalitions M of countries favoring the reform (under truthful voting), induced by the probability distribution μ . Formally:

$$\forall M \subseteq N, \quad P(M) = \mathbb{P}_\mu(\{i | u_i > 0\} = M).$$

By assumption, we have that for all $M \subseteq N$, $P(M) > 0$. As countries' utilities are independent, the expected utility of a country $i \in N$ under a rule v writes

$$\begin{aligned} U_i(v) &= \mathbb{E}_\mu \left[v((\mathbb{1}_{u_j > 0})_{j \in N}) u_i \right] \\ &= \sum_{M, i \in M} P(M) v(M) \mathbb{E}_\mu [u_i | u_i > 0] + \sum_{M, i \notin M} P(M) v(M) \mathbb{E}_\mu [u_i | u_i < 0] \\ &= \sum_{M, i \in M} P(M) v(M) w_i^+ - \sum_{M, i \notin M} P(M) v(M) w_i^-. \end{aligned}$$

The Lagrangian of the problem (\mathcal{P}) writes

$$\mathcal{L}(v) = \sum_{i \in N} U_i(v) + \sum_{i \in N} \lambda^i [U_i(v) - U_i^\emptyset] + \sum_{M \subseteq N} [\eta^M v(M) + \nu^M (1 - v(M))].$$

Its partial derivative with respect to $v(M)$ (one of the 2^n variables) is

$$\frac{\partial \mathcal{L}}{\partial v(M)}(v) = P(M) \left(\sum_{i \in M} (1 + \lambda^i) w_i^+ - \sum_{i \notin M} (1 + \lambda^i) w_i^- \right) + \eta^M - \nu^M.$$

As v^* is a solution of (\mathcal{P}) , we can apply the first-order conditions of the Kuhn-Tucker theorem (the constraints are affine functions). There exist non-negative coefficients $(\lambda^i, \eta^M, \nu^M)_{i \in N, M \subseteq N}$ such that

$$\left\{ \begin{array}{l} (i) \quad \forall M \subseteq N, \quad \frac{\partial \mathcal{L}}{\partial v(M)}(v^*) = 0 \\ (ii) \quad \forall i \in N, \quad \lambda^i [U_i(v^*) - U_i^\emptyset] = 0 \\ (iii) \quad \forall M \subseteq N, \quad \eta^M v^*(M) = 0 \\ (iv) \quad \forall M \subseteq N, \quad \nu^M (1 - v^*(M)) = 0. \end{array} \right.$$

By the last two lines, η^M and ν^M cannot be simultaneously positive. Therefore, we have

$$\left\{ \begin{array}{l} \eta^M - \nu^M < 0 \Rightarrow \nu^M > 0 \Rightarrow v^*(M) = 1 \\ \eta^M - \nu^M > 0 \Rightarrow \eta^M > 0 \Rightarrow v^*(M) = 0. \end{array} \right.$$

By (i), and the formula for the derivative of the Lagrangian, we have that

$$\begin{aligned}
\eta^M - \nu^M < 0 &\Leftrightarrow \sum_{i \in M} (1 + \lambda^i) w_i^+ > \sum_{i \notin M} (1 + \lambda^i) w_i^- \\
&\Leftrightarrow \sum_{i \in M} (1 + \lambda^i) (w_i^+ + w_i^-) > \sum_{i \in N} (1 + \lambda^i) w_i^- \\
&\Leftrightarrow \sum_{i \in M} (1 + \lambda^i) w_i^e > \frac{\sum_{i \in N} (1 + \lambda^i) w_i^-}{\sum_{i \in N} (1 + \lambda^i) w_i^e} \sum_{i \in N} (1 + \lambda^i) w_i^e.
\end{aligned}$$

Setting $w_i^* = (1 + \lambda^i) w_i^e$ and

$$t^* = \frac{\sum_{i \in N} (1 + \lambda^i) w_i^-}{\sum_{i \in N} (1 + \lambda^i) w_i^e} = \frac{\sum_{i \in N} w_i^* t_i^e}{\sum_{i \in N} w_i^*},$$

we get that:

$$\forall M \subseteq N, \quad \begin{cases} \sum_{i \in M} w_i^* > t^* \sum_{i \in N} w_i^* &\Rightarrow v^*(M) = 1 \\ \sum_{i \in M} w_i^* < t^* \sum_{i \in N} w_i^* &\Rightarrow v^*(M) = 0. \end{cases}$$

Therefore we conclude that v^* , and thus any optimal rule in \mathcal{V}^* , is weighted and can be represented by $[\mathbf{w}^*; t^*]$. Moreover, by application of the Kuhn-Tucker conditions, we have that $U_i(v^*) = U_i^\emptyset \Rightarrow \lambda^i \geq 0 \Rightarrow w_i^* \geq w_i^e$ and $U_i(v^*) > U_i^\emptyset \Rightarrow \lambda^i = 0 \Rightarrow w_i^* = w_i^e$.

