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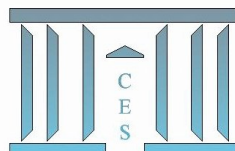Agnieszka Rusinowska[1] and Akylai Taalaibekova[2]

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Abstract. We consider a model of competitive opinion formation in which three persuaders characterized by (possibly unequal) persuasion impacts try to influence opinions in a society of individuals embedded in a social network. Two of the persuaders have the extreme and opposite opinions, and the third one has the centrist opinion. Each persuader chooses one individual to target, i.e., he forms a link with the chosen individual in order to spread his own "point of view" in the society and to get the average long run opinion as close as possible to his own opinion. We examine the opinion convergence and consensus reaching in the society. We study the existence and characterization of pure strategy Nash equilibria in the game played by the persuaders with equal impacts. This characterization depends on influenceability and centrality (intermediacy) of the targets. We discuss the effect of the centrist persuader on the consensus and symmetric equilibria, compared to the framework with only two persuaders having the extreme opinions. When the persuasion impacts are unequal with one persuader having a sufficiently large impact, the game has only equilibria in mixed strategies.

JEL Classification: D85, D72, C72

Keywords: social network, opinion formation, consensus, targeting, lobbying, extreme and centrist persuaders

1 Introduction

Social networks play a central role in most of our everyday activities, communicating and exchanging information, sharing knowledge, research and development, advertisement, among many others. A process that can perfectly be modeled by social networks is the one of opinion formation in a society. The opinions result from interactions with other individuals that hold views on given issues. In the seminal model on opinion formation introduced by DeGroot [22], individuals update their opinions by taking weighted averages of their "neighbors", i.e., people that they are connected to in the network. An accompanying question being particularly important, e.g., in lobbying, political campaigning, marketing, or counter-terrorism, is how to identify optimal targets to achieve social impact. Indeed, the reliance on others to form opinions lies at the heart of advertising [13],

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efforts to make people aware of different issues, preventing criminal social groups and organizations [7], or attempts of capturing votes in elections. In economics such models are used to study competition between firms and product differentiation. In political science, they are applied for determining equilibrium outcomes of electoral competitions.

Our point of departure for the present paper is [43]. We consider a game of competitive opinion formation in a society played by three competing persuaders that have different opinions on a certain issue. The society consists of individuals having their own opinions on that issue and updating them like in [22], i.e., by taking weighted averages of individuals' opinions that they listen to. The opinion is a real number between 0 and 1, and can be interpreted as the intensity of the opinion "yes". [43] extend the DeGroot model by introducing two persuaders (called external players in their paper) with the extreme opinions 0 and 1. In the present paper, we introduce a third persuader which has the centrist opinion $\frac{1}{2}$. Each persuader chooses one individual to target. Targeting in this setting means forming a link with that individual in order to make the average opinion in the society as close as possible to his own opinion. The persuaders are characterized by (possibly unequal) persuasion impacts. The higher the impact of a persuader targeting an individual, the more this individual takes the persuader's opinion into account when updating his own opinion.

The main objective of this work is to study the effects of entering the additional centrist persuader into competition between the two extremist persuaders. First, we examine the opinion convergence and consensus reaching in the society targeted by the three persuaders. Is it possible to obtain a limit opinion vector? Can the society reach a consensus meaning that every individual has the same opinion? If so, how does such a consensus look like? More specifically, how does the presence of the centrist persuader change the convergence and consensus reaching in the society? We focus on the competition between the three persuaders. We define a noncooperative game played by the persuaders with strategies being target individuals and study the existence and characterization of pure strategy Nash equilibria. [43] obtain a constant sum game where players have opposite interests. Our extended game cannot be considered as a constant sum game anymore, and hence we derive new expressions for the payoffs, appropriate for the extended setting. A number of new questions arises. How can the centrist persuader affect optimal strategies of the extreme persuaders determined in [43]? How do characteristics of the key (i.e., targeted) individuals change when the third persuader enters into the play? Which network structures appear to be consistent with the equilibrium in pure strategies?

The introduction of the centrist persuader with a specific position that involves balance, neutrality, and equal combination of the extreme positions makes the theoretical framework richer. It can give a more realistic explanation of the political and economic spectrum, and just daily life. A good example comes from the recent French Presidential Elections where the current President is largely seen as centrist. In political science there is a well known theory of spatial allocation. [25] represents the relative positioning of political parties and voters by using a spatial analogy built on the work of [46] that consists in representing the political preferences on a linear scale from left to right. [30] consider voters' behavior in three-candidate elections in a non-spatial context. In economics three parties can be seen as three main firms that differ from each other by production, work, and distribution. They can compete over marketing campaigns, product adoption, firm allocations, etc.

The extension of [43] by introducing the third persuader with the centrist opinion has a number of consequences on consensus reaching in the society and Nash equilibria of the noncooperative game played by the persuaders. The presence of the centrist persuader preserves opinion convergence but changes the long run opinions of the society and the consensus. When the three persuaders choose the same target, a consensus exists and is determined by the three persuasion impacts. If the impact of the centrist persuader is vanishing, we recover the consensus with only two extreme persuaders. When the impact of one of the persuaders is much larger than these of the others, the consensus approaches the opinion of the high-impact persuader. Moreover, when all three persuaders target the same individual, the presence of the centrist one improves the situation of the weaker extreme persuader in the sense that consensus moves closer to the opinion of the smaller-impact persuader.

By using some notions and definitions given in [43], we characterize equilibria in our three-persuader setting. We focus our analysis on the case with equal impacts and find that both intermediacy (centrality) and relative influenceability are important, and the target individuals are completely characterized by these two notions. More precisely, conditions for the existence of symmetric Nash equilibria of the game played by the three equal-impact persuaders is that the relative influence of a potential target must be at least twice higher than the one of any other individual in the network. Strong-impact persuaders must take into account the presence of the new centrist one. The persuaders are demanding higher centrality from their potential targets to compensate the impact of the new persuader. However, when the persuaders have weak impact, the conditions for Nash equilibria are the same as for the case with only two extreme persuaders. If the persuasion impacts are unequal and one persuader's impact is sufficiently large, then the game has only equilibria in mixed strategies.

The paper is organized as follows. The model is introduced in Section 2. Section 3 concerns the opinion convergence and consensus reaching. In Section 4 we define the noncooperative game played by the persuaders and present the equilibrium analysis. More precisely, we determine the equilibrium conditions for the case when the persuasion impacts are the same and briefly discuss the case of the unequal persuasion impacts. The related literature is surveyed in Section 5. Section 6 presents concluding remarks. The Appendix in Section 7 presents proofs of the main results.

2 The framework and preliminaries

The model with three persuaders We extend the model of strategic influence with two external players having extreme opinions [43] to a framework with three persuaders, by adding a persuader with the centrist opinion. The society consists of a set $N = \{1, ..., n\}$ of individuals who discuss a certain issue. Each individual $i \in N$ has an initial opinion on the issue, given by a real number $x_i(0) \in [0, 1]$ which can be interpreted as the intensity of i's personal opinion "yes" in time 0. The individuals interact with each other, that is, are embedded in a social network, and consequently update their opinions at discrete time $t \in \mathbb{N}$. The society is observed by three persuaders A, B and C who have the fixed opinions 1, $\frac{1}{2}$ and 0, respectively. Each of them chooses one individual in N to form a link with in order to influence the formation of opinions in the society. The individuals targeted by A, B and C are denoted by s_A, s_B and s_C, respectively.

The persuaders are characterized by possibly unequal (positive) persuasion impacts λ, γ and μ, respectively, to adjust influence in the society. When persuader A targets the individual s_A, a share λ of the attention of that individual is redirected to A. The same adjustment of influence holds for s_B and s_C being targeted by B and C, with impacts γ and μ, respectively.

It is assumed that in the absence of the persuaders, the individuals would update their opinion by using weighted averages of their neighbors' opinions [22], that is, according to the rule:

$$\mathbf{x}_N(t) = W\mathbf{x}_N(t-1) = W^t\mathbf{x}_N(0) \tag{1}$$

where $W = [w_{ik}]_{i,k \in N}$ is the interaction or influence matrix being row stochastic, i.e., $\sum_{k=1}^{n} w_{ik} = 1$ for every $i \in N$, w_{ik} denotes the weight or trust that individual i assigns to the current opinion of individual k in forming his own opinion in the next period, and $\mathbf{x}_N(t) = [x_1(t), \cdots, x_n(t)]'$ is the opinion (column) vector at time step t.[1]

Preliminaries on networks, influenceability and influence A directed graph G on N is associated to the matrix W such that there is an arc (i, k) from i to k meaning that i listens to k if and only if $w_{ik} > 0$. A walk from node i to node k is a sequence of nodes $(i_1 = i, i_2, \cdots, i_{j-1}, i_j = k)$ such that $w_{i_m i_{m+1}} > 0$ (i.e., there is an arc (i_m, i_{m+1})) for each $m \in \{1, \cdots, j-1\}$. A cycle around i is a walk from i to i which does not pass through i between the starting and ending nodes.[2] A path is a walk such that neither a node nor an arc appears more than once in the sequence. To be consistent with the DeGroot framework [22] we assume that the social network defined by the adjacency matrix W is connected, i.e., for every pair of individuals $i, k \in N$ there exists a path from i to k.

