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# TECHNOLOGICAL PROGRESS, THE SUPPLY OF HOURS WORKED, AND THE CONSUMPTION-LEISURE COMPLEMENTARITY

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**Abstract:** At least since 1870 hours worked per worker declined and real wages increased in many of today's industrialized countries. The dual nature of technological progress in conjunction with a consumption-leisure complementarity explains these stylized facts. Technological progress drives real wages up and expands the amount of available consumption goods. Enjoying consumption goods increases the value of leisure. Therefore, individuals demand more leisure and supply less labor. This mechanism appears in an OLG-model with two-period lived individuals equipped with per-period utility functions of the generalized log-log type proposed by Boppart-Krusell (2016). The optimal plan is piecewise defined and hinges on the wage level. Technological progress moves a poor economy out of a regime with low wages and an inelastic supply of hours worked into a regime where wages increase further and hours worked continuously decline.

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# 1 Introduction

Many of today's industrialized countries have seen a significant decline in the amount of hours worked per worker at least since 1870. According to recent estimates by Huberman (2004) and Huberman and Minns (2007), a full-time job of a US production worker in 1870 required an annual workload of 3096 hours of work. In the year 2000 this had come down to 1878 hours of work, an absolute decline of roughly 40%. According to these authors a similar tendency can be found in Australia, Belgium, Canada, Denmark, France, Germany, Ireland, Italy, the Netherlands, Spain, Sweden, Switzerland, and the UK.<sup>1</sup> At the same time, these countries experienced sustained increases in real wages, output, and consumption per capita.

What explains these stylized facts? My answer starts with the assertion that the observed decline in hours worked reflects a fall in the voluntary individual supply of labor. Then, I argue that the dual nature of technological progress in conjunction with a consumption-leisure complementarity caused the individual supply of hours worked to fall. On the one hand, technological progress drives productivity and the growth of real wages and real incomes. On the other hand, it expands the supply of consumption goods that individuals buy and enjoy during their leisure time. The consumption-leisure complementarity implies that the value of leisure increases with the amount of available consumption goods.<sup>2</sup> This affects the intensive margin of the individual labor supply and motivates individuals to supply fewer hours worked if their real wages and real incomes increase.

The present paper develops this argument and studies its implications for the comparative economic development of countries over the long run. To accomplish this I use a standard OLG-model with two-period lived individuals. The novel feature is the individual lifetime utility that features per-period utility functions of the generalized log-log type recently proposed by Boppart and Krusell (2016). My contribution to economic theory includes the detailed analysis of the decision-theoretic implications of this utility function for two-period lived overlapping generations. Moreover, I show that a *new*

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<sup>1</sup>Boppart and Krusell (2016), p. 75, use the data of these countries to estimate for the time span 1870-2000 that on average annual hours of work per worker declined at roughly 0.57% percent per year.

<sup>2</sup>According to Gordon (2016), p. 9, technological progress in home entertainment gives rise to a consumption-leisure complementarity:

Added household equipment, such as TV sets, and technological change, such as the improvement in the quality of TV-set pictures, increase the marginal product of home time devoted to household production and leisure. For instance, the degree of enjoyment provided by an hour of leisure spent watching a TV set in 1955 is greater than that provided by an hour listening to the radio in the same living room in 1935.

Video games constitute a more recent technological advance with similar consequences for the appreciation of leisure and the labor supply of (young) men (see, e. g., Aguiar, Bils, Charles, and Hurst (2017) or Avent (2017)).

canonical OLG-model emerges that accounts for the observed decline in hours worked.<sup>3</sup> Its dynamical system is available in closed form and gives rise to a stable steady state. To the best of my knowledge, my research is the first that employs a utility function of the Boppart-Krusell class in a full-fledged neoclassical growth model.<sup>4</sup>

In more detail, my main findings include the following. First, I show that the generalized log-log utility function, henceforth gll utility function, of Boppart and Krusell (2016) gives rise to a consumption-leisure complementarity of the kind discussed above, i. e., the marginal utility of leisure increases in the amount of contemporaneous consumption. The complementarity hinges crucially on a preference parameter  $\nu \in (0, 1)$ . It disappears in the limit  $\nu \rightarrow 0$ .

Second, I show that the optimal plan of each individual hinges on the level of the real wage. This gives rise to two regimes. Roughly speaking, in Regime 1 wages are high. The individual supply of hours worked declines in response to a wage hike, and  $(-\nu)$  is the wage elasticity. Here, the consumption-leisure complementarity is at work. A higher wage allows for the purchase of more consumption goods. This increases the value of leisure. Accordingly, individuals demand more leisure and supply less of their time endowment to the labor market. In technical parlance, due to the consumption-leisure complementarity the income effect on the Marshallian demand for leisure dominates the substitution effect.

In Regime 2 wages are low, and individuals are poor. In spite of the consumption-leisure complementarity the individual supply of hours worked does not respond to a higher real wage. The prospect of a low income induces individuals to supply their entire time endowment to the labor market. While a higher wage means a higher income any additional purchasing power is spent on consumption goods rather than on leisure. This makes intuitive sense. Poor people spend their entire income on consumption goods to satisfy basic needs. The demand for leisure will only become positive once these needs are adequately satisfied.

The two regimes suggest that the presence of a consumption-leisure complementarity is not sufficient for a declining individual supply of hours worked. In addition, the wage

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<sup>3</sup>In the literature, the canonical OLG-model has two defining properties (see, e. g., Acemoglu (2009), Chapter 9.3, or de la Croix and Michel (2002)). First, two-period lived individuals have preferences over levels of consumption when young and old that are represented by an additively separable life-time utility function with per-period log-utility. Second, the neoclassical production sector is Cobb-Douglas. The first property implies that individual savings and, hence, the process of capital accumulation does not hinge on the real rental rate of capital. In conjunction with the second property, this excludes multiple steady states that may arise if the elasticity of substitution between capital and labor is smaller than unity (see, e. g., Galor and Ryder (1989) and Galor (1996)). Since there is no appreciation for leisure in the life-time utility function young individuals supply their entire time endowment to the labor market. Hence, in contrast to the empirical evidence the canonical OLG-model cannot account for the observed decline in hours worked.

<sup>4</sup>Appendix B.2 of Boppart and Krusell (2016) sketches a closed-form solution for the Ramsey model under a generalized log-log utility, Cobb-Douglas production, and a rate of capital depreciation equal to 100%.

level has to be sufficiently high.

Third, I establish the dynamical system and study the evolution of the economy over the long run. This highlights the role of technological progress as an engine that liberated poor people from the necessity to assure a subsistence income by supplying an amount of working hours that touched upon their physical limits, often deteriorated their health, and led to premature death. This assertion is based on two arguments. The first argument looks at an economy void of technological progress that finds itself in Regime 2. I show that over time the economy remains in this regime and converges towards a steady state with a stationary real wage. Throughout this evolution individuals supply their entire time endowment to the labor market. Hence, a poor economy stays poor even though capital accumulation may induce the real wage to increase along the transition.

According to the second argument, the picture drastically changes if the same economy is exposed to sustained technological progress that becomes the main driver of real wage growth. Then, an initially poor economy must leave Regime 2 in finite time. It necessarily switches into Regime 1 where the individual supply of hours worked declines as wages increase further. Eventually, this economy converges towards a steady state along which real wages increase and hours worked decline at constant, yet different rates.

Fourth, I establish the properties of the steady state with sustained technological progress. Here, real wages grow at a constant rate,  $g_w$ , equal to the exogenous growth rate of technological knowledge,  $g_A > 0$ . The individual supply of hours worked grows approximately at rate  $-\nu g_w = -\nu g_A$  since  $(-\nu)$  is its wage elasticity. The growth rate of economic aggregates is given by the growth rate of aggregate efficient hours worked. The latter is the sum of the growth rates of (labor-augmenting) technological knowledge,  $g_A$ , of the labor force,  $g_L$ , and of individual hours worked,  $-\nu g_A$ . The last two rates represent, respectively, the growth rates of the extensive and the intensive margin of the aggregate labor supply. Accordingly, per-capita variables grow at rate  $g_A - \nu g_w = (1 - \nu) g_A$ .

Hence, even though technological knowledge grows at an exogenous rate, the economy's growth rate is endogenous as it depends on the preference parameter  $\nu$ . Intuitively, a growing stock of technological knowledge applies to an ever declining amount of hours of work. Therefore, the growth rate of per-capita variables falls short of  $g_A$ . Moreover, the greater  $\nu$  the lower the economy's growth rate since a hike in  $\nu$  speeds up the decline of the supply of hours worked.

The fifth set of results concerns the choice theoretic properties of a lifetime utility featuring the gll utility function. I show that this per-period utility function does not represent preferences for consumption and leisure defined over the usual domain  $\mathbb{R}_{++} \times [0, 1]$ . In particular, for bundles where the level of consumption is high relative to the amount of leisure the gll utility function ceases to give information about how individuals rank available alternatives. This property is related to the consumption-leisure complementarity build into this function. Another important analytical consequence of this complementarity is that parameter restrictions are needed to assure monotonicity and concavity

of the lifetime utility function. I establish that these restrictions diminish the individual choice set even further. However, in light of the empirical evidence I show by example that there are sets of reasonable parameter values that validate the model.

The sixth set of results sheds light on the underlying analytical structure of the OLG-model under scrutiny. I argue that my analysis of Regime 1 gives rise to a *new* canonical OLG-model that incorporates a declining supply of hours worked. In spite of the complications related to the inclusion of a labor-leisure decision the model retains its tractability. In particular, the individual supply of hours worked and individual savings do not depend on the real rental rate of capital.<sup>5</sup> I show that the reason for these simplifications is the log-utility of consumption when old. Moreover, I establish that the evolution under Regime 1 is described by a non-linear difference equation available in closed form. It gives rise to a unique, globally stable steady state.

The present paper is related to several strands of the literature. First, it fills a gap in the modern growth literature that largely neglects the fact that hours worked per worker have been falling for the last 130 years or so.<sup>6</sup> Instead, this literature relies on the assumption of an inelastic supply of hours worked. My findings concerning the mechanics of Regime 1 and 2 suggest that this assumption applies best to low-income countries.

Second, my research relates to a growing empirical literature that documents and interprets the decline of hours worked per worker or per capita across countries and/or over the long run (see, e. g., Bick, Fuchs-Schuendeln, and Lagakos (2018), Boppart and Krusell (2016), Greenwood and Vandenbroucke (2008), or Rogerson (2006)). In line with my theoretical predictions this literature finds that hours worked per worker are higher in poor than in rich countries.<sup>7</sup> Moreover, Bick, Fuchs-Schuendeln, and Lagakos (2018), Section 6.2, provide evidence showing that individuals work indeed more hours because of their low wages. This suggests that preferences for which the income effect dominates the substitution effect is an appropriate way to think about a supply of hours worked that declines in the wage.

Third, the present paper relates to the literature on discrete-time models with overlapping generations (de la Croix and Michel (2002)). As explained above, my findings for Regime 1 suggest that the OLG-model with two-period lived individuals endowed with

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<sup>5</sup>See Bloom, Canning, and Graham (2003) for empirical evidence suggesting that the effect of the real interest rate on the savings rate is small.

<sup>6</sup>As an indication of this neglect observe that the subject index of Daron Acemoglu's comprehensive treatment of modern growth theory has no entry "hours worked" or "individual labor supply". There is, however, an entry "inelastic supply of labor" which is indeed the standard assumption made in this literature (see Acemoglu (2009), pp. 977-990).

<sup>7</sup>Related contributions study hours worked and the role of the extensive margin of the labor supply of individuals and households using models that allow for home production in the spirit of Reid (1934) and Becker (1965) (see, e. g., McDaniel (2011) or, Greenwood, Seshadri, and Yorukoglu (2005)). Boppart, Krusell, and Olsson (2017) develop a novel framework to address these issues.

per-period gll utility functions and firms operating under Cobb-Douglas gives rise to a suitable *new* canonical OLG model. It captures the salient empirical finding of declining individual hours of work. Due to its tractability it may well become the workhorse model for various other economic applications.

This paper is organized as follows. Section 2 presents the model. Section 2.1 has a detailed discussion of the consumption-leisure complementarity in the lifetime utility function of individuals, derives the optimal plan of each cohort, and discusses the order of magnitude of some relevant model parameters. Section 2.2 introduces the firm sector. Section 3 studies the intertemporal general equilibrium. Its definition is given and explained in Section 3.1. Here, a particular focus is on the existence and the uniqueness of the equilibrium in the labor market. Section 3.2 sets up the dynamical system in Regime 1 and provides the analysis of the steady state. The focus of Section 3.3 is on the global dynamics. Here, I develop the idea that sustained technological progress is an engine that liberates people from the necessity to work very long hours. Section 4 studies more general lifetime utility functions to clarify the role of the consumption-leisure complementarity for the individual supply of hours worked (Section 4.1) and for individual savings (Section 4.2). Section 5 concludes. All proofs are contained in Section A, the Appendix.

## 2 The Model

The economy has a household sector and a production sector in an infinite sequence of periods  $t = 1, 2, \dots, \infty$ . The household sector comprises overlapping generations of individuals who live for two periods, youth and old age. The individual lifetime utility function features a Boppart-Krusell gll utility function that gives rise to a consumption-leisure complementarity. Moreover, this lifetime utility function is compatible with a steady-state path along which per-capita income and consumption grow and the individual labor supply declines at a constant rate.

The production sector has competitive firms producing a single good using physical capital, technology, and labor hours as inputs. This good may be either consumed or invested. In the latter case, it serves as future capital. Henceforth, I shall refer to the single produced good as the *manufactured good*. If consumed it is referred to as the *consumption good*, if invested as *capital*.

In all periods, there are three objects of exchange, the *consumption good*, *labor* and *capital*. Capital at  $t$  is built from the savings of period  $t - 1$ , and, without loss of generality, depreciates after use. Households supply labor and capital. Labor is “owned” by the young; the old own the capital stock. Each period has markets for all three objects of exchange. Capital is the only asset in the economy. The manufactured good serves as numéraire.



Throughout, I denote the time-invariant growth rate of some variable  $x_t$  between two adjacent periods by  $g_x$ . Moreover, I often use subscripts to write first- and second-order derivatives. For instance, the notation for the derivatives of some function  $G(x, y)$  would be  $G_2(x, y) \equiv \partial G(x, y) / \partial y$  or  $G_{21}(x, y) \equiv \partial^2 G(x, y) / \partial y \partial x$ . I shall also write  $G$  instead of  $G(x, y)$  or  $G(\cdot)$  whenever this does not cause confusion.

## 2.1 The Household Sector

The population at  $t$  consists of  $L_t$  young (cohort  $t$ ) and  $L_{t-1}$  old individuals (cohort  $t - 1$ ). Due to birth and other demographic factors the number of young individuals between two adjacent periods grows at rate  $g_L < (-1)$ . For short, I shall refer to  $g_L$  as the population growth rate.