7.3 Proof of Proposition 2

The proof is divided in two steps. First, we construct a profile of the repeated game, with cooperation at each stage on the equilibrium path, and we show that it is a perfect public equilibrium if the constraints of the proposition are satisfied. Second, we show that if one constraint is not satisfied, no perfect public equilibrium can sustain cooperation on the equilibrium path.

Step 1: Let v be a voting rule that satisfies the constraints of Proposition 2. We consider the following profile of strategies:

- At each stage T , play the cooperative profile of the game $\Gamma_{NE}(v)$.
- If exactly one country is observed to deviate at time T (either $d_i^T = 0$ or $a_i^T \neq \hat{v}^T(\mathbf{m}^T)$ for some $i \in N$), then all countries choose to remain sovereign in any subsequent stage $T' > T$.

Consider a potential deviation from the previous profile, for some country i , and assume it is a (strict) best reply. We note (d_i^0, m_i^0, a_i^0) the first stage of this deviation.

If $d_i^0 = 0$, the deviation yields a stage payoff U_i^θ , and a future payoff U_i^θ (given the trigger strategies employed by other players). As $U_i(v) \geq U_i^\theta$ (each country satisfies at least the participation constraint), this deviation is not profitable.

If $d_i^0 = 1$, and the deviation is such that $\exists u_i \in \mathbb{R}, m_i^0(u_i) \neq \mathbf{1}_{u_i > 0}$, then it is (weakly) dominated by the strategy $(d_i^0, \mathbf{1}_{u_i > 0}, a_i^0)$. Indeed, as the rule v is non-decreasing, lying only makes it more likely for the action plan $\hat{v}(\mathbf{m})$ to go against the country's will, which is never beneficial, and it doesn't change what happens at subsequent stages as it cannot be detected. We may thus assume that $m_i^0 = \mathbf{1}_{u_i > 0}$.

Let $u_i \in \mathbb{R}$ and $\mathbf{m}_{-i} \in \{0, 1\}^{N \setminus \{i\}}$ be such that $\mu(u_i) > 0$ and $a_i^0 \neq \hat{v}(\mathbf{1}_{u_i > 0}, \mathbf{m}_{-i})$. As the deviation will be detected, it must yield a stage-payoff of more than

$$\hat{v}(\mathbf{1}_{u_i > 0}, \mathbf{m}_{-i})u_i + \frac{\delta}{1 - \delta}(U_i(v) - U_i^\theta)$$

(because of the trigger strategies). This deviation can only be profitable if $u_i < 0$. We then distinguish two cases:

- If i has veto power, since $\mathbf{1}_{u_i > 0} = 0$, we have $\hat{v}(\mathbf{1}_{u_i > 0}, \mathbf{m}_{-i}) = 0$. Therefore, the deviation is not profitable.
- If i has no veto, we have:

$$\hat{v}(\mathbf{1}_{u_i > 0}, \mathbf{m}_{-i})u_i + \frac{\delta}{1 - \delta}(U_i(v) - U_i^\theta) \geq -w_i^D + \frac{\delta}{1 - \delta}(U_i(v) - U_i^\theta) \geq 0$$

because i satisfies the compliance constraint. Therefore, the deviation is not profitable.

Finally, we conclude that the proposed profile of strategies is a perfect public equilibrium.

Step 2: Now suppose that there exists $i \in VE(v)$ such that $U_i(v) < U_i^\theta$. Consider a profile such that cooperation is chosen at any stage, and $(\hat{v}^T(\mathbf{1}_{u_i^T > 0})_{i \in N})$ is always implemented. Country i 's expected utility is $U_i(v)$. Therefore, playing $d_i^T = 0$ for all T is a profitable deviation.