We recall some crucial concepts used in [43]. For any walk $p = (i_1, ..., i_m)$ in G, we denote by $w(p)$ its "weight" measured according to W, i.e.,

$$w(p) := \prod_{j=1}^{m-1} w_{i_j, i_{j+1}} \tag{2}$$

Moreover, for any two individuals i, k in the society N, let C_i^k denote the set of cycles around i that pass through k, and B_i^k the set of walks that start from any node $\neq i$, end up in i, and go through k. Let

$$c_i^k := \sum_{p \in C_i^k} w(p) \quad b_i^k := \sum_{p \in B_i^k} w(p) \tag{3}$$

The quantity c_i^k accounts for the self-feedback (echo) that individual i receives of his opinion through the network. The larger c_i^k is, the more individual k interferes with this self-reinforcement process and hence, the lesser is the influence that can be exerted on i by a persuader. Furthermore, the quantity $d_i c_i^k$ measures the *influenceability of individual i*, given that k is targeted by another persuader, where d_i is i's out-degree, i.e., the number of individuals that i listens to. The larger d_i is, the more opinions individual i takes into

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[1] Transposition of column vectors is denoted by $\prime$, and therefore $\mathbf{x}'_N(t)$ is a row vector.

[2] This definition of a cycle differs from the usual one, which does not allow repetition of any node between the starting and ending nodes.

4

account and the lesser/slower he can be influenced by an additional opinion. Hence, the lower $d_i c_i^k$ is, the more influenceable i is.

The quantity b_i^k accounts for the *influence (centrality, intermediacy) of k relatively to i*, i.e., it measures the extent to which k can interpose himself between i and other individuals, i.e., the extent to which the influence of individual k reaches the network before this of i.

c_i^k and b_i^k have some probabilistic interpretations. If the influence travels across the network according to the probabilities given by W, then c_i^k is the probability for i to be reached by the influence of k before he receives the self-feedback of his own opinion. Accordingly, b_i^k is the sum of the probabilities for the $n-1$ individuals other than i to be reached by the influence of k before this of i.

The extended matrix of influence and updating rule In the presence of the persuaders who choose the targets $\mathbf{s} = (s_A, s_B, s_C)$, the $n \times n$ matrix of influence W is extended to a $(n+3) \times (n+3)$ matrix $M_{\lambda,\gamma,\mu}(\mathbf{s})$ such that:

$$M_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \hline \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s}) & & & \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W \end{bmatrix} \tag{4}$$

which similarly to [43] accounts for two effects:

(i) the weight renormalization in the presence of the persuaders, given by the weight renormalization matrix $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})$ which is a diagonal matrix with diagonal elements equal to

$$\frac{d_1}{d_1 + \lambda\delta_{1,s_A} + \gamma\delta_{1,s_B} + \mu\delta_{1,s_C}}, \cdots, \frac{d_n}{d_n + \lambda\delta_{n,s_A} + \gamma\delta_{n,s_B} + \mu\delta_{n,s_C}} \tag{5}$$

with $\delta_{i,s_j} = 1$ if $i = s_j$ for all $i \in N$, $s_j \in \{s_A, s_B, s_C\}$ and 0 otherwise;

(ii) the strategic influence given by a matrix

$$E_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} \dfrac{\lambda}{d_{s_A}}e_{s_A} & \dfrac{\gamma}{d_{s_B}}e_{s_B} & \dfrac{\mu}{d_{s_C}}e_{s_C} \end{bmatrix} \tag{6}$$

where e_i denotes the unit vector with coordinate 1 at i.

In the influence matrix $M_{\lambda,\gamma,\mu}(\mathbf{s})$ the first three rows correspond to the weights of the persuaders A, B and C: since they do not listen to the individuals in the society, they put weight 1 for themselves and 0 otherwise. The next n rows correspond to the new weights of the individuals in N adjusted to the extended framework. The individuals targeted by the persuaders redistribute their trust among their neighbors and the targeting persuaders: the weights put for the persuaders depend on the persuaders' impacts and are given by $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})$, while the new weights put for the other individuals are $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})W$ instead of W.

The vector of opinions is extended to $\mathbf{x}(t) = [1 \ \frac{1}{2} \ 0 \ \mathbf{x}_N(t)]'$ where the first three coordinates correspond to the fixed opinions of the persuaders. The opinion updating rule is now determined by

$$\mathbf{x}(t+1) = M_{\lambda,\gamma,\mu}(\mathbf{s})\mathbf{x}(t) = (M_{\lambda,\gamma,\mu}(\mathbf{s}))^{t+1}\mathbf{x}(0) \tag{7}$$

5

which leads to the evolution law for the opinions of the individuals in N given by

$$\mathbf{x}_N(t+1) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s}) \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\mathbf{x}_N(t) \tag{8}$$

In the next sections we provide the consensus and equilibrium analysis in the extended framework.

3   Convergence of opinions and consensus reaching

Our first result concerns the convergence of opinions in the influence model with three persuaders. When the society gets a new persuader, the one with the centrist position, the opinion convergence is preserved in the society, i.e., opinions of the individuals do converge in long run. However, the long run opinions are obviously different from the ones reached in a society with only two persuaders having the extreme positions. More precisely, the following proposition holds:

Proposition 1 *For any initial vector of opinions $\boldsymbol{x}(0) := [1 \; \frac{1}{2} \; 0 \; \boldsymbol{x}_N(0)]'$, we have*

$$\lim_{t \to +\infty} (M_{\lambda,\gamma,\mu}(\boldsymbol{s}))^t \left[1 \; \tfrac{1}{2} \; 0 \; \boldsymbol{x}_N(0)\right]' = \left[1 \; \tfrac{1}{2} \; 0 \; \overline{\boldsymbol{x}}_N(\boldsymbol{s})\right]' \tag{9}$$

where

$$\overline{\boldsymbol{x}}_N(\boldsymbol{s}) = [I - \Delta_{\lambda,\gamma,\mu}(\boldsymbol{s})W]^{-1}\Delta_{\lambda,\gamma,\mu}(\boldsymbol{s}) \left(\frac{\lambda}{d_{s_A}}e_{s_A} + \frac{\gamma}{2d_{s_B}}e_{s_B}\right) \tag{10}$$

In the model with two persuaders having the opinions 1 and 0, and the impacts λ and μ, respectively, [43] prove the convergence result with

$$\overline{\mathbf{x}}_N(\mathbf{s}) = [I - \Delta_{\lambda,\mu}(\mathbf{s})W]^{-1}\frac{\lambda}{d_{s_A}}\Delta_{\lambda,\mu}(\mathbf{s})e_{s_A}$$

Similarly to [43] in our extended framework with three persuaders the asymptotic opinions of the individuals are independent of their vector of initial opinions. They are determined by the respective targets of the persuaders, since $\overline{\mathbf{x}}_N(\mathbf{s}) \in [0,1]^n$ depends on the vector $\mathbf{s}$. Compared to [43], the presence of the centrist persuader leads to some differences in $\overline{\mathbf{x}}_N(\mathbf{s})$ determined by (10). The coefficient $\frac{1}{2}$ comes from the vector of opinions where the first three coordinates are fixed points of the persuaders and $\frac{1}{2}$ indicates the "ideal" point of the centrist persuader.

The next issue concerns the effect of the centrist persuader on reaching a consensus among the society members. In other words, can all individuals end up with the same opinion in long run, and if so, how does their opinion look like? It appears that if the three persuaders choose the same target, then the long run opinion in the society converges towards a consensus $\alpha \in [0,1]$ among the individuals. The consensus is determined by the three persuasion impacts.

Proposition 2 *If $s_A = s_B = s_C$, then the individuals in N reach a consensus α given by*

$$\alpha = \frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)} \tag{11}$$

In particular, if $\lambda = \mu$, then the consensus is $\alpha = \frac{1}{2}$.

6

We can draw a number of intuitive conclusions from Proposition 2 which considers the case when all three persuaders target the same individual. In this case, the society reaches a consensus which depends on the persuaders' impacts. In particular, if the extreme persuaders have the equal impact $\lambda = \mu$, then the individual who receives three "types" of information from each of the persuaders, takes equally into account the opinions 0 and 1 of the extreme persuaders. At the end, the consensus of $\frac{1}{2}$ occurs in the society, independently of the impact of the centrist persuader B.