When young, individuals supply labor, earn wage income, save, and enjoy leisure as well as the consumption good. When old, they retire and consume their wealth.

### 2.1.1 Preferences, Utility, and the Optimal Plan of Cohort $t$

For cohort  $t$ , denote consumption when young and old by  $c_t^y$  and  $c_{t+1}^o$ , and leisure time enjoyed when young by  $l_t$ . I normalize the maximum per-period time endowment supplied to the labor market to unity. Then,  $1 - l_t = h_t$ , where  $h_t \in [0, 1]$  is hours worked when young. Individuals of all cohorts assess bundles  $(c_t^y, l_t, c_{t+1}^o)$  according to a lifetime utility function

$$U(c_t^y, l_t, c_{t+1}^o) = \ln c_t^y + \ln \left( 1 - \phi (1 - l_t) (c_t^y)^{\frac{\nu}{1-\nu}} \right) + \beta \ln c_{t+1}^o, \quad (2.1)$$

where  $\phi > 0$ ,  $\nu \in (0, 1)$ , are parameters to be interpreted below, and  $\beta \in (0, 1)$  is the discount factor. Hence, in both periods of life, consumption and leisure are evaluated according to a Boppart-Krusell gll utility function. Since retirement is legally enforced so that leisure when old,  $l_{t+1}^o$ , is equal to unity, the term  $\beta \ln \left( 1 - \phi (1 - l_{t+1}^o) (c_{t+1}^o)^{\frac{\nu}{1-\nu}} \right)$  disappears from  $U$ . For ease of notation, I follow Boppart and Krusell and use henceforth

$$x_t \equiv (1 - l_t) (c_t^y)^{\frac{\nu}{1-\nu}}. \quad (2.2)$$

The term  $\ln(1 - \phi x_t)$  reflects the disutility of labor when young. It is more pronounced the greater is  $\phi$ . This parameter represents properties of the labor market that affect the disutility of labor in the population irrespective of the amount of hours worked and the level of consumption. For instance, in an economy with demanding occupational safety regulations  $\phi$  may be lower than in an economy without such regulations. Similarly, if the labor market gives rise to a good matching between individual career aspirations and actual occupations then  $\phi$  ought to be low, too. Finally, as suggested by Landes (1998),  $\phi$  may reflect the climatic conditions under which labor is done.

The disutility of labor is also more pronounced the greater is  $c_t^y$ . This implies a key property of  $U$ , namely, the complementarity between consumption when young and leisure in the sense that  $U_{12} > 0$ . Since

$$U_{12} = \frac{\nu\phi}{(1-\nu)(c_t^y)^{\frac{1-2\nu}{1-\nu}}(1-\phi x_t)^2}$$

complementarity requires  $\phi > 0$  and  $\nu \in (0, 1)$ . Hence, for a young individual the marginal utility of leisure increases in the amount of the consumption good. This is how the utility function (2.1) captures the gist of the logic sketched out in the Introduction.<sup>8</sup>

Since the natural logarithmic function requires a strictly positive argument, the domain of  $U$  cannot include all bundles  $(c_t^y, l_t, c_{t+1}^o) \in \mathbb{R}_{++}^2 \times [0, 1]$ . Those for which  $1 - \phi x_t \leq 0$  must be excluded. Figure 2.1 depicts a typical set of pairs  $(l_t, c_t^y)$  included in the domain of  $U$ .<sup>9</sup> Intuitively, to be in the domain the disutility of labor must not be too large, i.e.,  $c_t^y$  must not be too high given  $l_t$ . Accordingly,  $U$  is not a utility function that represents a preference relation  $\succeq$  over all bundles  $(c_t^y, l_t, c_{t+1}^o) \in \mathbb{R}_{++}^2 \times [0, 1]$ .<sup>10</sup>

The complementarity between consumption and leisure has an “analytical cost”:  $U$  is not necessarily increasing in  $c_t^y$  and not necessarily concave in  $(c_t^y, l_t, c_{t+1}^o)$ . As to monotonicity, one readily verifies that

$$U_1 > 0 \quad \Leftrightarrow \quad 1 - \nu - \phi x_t > 0. \quad (2.3)$$

The second term in  $U$  is responsible for this property. For small values of  $c_t^y$  condition (2.3) will hold since  $U$  satisfies the Inada condition  $\lim_{c_t^y \rightarrow 0} U_1 = \infty$ . However, as shown in Figure 2.2, for large values of  $c_t^y$  condition (2.3) is violated: the marginal utility of  $c_t^y$  becomes negative for pairs  $(l_t, c_t^y)$  above the dark blue curve. Figure 2.2 also shows that the set of bundles for which (2.3) holds, i.e., the blue shaded area, is a subset of the domain of  $U$ . Intuitively,  $x_t$  increases in  $c_t^y$  since  $\nu > 0$ . Hence, given  $l_t$  the set of values  $c_t^y > 0$  compatible with (2.3) must be smaller than and included in the set that satisfies  $1 - \phi x_t > 0$ .

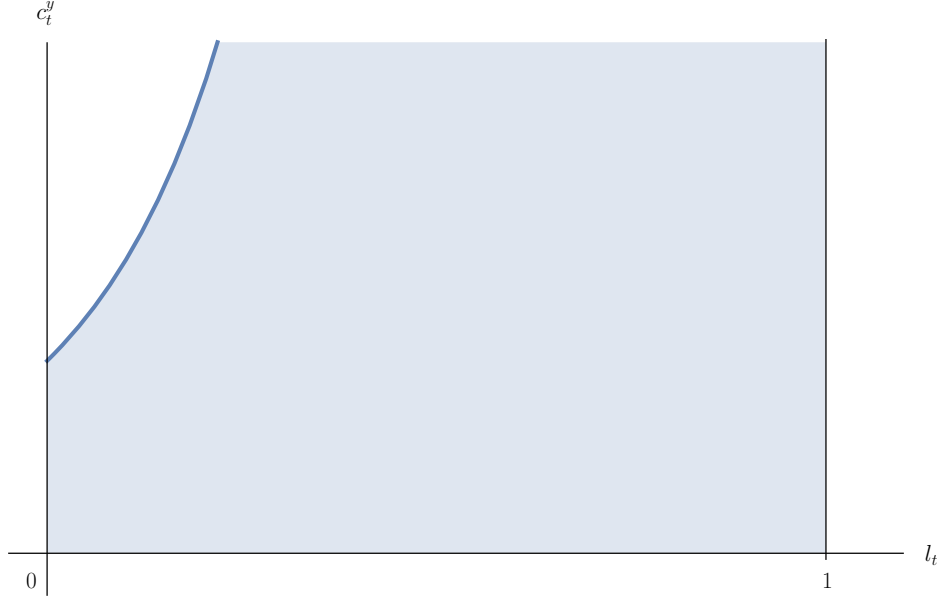
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<sup>8</sup>The utility function (2.1) is also valid with  $\nu \leq 0$ . For  $\nu < 0$  one has  $U_{12} < 0$ , and leisure and consumption are substitutes. For  $\nu = 0$ , consumption and leisure are neither complements nor substitutes and  $U$  boils down to the log-specification discussed in King, Plosser, and Rebelo (1988). Here, a reasonable solution requires in addition that  $\phi \geq (1 + \beta) / (2 + \beta)$ . Finally,  $U$  also nests the case where  $\phi = 0$  so that no utility is associated with leisure. This assumption is made in the canonical OLG-model (see, e.g., Acemoglu (2009), Chapter 9.3, or de la Croix and Michel (2002)) where young individuals supply their entire labor endowment inelastically to maximize the wage income.

<sup>9</sup>Figures 2.1 - 2.5 use the following parameter values:  $\nu = 0.2024$ ,  $\phi = 0.4$ , and  $\beta = 0.294$ .

<sup>10</sup>See Definition 1.B.2 of Mas-Colell, Whinston, and Green (1995), p. 9, for the definition of a utility function. A natural way to search for a utility function that represents  $\succeq$  over all bundles  $(c_t^y, l_t, c_{t+1}^o) \in \mathbb{R}_{++}^2 \times [0, 1]$  is to consider the strictly monotonic transformation  $\bar{U} = \exp[U]$ . Then, one readily verifies that  $\bar{U}$  is indeed a utility function since  $\bar{U} : \mathbb{R}_{++}^2 \times [0, 1] \rightarrow \mathbb{R}$ . However,  $\bar{U}$  is not a concave function for any pair  $(c_t^y, c_{t+1}^o) \in \mathbb{R}_{++}^2$ .

Figure 2.1: The area shaded in light blue shows pairs  $(l_t, c_t^y) \in [0, 1] \times \mathbb{R}_{++}$  that are in the domain of  $U$ , i. e., they satisfy  $1 - \phi x_t > 0$ .



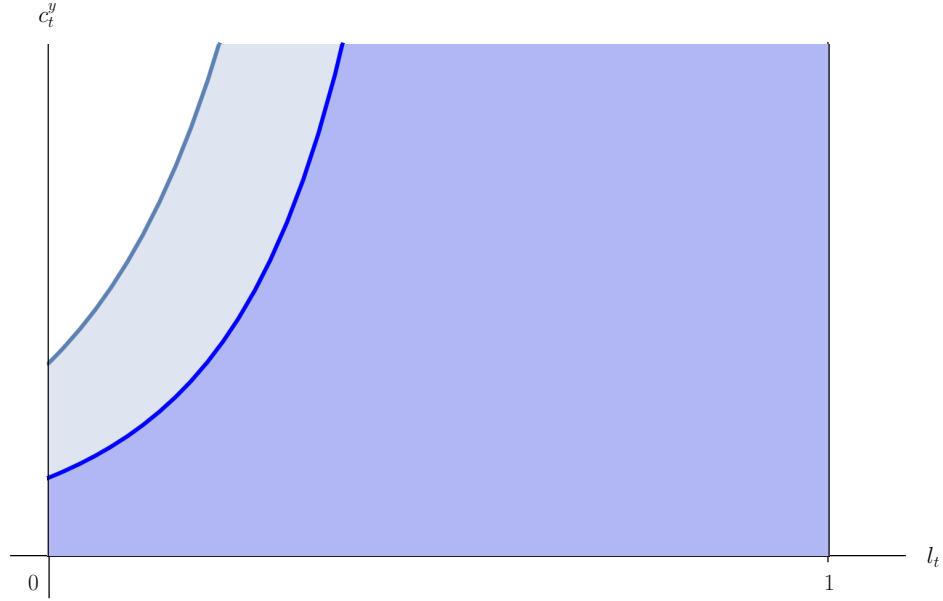
As to concavity the proof of Proposition 1 below lays open that  $U$  is strictly concave if and only if

$$1 - 2\nu - (1 - \nu)\phi x_t > 0. \quad (2.4)$$

This condition requires  $\nu < 1/2$ . Moreover, as depicted in Figure 2.3, the set of bundles for which (2.4) holds, i. e., the area shaded in dark blue, is a subset of the set of bundles in the domain of  $U$  for which (2.3) holds. Intuitively, this makes sense since condition (2.4) may be expressed as  $(1 - \nu - \phi x_t) - \nu(1 - \phi x_t) > 0$ . Both terms in parenthesis are positive if  $(l_t, c_t^y)$  is in the domain of  $U$  and  $U_1 > 0$ . Hence,  $1 - \nu - \phi x_t > \nu(1 - \phi x_t) > 0$ . Accordingly, given  $l_t$  the set of values  $c_t^y > 0$  compatible with (2.4) must be smaller than and included in the set that satisfies (2.3).

Henceforth, I shall refer to the set of bundles  $(c_t^y, l_t, c_{t+1}^o) \in \mathbb{R}_{++}^2 \times [0, 1]$  that satisfies (2.4) as the *set of permissible bundles* and denote this set by  $\mathcal{P}$ . From the preceding discussion it should be evident that permissible bundles are physically feasible, lie in the domain of  $U$ , are associated with a strictly positive marginal utility of  $c_t^y$ , and imply that  $U$  is strictly concave.

Figure 2.2: The blue shaded area shows pairs  $(l_t, c_t^y)$  that satisfy  $1 - v - \phi x_t > 0$  so that the marginal utility of  $c_t^y$  is strictly positive.



Let  $w_t > 0$  denote the real wage per hour worked and  $R_{t+1} > 0$  the perfect foresight real rental rate paid per unit of savings. I refer to  $(c_t^y, l_t, c_{t+1}^o, s_t)$  as the plan of cohort  $t$ . Then, the optimal plan of cohort  $t$  solves

$$\max_{(c_t^y, l_t, c_{t+1}^o, s_t) \in \mathcal{P} \times \mathbb{R}} \ln c_t^y + \ln \left( 1 - \phi (1 - l_t) (c_t^y)^{\frac{v}{1-v}} \right) + \beta \ln c_{t+1}^o \quad (2.5)$$

subject to the per-period budget constraints

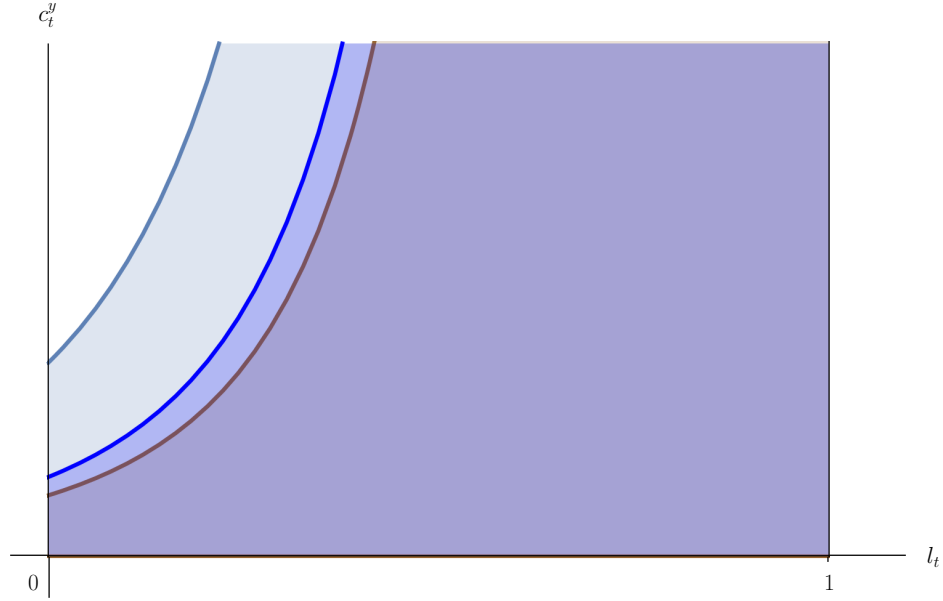
$$c_t^y + s_t \leq w_t(1 - l_t) \quad \text{and} \quad c_{t+1}^o \leq R_{t+1}s_t. \quad (2.6)$$

Before I fully characterize the solution to this problem the following assumption must be introduced.