Alternatively, suppose that there exists $i \notin VE(v)$ such that $U_i(v) < U_i^\theta + \frac{1 - \delta}{\delta}w_i^D$. Consider a profile such that cooperation is chosen at any stage, and $(\hat{v}^T(\mathbf{1}_{u_i^T > 0})_{i \in N})$ is always implemented. Consider the following deviation of player i : follow the first profile, and if there exists some T such that $u_i = -w_i^D$, then play $a_i^T = 0$, and $d_i^{T'} = 0$ for all $T' > T$. The event $\{u_i^T = -w_i^D\}$ occurs almost surely in finite time (for $T < +\infty$), and yields a superior payoff when it occurs. This is thus a profitable deviation.

7.4 Proof of **Theorem 3**

Let v^{**} be an optimal self-enforcing rule, and let $V^{**} = VE(v^{**})$ be its set of veto countries. By definition, v^{**} is solution of the problem $\max_{v \in \mathcal{SE}} W(v)$, which is equivalent to $\max_{V \subseteq N} \max_{v \in \mathcal{SE}, VE(v)=V} W(v)$. Therefore, v^{**} is solution of $\max_{v \in \mathcal{SE}, VE(v)=V^{**}} W(v)$, and by the argument made at the beginning of the proof of **Theorem 2**, this problem is equivalent to:

$$(\mathcal{P}^{V^{**}}) : \begin{cases} \max_{\{v(M)\}_{V^{**} \subseteq M \subseteq N}} \sum_{i \in N} U_i(v) \\ s.t. \quad \forall i \in V^{**}, \quad U_i(v) \geq U_i^\emptyset \\ s.t. \quad \forall i \in N \setminus V^{**}, \quad U_i(v) \geq U_i^\emptyset + \frac{1-\delta}{\delta} w_i^D. \\ s.t. \quad \forall M, \quad V^{**} \not\subseteq M \Rightarrow v(M) = 0. \end{cases}$$

Now, by the arguments made in the proof of **Theorem 2**, if λ^i denotes the Lagrange multiplier associated to country i 's constraint in $(\mathcal{P}^{V^{**}})$, and if we note $w_i^0 = (1 + \lambda^i)w_i^e$ and

$$t^0 = \frac{\sum_{i \in N} w_i^0 t_i^e}{\sum_{i \in N} w_i^0}$$

we obtain:

$$\forall M \subseteq N, V^{**} \subseteq M, \quad \begin{cases} \sum_{i \in M} w_i^0 > t^0 \sum_{i \in N} w_i^0 \Rightarrow v^{**}(M) = 1 \\ \sum_{i \in M} w_i^0 < t^0 \sum_{i \in N} w_i^0 \Rightarrow v^{**}(M) = 0. \end{cases}$$

Moreover, we know that $V^{**} \not\subseteq M \Rightarrow v^{**}(M) = 0$. Now, we define $w_i^{**} = w_i^0 + K \mathbf{1}_{i \in V^{**}}$, where K is defined as a sufficiently large number, for instance $K = 1 + \sum_{i \in N} w_i^0$. We obtain that if $\sum_{i \in M} w_i^{**} > t^0 \sum_{i \in N} w_i^0 + K \#V^{**}$, then $V^{**} \subseteq M$ and $\sum_{i \in M} w_i^0 > t^0 \sum_{i \in N} w_i^0$, and therefore $v^{**}(M) = 1$. We also have that if $\sum_{i \in M} w_i^{**} < t^0 \sum_{i \in N} w_i^0 + K \#V^{**}$, then $V^{**} \not\subseteq M$ or $\sum_{i \in M} w_i^0 < t^0 \sum_{i \in N} w_i^0$, and therefore $v^{**}(M) = 0$. Finally, denoting

$$t^{**} = \frac{1}{\sum_{i \in N} w_i^{**}} \left(t^0 \sum_{i \in N} w_i^0 + K \#V^{**} \right)$$

We obtain that:

$$\forall M \subseteq N, \quad \begin{cases} \sum_{i \in M} w_i^{**} > t^{**} \sum_{i \in N} w_i^{**} \Rightarrow v^{**}(M) = 1 \\ \sum_{i \in M} w_i^{**} < t^{**} \sum_{i \in N} w_i^{**} \Rightarrow v^{**}(M) = 0. \end{cases}$$