In the model [43] with two extreme persuaders targeting the same individual, the society reaches a consensus given by $\alpha = \frac{\lambda}{\lambda+\mu}$. We recover this result from (11) when the centrist persuader in the extended model has the vanishing impact $\gamma \to 0$. On the contrary, if the centrist persuader is much stronger than the two extreme ones, i.e., if $\gamma \to +\infty$ and $\lambda, \mu \in \mathbb{R}_+$, then the consensus is equal to $\frac{1}{2}$, the opinion of the centrist persuader. Similarly, when $\lambda \to +\infty$ and $\gamma, \mu \in \mathbb{R}_+$, the consensus is equal to 1 (A's opinion), while under $\mu \to +\infty$ and $\lambda, \gamma \in \mathbb{R}_+$, the consensus approaches 0 (C's opinion). Furthermore, note that

$$\frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)} > \frac{\lambda}{\lambda + \mu} \quad \text{if and only if} \quad \lambda < \mu$$

This means that when all three persuaders target the same individual, independently of the impact of the centrist persuader, his presence in the society always improves the situation of the weaker extreme persuader, i.e., it moves the consensus opinion closer to the ideal point of the persuader with the smaller impact.

When persuaders A and C target the same individual, then the society ends up in a consensus, even if the centrist persuader targets another individual and independently of his own impact, but only if the extreme persuaders are equally strong. In this case, the consensus is equal to $\frac{1}{2}$. More precisely, the following holds true.

Proposition 3 *If $s_A = s_C$ then the individuals in N reach a consensus $\alpha = \frac{1}{2}$ with $\lambda = \mu$.*

The individual targeted by persuaders A and C listens to both of them. He recounts his trust weights, and since impacts are equal ($\lambda = \mu$), spreads the opinion of $\frac{1}{2}$. At the same time, the individual targeted by persuader B shares the same opinion. Consequently, the society reaches the consensus $\frac{1}{2}$. We present an illustrative example:

Example 1 Let us consider a five-individual society with the following weight matrix:

$$W = \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Situation 1: First, we assume that the three persuaders are equally strong, $\lambda = \gamma = \mu = 1$, and they all target individual 2, i.e., $s_A = s_B = s_C = 2$. The vector $\bar{\mathbf{x}}_N(\mathbf{s})$ is

obtained from (8), letting $\mathbf{x}_N(t+1) = \mathbf{x}_N(t) = \overline{\mathbf{x}}_N(\mathbf{s})$. The solution of

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix}' + \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \overline{\mathbf{x}}_N(\mathbf{s})$$

is $\overline{x}_i(\mathbf{s}) = \frac{1}{2}$ for $i \in \{1, 2, 3, 4, 5\}$, i.e., the society converges to a consensus $\alpha = \frac{1}{2}$. Obviously, the solution is consistent with Proposition 2.

Situation 2: If the extreme persuaders target individual 2 while the centrist persuader targets individual 4, i.e., if $s_A = s_C = 2$, $s_B = 4$, then we obtain the solution of

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix}' + \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \overline{\mathbf{x}}_N(\mathbf{s})$$

equal to $\overline{x}_i = \frac{1}{2}$ for $i \in \{1, 2, 3, 4, 5\}$. The payoffs of the persuaders are the same as in the previous case. Consistently with Proposition 3, since $\lambda = \mu$, the society reaches the consensus equal to $\alpha = \frac{1}{2}$, despite the fact that the impact of the centrist persuader is different from the one of the extreme persuaders.

Situation 3: Assume now that $s_A = s_C = 2$, $s_B = 4$, but the persuaders have different impacts λ, γ and μ. Take, for instance, $\lambda = 4$, $\gamma = 3$ and $\mu = 8$. The solution of

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{4}{15} & 0 & \frac{8}{15} \\ 0 & 0 & 0 \\ 0 & \frac{3}{8} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix}' + \begin{bmatrix} 0 & \frac{3}{4} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \overline{\mathbf{x}}_N(\mathbf{s})$$

is equal to $\overline{x}_1(\mathbf{s}) = 0.3511$, $\overline{x}_2(\mathbf{s}) = 0.3366$, $\overline{x}_3(\mathbf{s}) = 0.3614$, $\overline{x}_4(\mathbf{s}) = 0.4173$, and $\overline{x}_5(\mathbf{s}) = 0.3718$. Since the impact μ of the third persuader is twice of the impact λ of the first one, the long run average opinion is biased toward the first half of opinion domain. This confirms Proposition 3: when $\lambda \neq \mu$, we get the long run opinions convergence, but there is no consensus $\overline{x}_i(\mathbf{s}) \neq \overline{x}_k(\mathbf{s})$ for some $i, k \in N$.

4 Nash equilibrium of the model

Payoffs and the aggregate opinion We consider a game $\mathcal{G}_{\lambda,\gamma,\mu}$ played by the three persuaders, with their set of strategies being N, i.e., the strategies of A, B and C are the targeted individuals s_A, s_B and s_C, respectively. Each persuader aims at bringing the asymptotic average opinion in the society as close as possible to his own opinion (1 for persuader A, $\frac{1}{2}$ for persuader B and 0 for persuader C), i.e., at minimizing the distance between the asymptotic average opinion in the society and his own "ideal" point. In other words, our game-theoretic model of competition between the persuaders is a system of

8

minimization problems, where the persuaders' goal is to minimize their payoffs, given a strategy profile $\mathbf{s} = (s_A, s_B, s_C) \in N \times N \times N$, defined in the following way:

$$\pi_{\lambda,\gamma,\mu}^A(s_A, s_B, s_C) = \left(1 - \frac{1}{n}\mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s})\right)^2$$

$$\pi_{\lambda,\gamma,\mu}^B(s_A, s_B, s_C) = \left(\frac{1}{2} - \frac{1}{n}\mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s})\right)^2 \tag{12}$$

$$\pi_{\lambda,\gamma,\mu}^C(s_A, s_B, s_C) = \left(\frac{1}{n}\mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s})\right)^2$$

where $\overline{\mathbf{x}}_N(\mathbf{s})$ is given by (10). For convenience, we introduce the notation

$$\tilde{x}_N(\mathbf{s}) := \mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s}) = \sum_{i \in N} \overline{x}_i(\mathbf{s})$$

for the aggregate opinion formed in the society. The following results determine $\tilde{x}_N(\mathbf{s})$, i.e. equivalently the persuaders' payoffs, for some strategy profiles, in terms of the persuaders' impacts and the individuals' influence (centrality, intermediacy) and influenceability recalled in the network preliminaries.

Theorem 1 *The payoffs of persuaders A, B and C, given the strategy profile $\boldsymbol{s} = (s_A, s_B, s_C)$ are as follows:*

(i) If $s_A = s_B = s_C = i$, i.e., if all three persuaders target the same individual i, then

$$\tilde{x}_N(i,i,i) = \frac{n(2\lambda + \gamma)}{2(\lambda + \gamma + \mu)} \tag{13}$$

(ii) If $s_A = s_C = i$ and $s_B = k \neq i$, i.e., if the two extreme persuaders target the same individual i and the centrist one targets a different individual k then:

$$\tilde{x}_N(i,k,i) = \frac{2\lambda(\gamma b_k^i + d_k c_k^i n) + \gamma((\lambda + \mu)b_i^k + d_i c_i^k n)}{2(\gamma d_i c_i^k + (\lambda + \mu)(d_k c_k^i + \gamma))} \tag{14}$$

(iii) If $s_A = k$ and $s_B = s_C = i \neq k$, i.e., if the persuader with the opinion 1 targets an individual k, while the remaining persuaders target the same individual i but different from k, then:

$$\tilde{x}_N(k,i,i) = \frac{2\lambda((\gamma + \mu)b_i^k + d_i c_i^k n)) + \gamma(\lambda b_k^i + d_k c_k^i n)}{2(\lambda d_i c_i^k + (\gamma + \mu)(d_k c_k^i + \lambda))} \tag{15}$$

(iv) If $s_A = s_B = i$ and $s_C = k \neq i$, i.e., if the first two persuaders target i and the one with the opinion 0 targets a different individual k, then:

$$\tilde{x}_N(i,i,k) = \frac{(2\lambda + \gamma)(\mu b_k^i + d_k c_k^i n)}{2(\mu d_i c_i^k + (\lambda + \gamma)(d_k c_k^i + \mu))} \tag{16}$$

The first result (i) of Theorem 1 is consistent with Proposition 2. If the three persuaders target the same individual i, then the aggregate opinion in the society is equal to

9

$\tilde{x}_N(i, i, i) = \frac{n(2\lambda + \gamma)}{2(\lambda + \gamma + \mu)}$, where $\frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)}$ is the consensus reached in the society. In this case, the payoffs are equal to:

$$\pi_\lambda^A(i, i, i) = \left(\frac{2\mu + \gamma}{2(\lambda + \gamma + \mu)} \right)^2$$

$$\pi_\lambda^B(i, i, i) = \left(\frac{\mu - \lambda}{2(\lambda + \gamma + \mu)} \right)^2 \tag{17}$$

$$\pi_\lambda^C(i, i, i) = \left(\frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)} \right)^2$$

Moreover, the second result (ii) of Theorem 1 is consistent with Proposition 3. If $s_A = s_C = i$, $s_B = k \neq i$, and $\lambda = \mu$, then the society reaches the consensus $\frac{1}{2}$. Applying $\lambda = \mu$ to (14) gives the aggregate opinion $\tilde{x}_N(i, k, i) = \frac{n}{2}$.