**Assumption 1** *It holds that*

$$0 < v < \bar{v}(\beta) \equiv \frac{3 + \beta - \sqrt{5 + \beta(2 + \beta)}}{2(1 + \beta)}.$$

Figure 2.3: The area shaded in dark blue shows permissible pairs  $(l_t, c_t^y)$  that satisfy  $1 - 2\nu - (1 - \nu)\phi x_t > 0$  so that  $U$  is strictly concave.



As will become clear in the Proof of Proposition 1, Assumption 1 assures that the bundle identified by the Lagrangian associated with the choice problem (2.5) satisfies condition (2.4). The function  $\bar{\nu}(\beta)$  is strictly positive and declining on  $\beta \in [0, 1]$  with  $\bar{\nu}(0) \approx 0.382$  and  $\bar{\nu}(1) \approx 0.293$ . Hence, Assumption 1 imposes a tighter constraint on  $\nu$  than just  $\nu < 1/2$  which is necessary for (2.4) to hold.

To simplify the notation define

$$w_c \equiv \left( \frac{(1 + \beta)(1 - \nu)}{z(\phi, \beta, \nu)} \right)^{\frac{1}{\nu}},$$

where

$$z(\phi, \beta, \nu) \equiv [\phi(1 + (1 + \beta)(1 - \nu))]^{1-\nu} [1 - \nu(1 + \beta)]^\nu.$$

Then the following proposition can be stated and proved.

**Proposition 1** (*Optimal Plan of Cohort  $t$* )

Suppose Assumption 1 holds. Then, for cohorts  $t = 1, 2, \dots, \infty$  and prices  $(w_t, R_{t+1}) \in \mathbb{R}_{++}^2$  the optimal plan involves continuous, piecewise defined functions

$$h_t = h(w_t), \quad c_t^y = c^y(w_t), \quad c_{t+1}^o = c^o(w_t, R_{t+1}), \quad \text{and} \quad s_t = s(w_t). \quad (2.7)$$

**Regime 1:** If  $w_t \geq w_c$  then  $l_t \geq 0$ ,  $h_t \leq 1$  and

$$\begin{aligned} h_t &= w_c^\nu w_t^{-\nu}, \\ c_t^y &= \frac{1 - \nu(1 + \beta)}{(1 + \beta)(1 - \nu)} w_c^\nu w_t^{1-\nu}, \\ c_{t+1}^o &= \frac{\beta R_{t+1}}{(1 + \beta)(1 - \nu)} w_c^\nu w_t^{1-\nu}, \\ s_t &= \frac{\beta}{(1 + \beta)(1 - \nu)} w_c^\nu w_t^{1-\nu}. \end{aligned}$$

**Regime 2:** If  $0 < w_t \leq w_c$  then  $l_t = 0$ ,  $h_t = 1$ , and  $c^y(w_t)$  is implicitly given by

$$c_t^y \left( 1 + \beta \frac{(1 - \nu) \left( 1 - \phi(c_t^y)^{\frac{\nu}{1-\nu}} \right)}{1 - \nu - \phi(c_t^y)^{\frac{\nu}{1-\nu}}} \right) = w_t.$$

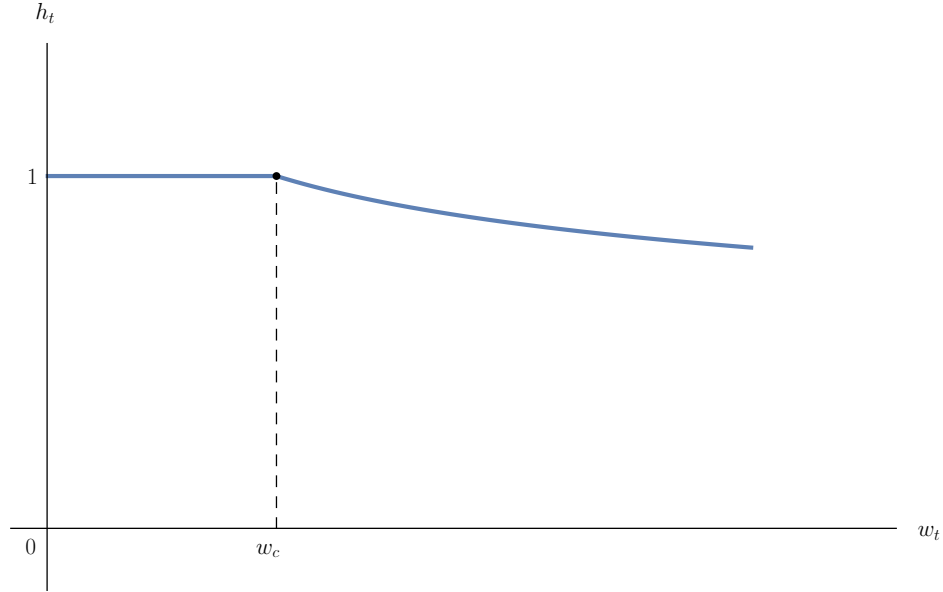
Moreover,  $s_t = w_t - c^y(w_t) \equiv s(w_t)$  and  $c_{t+1}^o = R_{t+1}s(w_t) \equiv c^o(w_t, R_{t+1})$ .

Finally, for members of cohort 0, we have  $c_1^o = R_1 s_0 > 0$  where  $s_0 > 0$  is given.

Proposition 1 is a central result of this paper. It makes two important points. First, it establishes that the optimal plan hinges on the level of the real wage. In particular, this suggests that the standard assumption of an inelastic labor supply made in almost all growth models is only plausible if the real wage is sufficiently low. Second, it shows that the individual supply of hours worked, consumption when young, and individual savings are independent of the real rental rate of capital.

There are two regimes. In Regime 1 the real wage exceeds the critical level  $w_c$ , and individuals supply less than their time endowment to the labor market. As the real wage increases, the supply of hours worked declines at a constant proportionate rate equal to  $\nu \in (0, 1)$ . In Regime 2 the real wage is below  $w_c$ , and individuals supply their entire time endowment to the labor market. Hence, the individual labor supply is indeed piecewise defined (see Figure 2.4 for an illustration). Since  $U$  is continuous and the set of feasible values  $(c_t^y, l_t, c_{t+1}^o)$  does not drastically expand or shrink the optimal plan involves continuous functions.

Figure 2.4: The individual supply of hours worked  $h_t = h(w_t)$ . For  $0 < w \leq w_c$  individuals supply their entire time endowment to the labor market. For  $w \geq w_c$  the wage elasticity of the individual supply of hours worked is  $(-\nu) \in (-1, 0)$ .



To understand why the individual labor supply is piecewise defined recall that the utility-maximizing plan involving  $(c_t^y, l_t, c_{t+1}^o) \gg \mathbf{0}$  satisfies the first-order condition  $U_2 = w_t U_1$ , i. e., the marginal utility of leisure is equal to its opportunity cost in terms of foregone consumption when young. However, for low values of  $w_t$  this condition cannot be satisfied as an equality. Intuitively, if the real wage is small then the demand for consumption when young and for leisure will be small, too. Since  $U$  satisfies the Inada condition  $\lim_{c_t^y \rightarrow 0} U_1 = \infty$  whereas  $\lim_{l_t \rightarrow 0} U_2 < \infty$  it holds for  $0 < w_t \leq w_c$  that  $U_2 \leq w_t U_1$ . The latter inequality is strict whenever  $w_t < w_c$ . Then, the individual demand for leisure is indeed equal to zero whereas  $c_t^y > 0$  (see Figure 2.5 for an illustration).

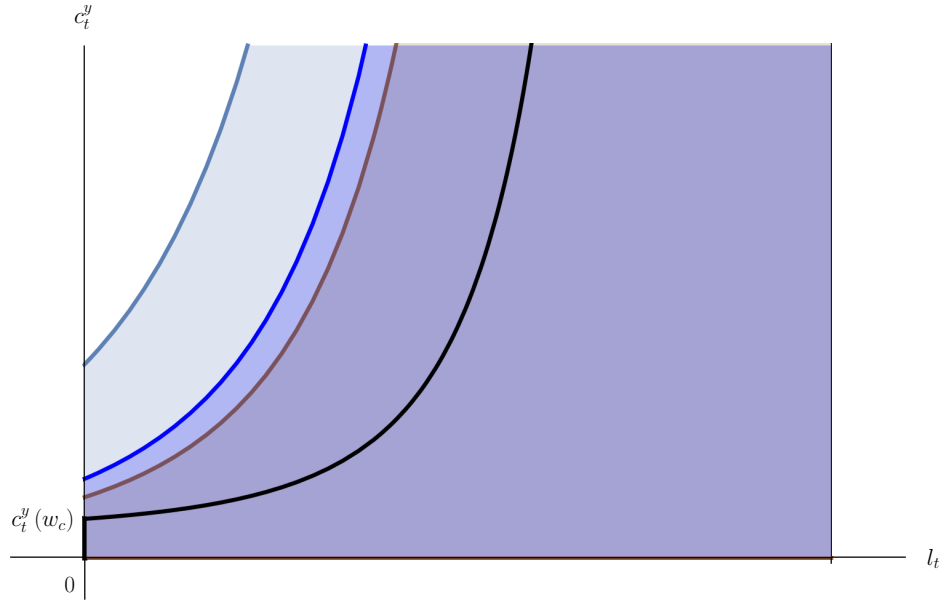
This behavior makes intuitive sense. When wages and incomes are low then the individual demand for consumption goods is positive to satisfy basic needs. The demand for leisure is zero since the only way to earn a decent income is by working the maximum of available hours. Rising wages and incomes allow people to adequately satisfy their consumption needs and to work less. Accordingly, the demand for leisure becomes positive.

For Regime 1, the Proof of Proposition 1 reveals that the optimal plan of cohort  $t$  involves

$$\phi x_t = \phi x = \frac{(1 + \beta)(1 - \nu)}{1 + (1 + \beta)(1 - \nu)} \in (0, 1). \quad (2.8)$$

From the definition of  $x_t$  in (2.2) it is obvious that condition (2.8) ties the optimal choices of  $c_t^y$  and  $l_t$  for all  $t$ . The black curve in Figure 2.5 starting at  $(0, c_t^y(w_c))$  depicts a typical set of pairs  $(l_t, c_t^y)$  satisfying this condition. It shows that less work goes along with more consumption when young. Assumption 1 assures that the entire black line lies inside the set  $\mathcal{P}$ .

Figure 2.5: The black line shows the optimal pairs  $(l_t, c_t^y)$  for all  $t$ . Points on the vertical line  $[0, c_t^y(w_c)]$  correspond to optimal pairs under Regime 2. Points on the black curve satisfy (2.8) and represent optimal pairs under Regime 1.



To see that in Regime 1 savings are independent of the real rental rate consider the intertemporal trade-off between consumption when young and consumption when old. It is governed by the first-order condition  $U_1 = R_{t+1}U_3$ . It says that, for any given  $l_t$ , the marginal utility of consumption when young is equal to its opportunity cost in terms of consumption when old. Evaluated at optimal pairs  $(c_t^y, l_t)$  that satisfy (2.8) this trade-off



delivers the Euler equation

$$\frac{c_{t+1}^o}{c_t^y} = \frac{\beta R_{t+1}}{1 - \nu(1 + \beta)}. \quad (2.9)$$

The latter states the desired consumption growth factor of a member of cohort  $t$ . The parameter  $\nu$  reflects the disutility of consumption when young associated with the labor supply that shows up in the second term of  $U$ . Its presence weakens the tendency to smooth consumption over the life-cycle. In addition, equation (2.9) reveals that the *inter-temporal elasticity of substitution* associated with the optimal plan of all cohorts  $t$  is equal to unity, i. e.,

$$\frac{d \ln \left( \frac{c_{t+1}^o}{c_t^y} \right)}{d \ln R_{t+1}} = 1. \quad (2.10)$$

The Euler equation (2.9) also implies for Regime 1 that  $h_t$ ,  $c_t^y$ , and  $s_t$  are independent of  $R_{t+1}$ . At a deeper level, this finding can be traced back to the fact that individuals value consumption when old according to the logarithmic utility function (see Section 4 for details).

Regime 1 of Proposition 1 exhibits another intuitive property of the optimal plan:  $c_t^y$ ,  $s_t$ , and  $c_{t+1}^o$  are proportionate to the wage income,  $w_t h_t$ . In particular, one finds that

$$c_t^y = \frac{1 - \nu(1 + \beta)}{(1 + \beta)(1 - \nu)} w_t h_t \quad \text{and} \quad s_t = \frac{\beta}{(1 + \beta)(1 - \nu)} w_t h_t. \quad (2.11)$$

Hence, *ceteris paribus*, the marginal (and average) propensity to consume when young declines in  $\nu$ , the marginal propensity to save out of wage income increases in  $\nu$ .

For Regime 2, the optimal plan is no longer available in closed form. This complication is due to the presence of the disutility of consumption when young associated with the supply of  $h_t = 1$  hours of work.<sup>11</sup> Since the utility function of consumption when old is logarithmic individual savings do not hinge on the real rental rate of capital (again, see Section 4 for details).

Next, I show that the optimal plan has intuitive behavioral properties.

**Proposition 2** (*Factor Prices and the Optimal Plan*)

*Consider the optimal plan of Proposition 1. It holds that*

$$h'(w_t) \leq 0, \quad (c^y)'(w_t) > 0, \quad c_1^o(w_t, R_{t+1}) > 0, \quad c_2^o(w_t, R_{t+1}) > 0, \quad \text{and} \quad s'(w_t) > 0.$$

---

<sup>11</sup> As  $\nu \rightarrow 0$ , the expressions of Regime 2 converge, respectively, to  $c_t^y = w_t / (1 + \beta)$ ,  $s_t = \beta w_t / (1 + \beta)$ , and  $c_{t+1}^o = \beta R_{t+1} w_t / (1 + \beta)$ , i. e., they coincide with the solution of the canonical OLG-model.

For Regime 2, this result is to be expected. As  $h'(w_t) = 0$ , a higher real wage increases real income one-to-one. Then, consumption smoothing requires that the higher income is used to increase consumption when young and old, hence savings. For Regime 1, a similar intuition holds since

$$\frac{d \ln(w_t h_t)}{d \ln w_t} = 1 - \nu > 0,$$

i. e., the proportionate increase in the wage income induced by a higher wage is still positive even though the labor supply declines. Clearly, only  $c_{t+1}^o$  increases in response to a higher  $R_{t+1}$ .

The optimal plan of Proposition 1 lends itself to the following intuitive comparative statics.