This means that v^{**} is represented by $[\mathbf{w}^{**}; t^{**}]$. Furthermore, note that:

$$\begin{aligned} t^{**} &= \frac{1}{\sum_{i \in N} w_i^{**}} \left(\sum_{i \in N} w_i^0 t_i^e + K \#V^* \right) \\ &= \frac{1}{\sum_{i \in N} w_i^{**}} \left(\sum_{i \in N} w_i t_i^e + K \sum_{i \in V^*} (1 - t_i) \right) \geq \frac{\sum_{i \in N} w_i^{**} t_i^e}{\sum_{i \in N} w_i^{**}}. \end{aligned}$$

Finally, let $i \in N$ be a country such that $U_i(v^{**}) < U_i^\emptyset + \frac{1-\delta}{\delta} w_i^D$. As we have assumed that v^{**} is self-enforcing, it must be that $i \in VE(v^{**})$. Then, by construction, we obtain that $w_i^{**} > w_i^0 \geq w_i^e$.

Conversely, let $i \in N$ be a country such that $U_i(v^{**}) > U_i^\emptyset + \frac{1-\delta}{\delta} w_i^D$. We then show that i has no veto power under v^{**} . By contradiction, suppose that i has veto power. Then, $v^{**}(N \setminus \{i\}) = 0$. For $\varepsilon > 0$, consider v^ε defined by:

$$\begin{cases} v^\varepsilon(N \setminus \{i\}) = \varepsilon \\ \forall M \neq N \setminus \{i\}, \quad v^\varepsilon(M) = v^{**}(M). \end{cases}$$

We have $U_i(v^\varepsilon) = U_i(v^{**}) - \varepsilon P(N \setminus \{i\}) w_i^-$ and $\forall j \neq i, U_j(v^\varepsilon) = U_j(v^{**}) + \varepsilon P(N \setminus \{i\}) w_j^+$. By Assumption NEV, we get $W(v^\varepsilon) > W(v^{**})$. Moreover, v^ε is self-enforcing for ε small enough, hence a contradiction. We obtain that $i \notin VE(v^{**})$. It follows that $w_i^{**} = w_i^0 = w_i^e$.

7.5 Proof of **Theorem 4**

We now have:

$$U_i(v) = \frac{(2q-1)p_i e}{2^n} \left(\sum_{M, i \in M} v(M) - \sum_{M, i \notin M} v(M) \right).$$

and $U_i^\emptyset = (2q-1)p_i$. Let v^* be an optimal rule maximizing the recourse to probabilistic tie-breaking (i.e. chosen as in the proof of **Theorem 2**, as we know that any representation of v^* will be a representation of any optimal rule). We know from **Theorem 2** that v^* is a weighted majority rule with threshold $t = \frac{1}{2}$, and with weights $(w_i)_{i \in N}$ satisfying³⁹

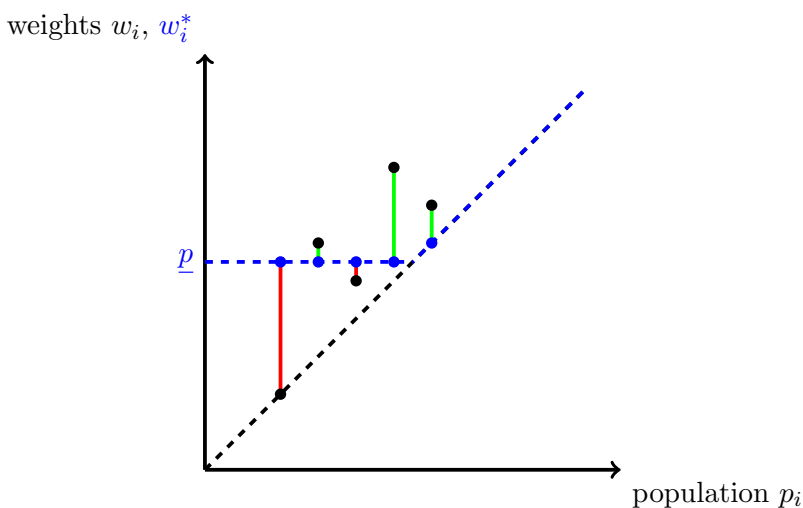
$$\begin{cases} \forall i \in N, \quad w_i = (1 + \lambda^i) p_i \geq p_i \\ \forall i \in N, \quad \lambda^i [U_i(v^*) - U_i^\emptyset] = 0. \end{cases}$$

³⁹For simplicity, the weights obtained in the proof of **Theorem 2** are re-scaled by a factor p_i/w_i^e , independent of i .