Equal persuasion impacts We focus our analysis on pure strategy Nash equilibria in the case when all three persuaders have the same impact, i.e., $\lambda = \gamma = \mu$. We replace $\mathcal{G}_{\lambda,\gamma,\mu}$ by the simplified notation $\mathcal{G}_\lambda$ for the game, and $\pi_{\lambda,\gamma,\mu}^A$, $\pi_{\lambda,\gamma,\mu}^B$, $\pi_{\lambda,\gamma,\mu}^C$ by π_λ^A, π_λ^B, π_λ^C for the payoffs. From equations (12), (13) and (14) we get direct conclusions of Theorem 1. Indeed, one has:

Fact 1 *If $\lambda = \mu$, then one has for all $i, k \in N$:*

$$\pi_\lambda^A(i, i, i) = \frac{1}{4}, \quad \pi_\lambda^B(i, i, i) = 0 \quad \pi_\lambda^C(i, i, i) = \frac{1}{4} \tag{18}$$

$$\pi_\lambda^A(i, k, i) = \frac{1}{4}, \quad \pi_\lambda^B(i, k, i) = 0, \quad \pi_\lambda^3(i, k, i) = \frac{1}{4} \tag{19}$$

This fact states that the centrist persuader is indifferent between targeting individual i or individual k, because in both cases the outcome is the same, and the average opinion is equal to his "ideal" opinion. Note that it is true only when the two extreme persuaders have the equal impacts $\lambda = \mu$.

The following result provides necessary and sufficient conditions for (i, i, i) to be an equilibrium.

Theorem 2 *A profile of strategies (i, i, i) is an equilibrium of the game $\mathcal{G}_\lambda$ if and only if for all $k \in N \setminus \{i\}$*

$$b_k^i - 2b_i^k \geq \frac{n}{\lambda} \left(d_i c_i^k - d_k c_k^i \right) \tag{20}$$

The equilibrium condition depends both on the intermediacy and the influenceability of the target i relative to any other individual: (i, i, i) is an equilibrium if for all $k \neq i$, the difference between the influence (intermediacy) of i over k and the double influence of k over i is not smaller than the difference between the influenceability of i and the influenceability of k, scaled by the factor $\frac{n}{\lambda}$. In the model with only two extreme persuaders, [43] get a similar condition for (i, i) to be an equilibrium, but with the expression $\left(b_k^i - b_i^k\right)$ on the left hand side of the inequality. In the extended three-persuader model, the condition to reach the equilibrium (i, i, i) requires more from the intermediacy of i over k than in the framework [43] with only two extreme persuaders: i must be even more influential (central) among other individuals to compensate impact of two other persuaders.

Condition (20) also shows what happens under different multiplier $\frac{n}{\lambda}$. As the number of individuals in the society increases, the relative importance of intermediacy compared to influenceability goes down. Conversely, the relative importance of intermediacy goes up with the level of λ, the impact of the persuaders.

When the persuaders have the same impact, pure Nash equilibria can exist in types of networks that are structurally very different. Common feature of such networks is the presence of an individual or a group of individuals with either high intermediacy or high influenceability. For the game with three competitive persuaders there exist networks with Nash equilibria in pure strategies, e.g., star networks, and also there are networks where no symmetric equilibria in pure strategies can be found, e.g., symmetric and circular networks. We show it in the following examples.

Example 2 Consider a perfectly symmetric society, i.e., a network structure such that for all distinct $i, k \in N$, $d_i = d_k$, $c_i^k = c_k^i$, and $b_i^k = b_k^i$. While (i, i, i) was always an equilibrium in the model with only two extreme persuaders, condition (20) does not hold in the extended framework, so that (i, i, i) is not an equilibrium of the game $\mathcal{G}_\lambda$ in perfectly symmetric networks. It is not worth targeting i and sharing the attention of the individual with two other persuaders, since there are other individuals with the same characteristics whose targeting can lead to a better payoff.

Example 3 Condition (20) means that a network has to contain a very "powerful" individual in order to get an equilibrium. We consider a star society, where one central individual is connected to any other individual in the network, i.e., the structure given by $d_i = n - 1$ and $d_k = 1$, with individual i being central and all individuals $k \neq i$ being peripheral. We have $c_k^i = 1$, $c_i^k = \frac{1}{n-1}$, $b_k^i = n - 1$, $b_i^k = 1$. Hence, (20) is always satisfied in such star networks (unless the number of individuals in the society is less than 3).

Example 4 Consider a society interacted in a directed circle, where every individual listens to the next one, and only to him. We have $d_i = 1$ for every $i \in N$. Moreover, for any $k \neq i$, $c_i^k = c_k^i = 1$, $b_k^i = l(k, i)$ and $b_i^k = l(i, k)$, where $l(k, i)$ and $l(i, k)$ are the lengths of the (unique) shortest walk from k to i, and from i to k, respectively. If $\lambda = \gamma = \mu$ then no symmetric equilibrium in pure strategies can exist in such a circular network, similarly to the case with only two extreme persuaders.

Let us consider the two polar cases where the impact of the persuaders is either infinitely large or infinitely small with respect to the normalized influence within the network. We get the following result.

Proposition 4 *(i) For distinct $i, k \in N$:*

$$\lim_{\lambda \to 0} \pi_\lambda^A(k, i, i) = \lim_{\lambda \to 0} \pi_\lambda^C(i, i, k) = \left(\frac{3 d_k c_k^i}{2(d_i c_i^k + 2 d_k c_k^i)} \right)^2$$

$$\lim_{\lambda \to 0} \pi_\lambda^B(i, k, i) = 0$$

so that (i, i, i) is an equilibrium of the game $\mathcal{G}_\lambda$ as $\lambda \to 0$ if and only if for all $k \in N$

$$d_k c_k^i \geq d_i c_i^k \tag{21}$$

11

(ii) For distinct $i, k \in N$:

$$\lim_{\lambda \to +\infty} \pi_\lambda^A(k, i, i) = \lim_{\lambda \to +\infty} \pi_\lambda^C(i, i, k) = \left(\frac{3b_k^i}{4n} \right)^2$$

$$\lim_{\lambda \to +\infty} \pi_\lambda^B(i, k, i) = 0$$

so that (i, i, i) is an equilibrium of the game $\mathcal{G}_\lambda$ as $\lambda \to +\infty$ if and only if for all $k \in N$

$$b_k^i \geq 2b_i^k \tag{22}$$

The first part of Proposition 4 is interpreted in terms of influenceability, and the second part – in terms of influence (centrality, intermediacy). Hence, Proposition 4 says the following:

(i) The strategy profile (i, i, i) is a Nash equilibrium of the game $\mathcal{G}_\lambda$ for a vanishingly small level of impact λ if and only if i is at least as influenceable as any other individual $k \in N$. When the persuaders are of the weak impact, they should rather target highly influenceable individuals, i.e., i with the lower $d_i c_i^k$. Such i does not listen to many other individuals and it is easier and quicker to convince him to follow a new opinion.

(ii) The strategy profile (i, i, i) is a Nash equilibrium of the game $\mathcal{G}_\lambda$ for an arbitrarily large level of impact λ if and only if the relative influence of i is not smaller than the double relative influence of any other individual k. When the level of impact increases, the persuaders should target highly influential (central) individuals.

When comparing Proposition 4 to the corresponding result in the presence of only two extreme persuaders [43], the strategies for weak-impact persuaders are the same even with a growing number of individuals, but strong-impact persuaders should take into account the presence of the new centrist persuader, since we have the condition $b_k^i \geq 2b_i^k$ instead of $b_k^i \geq b_i^k$ (condition in the case of only two extreme persuaders).