**Proposition 3** (*Comparative Statics of the Optimal Plan*)

Consider the optimal plan of Proposition 1.

For Regime 1, it holds that

$$\begin{aligned} \frac{\partial h_t}{\partial \phi} < 0, \quad \frac{\partial c_t^y}{\partial \phi} < 0, \quad \frac{\partial c_{t+1}^o}{\partial \phi} < 0, \quad \frac{\partial s_t}{\partial \phi} < 0, \\ \frac{\partial h_t}{\partial \beta} > 0, \quad \frac{\partial c_t^y}{\partial \beta} < 0, \quad \frac{\partial c_{t+1}^o}{\partial \beta} > 0, \quad \frac{\partial s_t}{\partial \beta} > 0. \end{aligned}$$

For Regime 2, it holds that

$$\begin{aligned} \frac{\partial c_t^y}{\partial \phi} < 0, \quad \frac{\partial c_{t+1}^o}{\partial \phi} > 0, \quad \frac{\partial s_t}{\partial \phi} > 0, \\ \frac{\partial c_t^y}{\partial \beta} < 0, \quad \frac{\partial c_{t+1}^o}{\partial \beta} > 0, \quad \frac{\partial s_t}{\partial \beta} > 0. \end{aligned}$$

Proposition 3 shows that the comparative statics properties of the optimal plan hinge on whether the supply of hours worked responds to the respective parameter change or not. First, consider Regime 1. For a greater  $\phi$  the disutility of labor is more pronounced. Accordingly, the labor supply falls. Consumption smoothing dictates that the concomitant decline in the wage income reduces consumption in both periods of life, hence, savings. Consumption when young is further reduced since the marginal utility of  $c_t^y$  falls in  $\phi$ . A greater  $\beta$  increases the value of consumption when old. Therefore,  $c_{t+1}^o$  increases at the expense of the demand of leisure and of consumption when young. Accordingly, the labor supply and savings increase.

In Regime 2 parameter changes do not affect the labor supply. However, a greater  $\phi$  reduces the marginal utility of consumption when young whereas a greater  $\beta$  increases the value of consumption when old. Hence, in both cases,  $c_t^y$  falls whereas  $s_t$  and  $c_{t+1}^o$  increase.

### 2.1.2 Some Orders of Magnitude

The validity of Proposition 1 hinges on the parameter restrictions summarized under Assumption 1. The main purpose of this section is to show by example that the optimal plan is consistent with reasonable magnitudes for key parameters of the model. In addition, I present a set of parameter values that delivers particularly simple functional forms for Regime 1. The latter will be used in Section 3 below.

To derive reasonable parameter values let a generation correspond to 30 years. Then,  $\beta$  is the discount factor over 30 years. The annual discount factor is often estimated to be around 0.96 (see, e. g., Prescott (1986)).<sup>12</sup> This implies  $\beta = 0.294$ , and, from Assumption 1, a corresponding critical value  $\bar{\nu} = 0.352$ .

Suppose that hours worked per worker and the real wage grow at constant annual rates. I take the constant annual growth rate of hours worked per worker to be 0.57% (see Footnote 1). Moreover, suppose the annual growth rate of the real wage is 2%. To match these data with the model the growth factors of hours worked and of the real wage should satisfy

$$\frac{h_{t+1}}{h_t} = 0.9943^{30} \quad \text{and} \quad \frac{w_{t+1}}{w_t} = 1.02^{30}.$$

According to Proposition 1 these growth factors are linked, i. e.,

$$\frac{h_{t+1}}{h_t} = \left( \frac{w_{t+1}}{w_t} \right)^{-\nu} \quad \text{or} \quad 0.9943^{30} = 1.02^{-30\nu}.$$

This gives an estimate of  $\nu$  as,

$$\nu = -\frac{\ln .9943}{\ln 1.02} = 0.288 < \bar{\nu},$$

and Assumption 1 is satisfied. However, observe that these calculations are sensitive to the value used for the annual growth rate of real wages. The smaller this growth rate the more difficult is it to satisfy Assumption 1. For instance, keeping all other parameter values as stated, Assumption 1 will be violated if the growth rate of wages falls short of roughly 1.6%.

A particular simple calibration obtains if I set

$$\nu = \frac{1}{4}, \quad \beta = \frac{1}{3}, \quad \text{and} \quad \phi = \frac{1}{2} \left( \frac{3}{2} \right)^{\frac{1}{3}}. \quad (2.12)$$

---

<sup>12</sup>The annual discount factor chosen by Blanchard and Fischer (1989), p. 147, is 0.97 which delivers a value  $\beta = 0.40$ . Barro and Sala-i-Martin (2004), p. 197, use an annual discount factor of 0.98 which corresponds to a value  $\beta = 0.55$ . Hence, the value of  $\beta$  is quite sensitive to the chosen value of the annual discount factor. However, it hardly impacts on the value of  $\bar{\nu}$  which is equal to 0.342 and 0.33 for an annual discount factor of .97 and .98, respectively.

While the values for  $\nu$  and  $\beta$  are not far away from those used above, this calibration involves a judicious choice for  $\phi$  so that  $z(\phi, \beta, \nu) = w_c = 1$ . Assumption 1 is satisfied since  $1/4 < (5 - \sqrt{13})/4$ . Moreover, one readily verifies that the optimal plan for Regime 1 involves

$$h_t = w^{-\frac{1}{4}}, \quad c_t^y = \frac{2}{3}w^{\frac{3}{4}}, \quad c_{t+1}^o = \frac{R_{t+1}}{3}w_t^{\frac{3}{4}}, \quad \text{and} \quad s_t = \frac{w^{\frac{3}{4}}}{3}.$$

## 2.2 Firms

At all  $t$ , the production sector can be represented by a single competitive firm with access to the production function

$$Y_t = \Gamma \cdot K_t^\gamma \cdot (A_t H_t)^{1-\gamma}, \quad \Gamma > 0, \quad 0 < \gamma < 1. \quad (2.13)$$

Here,  $K_t$  is physical capital and  $H_t$  the amount of hours of work employed by the firm. Technological knowledge is represented by  $A_t$  and advances exogenously at rate  $g_A > 0$ . Accordingly,  $A_t = (1 + g_A)^{t-1} A_1$ , with  $A_1 > 0$  given. The productivity parameter  $\Gamma > 0$  may reflect cross-country differences in geography, technical and social infrastructure that affect the transformation of capital and efficient hours worked into the manufactured good.

In each period, the firm chooses the amounts of capital,  $K_t$ , and of hours of work,  $H_t$ , to maximize the net-present value of profits. Doing so, it takes the evolution of  $A_t$  as given. Void of inter-temporal considerations, the respective first-order conditions read

$$w_t = \Gamma(1 - \gamma)A_t k_t^\gamma \quad \text{and} \quad R_t = \Gamma \gamma k_t^{\gamma-1}, \quad (2.14)$$

where  $k_t = K_t / A_t H_t$  is the efficient capital intensity at  $t$ , i.e., the amount of capital per efficient hours worked.

## 3 Intertemporal General Equilibrium

### 3.1 Definition

A *price system* corresponds to a sequence  $\{w_t, R_t\}_{t=1}^\infty$ . An *allocation* is a sequence

$$\{c_t^y, l_t, c_t^o, s_t, Y_t, H_t, K_t\}_{t=1}^\infty.$$

It comprises a plan  $\{c_t^y, l_t, c_{t+1}^o, s_t\}_{t=1}^\infty$  for all cohorts, consumption of the old at  $t = 1$ ,  $c_1^o$ , and a strategy for the production sector  $\{Y_t, H_t, K_t\}_{t=1}^\infty$ .

For an exogenous evolution of the labor force,  $L_t = L_1 (1 + g_L)^{t-1}$  with  $L_1 > 0$ , an exogenous evolution of technological knowledge,  $A_t = A_1 (1 + g_A)^{t-1}$  with  $A_1 > 0$ ,

and a given initial level of capital,  $K_1 > 0$ , an intertemporal general equilibrium with perfect foresight corresponds to a price system and an allocation that satisfy the following conditions for all  $t = 1, 2, \dots, \infty$ :

(E1) The plan of each cohort satisfies Proposition 1.

(E2) The production sector satisfies (2.14).

(E3) The market for the manufactured good clears, i. e.,

$$L_{t-1}c_t^0 + L_t c_t^y + I_t = Y_t, \quad (3.1)$$

where  $I_t$  is aggregate capital investment.

(E4) There is full employment of labor, i. e.,

$$H_t = L_t h_t. \quad (3.2)$$

(E1) guarantees the optimal behavior of the household sector under perfect foresight. Since the old own the capital stock, their consumption at  $t = 1$  is  $L_0 c_1^0 = R_1 K_1$  and  $s_0 = K_1 / L_0$ . (E2) assures the optimal behavior of the production sector and zero profits. (E3) states that the aggregate demand for the manufactured good produced at  $t$  is equal to its supply. It reflects the fact that capital fully depreciates after one period. Alternatively, (3.1) may be interpreted as the resource constraint of the economy at  $t$ . According to (E4) the demand for hours worked must be equal to the supply.

The labor market requires a special treatment. Since both the aggregate demand for hours worked and the aggregate supply of hours worked are decreasing in the real wage there may be none, one, or multiple wage levels at which demand is equal to supply. To address this issue let me refer to the first condition of (2.14) as the firms' aggregate demand for hours worked at  $t$  and restate it as

$$H_t^d = K_t A_t^{\frac{1-\gamma}{\gamma}} \left( \frac{\Gamma(1-\gamma)}{w_t} \right)^{\frac{1}{\gamma}} \equiv H_t^d(w_t). \quad (3.3)$$

Let  $H_t^s = L_t h_t$  denote the aggregate supply of hours worked at  $t$ . Then, using Proposition 1 I have

$$H_t^s = \left\{ \begin{array}{ll} L_t w_c^v w_t^{-v} & \text{if } w_t \geq w_c \\ L_t \cdot 1 & \text{if } 0 < w_t \leq w_c \end{array} \right\} \equiv H_t^s(w_t), \quad (3.4)$$

and the labor market equilibrium can be characterized as follows.

**Proposition 4** (*Labor-Market Equilibrium at  $t$* )

At all  $t$ , there is a unique labor market equilibrium,  $(\hat{w}_t, \hat{H}_t)$ . The equilibrium real wage is

$$\hat{w}_t = \begin{cases} \left( \frac{K_t}{A_t L_t} \right)^{\frac{\gamma}{1-\gamma\nu}} (A_t \cdot \Gamma(1-\gamma))^{\frac{1}{1-\gamma\nu}} \cdot (w_c)^{\frac{-\gamma\nu}{1-\gamma\nu}} & \text{if } \left( \frac{K_t}{A_t L_t} \right)^{\gamma} A_t \cdot \Gamma(1-\gamma) \geq w_c, \\ \left( \frac{K_t}{A_t L_t} \right)^{\gamma} A_t \cdot \Gamma(1-\gamma) & \text{if } \left( \frac{K_t}{A_t L_t} \right)^{\gamma} A_t \cdot \Gamma(1-\gamma) \leq w_c. \end{cases} \quad (3.5)$$

The equilibrium amount of hours worked is

$$\hat{H}_t = \begin{cases} \left[ L_t \left( \frac{w_c}{K_t^{\gamma} A_t^{1-\gamma} \Gamma(1-\gamma)} \right)^{\nu} \right]^{\frac{1}{1-\gamma\nu}} & \text{if } \left( \frac{K_t}{A_t L_t} \right)^{\gamma} A_t \cdot \Gamma(1-\gamma) \geq w_c, \\ L_t \cdot 1 & \text{if } \left( \frac{K_t}{A_t L_t} \right)^{\gamma} A_t \cdot \Gamma(1-\gamma) \leq w_c. \end{cases} \quad (3.6)$$

Hence, at all  $t$ , the labor market equilibrium exists and is unique. This is due to two properties of the labor market that are illustrated in Figure 3.1. First, since  $\nu$  is quite small the individual and the aggregate supply of hours worked,  $h(w_t)$  and  $H_t^s(w_t)$ , is fairly flat. Second, the image of the aggregate demand for hours worked,  $H_t^d(w_t)$ , is  $\mathbb{R}_{++}$  since the aggregate production function satisfies both Inada conditions.

For a given aggregate supply of hours worked the labor market equilibrium falls into Regime 1 if the aggregate demand for hours worked is large (see  $(\hat{w}_t^1, \hat{H}_t^1)$  in Figure 3.1). From (3.3) this is more likely the greater  $K_t$ ,  $A_t$ , or  $\Gamma$ . Intuitively, modern industrialized economies should possess these features. Conversely, economies with a low demand for hours worked would find their labor market equilibrium in Regime 2 (see  $(\hat{w}_t^2, \hat{H}_t^2)$  where  $\hat{H}_t^2 = L_t \cdot 1$  in Figure 3.1).

For a given aggregate demand for hours worked the labor market equilibrium is more likely to be in Regime 1 the smaller the total amount of workers, i.e., the smaller  $L_t$ . Intuitively, when  $L_t$  falls then the aggregate supply of hours worked shifts downwards. Labor becomes scarcer so that the equilibrium wage increases. Then, even for a low aggregate demand of hours worked such as  $H_t^{d2}$  an equilibrium wage in Regime 1 is possible.

### 3.2 Dynamical System for Regime 1 and Steady-State Analysis

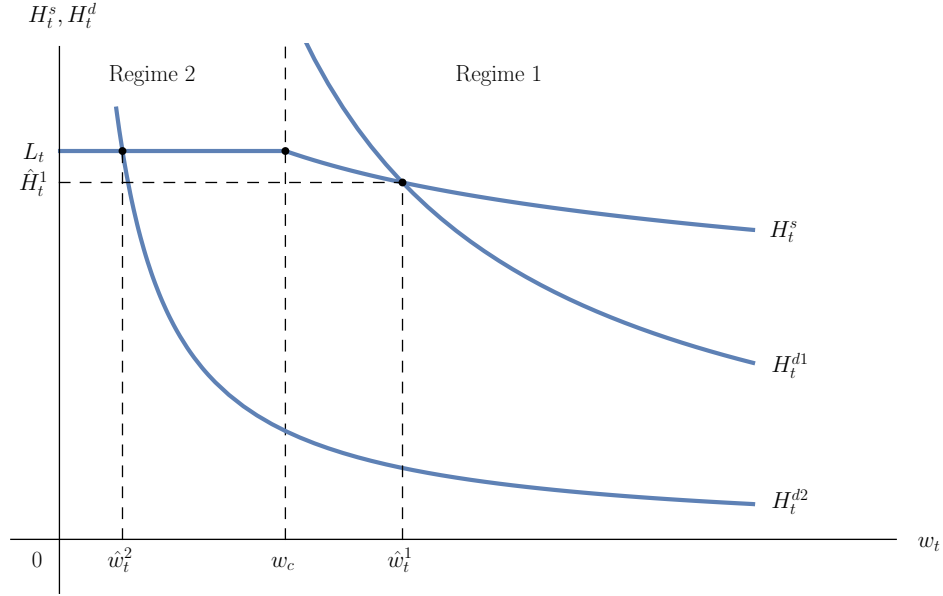
The equilibrium conditions (E1) - (E4) require for all  $t$  that aggregate saving equals capital investment, i.e.,<sup>13</sup>

$$s_t L_t = I_t = K_{t+1}. \quad (3.7)$$

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<sup>13</sup>More precisely, if firms and households optimize, the capital market in period  $t-1$ , the labor market in period  $t$ , and market for the manufactured good in period  $t$  clear, then the market for capital in period  $t$  will

Figure 3.1: The labor market equilibrium of period  $t$ . If the aggregate demand for hours worked is  $H_t^{d1}$  then the labor market equilibrium is  $(\hat{w}_t^1, \hat{H}_t^1)$ . The individual supply of hours worked is in Regime 1, i. e., it falls in  $w_t$ , and aggregate demand for hours worked is high. If the aggregate demand for hours worked is  $H_t^{d2}$  then the labor market equilibrium is  $(\hat{w}_t^2, \hat{H}_t^2)$  where  $\hat{H}_t^2 = L_t \cdot 1$ . The individual supply of hours worked is in Regime 2, i. e., it does not hinge on  $w_t$ , and aggregate demand for hours worked is low.