Let $S^1 = \{i \in N | U_i(v^*) > U_i^0\}$ and $S^0 = \{i \in N | U_i(v^*) = U_i^0\}$, so that $N = S^0 \cup S^1$ is a partition. From the previous system of equations, we have $w_i = p_i$ for all $i \in S^1$. If all parameters λ^i are null, we have $w_i = p_i$ for each country $i \in N$ and the result is obtained. If not, there are some countries in S^0 for which $w_i > p_i$, and the following equation has a unique solution $\underline{p} \in \mathbb{R}$ (as illustrated in Figure 4):⁴⁰

$$\sum_{i \in S^0} \max(p_i, \underline{p}) = \sum_{i \in S^0} w_i.$$

Figure 4: Definition of \underline{p} (total red length=total green length)



Let us show that v^* is represented by the modified system of weights $[\mathbf{w}^*; 1/2]$ defined by:

$$\begin{cases} \forall i \in S^0, & w_i^* = \max(p_i, \underline{p}) \\ \forall i \in S^1, & w_i^* = p_i. \end{cases}$$

Note that the vector \mathbf{w}^* can be obtained from \mathbf{w} by a finite sequence of (Pigou-Dalton) transfers of the form $(w_i \rightarrow w_i + \alpha, w_j \rightarrow w_j - \alpha)$ with $i, j \in S^0$ and $w_i < w_i + \alpha \leq w_j - \alpha < w_j$. Let us show that if v^* is represented by a vector $[\mathbf{w}; t]$, it is represented by the vector $[\mathbf{w}'; t]$, when \mathbf{w}' has been obtained from \mathbf{w} by such a transfer.

Note $Q = t \sum_{i \in N} w_i = t \sum_{i \in N} w'_i$ and let R be a coalition such that $\sum_{k \in R} w'_k > Q$. We have two cases:

⁴⁰Note that the function $f : x \mapsto f(x) = \sum_{i \in S^0} \max(p_i, x)$ is increasing and continuous, with $f(0) < \sum_{i \in S^0} w_i$ and $\lim_{x \rightarrow \infty} f(x) = +\infty$.

- In all cases but $(i \in R, j \notin R)$, we have

$$\sum_{k \in R} w_k \geq \sum_{k \in R} w'_k > Q = t \sum_{k \in N} w_k,$$

and we get $v^*(R) = 1$, as v^* is represented by $[\mathbf{w}; t]$.

- If $i \in R$ and $j \notin R$, we may have $\sum_{k \in R} w_k \leq Q$. Let us show that $v^*(R) = 1$. Assume by contradiction that $v^*(R) < 1$. Let $\sigma : N \rightarrow N$ be the transposition between i and j . We have $v^*(\sigma(R)) = 1$ (by the previous argument, since $j \in \sigma(R)$). Moreover, since v^* is represented by the system of weights $[\mathbf{w}; t]$ with $w_i < w_j$, we have for any coalition M :

$$\begin{cases} i, j \in M & \Rightarrow v^*(\sigma(M)) = v^*(M) & (\text{since } \sigma(M) = M) \\ i, j \notin M & \Rightarrow v^*(\sigma(M)) = v^*(M) & (\text{since } \sigma(M) = M) \\ i \in M, j \notin M & \Rightarrow v^*(\sigma(M)) \geq v^*(M) \\ i \notin M, j \in M & \Rightarrow v^*(\sigma(M)) \leq v^*(M). \end{cases}$$

We obtain:

$$\begin{aligned} \frac{U_j(v^*)}{U_j^\emptyset} &= \frac{e}{2^n} \left(v^*(\sigma(R)) + \sum_{M, j \in M, M \neq \sigma(R)} v^*(M) - \sum_{M, j \notin M} v^*(M) \right) \\ &\geq \frac{e}{2^n} \left(v^*(\sigma(R)) + \sum_{M, j \in M, M \neq \sigma(R)} v^*(\sigma(M)) - \sum_{M, j \notin M} v^*(\sigma(M)) \right) \\ &> \frac{e}{2^n} \left(v^*(R) + \sum_{M, i \in M, M \neq R} v^*(M) - \sum_{M, i \notin M} v^*(M) \right) \\ &> \frac{U_i(v^*)}{U_i^\emptyset}. \end{aligned}$$

We get a contradiction with the assumption that $i, j \in S^0$. Finally, it must be that $v^*(R) = 1$.