Unequal persuasion impacts Next we briefly discuss the case of unequal impacts of the persuaders. According to Theorem 1, at each symmetric strategy profile (i, i, i) the payoffs are given by (17) and the aggregate opinion by (13). Assume that $\lambda > \gamma > \mu$. It is clear that as the persuasion impact λ of A increases, the aggregate opinion $\tilde{x}_N$ gets closer to n. It means that all the influence in the network is going under control of persuader A. In such a situation persuaders B and C have to conceal their intentions in order to keep their fraction of influence among the individuals in the society. In this case, they are using mixed strategies, which can guarantee that, with a positive probability, they will be the only influencers of one individual and the aggregate opinion will be less than $\frac{n(2\lambda + \gamma)}{2(\lambda + \gamma + \mu)}$.

As in the framework of [43], if the impact levels $\gamma, \mu > 0$ of persuaders B and C are fixed and the impact λ of persuader A is sufficiently large, then the game $\mathcal{G}_{\lambda, \gamma, \mu}$ has only equilibria in mixed strategies.

As the impact of persuader A tends towards infinity, one has:

$$\lim_{\lambda \to +\infty} \tilde{x}_N(k, i, i) = \frac{2\big((\gamma + \mu)b_i^k + d_i c_i^k n\big) + \gamma b_k^i}{2(d_i c_i^k + \gamma + \mu)} = n \left(\frac{(\gamma + 2\mu)\frac{b_i^k}{n} + \gamma + 2d_i c_i^k}{2(d_i c_i^k + \gamma + \mu)} \right) \tag{23}$$

12

When λ, μ are fixed, and persuader B has the infinite impact γ, then:

$$\lim_{\gamma \to +\infty} \tilde{x}_N(i, k, i) = \frac{2\lambda n - (\lambda - \mu)b_i^k + d_i c_i^k n}{2(d_i c_i^k + \lambda + \mu)} = \frac{n}{2}\left(\frac{2\lambda - (\lambda - \mu)\frac{b_i^k}{n} + d_i c_i^k}{d_i c_i^k + \lambda + \mu}\right) \qquad (24)$$

and for persuader C with very large μ, when the impacts of A and C are fixed:

$$\lim_{\mu \to +\infty} \tilde{x}_N(i, i, k) = \frac{(2\lambda + \gamma)n - (2\lambda + \gamma)b_i^k}{2(d_i c_i^k + \lambda + \gamma)} = n\left(\frac{(2\lambda + \gamma)(1 - \frac{b_i^k}{n})}{2(d_i c_i^k + \lambda + \gamma)}\right) \qquad (25)$$

(23), (24) and (25) determine the aggregate opinions when one of the persuaders dominates by exerting infinite impact while the others have fixed impacts and target the same individual. We have $\frac{b_i^k}{n} < 1$. Persuader A wants the aggregate opinion to be as close as possible to the total number of individuals in the society, persuader B – as close as possible to $\frac{n}{2}$, and persuader C aims at having the aggregate opinion as close as possible to 0. It follows that the payoffs of the persuaders will be optimal as $d_i c_i^k$ increases. In all three cases the dominant persuader is better off by not only targeting highly influential individuals but also by reducing the influence that his opponents have on their target, i.e., by preventing the opponents' target to escape from the influence of the dominant persuader.

5 Related literature

There is a vast literature on social networks devoted to modeling and analyzing opinion formation and diffusion; for surveys, see e.g. [3], [16], [47]. A society is usually described as a network whose nodes represent the individuals and the edges represent their social relations. Each node has an opinion on a certain issue. The opinion can be a binary variable (or vector) which is a good description for a variety of situations (e.g., [21], [29], [38], [42]). However, in some cases, e.g. concerning political issues, a continuous variable might be more appropriate for representing an opinion (e.g., [22], [43], [44]). The updating of individuals' opinions can be based on various rules, e.g., by taking into account opinions of neighbors. Moreover, independently of the opinion updating rule, different approaches to opinion diffusion in a society can be used. For instance, diffusion of opinion can accelerate when opinion leaders or key players are engaged [15]. Opinions can also be led by informed agents, since finding the opinion leaders needs global knowledge about the topology of the network [5]. A phenomenon closely related to influence and opinion conformity is that of persuasion, which can attempt to influence a person's beliefs, attitudes, intentions, motivations, or behaviors; for surveys and different persuasion methods, see e.g. [19], [20], [37]. Our paper also contributes to the literature on consensus reaching, the topic studied extensively in different scientific fields; see e.g. [27], [31], [48]; for surveys see e.g. [45].

There are essentially two methods of modeling social learning through networks: Bayesian learning, where agents use Bayes' rule to assess the state of the world (e.g., [2], [6], [8], [11], [12], [28], [34]) and non-Bayesian approach, like imitation models, where agents instead consider a weighted average of their neighbors' opinions or actions in a previous period (e.g., [22], [39], [47]). The DeGroot model is such an imitation framework:

13

it involves repeated communication, where people can keep talking to each other and taking weighted averages of information that they get from their friends. There exist various extensions of [22], e.g., works with the updating varying in time and circumstances (e.g., [23], [32], [33], [51]) and the misrepresenting own opinions [17].

The literature closely related to influence and opinion formation is the one concerning targeting. In computer science literature usually an algorithmic perspective is used to study the target selection for the optimal adoption and diffusion of innovation (e.g., [24], [49], [50], [56]). Also in economics and marketing there is a growing literature that concerns targeting in social networks. [59] studies the optimal targeting strategy in diffusion based on social imitation. [36] model networks in terms of degree distributions and study influence strategies in the presence of local interaction. They consider two groups of agents, where the one group influences the another one, and optimal influence strategies depend of the connectivity of targeted individuals. [60] assume that some agents are "stubborn", i.e., their opinion is fixed at one of the two values. "Stubborn" players are also considered in studies of competitive strategy in network environments. [1] and [4] analyze an opinion dynamics model with two types of agents: regular, and stubborn or forceful. The competition between firms aiming at maximizing product adoption by consumers located in a social network is also studied in [13], [26], and [41]. While social influence is often blamed for unpredictability, inequalities and inefficiencies in markets, [53] show that with a proper social signal and position assignment for the products, the market can become predictable, and inequalities and inefficiencies can be controlled. [35] propose a framework to study optimal interventions, when individuals interact strategically with their neighbors. They solve such intervention problems by exploiting the singular value decomposition of strategic interaction matrices. [54] apply another approach, based on cooperative games, to influence. More precisely, they model an influence game as a cooperative simple game in which a team of agents succeeds if it is able to convince enough agents to participate in the task, e.g., to vote in favor of a decision.

Opinion formation is crucial for the analysis of voting and political campaigns. [55] develop a voting advice model to match voters with political candidates, that accounts for political power, media visibility, and proximity of opinions. They apply their model to Parliamentary Elections in Finland. To the best of our knowledge, the network approach has not been frequently used to study lobbying and political campaigning, despite the fact that targeting in networks can be a quite useful and natural tool for modeling voting competitions in political science. An exception is the work of [52] who analyzes the strategic campaign spending in elections by using the network perspective. He considers a framework with two persuaders (political parties, competing lobbies) who allocate resources to sway voters, and shows that the unique pure strategy Nash equilibrium is such that the spending on each voter is proportional to his eigenvector centrality. This confirms the well known fact that the structure of the social network can influence results.

There exist several other studies that link network centrality with economic outcomes, see e.g. [7] and [18]. Their main result is the characterization of the Nash equilibrium with a player's action being proportional to his Bonacich centrality [14]. The key player in [7] is identified by an intercentrality measure that takes into account both a player's centrality and his contribution to the centrality of the others. [58] analyses targeting in the context of viral marketing and shows that the optimal targeting strategy involves the individuals' decay centrality. Also [9] and [10] study the problem of identifying the most influential

14

agents in a process of information transmission. They introduce diffusion centrality which measures how extensively the information spreads from a given player. The diffusion centrality nests the degree centrality (if there is one time period of communication), the eigenvector centrality and Katz-Bonacich centrality (if there are unlimited periods of communication). The best targets in [43] are characterized by another (new) network centrality called intermediacy, which is also the key concept in the present paper. Also [40] study the issue of ranking the nodes in terms of closeness, betweenness and influence. They extend the betweenness centrality measure to take into account several dimensions (criteria) and analyze a case study of the Iranian Government to detect the key members.

6 Conclusions

In this paper we studied a model of competitive opinion formation in a social network. Our point of departure was the model of influence [43] with two strategic players having opposite opinions and targeting non-strategic agents in a network. We extended that framework by adding a "centrist persuader" and focused on the effects of the presence of the third persuader on opinion convergence and consensus reaching in the society, on conditions for Nash equilibria in the game played by the persuaders, and on characterizations of targets in the extended model.