Using Proposition 1 and (2.14) in (3.7) reveals that the intertemporal general equilibrium may be studied by means of a sequence of a single state variable, the efficient capital

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also clear, i. e., (3.7) will hold. To see this consider the market clearing condition of the manufactured good (3.1). (E1) means that  $c_t^o$  and  $c_t^y$  may be replaced by the respective budget constraints of cohort  $t - 1$  and  $t$ . This gives

$$R_t s_{t-1} L_{t-1} + w_t h_t L_t - s_t L_t + I_t = Y_t.$$

Clearing of the capital market in  $t - 1$  means that  $s_{t-1} L_{t-1} = I_{t-1} = K_t$ , clearing of the labor market in  $t$  means that  $h_t L_t = H_t$ . Then, the market clearing condition of the manufactured good may be written as

$$R_t K_t + w_t H_t - s_t L_t + I_t = Y_t.$$

Finally, (E2) implies that firms make zero profits, i. e.,  $R_t K_t + w_t H_t = Y_t$ , and, by definition, we have  $I_t = K_{t+1}$ . Hence, (3.7) follows.

intensity,  $k_t$ . This variable will be constant in steady state. To analyze the steady state and the dynamics in its neighborhood I focus this section on Regime 1 and postpone the discussion of the dynamics involving Regime 2 to Section 3.3.

Define

$$k_c \equiv \left[ \frac{w_c}{\Gamma(1-\gamma)A_1} \right]^{\frac{1}{\gamma}} \quad (3.8)$$

and

$$k^* \equiv \left[ \frac{\beta\Gamma(1-\gamma)}{(1+\beta)(1-\nu)(1+g_L)(1+g_A)^{1-\nu}} \right]^{\frac{1}{1-\gamma}}. \quad (3.9)$$

**Assumption 2** *It holds that  $k_1 \geq k_c$  and  $k^* > k_c$ .*

The intuition behind Assumption 2 is the following. The parameter  $k_c$  assures that  $w_1 \geq w_c$ . Hence, if  $k_1 \geq k_c$  the economy starts with a real wage compatible with Regime 1. As will become clear from the following proposition,  $k^*$ , is the steady state of the equilibrium sequence,  $\{k_t\}_{t=1}^{\infty}$ . Then,  $k^* > k_c$  assures that over time the equilibrium sequence remains in Regime 1.

**Proposition 5** *(Dynamical System - Regime 1)*

*Suppose the initial conditions  $(K_1, L_1, A_1)$  are such that Assumption 2 holds. Then, the transitional dynamics of the intertemporal general equilibrium is given by a unique and monotonous equilibrium sequence  $\{k_t\}_{t=1}^{\infty}$ , generated by the difference equation*

$$k_{t+1} = \left[ \frac{\beta\Gamma(1-\gamma)}{(1+\beta)(1-\nu)(1+g_L)(1+g_A)^{1-\nu}} \right]^{\frac{1}{1-\gamma\nu}} \times k_t^{\frac{\gamma(1-\nu)}{1-\gamma\nu}}, \quad (3.10)$$

where

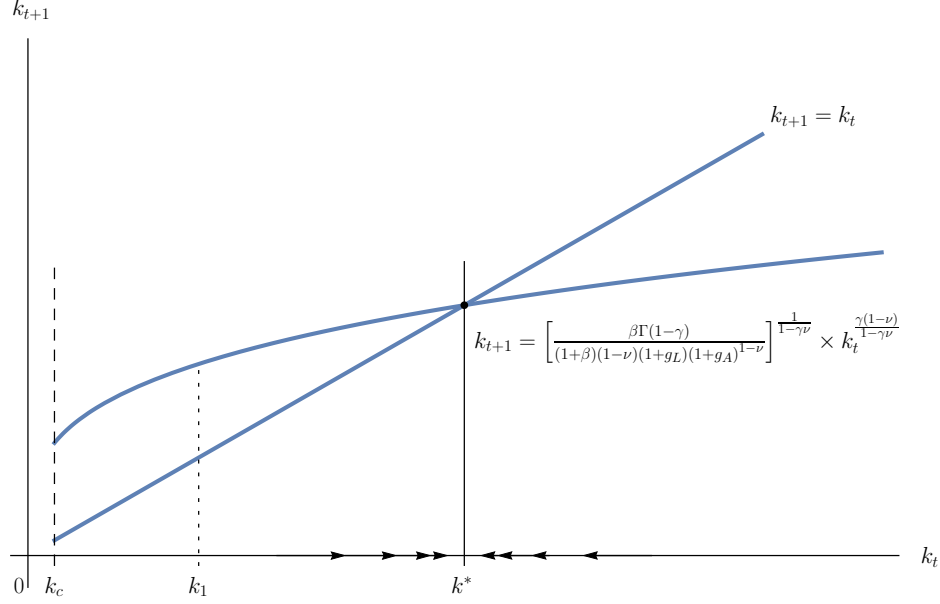
$$k_1 = \frac{K_1}{A_1 \hat{H}_1} \quad \text{and} \quad \lim_{t \rightarrow \infty} k_t = k^*. \quad (3.11)$$

Since  $\gamma(1-\nu)/(1-\gamma\nu) < 1$  the equilibrium sequence generated by (3.10) is monotonous and the steady state is stable, i. e., for  $k_1 < k^*$  the equilibrium sequence  $\{k_t\}_{t=1}^{\infty}$  increases monotonically, for  $k_1 > k^*$  it falls monotonically. This is illustrated in Figure 3.2.

The key element in the difference equation (3.10) is the parameter  $\nu$ . It affects the equilibrium sequence through three channels. First, the factor  $(1-\nu)$  in the denominator of the bracketed term shows the effect of  $\nu$  on the marginal propensity to save (see, equation



Figure 3.2: The Dynamical System under Regime 1.



(2.11)). A greater  $\nu$  increases the fraction of the wage income that is saved and invested, hence,  $k_{t+1}$  increases.

The second and the third channel are related and reflect the acceleration in the evolution of the efficient capital intensity due to a decline in hours worked. In fact, given  $k_t$  one finds from the proof of Proposition 5 that  $k_{t+1}$  takes on a larger value the greater  $h_t/h_{t+1}$ . From Proposition 1 and equation (2.14) the latter is equal to

$$\frac{h_t}{h_{t+1}} = \left( \frac{w_t}{w_{t+1}} \right)^{-\nu} = \frac{1}{(1+g_A)^{-\nu}} \left( \frac{k_t}{k_{t+1}} \right)^{-\gamma\nu}.$$

This shows that the decline in hours worked is eventually due to the growth of  $A_t$  and the growth of  $k_t$ . The presence of  $(1+g_A)^{-\nu}$  in the denominator of the bracketed term can be traced back to the effect of technological progress on the evolution of the real wage and the concomitant decline in hours worked. This is the second channel. The third channel is represented by the factor  $(k_t/k_{t+1})^{-\gamma\nu}$ . It reflects the effect of a growing efficient capital intensity on the evolution of the real wage and its impact on hours worked. This channel modifies the exponent of  $k_t$  on the right-hand side of (3.10) and accelerates the process of

convergence towards the steady state. Indeed, the speed of convergence defined as

$$-\frac{\partial \ln \left( \frac{k_{t+1}}{k_t} \right)}{\partial \ln k_t} = \frac{1 - \gamma}{1 - \gamma \nu} > 0$$

increases in  $\nu$ .

In the limit  $\nu \rightarrow 0$  the difference equation (3.10) boils down to the one of the canonical OLG-model. If, in addition  $\phi = (1 + \beta)/(2 + \beta)$ , then in this limit I also have  $h_t = 1$ ,  $w_c = 0$ , and  $k_c = 0$ . Hence, Regime 1 encompasses the canonical growth model with or without technological progress as special cases.

The following proposition characterizes the steady state.

**Proposition 6** (*Properties of the Steady State*)

*Along the steady-state path the growth rate of the real wage is  $g_w = g_A > 0$ , the real rental rate of capital is constant. Moreover, it holds that*

$$\begin{aligned} a) \quad & \frac{h_{t+1}}{h_t} = (1 + g_A)^{-\nu}, \quad \frac{\hat{H}_{t+1}}{\hat{H}_t} = (1 + g_A)^{-\nu} (1 + g_L), \\ b) \quad & \frac{c_{t+1}^y}{c_t^y} = \frac{c_{t+1}^o}{c_t^o} = \frac{s_{t+1}}{s_t} = (1 + g_A)^{1-\nu}, \\ c) \quad & \frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = (1 + g_A)^{1-\nu} (1 + g_L), \\ d) \quad & \frac{\partial (1 + g_A)^{1-\nu}}{\partial \nu} < 0 \quad \text{and} \quad \frac{\partial k^*}{\partial \nu} > 0. \end{aligned}$$

Hence, in steady state the individual supply of hours worked declines at an approximate rate  $\nu g_A$  since  $(-\nu)$  is the wage elasticity of  $h(w_t)$ . The steady-state growth rate of the aggregate supply of hours worked is approximately equal to  $-\nu g_A + g_L$ . It reflects the intensive and the extensive margin of the labor supply. Depending on which margin dominates it may be positive or negative. The growth rates under b) follow from Proposition 1 as the wage elasticity of  $c_t^y$ ,  $c_{t+1}^o$ , and  $s_t$  is  $1 - \nu$ .

The findings under a) and b) highlight why the optimal plan of Proposition 1 is consistent with a steady state equilibrium. The steady-state growth factor of individual hours worked is  $(1 + g_h) = (1 + g_A)^{-\nu}$ , the one of  $c_t^y$ ,  $c_{t+1}^o$ , and  $s_t$  is  $(1 + g_A)^{1-\nu}$ . In steady state, individual wage income,  $w_t h_t$ , grows at a factor  $(1 + g_h)(1 + g_w) = (1 + g_A)^{1-\nu}$  that coincides with the growth factor of  $c_t^y$ ,  $c_{t+1}^o$ , and  $s_t$ . Therefore, these growth patterns are indeed consistent with the individual and the economy-wide budget constraints.

At the level of economic aggregates we obtain from (3.7) that in steady state  $(1 + g_K) = (1 + g_A)^{1-\nu} (1 + g_L)$ . Then, the production function delivers  $g_Y = g_K$ .

Overall, the rule is that the steady-state growth factor of economic aggregates like  $Y_t$ ,  $K_t$ , or aggregate consumption,  $L_t c_t^y + L_{t-1} c_t^o$ , is the growth factor of aggregate efficient hours worked,  $A_t H_t = A_t L_t h_t$ . The growth factor of per-capita variables like  $c_t^y$ ,  $c_{t+1}^o$ ,  $s_t$ , or output per worker,  $Y_t/L_t$ , is the one of efficient individual hours worked,  $A_t H_t/L_t = A_t h_t$ . The latter growth factor is  $(1 + g_A)^{1-\nu}$  and reflects the attenuating effect of a declining individual supply of hours worked on the growth rate of per-capita variables.

According to d), the attenuation of the growth factor is more pronounced the greater is  $\nu$ . Hence, the growth rate of per-capita variables declines in  $\nu$ , and, ceteris paribus, an economy with a greater  $\nu$  is predicted to grow slower in per-capita terms. However, a greater  $\nu$  has a positive level effect. Ceteris paribus, an economy with a greater  $\nu$  has a greater steady-state efficient capital intensity. Intuitively, form (3.9) changing  $\nu$  induces two reinforcing effects on  $k^*$ . First, greater  $\nu$  means that the marginal propensity to save increases. Second,  $(1 + g_A)^{1-\nu}$  becomes smaller. An important implication of this result is that an economy with a greater  $\nu$  has a greater steady-state output of the manufactured good.<sup>14</sup> Hence, more will be consumed and invested.

Finally, it is worth mentioning that for all adjacent periods  $t$  and  $t + 1$  hours worked per worker and hours worked per capita grow at the same rate. To see this, denote the population at  $t$  by  $N_t = L_t + L_{t-1}$ . Then, hours worked per capita at  $t$  is the product of hours worked per worker and the labor-market participation rate, i. e.,

$$\frac{H_t}{N_t} = h_t \times \frac{L_t}{L_t + L_{t-1}} = h_t \times \frac{1 + g_L}{2 + g_L}.$$

Hence, in line with the cross-country evidence over the long run the participation rate is constant (Boppart and Krusell (2016)). Moreover, the growth factor of hours worked is  $h_{t+1}/h_t$  and, in steady state, equal to  $(1 + g_A)^{-\nu}$ .

### 3.3 Global Dynamics: Technological Progress as an Engine of Liberation

This section studies the global dynamics of the economy of Section 2. The analysis reveals that sustained technological progress is the main cause for why workers have enjoyed more and more leisure over time. It liberated poor individuals from the necessity to supply long hours of work to assure a subsistence income. In this sense, technological progress has been an engine of liberation.<sup>15</sup> In the present analytical framework the intuition behind this assertion is the following.

<sup>14</sup>**Proof:** In steady state (2.13) delivers  $Y_t = \Gamma A_t H_t (k^*)^\gamma = \Gamma A_t L_t h (w(k^*)) (k^*)^\gamma$ . Then,

$$\frac{\partial Y_t}{\partial \nu} = \Gamma A_t L_t (k^*)^{\gamma-1} h (w(k^*))^\gamma (1 - \nu) \frac{\partial k^*}{\partial \nu} > 0$$

since  $\partial k^*/\partial \nu > 0$ . ■

<sup>15</sup>In a related sense, the metaphor “engine of liberation” is also used by Greenwood, Seshadri, and Yorukoglu (2005) to describe the role of technological change for the liberation of women from the home. Galor and Weil (1996) develop a related idea.