Similarly, one can show that $\sum_{k \in R} w'_k < Q$ implies $v^*(R) = 0$. Finally, v^* is represented by $[\mathbf{w}'; t]$. By induction, v^* (and thus any optimal rule) is represented by $[\mathbf{w}^*; 1/2]$.

Finally, let us show that $w_i^* = \max(p_i, \underline{p})$ for any $i \in S^1$. Let $i \in S^1$ and $j \in S^0$. As $\frac{U_i(v^*)}{U_i^\emptyset} > \frac{U_j(v^*)}{U_j^\emptyset}$, and v^* is represented by $[\mathbf{w}^*; 1/2]$, it must be that $w_i^* \geq w_j^*$ (by an argument similar to the previous computation). We have $w_j^* = \max(p_j, \underline{p}) \geq \underline{p}$, and thus $w_i^* \geq \underline{p}$. As we already know that $w_i^* = p_i$, we can write $w_i^* = \max(p_i, \underline{p})$.

7.6 Proof of Theorem 5

We introduce the notion of relative utility of a country under a rule v as the ratio between its utility under v and the utility it would get if it was a dictator:

$$\forall i \in N, \quad u_i(v) = \frac{U_i(v)}{w_i^e/2} = \frac{U_i(v)}{(2q-1)ep_i} = \frac{U_i(v)}{eU_i^\emptyset}.$$

With this notation, i 's compliance constraint can be written as: $u_i(v) \geq \frac{1}{e} + \frac{1-\delta}{\delta}$.

Claim 1 : The optimal rule is either unanimous or a weighted majority rule with threshold $1/2$.

By application of Theorem 3, it suffices to show that the optimal self-enforcing rule v cannot have a set of veto players $V = VE(v)$ such that $\emptyset \subsetneq V \subsetneq N$. Assume by contradiction that it is the case, and take $i \in V$ and $j \notin V$. We have:

$$u_i(v) = \frac{1}{2^n} \sum_{M, V \subseteq M} v(M)$$

$$u_j(v) = \frac{1}{2^n} \left(\sum_{M, V \subseteq M, j \in M} v(M) - \sum_{M, V \subseteq M, j \notin M} v(M) \right).$$

Since $j \notin V$, there exists a coalition M with $V \subseteq M$, $j \notin M$ and $v(M) > 0$. Therefore, $u_i(v) > u_j(v)$. As v is self-enforcing, we have $u_i(v) > u_j(v) \geq \frac{1}{e} + \frac{1-\delta}{\delta}$: i 's constraint is not binding. For $\varepsilon > 0$, consider now v^ε defined by:

$$\begin{cases} v^\varepsilon(N \setminus \{i\}) = \varepsilon \\ \forall M \neq N \setminus \{i\}, \quad v^\varepsilon(M) = v(M). \end{cases}$$

We have $u_i(v^\varepsilon) = u_i(v) - \frac{\varepsilon}{2^n}$ and $\forall j \neq i, u_j(v^\varepsilon) = u_j(v) + \frac{\varepsilon}{2^n}$. As $p_i < \sum_{j \neq i} p_j$, we have $W(v^\varepsilon) > W(v)$. Moreover, v^ε is self-enforcing for ε small enough, hence a contradiction.

Claim 2 : If simple majority is self-enforcing, the optimal self-enforcing rule is a weighted majority rule with threshold $1/2$. If unanimity is self-enforcing, but simple majority is not, then unanimity is the optimal self-enforcing rule. If neither unanimity nor simple majority is self-enforcing, there is no self-enforcing rule.

Let us note v^m the simple majority rule. If v^m is self-enforcing, as unanimity is strictly welfare-dominated by v^m ,⁴¹ we get from Claim 1 that the optimal self-enforcing rule is a

⁴¹The relative utility of any country is $\frac{1}{2^{n-1}} \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$ under simple majority and $1/2^n$ under unanimity.

weighted majority rule with threshold $1/2$.