We showed that due to the basic assumptions of the DeGroot model, opinion convergence is preserved, although obviously the long run opinion in the society is different from the one reached in the presence of only two persuaders. Furthermore, consensus can emerge in the society if the three persuaders target the same individual. The study reveals that in this case, if additionally the persuaders are of the equal impact, then the centrist persuader has no effect on the social opinion, but the outcome turns out to be the best and ideal for him. The same "middle outcome" is already obtained when only the extreme persuaders target the same individual and have the equal impact, independently of the behavior and impact of the centrist persuader.

In the presence of the new (centrist) persuader, the game is not constant sum anymore, as it was the case in [43], and the payoffs are defined in a new way. For each persuader we set a payoff function as a difference between his own opinion and the one formed in the society. So, the game is defined as a system of minimization problems. Furthermore, we considered equilibria of the game. Our illustrative examples showed that the existence of a pure strategy Nash equilibrium does depend on the structure of the network. For example, there is no equilibrium in pure strategies in circular networks. Similarly, no equilibrium in pure strategies exists in perfectly symmetric networks. We showed that there exist influence networks that admit equilibria in pure strategies, i.e., star networks. This type of networks have an individual with outstanding characteristics that makes it possible to have a symmetric equilibria in pure strategies.

We used [43] as a baseline and our results are framed by using some notions and definitions introduced in [43]. Adding the third persuader into the model leads naturally to some differences between the sample model and the extended framework. We showed that a symmetric equilibrium in pure strategies emerges when the persuaders exert an equal impact. We gave a general condition for the existence of the equilibrium. It is characterized by two features of the targets: their influenceability and centrality. Our result in this respect is similar to the result obtained in [43] with an exception that in our framework

the relative influence of a potential target has to be at least twice higher than the one of any other individual in the network. In other words, the persuaders are demanding higher influence/centrality from the individual they want to target to compensate the impact of the additional persuader. Influenceability gains importance versus intermediacy as the size of the network grows or the impact of the persuaders decreases. In the case when the persuasion impacts are unequal, the high-impact persuader aims at ensuring preeminence on the network by increasing his centrality and diminishing the influenceability of his opponents' target. As for the low-impact persuaders, they seek to keep a minimal level of influence by hiding their target from the opponent's impact. Therefore, the low-impact persuaders must use mixed strategies in order to hide their target from the dominant opponent. A growing number of the persuaders does not affect too much the game when the persuaders are weak, similarly to [43].

7 Appendix

7.1 Proof of Proposition 1

We consider a society represented by a directed graph G. Due to our assumption that the social network defined by the adjacency matrix W is connected, the convergence of opinions in our targeting model with three persuaders is a direct consequence of the DeGroot model. It means that the only essential classes, such that no arc is going outside, are the persuaders $\{A\}$, $\{B\}$, and $\{C\}$. Consider a steady state vector $\overline{\mathbf{x}}_N$ such that $\mathbf{x}_N(t+1) = \mathbf{x}_N(t) = \overline{\mathbf{x}}_N(\mathbf{s})$. From (8) we have

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})\left[1\ \tfrac{1}{2}\ 0\right]' + \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\overline{\mathbf{x}}_N(\mathbf{s})$$

$$\left[I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\right]\overline{\mathbf{x}}_N(\mathbf{s}) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})\left[1\ \tfrac{1}{2}\ 0\right]'$$

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \left[I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\right]^{-1}\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})\left[1\ \tfrac{1}{2}\ 0\right]'$$

$$\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})\left[1\ \tfrac{1}{2}\ 0\right]' = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})\left[\tfrac{\lambda}{d_{s_A}}e_{s_A}\ \tfrac{\gamma}{d_{s_B}}e_{s_B}\ \tfrac{\mu}{d_{s_C}}e_{s_C}\right]\begin{bmatrix}1\\\tfrac{1}{2}\\0\end{bmatrix}$$

Hence,

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \left[I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\right]^{-1}\Delta_{\lambda,\gamma,\mu}(\mathbf{s})\left(\frac{\lambda}{d_{s_A}}e_{s_A} + \frac{\gamma}{2d_{s_B}}e_{s_B}\right)$$

$\left[I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\right]^{-1}$ is always invertible, and therefore the steady state vector $\overline{\mathbf{x}}_N(\mathbf{s})$ always exists.

7.2 Proof of Proposition 2

Suppose $s_A = s_B = s_C = i$. Then

$$\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \frac{\lambda}{d_i+\lambda+\gamma+\mu} & \frac{\gamma}{d_i+\lambda+\gamma+\mu} & \frac{\mu}{d_i+\lambda+\gamma+\mu} \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \tag{26}$$

16

with $0 < \frac{\lambda}{d_i+\lambda+\gamma+\mu} + \frac{\gamma}{d_i+\lambda+\gamma+\mu} + \frac{\mu}{d_i+\lambda+\gamma+\mu} \leq 1$.

Since the solution of $\overline{\mathbf{x}}_N(\mathbf{s}) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})[1\frac{1}{2}0]' + \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\overline{\mathbf{x}}_N(\mathbf{s})$ is unique, we can check if a consensus vector $\overline{\mathbf{x}}_N(\mathbf{s}) = [\alpha \cdots \alpha]'$ is a solution. We have for all rows $j \neq i$:

$$\alpha = 0 + 1 \cdot \alpha = \alpha$$

Since $M_{\lambda,\gamma,\mu}(\mathbf{s})$ is row-stochastic, for i-targeted individual we have:

$$\alpha = \frac{\lambda}{d_i+\lambda+\gamma+\mu} + \frac{\gamma}{2(d_i+\lambda+\gamma+\mu)} + \left(1 - \frac{\lambda}{d_i+\lambda+\gamma+\mu} - \frac{\gamma}{d_i+\lambda+\gamma+\mu} - \frac{\mu}{d_i+\lambda+\gamma+\mu}\right)\alpha$$
$$\alpha = \frac{2\lambda+\gamma}{2(d_i+\lambda+\gamma+\mu)}\frac{d_i+\lambda+\gamma+\mu}{\lambda+\gamma+\mu}$$
$$\alpha = \frac{2\lambda+\gamma}{2(\lambda+\gamma+\mu)}$$

7.3 Proof of Proposition 3

Suppose $s_A = s_C = i$ and $s_B = k \neq i$. Then

$$\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s}) = \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ \frac{\lambda}{d_i+\lambda+\mu} & 0 & \frac{\mu}{d_i+\lambda+\mu} \\ 0 & \frac{\gamma}{d_k+\gamma} & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

Since the solution of $\overline{\mathbf{x}}_N(\mathbf{s}) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})[1\frac{1}{2}0]' + \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\overline{\mathbf{x}}_N(\mathbf{s})$ is unique, we can check if a consensus vector $\overline{\mathbf{x}}_N(\mathbf{s}) = [\alpha \cdots \alpha]'$ is a solution. We have for all rows $j \neq i$ and $j \neq k$:

$$\alpha = 0 + 1 \cdot \alpha = \alpha$$

Since $M_{\lambda,\gamma,\mu}(\mathbf{s})$ is row-stochastic, for i-targeted individual, we get:

$$\alpha = \frac{\lambda}{d_i+\lambda+\mu} + \left(1 - \frac{\lambda}{d_i+\lambda+\mu} - \frac{\mu}{d_i+\lambda+\mu}\right)\alpha$$
$$\alpha = \frac{\lambda}{d_i+\lambda+\mu}\frac{d_i+\lambda+\mu}{\lambda+\mu}$$
$$\alpha = \frac{\lambda}{\lambda+\mu}$$

and for k individual – the target of the centrist persuader:

$$\alpha = \frac{\gamma}{2(d_k+\gamma)} + \left(1 - \frac{\gamma}{d_k+\gamma}\right)\alpha$$
$$\alpha = \frac{\gamma}{2(d_k+\gamma)}\frac{d_k+\gamma}{\gamma}$$
$$\alpha = \frac{1}{2}$$

7.4 Proof of Theorem 1

First, we recall from [43] some additional notations and two lemmas. Let $\overline{C}_i^k$ be the set of cycles around i that do not pass through k, $\overline{B}_i^k$ be the set of walks starting from any individual $\neq i$ that reach i before going through k, and let $F_{i,k}$ be the set of direct walks from i to k, i.e., the set of walks that start in i, end up in k and do not pass through i nor k in between. Moreover, let:

$$\overline{c}_i^k := \sum_{p\in\overline{C}_i^k} w(p), \ \overline{b}_i^k := \sum_{p\in\overline{B}_i^k} w(p), \ f_{ik} := \sum_{p\in F_{i,k}} w(p) \tag{27}$$

17

We have $c_i^k + \bar{c}_i^k = 1$ for all distinct $i, k \in N$. The corresponding set of walks and measures for k are denoted by $\overline{C}_k^i$, $\overline{B}_k^i$, $F_{k,i}$, $\bar{c}_k^i$, $\bar{b}_k^i$ and $f_{k,i}$.