On the supply side, technological progress increases the marginal product of labor. Accordingly, real wages increase. During the transition to the steady state the growth rate of real wages reflects the growth rates of the physical capital stock and of the labor force, too. However, in the long run it is technological progress alone that drives real wage growth. In addition, with technological progress the total factor productivity and the aggregate output of the manufactured good increases.

On the household side, individuals who see their real income increase want to buy more of the consumption good and consider working less. Additional purchases of the consumption good become feasible since technological progress allows for the total output of the consumption good to increase. The desire to work less is due to the consumption-leisure complementarity. As consumption per capita increases the valuation of leisure increases. Accordingly, individuals want to enjoy more leisure and supply less labor.

I start out in Section 3.3.1 with the analysis of an economy void of technological progress. Initially, the economy is in Regime 2. Hence, real wages are low and individuals are poor. As a consequence, they supply their entire time endowment to the labor market. For the chosen parameter values, I establish that this economy converges towards a steady state with a constant real wage below  $w_c$ . Hence, while real wages may grow over time due to capital accumulation individuals remain poor and supply their entire time endowment to the labor market.

Section 3.3.2 adds sustained technological progress to an otherwise identical economy. The initial state of the economy is again in Regime 2. However, due to technological progress the economy evolves from Regime 2 into Regime 1 where the supply of individual hours worked continuously declines. Eventually, there is convergence towards the steady state of Proposition 5.

To straighten the presentation I choose particular parameter values and make the following simplifying assumptions. On the production side, let  $\Gamma = 3/2$  and  $\gamma = 1/3$ . Then, from (2.14) the inverse aggregate demand for hours worked is

$$w_t = A_t^{\frac{2}{3}} \left( \frac{K_t}{H_t^d} \right)^{\frac{1}{3}}. \quad (3.12)$$

On the household side, I use the parameter values of (2.12). To simplify further, I abstract from population growth and set  $L_t = 1$  for all  $t$ . Then, the evolution of the capital stock (3.7) simplifies to

$$s(w_t) = K_{t+1}. \quad (3.13)$$

### 3.3.1 No Technological Progress: The Equilibrium Dynamics in Regime 2

Consider the economy of Section 2 void of technological progress, i. e.,  $A_t = 1$  for all  $t$ . The initial state of the economy is in Regime 2. Hence, the aggregate supply of hours

worked is  $H_t^s = 1 \cdot 1$ . Then, (3.12) delivers the equilibrium real wage as  $\hat{w}_t = K_t^{1/3}$ . Combining the latter with (3.13) delivers the evolution of the equilibrium real wage in Regime 2 as

$$\hat{w}_{t+1} = [s(\hat{w}_t)]^{\frac{1}{3}}. \quad (3.14)$$

The following proposition characterizes the steady state and the transitional dynamics of Regime 2.

**Proposition 7** (*Dynamical System - Regime 2*)

*The difference equation (3.14) gives rise to a unique, strictly positive steady-state equilibrium real wage  $\hat{w}^{**} < w_c$  given by*

$$\hat{w}^{**} = [s(\hat{w}^{**})]^{\frac{1}{3}}. \quad (3.15)$$

*Suppose  $0 < \hat{w}_1 < w_c$  then the sequence  $\{\hat{w}_t\}_{t=1}^{\infty}$  generated by (3.14) converges monotonically with  $\lim_{t \rightarrow \infty} \hat{w}_t = \hat{w}^{**}$ .*

The point of Proposition 7 is that a poor economy may not escape from poverty without technological progress but remain forever stuck in Regime 2. The reason is that wage growth is driven by the process of capital accumulation alone. Due to diminishing returns the latter eventually peters out and the growth of wages comes to halt. This tendency cannot be outweighed by the consumption-leisure complementarity that reduces the marginal utility of consumption when young and, thus, implies higher savings per worker.<sup>16</sup> The following section shows that sustained technological progress annihilates the possibility of a steady state involving a stationary real wage.

### 3.3.2 Global Dynamics with Sustained Technological Progress

Sustained technological progress means that  $A_t$  grows over time at a constant rate  $g_A > 0$ . Then, equations (3.12) and (3.13) deliver the evolution of the equilibrium real wage in Regime 2 as

$$\hat{w}_{t+1} = A_t^{\frac{2}{3}} [s(\hat{w}_t)]^{\frac{1}{3}}. \quad (3.16)$$

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<sup>16</sup>To be precise, it is not difficult to show that in Regime 2 consumption when young,  $c^y(w_t)$ , is strictly smaller than  $c^y(w_t) = w_t/(1 + \beta)$  that results for  $\nu = 0$ . Then, the budget constraint when young implies that  $s(w_t)$  for  $\nu > 0$  must exceed  $s(w_t) = \beta w_t / (1 + \beta)$  obtained for  $\nu = 0$ .

The economy starts in Regime 2 with an equilibrium real wage  $\hat{w}_1 < \hat{w}^{**} < w_c$ . Then, as seen above, even without technological progress  $\hat{w}_t$  increases over time. However, technological progress prevents the economy from converging to the steady state of Proposition 7 since the right-hand side of (A.10) shifts up by a factor  $(1 + g_A)^{2/3}$  between any pair of periods  $t$  and  $t + 1$ .

Instead, there is a finite  $t_c$  at which the difference equation (3.16) prescribes a real wage  $w_{t_c+1} > w_c$ . This is illustrated in Figure 3.3. However, since  $w_{t_c+1}$  falls into Regime 1 it is not the equilibrium wage of period  $t_c + 1$ . Intuitively, at  $t_c + 1$  individuals realize that the real wage is so high that they want to reduce their supply of hours worked. Accordingly, the aggregate supply of hours worked becomes  $H_{t_c+1}^s$  of (3.4) for  $w_{t_c+1} > w_c$ . Equating the latter with the aggregate demand for hours worked,  $H_{t_c+1}^d$  of (3.3), delivers the labor market equilibrium  $(\hat{w}_{t_c+1}, \hat{H}_{t_c+1})$ .

Observe that at  $t_c + 1$  the capital stock and the level of technology will be  $K_{t_c+1}$  and  $A_{t_c+1}$ , respectively. Hence, the efficient capital intensity  $k_{t_c+1} = K_{t_c+1} / (A_{t_c+1} \hat{H}_{t_c+1})$  is the initial condition for the dynamical system of Regime 1 as outlined in Proposition 5. It satisfies  $k_{t_c+1} > k_c$ . Hence, the economy will converge towards  $k^*$  with ever declining levels of hours worked.<sup>17</sup> Along the transition capital grows faster than efficient labor. Technological progress provides the countervailing force against a falling marginal product of capital. Accordingly, wages keep growing and individual hours worked decline.

Hence, technological progress drives the economy out of the poverty Regime 2. The advantages of productivity growth are not confined to the possibility to buy larger amounts of the consumption good. They also open the opportunity to enjoy more leisure.

## 4 General Functional Forms and the Consumption-Leisure Complementarity

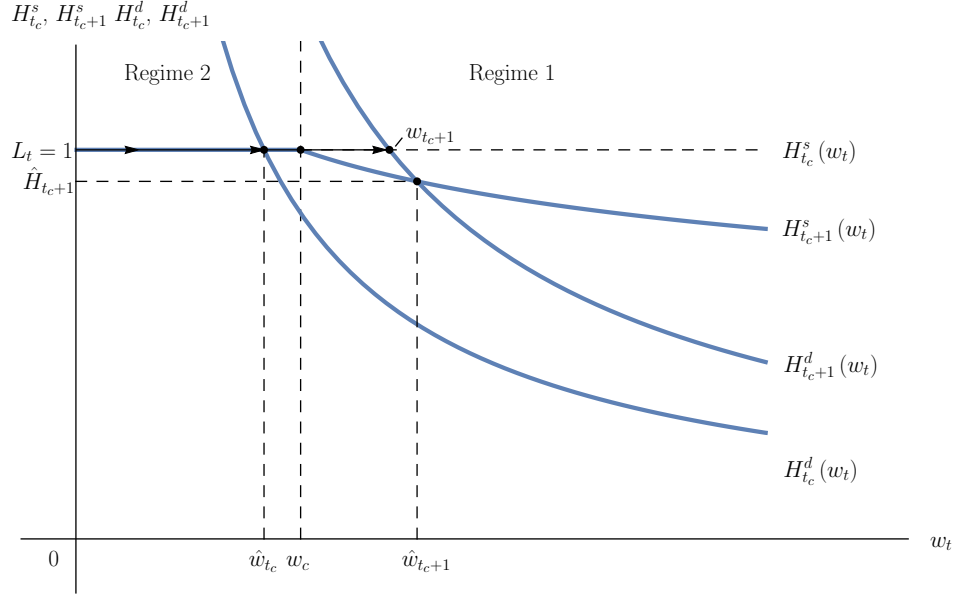
The objective of this section is to deepen the understanding of some key properties of the optimal plan presented in Proposition 1. In particular, I ask three questions. First, what is the role of the consumption-leisure complementarity for a negative response of the individual labor supply to a wage increase? Second, under what conditions will the individual labor supply be independent of the rental rate of capital? Finally, under what conditions will individual savings be independent of the rental rate of capital?

To address these issues I consider a somewhat more general lifetime utility function  $U(c^y, l, c^o)$  which is differentiable, increasing and jointly concave in its arguments. The

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<sup>17</sup>I assume here that the jump from Regime 2 into Regime 1 is not too large in the sense that  $k_{t_c+1} > k^*$ . A purely qualitative argument cannot exclude this case in which convergence may initially include periods of a declining real wage. This however requires that the effect of a declining efficient capital intensity on the real wage exceeds the effect of an increasing level of technology.

Figure 3.3: The switch between Regime 2 and 1. At  $t_c$  the equilibrium in the labor market is  $(\hat{w}_{t_c}, \hat{H}_{t_c})$ . It results from the intersection between  $H_{t_c}^d$  and  $H_{t_c}^s = L_{t_c} = 1$ . Then, the difference equation (3.16) delivers  $w_{t_c+1} > w_c$  which is the intersection between  $H_{t_c+1}^d$  and  $H_{t_c}^s = L_{t_c} = 1$ . However, for wage levels greater than  $w_c$  the equilibrium expression for the aggregate supply of hours worked is  $H_{t_c+1}^s$ . Accordingly, the labor market equilibrium in period  $t_c + 1$  is  $(\hat{w}_{t_c+1}, \hat{H}_{t_c+1})$  where  $\hat{w}_{t_c+1} > w_{t_c+1}$ .



complementarity between consumption when young and leisure means that  $U_{12} > 0$ . Moreover, I assume that  $U_{13} = U_{23} = 0$ , i. e., there is time-separability in consumption, and the consumption-leisure complementarity has no bite when old.

Throughout, I suppress time arguments. Let  $(c^y, l, c^o, s)$  denote the optimal plan that results as the interior solution to the maximization of  $U(c^y, l, c^o)$  subject to the constraints (2.6), i. e.,  $c^y + s = w(1 - l)$  and  $c^o = Rs$ . Moreover, let me introduce the following elasticities

$$\eta_1^U \equiv -c^y \frac{U_{11}}{U_1} > 0 \quad \text{and} \quad \eta_3^U \equiv -c^o \frac{U_{33}}{U_3} > 0. \quad (4.1)$$

They measure the curvature of  $U$  with respect to  $c^y$  and  $c^o$  at the optimal plan. Since  $U_{12} > 0$  whereas  $U_{23} = 0$ , these elasticities may differ. Moreover,  $1/\eta_3^U$  is the intertemporal elasticity of substitution defined as the proportionate change in  $c^o/c^y$  to a change in  $R$

given  $c^y$  and  $l$ .<sup>18</sup>

#### 4.1 Consumption-Leisure Complementarity and the Individual Supply of Hours Worked

The following proposition has the answer to the first and the second question.

**Proposition 8** (*Properties of the Marshallian Demand for Leisure*)

The optimal plan  $(c^y, l, c^o, s)$  involves a demand for leisure that satisfies

$$\frac{dl}{dw} \gtrless 0 \Leftrightarrow -1 + \eta_1^U \left( 1 + \left( \frac{c^o}{Rc^y} \right) \left( 1 - \frac{1}{\eta_3^U} \right) \right) + \frac{U_{21}}{U_1} (1 - l) \gtrless 0, \quad (4.2)$$

$$\frac{dl}{dR} \gtrless 0 \Leftrightarrow - \left( 1 - \eta_3^U \right) \left( \frac{U_2}{c^y} \eta_1^U + U_{21} \right) \gtrless 0. \quad (4.3)$$

These comparative statics state the response of the Marshallian demands for leisure to an increase in the real wage and in the real rental rate. Both responses feature a substitution and an income effect. First, consider the response to a wage hike (4.2). According to the substitution effect, represented by  $(-1)$ , the demand for leisure falls when its opportunity costs increase. The remaining two terms make up for the income effect. It features  $\eta_1^U$ ,  $\eta_3^U$ , and  $U_{21}$ . Interestingly, since both elasticities may differ,  $\eta_1^U > 1$  is no longer sufficient for  $dl/dw > 0$  even if  $U_{21} > 0$  (for instance, think of  $\eta_3^U = 1/2$ ).

However, for the lifetime utility function (2.1) we have  $\eta_3^U = 1$ . Then, (4.2) boils down to

$$\frac{dl}{dw} \gtrless 0 \Leftrightarrow -1 + \eta_1^U + \frac{U_{21}}{U_1} (1 - l) \gtrless 0. \quad (4.4)$$

Here, the income effect is unequivocally positive, and leisure is a normal good. At the optimal plan, i. e., given  $c^y, l, c^o$ , and  $U_1$ , the income effect is more pronounced the greater

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<sup>18</sup>**Proof:** Let  $c \equiv c^o/c^y$ . Then, given  $c^y$  and  $l$ , the consumption-savings trade-off gives rise to the first-order condition

$$U_1(c^y, l, c^y \cdot c) - RU_3(c^y, l, c^y \cdot c) = 0.$$

Total differentiation with respect to  $c$  and  $R$  gives

$$-R \frac{c^o}{c} U_{33}(c^y, l, c^y \cdot c) dc - U_3(c^y, l, c^y \cdot c) dR = 0.$$

Rearranging delivers

$$\frac{R \cdot dc}{c \cdot dR} = \frac{d \ln c}{d \ln R} = \frac{-U_3(c^y, l, c^o)}{c^o \cdot U_{33}(c^y, l, c^o)} = \frac{1}{\eta_3^U}.$$

■



$\eta_1^U$  and  $U_{12}$ . Intuitively, if  $(-U_{11})$ , hence  $\eta_1^U$ , is large then the curvature of  $U$  with respect to consumption when young is strong. Accordingly, spending additional wage income on consumption when young does not add much utility. One would rather want to spend it on leisure. Moreover, if  $U_{12}$  is large then the consumption-leisure complementarity is strong. Spending additional wage income on leisure has a welcome side effect, i.e., it increases the marginal utility of consumption when young. Hence, spending additional wage income on leisure becomes more attractive.