If v^m is not self-enforcing, note that no country satisfies the compliance constraint under simple majority (simple majority yields the same relative utility for all countries, and they all face the same constraint). We show that no weighted majority v can then be self-enforcing. Indeed, we have:

$$\begin{aligned}
\sum_{i \in N} u_i(v) &= \frac{1}{2^n} \sum_{i \in N} \left(\sum_{M, i \in M} v(M) - \sum_{M, i \notin M} v(M) \right) \\
&= \frac{1}{2^n} \sum_{M \subseteq N} \left(\sum_{i \in M} v(M) - \sum_{i \notin M} v(M) \right) \\
&= \frac{1}{2^n} \sum_{M \subseteq N} (2\#M - n)v(M) \\
&\leq \sum_{i \in N} u_i(v^m).
\end{aligned}$$

At least one country has a (weakly) lower relative utility under v than under v^m , therefore v cannot be self-enforcing (as v does not grant veto to any country, the endogenous constraints are the same for v and v^m). To conclude, if simple majority is not self-enforcing, the only possible optimal self-enforcing rule is unanimity, and it can be optimal self-enforcing only when it is self-enforcing.

Let u^m be the relative utility of any country under simple majority. It is easy to see that: $u^m = \frac{1}{2^{n-1}} \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$. Simple majority is self-enforcing if and only if

$$\begin{aligned}
u^m \geq \frac{1}{e} + \frac{1-\delta}{\delta} &\Leftrightarrow \frac{1}{\delta} \leq 1 + u^m - \frac{1}{e} \\
&\Leftrightarrow \delta \geq \frac{1}{1 + u^m - \frac{1}{e}} := \delta^c(e).
\end{aligned}$$

Let $u^{eff} > 0$ be the relative utility of the smallest country under the efficient voting rule. The (efficient) population-weighted majority rule is self-enforcing, and is therefore the optimal self-enforcing rule, if and only if

$$\delta \geq \frac{1}{1 + u^{eff} - \frac{1}{e}} := \delta^{eff}(e).$$

By the proof of claim 2, it is easy to see that $u^{eff} \leq u^m$, thus $\delta^{eff}(e) \geq \delta^c(e)$. Moreover, as $u^{eff} > 0$, we have $\lim_{e \rightarrow \infty} \delta^{eff}(e) < 1$.

Finally, unanimity is self-enforcing if and only if $\frac{1}{2^n} \geq \frac{1}{e}$, that is if and only if $e \geq 2^n := \underline{e}$.

Claim 3 : For $\delta \geq \delta^c(e)$, there exists a minimal weight function $\underline{p}(e, \delta)$ non-increasing in both e and δ , such that, for each e and δ , any optimal self-enforcing rule is represented by $[\mathbf{w}; 1/2]$, with for all $i \in N$, $w_i = \max(p_i, \underline{p}(e, \delta))$.⁴²

We note $k(e, \delta) = \frac{1}{e} + \frac{1-\delta}{\delta}$, this function is decreasing in both e and δ . Consider the following problem:

$$(\mathcal{P}^k) : \begin{cases} \max_{\{v(M)\}_{M \subseteq N}} \sum_{i \in N} p_i u_i(v) \\ \text{s.t. } \forall i \in N, \quad u_i(v) \geq k \\ \text{s.t. } \forall M \subseteq N, \quad 0 \leq v(M) \leq 1. \end{cases}$$

Following the proofs of [Theorem 2](#) and [Theorem 4](#), we know that any solution v of (\mathcal{P}^k) is a weighted rule represented by a certain vector $[\mathbf{w}; 1/2]$ with:

- for all $i \in N$, $w_i = \max(p_i, \underline{p})$
- \underline{p} is the solution⁴³ of $\sum_{i \in N} \max(p_i, \underline{p}) = \sum_{i \in N} p_i(1 + \lambda_*^i)$
- for all $i \in N$, λ_*^i is the Lagrangian coefficient associated to country i 's constraint, related to a solution v^* of the problem (\mathcal{P}^k) .

With this definition, it is clear that \underline{p} increases with $\sum_{i \in N} p_i \lambda_*^i$. Let us show that this last quantity is non-decreasing as a function of k . The linear program (\mathcal{P}^k) can be re-written as follows (we multiplied each country i 's constraint by a factor $-p_i$):

$$(\mathcal{P}^k) : \begin{cases} \max_{\{v(M)\}_{M \subseteq N}} \frac{1}{2^n} \sum_{M \subseteq N} \left(\sum_{i \in M} p_i - \sum_{i \notin M} p_i \right) v(M) \\ \text{s.t. } \forall i \in N, \quad \frac{1}{2^n} \left(\sum_{M, i \notin M} p_i v(M) - \sum_{M, i \in M} p_i v(M) \right) \leq -kp_i \\ \text{s.t. } \forall M \subseteq N, \quad v(M) \leq 1 \\ \text{s.t. } \forall M \subseteq N, \quad v(M) \geq 0. \end{cases}$$

⁴²The claim may seem obvious, as increasing e and/or δ relaxes the self-enforcing constraints. Note however that the welfare attached to a rule weighted by $[\mathbf{w}; 1/2]$, with $w_i = \max(p_i, \underline{p})$, may be non-monotonic as a function of \underline{p} . One can construct such an example with $\mathbf{p} = (2, 4, 4, 5)$ and $\underline{p} = 2$ or 4 or 5.