Lemma 1 *For all distinct $i, k \in N$, one has:*

$$\bar{b}_i^k + \bar{b}_k^i = n - 2, \; b_i^k + \bar{b}_i^k = n - 1, \; b_i^k + b_k^i = n \tag{28}$$

Let Γ_i be the set of cycles around i (i.e., walks that start and finish in i and do not pass through i in between), and $y_i = \sum_{p \in \Gamma_i} w(p)$. Let Φ_i be the set of walks to i that have never passed through i before and $\phi_i := \sum_{p \in \Phi_i} w(p)$.

Lemma 2 *For all $i = 1, \cdots, n$, $y_i = 1$ and $\phi_i = n - 1$.*

From Proposition 1 we have (10):

$$\overline{\mathbf{x}}_N(\mathbf{s}) = [I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W]^{-1} \Delta_{\lambda,\gamma,\mu}(\mathbf{s}) \left(\frac{\lambda}{d_{s_A}} e_{s_A} + \frac{\gamma}{2d_{s_B}} e_{s_B} \right)$$

Using the results of [57] about non-negative matrices:

Lemma 3 *Let A be a finite $n \times n$ matrix such that $\lim_{k \to \infty} A^k = \mathbf{0}$. Then $[I - A]^{-1}$ exists and*

$$[I - A]^{-1} = \sum_{k=0}^{\infty} A^k$$

with $A^0 = I$.

We can modify the aggregate opinion in the following way:

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \sum_{m=0}^{\infty} \left(\Delta_{\lambda,\gamma,\mu}(\mathbf{s}) \cdot W \right)^m \Delta_{\lambda,\gamma,\mu}(\mathbf{s}) \left(\frac{\lambda}{d_{s_A}} e_{s_A} + \frac{\gamma}{2d_{s_B}} e_{s_B} \right) \tag{29}$$

(i) If $s_A = s_B = s_C = i$, i.e., if all three persuaders target the same individual i, then

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \sum_{m=0}^{\infty} \left(\Delta_{\lambda,\gamma,\mu}(\mathbf{s}) \cdot W \right)^m \frac{2\lambda + \gamma}{2(d_{s_A} + \lambda + \gamma + \mu)}$$

$P_{k,l}^m$ is the set of walks of length m from k to l in the graph G associated to W. As introduced earlier, for any walk $p = (i_1, \cdots, i_m)$, $w(p)$ denotes its weight measured according to W. $v_k(p)$ is the number of times the walk p passes through k (without taking into account the departure node). Equation (29) can then be rewritten as a sum representing the influence conveyed through each walk of the network and where each passage through one of the target nodes is re-weighted in order to account for the influence of the persuaders. That is, one has:

$$\overline{\mathbf{x}}_N(\mathbf{s}) = \left(\sum_{k \in N} \sum_{m=0}^{\infty} \sum_{p \in P_{k,s_A}^m} w(p) \left(\frac{d_{s_A}}{d_{s_A} + \lambda + \gamma + \mu} \right)^{v_{s_A}(p)} \right) \frac{2\lambda + \gamma}{2(d_{s_A} + \lambda + \gamma + \mu)} \tag{30}$$

The payoff of a link to individual i depends on the degree to which he is influenceable (through the number of outgoing links of the individual) and his influence measured by the (weighted and discounted) number of walks that pass through that individual.

18

Assume that all three persuaders target i and let Π_i^k denote the set of walks that end in i and have gone exactly k times through i before. Then one can decompose the set of walks ending up in i according to their total number of passages in i, so that (30) becomes:

$$\tilde{x}_N(i,i,i) = \left(\sum_{k=0}^{\infty} \sum_{p \in \Pi_i^k} w(p) \left(\frac{d_i}{d_i + \lambda + \gamma + \mu} \right)^k \right) \frac{2\lambda + \gamma}{2(d_i + \lambda + \gamma + \mu)}$$

A walk in Π_i^k consists of k cycles around i plus, possibly, a walk to i. Hence,

$$\sum_{p \in \Pi_i^k} w(p) = (y_i)^k (1 + \phi_i)$$

and therefore

$$\tilde{x}_N(i,i,i) = \left(\sum_{k=0}^{\infty} \left(y_i \frac{d_i}{d_i + \lambda + \gamma + \mu} \right)^k (1 + \phi_i) \right) \frac{2\lambda + \gamma}{2(d_i + \lambda + \gamma + \mu)}$$

Consequently, we have:

$$\tilde{x}_N(i,i,i) = \frac{(1 + \phi_i)(2\lambda + \gamma)}{2(d_i + \lambda + \gamma + \mu - y_i d_i)}$$

Using Lemma 2, we get the result given in (13):

$$\tilde{x}_N(i,i,i) = \frac{n(2\lambda + \gamma)}{2(\lambda + \gamma + \mu)}$$

Hence, given (12) the payoffs are:

$$\begin{aligned}
\pi_{\lambda,\gamma,\mu}^A(i,i,i) &= \left(\frac{2\mu + \gamma}{2(\lambda + \gamma + \mu)} \right)^2 \\
\pi_{\lambda,\gamma,\mu}^B(i,i,i) &= \left(\frac{\mu - \lambda}{2(\lambda + \gamma + \mu)} \right)^2 \\
\pi_{\lambda,\gamma,\mu}^C(i,i,i) &= \left(\frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)} \right)^2
\end{aligned} \tag{31}$$

(ii) Next, we consider the case where persuaders A and C target i and persuader B targets k, i.e., we assume that $s_A = s_C = i$ and $s_B = k$. Let us then denote (as in [43]) by ϕ_i^k the sum of weights of the walks to i but where each passage through k is weighted by the factor $\frac{d_k}{d_k + \gamma}$. Similarly, let us denote by y_i^k the sum of weights of walks that cycle around i but where each passage through k is weighted by the factor $\frac{d_k}{d_k + \gamma}$. An argument similar to the one in the symmetric case then yields:

$$\begin{aligned}
\tilde{x}_N(\mathbf{s}) = &\left(\sum_{k \in N} \sum_{m=0}^{\infty} \sum_{p \in P_{k,s_A}^m} w(p) \left(\frac{d_{s_A}}{d_{s_A} + \lambda + \mu} \right)^{v_{s_A}(p)} \right) \left(\frac{\lambda}{d_{s_A} + \lambda + \mu} \right) + \\
&\left(\sum_{k \in N} \sum_{m=0}^{\infty} \sum_{p \in P_{k,s_B}^m} w(p) \left(\frac{d_{s_B}}{d_{s_B} + \gamma} \right)^{v_{s_B}(p)} \right) \left(\frac{\gamma}{2(d_{s_B} + \gamma)} \right)
\end{aligned}$$

19

$$\tilde{x}_N(i,k,i) = \frac{\lambda(1+\phi_i^k)}{d_i + \lambda + \mu - y_i^k d_i} + \frac{\gamma(1+\phi_k^i)}{2(d_k + \gamma - y_k^i d_k)} \tag{32}$$

In order to decompose further equation (32) and derive formula (14), we use the auxiliary definitions recalled at the beginning of the proof:

The set of walks to i consists in walks to i that do not pass through k plus the set of direct walks from k to i preceded by an arbitrary number of cycles around k (not passing through i) preceded by a walk to k that does not pass through i. We then have:

$$\phi_i^k = \bar{b}_i^k + f_{k,i}\frac{d_k}{d_k+\gamma}\left(\sum_{j=0}^{+\infty}\left(\bar{c}_k^i\frac{d_k}{d_k+\gamma}\right)^j\right)(1+\bar{b}_k^i) = \bar{b}_i^k + f_{k,i}\frac{d_k}{d_k+\gamma-\bar{c}_k^i d_k}(1+\bar{b}_k^i) \tag{33}$$

In a similar way, the set of cycles around i consists in the set of cycles around i not passing through k together with the set of direct walks from k to i preceded by an arbitrary number of cycles around k (not passing through i) preceded by a direct walk from i to k. So that:

$$y_i^k = \bar{c}_i^k + f_{k,i}\frac{d_k}{d_k+\gamma}\left(\sum_{j=0}^{+\infty}\left(\bar{c}_k^i\frac{d_k}{d_k+\gamma}\right)^j\right)f_{i,k} = \bar{c}_i^k + f_{k,i}f_{i,k}\frac{d_k}{d_k+\gamma-\bar{c}_k^i d_k} \tag{34}$$