For the lifetime utility function (2.1) one readily verifies that

$$\eta_1^U = \frac{(1-\nu)[1-\nu(1+\beta)(1-\nu(1+\beta))] + \nu^2(1+\beta)}{(1-\nu)(1-\nu(1+\beta))} > 1$$

and

$$\frac{U_{21}(1-l)}{U_1} = \frac{\nu(1+\beta)(1+(1+\beta)(1-\nu))}{1-\nu(1+\beta)} > 0,$$

where  $\lim_{\nu \rightarrow 0} \eta_1^U = 1$  and  $\lim_{\nu \rightarrow 0} U_{21}(1-l)/U_1 = 0$ . Hence, as suggested by Proposition 1,  $dl/dw > 0$  ( $dh/dw < 0$ ) results since  $\nu > 0$  implies  $\eta_1^U > 1$  and  $U_{21} > 0$ . In the limit  $\nu \rightarrow 0$  the demand for leisure will no longer respond to changes in the real wage. However, as  $\nu$  switches to a strictly positive value the consumption-leisure complementarity appears as  $U_{21} > 0$  and spending on consumption when young becomes less attractive as  $\eta_1^U > 1$ . It is this sense that the consumption-leisure complementarity “causes” an income effect that dominates the substitution effect.

Second, consider the response of the Marshallian demand for leisure to a higher rental rate of capital (4.3). The substitution effect reduces the demand for leisure whereas the income effect increases it. Obviously, the total effect hinges on whether  $\eta_3^U \gtrless 1$ . Intuitively, if  $\eta_3^U > 1$  then  $(-U_{33})$  is large. Accordingly, spending income on consumption when old does not add much additional utility. Instead, the individual will demand more leisure (and consumption when young), hence  $dl/dR > 0$ . The lifetime utility function (2.1) has  $\eta_3^U = 1$ . Hence, in accordance with Proposition 1 I find  $dl/dR = 0$ .

## 4.2 Consumption-Leisure Complementarity and Individual Savings

To derive the comparative statics of  $s$  with respect to  $R$  recall that  $s = w(1-l) - c^y$  from the budget constraint when young. Hence, a change in  $R$  delivers

$$\frac{ds}{dR} = -w \frac{dl}{dR} - \frac{dc^y}{dR},$$

where both, the demand for leisure and the demand for consumption when young gives rise to a substitution and an income effect. The following proposition shows that in spite of this complication the comparative statics of individual savings with respect to the rental rate of capital can be readily characterized. This delivers the answer to the third question mentioned above.

**Proposition 9** (*Properties of Individual Savings*)

The optimal plan  $(c^y, l, c^o, s)$  involves savings that satisfies

$$\frac{ds}{dR} \gtrless 0 \quad \Leftrightarrow \quad (1 - \eta_3^U) \left( \frac{wU_2}{c^y} \eta_1^U + 2wU_{21} - U_{22} \right) \gtrless 0. \quad (4.5)$$

Since  $U_{21} > 0$ , the term in parenthesis is strictly positive, and the sign of  $ds/dR$  hinges on whether  $\eta_3^U \gtrless 1$ . As the lifetime utility function (2.1) has  $\eta_3^U = 1$ , the substitution and the income effects cancel out and  $ds/dR = 0$ .<sup>19</sup>

To sum up, this section shows that the lifetime utility function (2.1) exhibits two important (and related) assumptions. The first is  $\eta_3^U = 1$ . This assumption implies that the intertemporal elasticity of substitution is equal to unity. Moreover, the demand for leisure and individual savings are independent of the rental rate of capital. It also simplifies the response of the demand for leisure to changes in the real wage. The second assumption is the consumption-leisure complementarity, i. e.,  $\nu > 0$ . In conjunction with  $\eta_3^U = 1$  it implies that  $\eta_1^U > 1$  and  $U_{21} > 0$ . Accordingly, the demand for leisure increases in the real wage.

## 5 Concluding Remarks

The data for many of today's industrialized countries point to a significant decline in hours worked per worker at least over the period 1870-2000. At the same time, real wages, output, and consumption per capita increased. The present paper argues that the dual nature of technological progress in conjunction with a consumption-leisure complementarity explains these stylized facts. On the one hand, technological progress drives productivity and the growth of real wages and real incomes. On the other hand, it expands the supply of consumption goods that individuals buy and enjoy during their leisure time. This increases the value of leisure since consumption and leisure are complements. As a consequence, the individual supply of hours worked declines over time.

I make this point in an OLG-model with two-period lived individuals endowed with per-period utility functions of the generalized log-log type of Boppart and Krusell (2016). I

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<sup>19</sup>One readily verifies for Regime 2 that the result corresponding to (4.5) is  $ds/dR \gtrless 0 \Leftrightarrow (1 - \eta_3^U) \gtrless 0$ , where  $U$  is evaluated at the optimal choice  $(c^y, 0, c^o)$ . Hence, when  $\eta_3^U = 1$  individual savings are independent of the rental rate of capital.

show that the resulting lifetime utility function features a consumption-leisure complementarity when individuals are young.<sup>20</sup>

My analysis suggests that different experiences across countries may reflect an evolution through different stages of growth. Poor countries may find themselves in a stage where wages are so low that individuals supply their entire time endowment to the labor market. The desire for a declining supply of hours worked manifests itself only if wages are sufficiently high and grow further. These qualitative features are consistent with recent empirical findings suggesting that hours worked per worker are higher in poor than in rich countries.

On the theoretical front, my analysis derives the appropriate parameter restrictions so that the lifetime utility function (2.1) retains its intuitively appealing properties and delivers tractable expressions describing individual behavior. In conjunction with a production sector that operates under Cobb-Douglas it delivers for Regime 1 a new canonical OLG-model that is likely to have applications in other economically relevant contexts.

Naturally, the results I derive are subject to some caveats. For instance, the actual amount of hours worked per worker has often been determined in negotiations between an employers' association and a trade union, or by government legislation (Huberman and Minns (2007), p. 544). Technological progress may well create the necessary leeway for declining hours of work. Nevertheless, whether such a decline is implemented may well depend on the bargaining power of the negotiating parties or on the political representation of the working class. The role of these factors for the observed decline of hours worked is still open.

Moreover, my analysis is moot on the potential role of government activity and its effect on the individual supply of hours worked over the long run. However, a large literature following Prescott (2004) shows that payroll taxes, pension schemes, or differential labor market regulations help understand differences in hours worked across countries.

Finally, the pace of technological progress may itself respond to the decline of hours worked. Labor becomes scarcer when people work less. Accordingly, wages should

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<sup>20</sup>In the spirit of MaCurdy (1981), an alternative lifetime utility function generating an individual supply of hours worked that declines in the real wage even along a balanced growth path would be

$$\check{U}(c_t^y, l_t, c_{t+1}^o) = \frac{(c_t^y)^{1-\theta} - 1}{1-\theta} - \psi \frac{(1-l)^{1+\epsilon}}{1+\epsilon} + \beta \frac{(c_{t+1}^o)^{1-\theta} - 1}{1-\theta},$$

where  $\theta > 1$ ,  $\epsilon > 0$ ,  $\psi > 0$ , and  $\beta > 0$ . This lifetime utility function represents preferences over the entire domain  $\mathbb{R}_{++}^2 \times [0, 1]$ . However,  $\check{U}$  does not exhibit a consumption-leisure complementarity and, therefore, has little intuitive appeal. The wage response of the supply of hours worked stems from the ad hoc assumption  $\eta_1^U = \eta_3^U = \theta > 1$ . In light of Section 4 this assumption also implies that all elements of the optimal plan hinge on the rental rate of capital. As in Proposition 1, one can show that the optimal plan under  $\check{U}$  gives rise to continuous, piecewise defined functions. However, the critical wage level hinges on the rental rate of capital, too. For models with two-period lived overlapping generations this suggests that MaCurdy preferences are unlikely to provide a tractable alternative to the lifetime utility function of equation (2.1).

increase. However, as suggested by Hicks (1932), entrepreneurs may respond to such a tendency with additional investments that raise the productivity of labor. For the economy as a whole, this may result in accelerated productivity growth (see, e. g., Heer and Irmen (2014) and Irmen (2017)). The question is then whether productivity growth causes hours of work to decline as in the present paper, or whether the decline in hours worked per worker induces productivity growth. I leave these issues for future research.

## A Appendix - Proofs

### A.1 Proof of Proposition 1

For ease of notation I shall most often suppress the time argument. First, I characterize the unique interior candidate solution to the maximization problem (2.5). This delivers the expressions stated under Regime 1. Then, I show that these expressions are indeed the interior solution to the maximization problem of a two-period lived individual if Assumption 1 holds. Second, I turn to the corner solution of Regime 2 and the continuity of the functions stated in (2.7).

Consider an interior solution,  $(c^y, l, c^o, s) \in \mathbb{R}_{++}^2 \times (0, 1) \times \mathbb{R}$ , to problem (2.5) that satisfies the budget constraints (2.6). Since preferences are increasing in  $c^o$  both per-period budget constraints will hold as equalities and can be merged. Accordingly, such a solution will be identified by the Lagrangian

$$\mathcal{L} = \ln c^y + \ln \left( 1 - \phi (1 - l) (c^y)^{\frac{\nu}{1-\nu}} \right) + \beta \ln c^o + \lambda \left[ w(1 - l) - c^y - \frac{c^o}{R} \right]. \quad (\text{A.1})$$

With  $x \equiv (1 - l) (c^y)^{\frac{\nu}{1-\nu}}$  the respective first-order conditions read

$$\frac{\partial \mathcal{L}}{\partial c^y} = \frac{1 - \nu - \phi x}{c^y(1 - \nu)(1 - \phi x)} - \lambda = 0, \quad (\text{A.2})$$

$$\frac{\partial \mathcal{L}}{\partial l} = \frac{\phi (c^y)^{\frac{\nu}{1-\nu}}}{1 - \phi x} - \lambda w = 0, \quad (\text{A.3})$$

$$\frac{\partial \mathcal{L}}{\partial c^o} = \frac{\beta}{c^o} - \frac{\lambda}{R} = 0, \quad (\text{A.4})$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w(1 - l) - c^y - \frac{c^o}{R} = 0. \quad (\text{A.5})$$

Upon multiplication by  $(1 - l)$ , condition (A.3) may be written as

$$\frac{\phi x}{(1 - \phi x) w (1 - l)} = \lambda. \quad (\text{A.6})$$

Using the latter to replace  $\lambda$  in (A.2) and (A.4) delivers, respectively,

$$c^y = \left( \frac{1}{\phi x} - \frac{1}{1 - \nu} \right) w(1 - l) \quad (\text{A.7})$$

and

$$c^o = \beta R \left( \frac{1}{\phi x} - 1 \right) w(1 - l). \quad (\text{A.8})$$

With (A.7) and (A.8) in the budget constraint (A.5) I obtain (2.8), i. e.,  $\phi x$  is time-invariant. Using (2.8) in (A.7) and (A.8) delivers (2.11). Since the optimal plan satisfies (2.3) and (2.8) I have  $1 > \nu(1 + \beta)$ , hence,  $c^y > 0$ . Since  $1 - \nu(1 + \beta) < (1 + \beta)(1 - \nu)$ , the budget constraint when young,  $c^y + s = w(1 - l)$ , delivers strictly positive savings.

From the definition of  $x$  with  $h = 1 - l$ , I have  $c^y = (xh^{-1})^{\frac{1-\nu}{\nu}}$ . Replacing  $c^y$  with this expression in (2.11) and solving for  $h$  delivers  $h_t$  as stated under Regime 1. Clearly,  $h_t \leq 1$  as long as  $w_t \geq w_c$ . Finally, using  $h_t$  in (2.11) delivers  $c_t^y, s_t$ , and  $c_{t+1}^o = R_{t+1}s_t$ .

To see that the solution identified by the Lagrangian (A.1) is indeed a global maximum on  $\mathcal{P}$  consider the leading principal minors of the Hessian matrix of  $U(c^y, l, c^o)$ , i. e.,

$$\begin{aligned} D_1(c^y, l, c^o) &= -\frac{(1-\nu-\phi x)^2 + \nu\phi x(1-\phi x)}{(c^y(1-\nu)(1-\phi x))^2}, \\ D_2(c^y, l, c^o) &= \frac{\phi^2(1-2\nu-(1-\nu)\phi x)}{(c^y)^{\frac{2(1-2\nu)}{1-\nu}}(1-\nu)^2(1-\phi x)^3}, \\ D_3(c^y, l, c^o) &= -\frac{\beta}{(c^o)^2} D_2(c^y, l, c^o). \end{aligned}$$

First, we have  $-D_1(c^y, l, c^o) > 0$ . Second, observe that  $D_2(c^y, l, c^o) > 0$  and  $-D_3(c^y, l, c^o) > 0$  hold if and only if condition (2.4) holds. Hence,  $U$  is strictly concave for all  $(c^y, l, c^o) \in \mathcal{P}$ .

What remains to be shown is that the solution identified by the Lagrangian satisfies condition (2.4). With  $\phi x$  of (2.8) this is the case if and only if

$$\frac{1-2\nu}{1-\nu} > \frac{(1+\beta)(1-\nu)}{1+(1+\beta)(1-\nu)}$$

or

$$\nu^2(1+\beta) - \nu(3+\beta) + 1 > 0.$$

It is not difficult to show that the latter condition is satisfied if and only if Assumption 1 holds.

From the expression of  $h_t$  under Regime 1 it is obvious that  $h_t > 1$  if  $w_t < w_c$ . Hence, for a real wage below  $w_c$  the constraint,  $l \geq 0$  ( $h_t \leq 1$ ) becomes binding, and the optimal plan involves  $l = 0$  and  $h_t = 1$ . Moreover, consumption when young and old (and  $\lambda$ ) are determined by conditions (A.2), (A.3), and (A.5). Savings result from the budget constraint when young. This procedure delivers

$$c^y \left( 1 + \beta \frac{(1-\nu) \left( 1 - \phi x^{\frac{\nu}{1-\nu}} \right)}{1 - \nu - \phi x^{\frac{\nu}{1-\nu}}} \right) = w, \quad (\text{A.9})$$

which coincides with the expression stated under Regime 2 since  $x \equiv (c^y)^{\frac{\nu}{1-\nu}}$ . From (A.9) one readily verifies that  $c^y(w)$ , hence,  $s(w)$  and  $c^o(w, R)$  exists for all  $w > 0$ . Moreover, under Assumption 1,  $U(c^y, 1, c^o)$  is strictly concave for all  $(c^y, 1, c^o) \in \mathcal{P}$  since  $U_{11}(c^y, 0, c^o) = D_1(c^y, 0, c^o) < 0$ ,  $U_{33}(c^y, 0, c^o) = -\beta/(c^o)^2 < 0$ , and  $U_{13} = 0$ .