⁴³To be precise, \underline{p} is defined in the proof of [Theorem 5](#) as the solution of $\sum_{i \in S^0} \max(p_i, \underline{p}) = \sum_{i \in S^0} p_i(1 + \lambda_*^i)$, and it is shown at the end of the proof that for all $i \in S^1 = N \setminus S^0$, $\max(p_i, \underline{p}) = p_i = p_i(1 + \lambda_*^i)$. Therefore, the above definition is equivalent.

The dual of (\mathcal{P}^k) , of which the variables $(\lambda^i)_{i \in N}$ are solution, is the following linear program:

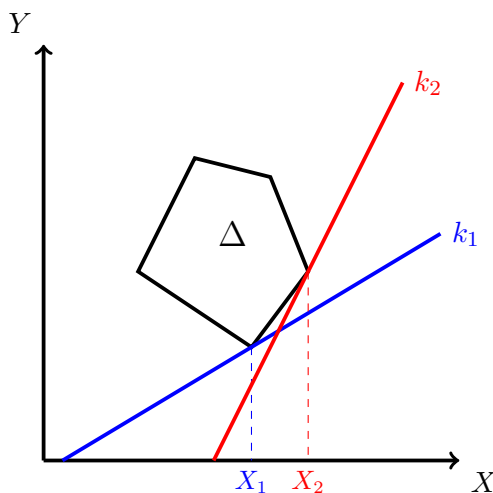
$$(\mathcal{D}^k) : \begin{cases} \min_{\{\lambda^i\}_{i \in N}, \{\nu^M\}_{M \subseteq N}} \sum_{M \subseteq N} \nu^M - k \sum_{i \in N} p_i \lambda^i \\ s.t. \quad \forall M \subseteq N, \quad \frac{1}{2^n} \left(\sum_{i \notin M} p_i \lambda^i - \sum_{i \in M} p_i \lambda^i \right) + \nu^M \geq \frac{1}{2^n} \left(\sum_{i \in M} p_i - \sum_{i \notin M} p_i \right) \\ s.t. \quad \forall i \in N, \quad \lambda^i \geq 0 \\ s.t. \quad \forall M \subseteq N, \quad \nu^M \geq 0. \end{cases}$$

Now, consider the mapping $\Phi : ((\lambda^i), (\nu^M)) \mapsto (X = \sum_{i \in N} p_i \lambda^i, Y = \sum_{M \subseteq N} \nu^M)$. It is a linear mapping from $\mathbb{R}^{2^n} \times \mathbb{R}^N$ into \mathbb{R}^2 , which transforms any convex polyhedron into a convex polyhedron. Therefore, for any solution $((\nu^M), (\lambda^i))$ of the program (\mathcal{D}^k) , there is a corresponding solution (X, Y) of a the 2-dimensional program $(\mathcal{D}^k)'$ defined by:

$$(\mathcal{D}^k)' : \begin{cases} \min_{X, Y} Y - kX \\ s.t. \quad (X, Y) \in \Delta \\ s.t. \quad X \geq 0 \\ s.t. \quad Y \geq 0, \end{cases}$$

where $\Delta \subset \mathbb{R}^2$ is a convex polyhedron. It is clear (see [Figure 5](#)) that X is non-decreasing as a function of k , in the following strong sense: for any $k_2 > k_1$ and any solutions (X_1, Y_1) of $(\mathcal{D}^{k_1})'$ and (X_2, Y_2) of $(\mathcal{D}^{k_2})'$, we have that $X_2 \geq X_1$.

Figure 5: Solutions of $(\mathcal{D}^{k_1})'$ and $(\mathcal{D}^{k_2})'$ for $k_2 > k_1$.



To conclude, there exists a non-decreasing function $\underline{p}(k)$, such that for each k , any solution of (\mathcal{P}^k) is represented by $[\mathbf{w}; 1/2]$, with for all $i \in N$, $w_i = \max(p_i, \underline{p}(k))$.