Continuity of the functions stated in (2.7) follows from Theorem of the Maximum.

Finally, observe that members of cohort 0 satisfy their budget constraint when old as equality, i. e., we have  $c_1^o = R_1 s_0 > 0$ . ■

## A.2 Proof of Proposition 2

Follows from a simple application of the implicit function theorem to the expressions of Regime 2 and by inspection of those of Regime 1. ■

## A.3 Proof of Proposition 3

Consider  $h_t$  of Regime 1. It holds that

$$\frac{\partial h_t}{\partial \phi} = -\frac{\partial z}{\partial \phi} \frac{h_t}{z} < 0, \quad \text{and} \quad \frac{\partial h_t}{\partial \beta} = \frac{(1+\nu)w_t^{-\nu}}{\phi(1+(1+\beta)(1-\nu))(1-\nu(1+\beta))} > 0$$

as  $\partial z / \partial \phi > 0$ .

As to  $c_t^y$  one finds from (2.11) that  $\partial c_t^y / \partial \phi < 0$  since  $\partial h_t / \partial \phi < 0$ . Moreover,

$$\frac{\partial c_t^y}{\partial \beta} = -\frac{w^{1-\nu}(1-\nu)}{(1+(1+\beta)(1-\nu))z} < 0.$$

As to  $s_t$  one finds from (2.11) that  $\partial s_t / \partial \phi < 0$  since  $\partial h_t / \partial \phi < 0$ . Moreover, since the marginal propensity to save and  $h_t$  increase in  $\beta$  we have  $\partial s_t / \partial \beta > 0$ .

Finally, consider  $c_{t+1}^o$ . Since  $c_{t+1}^o = R_{t+1}s_t$ , the qualitative results of the comparative statics for  $s_t$  apply here, too.

For Regime 2 one obtains the comparative statics for  $\phi$  and  $\beta$  from a straightforward application of the implicit function theorem. ■

## A.4 Proof of Proposition 4

Immediate from the aggregate demand for hours worked of (3.3) and the aggregate supply of hours worked of (3.4). ■

## A.5 Proof of Proposition 5

To derive (3.10) use Proposition 1 and (2.14) to express (3.7) in units of efficient hours worked in  $t+1$ . This gives,

$$\frac{\beta\Gamma(1-\gamma)}{(1+\beta)(1-\nu)(1+g_L)(1+g_A)} \left( \frac{h_t}{h_{t+1}} \right) k_t^\gamma = k_{t+1}.$$

Replacing  $h_t/h_{t+1}$  using again Proposition 1 delivers

$$\frac{\beta\Gamma(1-\gamma)}{(1+\beta)(1-\nu)(1+g_L)(1+g_A)} \left( \frac{w_t}{w_{t+1}} \right)^{-\nu} k_t^\gamma = k_{t+1}.$$

Finally, invoke (2.14) again to obtain

$$\frac{\beta\Gamma(1-\gamma)}{(1+\beta)(1-\nu)(1+g_L)(1+g_A)} \left( \frac{1}{(1+g_A)^{-\nu}} \right) \left( \frac{k_t}{k_{t+1}} \right)^{-\gamma\nu} k_t^\gamma = k_{t+1}.$$

Solving the latter for  $k_{t+1}$  gives (3.10). Given the initial conditions  $(K_1, L_1, A_1)$ ,  $k_1$  uses the equilibrium hours of work from Proposition 4. Since  $\gamma(1-\nu)/(1-\gamma\nu) < 1$  the equilibrium sequence is monotonous and the steady state is stable. ■

## A.6 Proof of Proposition 6

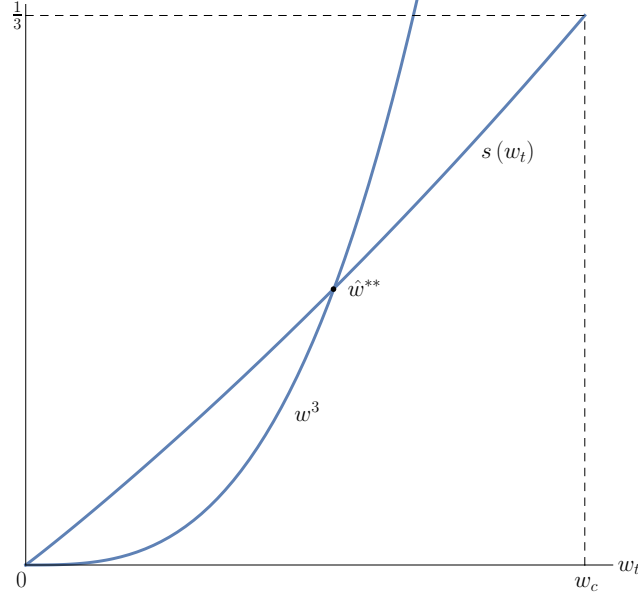
The proof of statement a) - c) follow from Proposition 1, Proposition 4, the capital market equilibrium condition (3.7), and the production function (2.13).

As to statement d) observe that for  $g_A > 0$

$$\begin{aligned} \frac{\partial g^*}{\partial \nu} &= -(1+g_A)^{1-\nu} \ln(1+g_A) < 0, \\ \text{sign} \left[ \frac{\partial k^*}{\partial \nu} \right] &= \text{sign} \left[ \frac{\partial}{\partial \nu} \left( \frac{1}{(1-\nu)(1+g_A)^{1-\nu}} \right) \right] = \text{sign} \left[ \frac{1+(1-\nu)\ln(1+g_A)}{(1-\nu)^2(1+g_A)^{1-\nu}} \right] > 0. \end{aligned}$$

■

Figure A.1: The steady state in Regime 2



## A.7 Proof of Proposition 7

As to the existence of a unique steady state  $0 < \hat{w}^{**} < w_c$  consider Figure A.1. For the indicated parameter values it depicts the function  $s(w)$  and the function  $w^3$ . From Proposition 1, individual savings are

$$s(w) = c^y(w) \left( \frac{1 - \frac{1}{2} \left( \frac{3c^y(w)}{2} \right)^{\frac{1}{3}}}{3 - 2 \left( \frac{3c^y(w)}{2} \right)^{\frac{1}{3}}} \right), \quad (\text{A.10})$$

where  $s(w)$  satisfies  $s(0) = 0$  and  $s(1) = 1/3$ .<sup>21</sup> Moreover, from Proposition 2 I have  $s'(w_t) > 0$ . Figure A.1 reveals a unique intersection at  $\hat{w}^{**} > 0$ . Approximatively, it occurs at  $\hat{w}^{**} = 0.55$  and  $s(\hat{w}^{**}) = 0.167$ .

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<sup>21</sup>This follows since  $c^y(w)$  is implicitly given by

$$c^y \left( 1 + \frac{1 - \frac{1}{2} \left( \frac{3c^y}{2} \right)^{\frac{1}{3}}}{3 - 2 \left( \frac{3c^y}{2} \right)^{\frac{1}{3}}} \right) = w$$

with  $c^y(0) = 0$  and  $c^y(1) = 2/3$ . Moreover, in Regime 2 the budget when young dictates  $s = w - c^y$ .



To prove the local stability of the steady state I derive from (3.14) that

$$\frac{d\hat{w}_{t+1}}{d\hat{w}_t} = \frac{1}{3} \left( \frac{s'(\hat{w}_t)}{[s(\hat{w}_t)]^{\frac{2}{3}}} \right).$$

The function  $s(w_t)$  satisfies

$$\frac{16(w_t - s) - 5 \cdot 2^{2/3} \cdot \sqrt[3]{3} \cdot (w_t - s)^{4/3}}{12 - 4 \cdot 2^{2/3} \cdot \sqrt[3]{3} \cdot \sqrt[3]{w_t - s}} - w_t = 0.$$

Then, implicit differentiation and evaluation at the steady state delivers  $s'(\hat{w}^{**}) = .339$ . Hence, with (3.14)

$$\left. \frac{d\hat{w}_{t+1}}{d\hat{w}_t} \right|_{\hat{w}_t = \hat{w}^{**}} = \frac{1}{3} \left( \frac{s'(\hat{w}^{**})}{[s(\hat{w}^{**})]^{\frac{2}{3}}} \right) = 0.372 < 1.$$

Accordingly, the steady state is locally stable.

Global stability over the domain  $w_t \in (0, 1)$  follows since  $s(0) = 0$ ,  $s(1) = 1/3$ ,  $s'(w) > 0$ , and

$$\lim_{\hat{w}_t \rightarrow 0} \frac{d\hat{w}_{t+1}}{d\hat{w}_t} = \infty.$$

■

## A.8 Proof of Proposition 8

The optimal plan  $(c^y, l, c^o, s)$  that maximizes  $U(c^y, l, c^o)$  subject to the constraints (2.6), i. e.,  $c^y + s = w(1 - l)$  and  $c^o = Rs$ , solves

$$\max_{c^y, l \in [0, 1]} U(c^y, l, R(w(1 - l) - c^y)).$$

The respective first-order sufficient conditions are

$$\begin{aligned} c^y : \quad & U_1 - RU_3 = 0, \\ l : \quad & U_2 - R w U_3 = 0, \end{aligned} \tag{A.11}$$

where  $U$  is evaluated at  $(c^y, l, R(w(1 - l) - c^y))$ .

Total differentiation of the two equations (A.11) delivers the comparative statics

$$\frac{dl}{dw} = \frac{D_w^l}{D} \quad \text{and} \quad \frac{dl}{dR} = \frac{D_R^l}{D},$$

where  $D$ ,  $D_w^l$ , and  $D_R^l$  are the following determinants

$$D = \begin{vmatrix} U_{11} + R^2 U_{33} & U_{12} + R^2 w U_{33} \\ U_{21} + R^2 w U_{33} & U_{22} + R^2 w^2 U_{33} \end{vmatrix}, \tag{A.12}$$

$$D_w^l = \begin{vmatrix} U_{11} + R^2 U_{33} & R^2 (1 - l) U_{33} \\ U_{21} + R^2 w U_{33} & R (U_3 + R w (1 - l) U_{33}) \end{vmatrix}, \tag{A.13}$$

$$D_R^l = \begin{vmatrix} U_{11} + R^2 U_{33} & U_3 + R (w(1 - l) - c^y) U_{33} \\ U_{21} + R^2 w U_{33} & w (U_3 + R (w(1 - l) - c^y) U_{33}) \end{vmatrix}, \tag{A.14}$$

all evaluated at the optimal plan  $(c^y, l, c^o, s)$  that satisfies (A.11) and the two constraints.

Since  $U$  is strictly concave it follows that

$$D = U_{11} (U_{22} + R^2 w^2 U_{33}) + R^2 U_{33} U_{22} - (U_{12})^2 - 2R^2 w U_{12} U_{33} > 0.$$

As to  $D_w^l$  I find from (A.13) that

$$\begin{aligned} D_w^l &= (U_{11} + R^2 U_{33}) R (U_3 + R w (1-l) U_{33}) - (U_{21} + R^2 w U_{33}) R^2 (1-l) U_{33}, \\ &= -R^2 U_{33} U_1 \left[ -1 + \eta_1^U \left( 1 + \frac{c^o}{R c^y} \left( 1 - \frac{1}{\eta_3^U} \right) \right) + \frac{U_{21}}{U_1} (1-l) \right]. \end{aligned}$$

Since the term preceding the brackets is strictly positive I obtain

$$D_w^l \geq 0 \Leftrightarrow -1 + \eta_1^U \left( 1 + \left( \frac{c^o}{R c^y} \right) \left( 1 - \frac{1}{\eta_3^U} \right) \right) + \frac{U_{21}}{U_1} (1-l) \geq 0.$$

Since,  $D > 0$  the sign of  $dl/dw$  is indeed determined by condition (4.2).

As to  $D_R^l$  I obtain from (A.14) that

$$\begin{aligned} D_R^l &= (U_{11} + R^2 U_{33}) w (U_3 + R (w(1-l) - c^y) U_{33}) - (U_{21} + R^2 w U_{33}) (U_3 + R (w(1-l) - c^y) U_{33}), \\ &= (U_3 + c^o U_{33}) (w U_{11} - U_{21}), \\ &= -U_3 \left[ \left( 1 - \eta_3^U \right) \left( \frac{U_2}{c^y} \eta_1^U + U_{21} \right) \right], \end{aligned}$$

where use is made of the budget constraints and (A.11). Since  $U_3 > 0$  and  $D > 0$  the sign of  $dl/dR$  is indeed determined by condition (4.3).  $\blacksquare$

## A.9 Proof of Proposition 9

To derive the comparative statics of  $s$  with respect to  $R$  recall that  $s = w(1-l) - c^y$ . Hence, a change in  $R$  delivers

$$\frac{ds}{dR} = -w \frac{dl}{dR} - \frac{dc^y}{dR} = -w \frac{D_R^l}{D} - \frac{D_R^{c^y}}{D},$$

where

$$D_R^{c^y} = \begin{vmatrix} U_3 + R (w(1-l) - c^y) U_{33} & U_{12} + R^2 w U_{33} \\ w (U_3 + R (w(1-l) - c^y) U_{33}) & U_{22} + R^2 w^2 U_{33} \end{vmatrix} \quad (\text{A.15})$$

evaluated at the triple  $(c^y, l, c^o)$  that satisfies (A.11) and  $c^o = R (w(1-l) - c^y)$ . One finds that

$$D_R^{c^y} = U_3 \left( 1 - \eta_3^U \right) (U_{22} - w U_{12}).$$

Hence,

$$\begin{aligned} \frac{ds}{dR} &= -w \frac{dl}{dR} - \frac{dc^y}{dR} \\ &= w \frac{U_3 (1 - \eta_3^U) \left( \frac{U_2}{c^y} \eta_1^U + U_{21} \right)}{D} - \frac{U_3 (1 - \eta_3^U) (U_{22} - w U_{12})}{D} \\ &= \frac{U_3}{D} \left( 1 - \eta_3^U \right) \left[ \frac{w U_2}{c^y} \eta_1^U + 2w U_{21} - U_{22} \right]. \end{aligned} \quad (\text{A.16})$$

Since  $U_3 > 0$  and  $D > 0$  the sign of  $ds/dR$  is determined by (4.5).  $\blacksquare$

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