Similarly:

$$\phi_k^i = \bar{b}_k^i + f_{i,k}\frac{d_i}{d_i+\lambda+\mu}\left(\sum_{j=0}^{+\infty}\left(\bar{c}_i^k\frac{d_i}{d_i+\lambda+\mu}\right)^j\right)(1+\bar{b}_i^k) = \bar{b}_k^i + f_{i,k}\frac{d_i}{d_i+\lambda+\mu-\bar{c}_i^k d_i}(1+\bar{b}_i^k) \tag{35}$$

$$y_k^i = \bar{c}_k^i + f_{i,k}\frac{d_i}{d_i+\lambda+\mu}\left(\sum_{j=0}^{+\infty}\left(\bar{c}_i^k\frac{d_i}{d_i+\lambda+\mu}\right)^j\right)f_{k,i} = \bar{c}_k^i + f_{i,k}f_{k,i}\frac{d_i}{d_i+\lambda+\mu-\bar{c}_i^k d_i} \tag{36}$$

Plugging these equations into (32) leads to:

$$\tilde{x}_N(i,k,i) = \frac{\lambda\left(1+\bar{b}_i^k + f_{k,i}\frac{d_k}{d_k+\gamma-\bar{c}_k^i d_k}(1+\bar{b}_k^i)\right)}{d_i+\lambda+\mu - d_i\left(\bar{c}_i^k + f_{k,i}f_{i,k}\frac{d_k}{d_k+\gamma-\bar{c}_k^i d_k}\right)}+$$
$$\frac{\gamma\left(1+\bar{b}_k^i + f_{i,k}\frac{d_i}{d_i+\lambda+\mu-\bar{c}_i^k d_i}(1+\bar{b}_i^k)\right)}{2\left(d_k+\gamma - d_k\left(\bar{c}_k^i + f_{i,k}f_{k,i}\frac{d_i}{d_i+\lambda+\mu-\bar{c}_i^k d_i}\right)\right)}$$

$$\tilde{x}_N(i,k,i) = \frac{\lambda\left[(1+\bar{b}_i^k)\gamma + d_k(1-\bar{c}_k^i)\left(1+\bar{b}_i^k + f_{k,i}\frac{1+\bar{b}_k^i}{1-\bar{c}_k^i}\right)\right]}{(d_i+\lambda+\mu)(d_k+\gamma-\bar{c}_k^i d_k) - d_i\left(\bar{c}_i^k\gamma + d_k(1-\bar{c}_k^i)\left(\bar{c}_i^k + f_{k,i}f_{i,k}\frac{1}{1-\bar{c}_k^i}\right)\right)}+$$
$$\frac{\gamma\left[(1+\bar{b}_k^i)(\lambda+\mu) + d_i(1-\bar{c}_i^k)\left(1+\bar{b}_k^i + f_{i,k}\frac{1+\bar{b}_i^k}{1-\bar{c}_i^k}\right)\right]}{2\left[(d_k+\gamma)(d_i+\lambda+\mu-\bar{c}_i^k d_i) - d_k\left(\bar{c}_k^i(\lambda+\mu) + d_i(1-\bar{c}_i^k)\left(\bar{c}_k^i + f_{i,k}f_{k,i}\frac{1}{1-\bar{c}_i^k}\right)\right)\right]}$$

20

We have:

$$\phi_i = \bar{b}_i^k + f_{k,i}\left(\sum_{k=0}^{+\infty}(\bar{c}_k^i)^k\right)(1+\bar{b}_k^i) = \bar{b}_i^k + f_{k,i}\frac{1}{1-\bar{c}_k^i}(1+\bar{b}_k^i)$$

$$y_i = \bar{c}_i^k + f_{k,i}\left(\sum_{k=0}^{+\infty}(\bar{c}_k^i)^k\right)f_{i,k} = \bar{c}_i^k + f_{i,k}f_{k,i}\frac{1}{1-\bar{c}_k^i}$$

We can get ϕ_k and y_k in a similar way. Hence, we have:

$$\tilde{x}_N(i,k,i) = \frac{\lambda\left[(1+\bar{b}_i^k)\gamma + d_k(1-\bar{c}_k^i)(1+\phi_i)\right]}{(d_i+\lambda+\mu)(d_k+\gamma-\bar{c}_k^i d_k) - d_i\left(\bar{c}_k^k\gamma + d_k(1-\bar{c}_k^i)y_i\right)} + \tag{37}$$
$$\frac{\gamma\left[(1+\bar{b}_k^i)(\lambda+\mu) + d_i(1-\bar{c}_i^k)(1+\phi_k)\right]}{2\left[(d_k+\gamma)(d_i+\lambda+\mu-\bar{c}_i^k d_i) - d_k\left(\bar{c}_i^i(\lambda+\mu) + d_i(1-\bar{c}_i^k)y_k\right)\right]}$$

According to Lemma 2, one has for all $i = 1, \cdots, n$ $y_i = 1$ and $\phi_i = n - 1$:

$$\tilde{x}_N(i,k,i) = \frac{\lambda\left[(1+\bar{b}_i^k)\gamma + d_k(1-\bar{c}_k^i)n\right]}{(d_i+\lambda+\mu)(d_k+\gamma-\bar{c}_k^i d_k) - d_i\left(\bar{c}_i^k\gamma + d_k(1-\bar{c}_k^i)\right)} +$$
$$\frac{\gamma\left[(1+\bar{b}_k^i)(\lambda+\mu) + d_i(1-\bar{c}_i^k)n\right]}{2\left[(d_k+\gamma)(d_i+\lambda+\mu-\bar{c}_i^k d_i) - d_k\left(\bar{c}_k^i(\lambda+\mu) + d_i(1-\bar{c}_i^k)\right)\right]}$$

$$\tilde{x}_N(i,k,i) = \frac{\lambda\left[\gamma b_k^i + d_k c_k^i n\right]}{\gamma d_i c_i^k + (\lambda+\mu)d_k c_k^i + \gamma(\lambda+\mu)} + \frac{\gamma\left[(\lambda+\mu)b_i^k + d_i c_i^k n\right]}{2\left((\lambda+\mu)d_k c_k^i + \gamma d_i c_i^k + \gamma(\lambda+\mu)\right)} \tag{38}$$

By following the same procedure, we can get the expressions for (15) and (16).

7.5 Proof of Theorem 2

According to the definition of Nash equilibrium, (i,i,i) is an equilibrium if and only if no individual has a profitable deviation on his own, that is for all $k \in N$:

$$\begin{cases} \pi_\lambda^A(i,i,i) \leq \pi_\lambda^A(k,i,i) \\ \pi_\lambda^B(i,i,i) \leq \pi_\lambda^B(i,k,i) \\ \pi_\lambda^C(i,i,i) \leq \pi_\lambda^C(i,i,k) \end{cases}$$

$$\begin{cases} \left(1-\frac{1}{n}\tilde{x}_N(i,i,i)\right)^2 \leq \left(1-\frac{1}{n}\tilde{x}_N(k,i,i)\right)^2 & (a) \\ \left(\frac{1}{2}-\frac{1}{n}\tilde{x}_N(i,i,i)\right)^2 \leq \left(\frac{1}{2}-\frac{1}{n}\tilde{x}_N(i,k,i)\right)^2 & (b) \\ \left(\frac{1}{n}\tilde{x}_N(i,i,i)\right)^2 \leq \left(\frac{1}{n}\tilde{x}_N(i,i,k)\right)^2 & (c) \end{cases}$$

For equation (b):

$$\frac{1}{4} - \frac{1}{n}\tilde{x}_N(i,i,i) + \frac{1}{n^2}\left(\tilde{x}_N(i,i,i)\right)^2 \leq \frac{1}{4} - \frac{1}{n}\tilde{x}_N(i,k,i) + \frac{1}{n^2}\left(\tilde{x}_N(i,k,i)\right)^2$$

$$\frac{1}{n}\left(\tilde{x}_N(i,i,i)\right)^2 - \frac{1}{n}\left(\tilde{x}_N(i,k,i)\right)^2 \leq \tilde{x}_N(i,i,i) - \tilde{x}_N(i,k,i)$$

21

$$\frac{1}{n} \left(\tilde{x}_N(i,i,i) - \tilde{x}_N(i,k,i) \right) \left(\tilde{x}_N(i,i,i) + \tilde{x}_N(i,k,i) \right) \le \tilde{x}_N(i,i,i) - \tilde{x}_N(i,k,i)$$

We have two possibilities:

1) if $\tilde{x}_N(i,i,i) < \tilde{x}_N(i,k,i)$ then $\frac{1}{n}\tilde{x}_N(i,i,i) \ge 1 - \frac{1}{n}\tilde{x}_N(i,k,i)$

2) if $\tilde{x}_N(i,i,i) > \tilde{x}_N(i,k,i)$ then $\frac{1}{n}\tilde{x}_N(i,i,i) \le 1 - \frac{1}{n}\tilde{x}_N(i,k,i)$

Hence, the system of inequalities becomes:

$$\begin{cases} \tilde{x}_N(i,i,i) \ge \tilde{x}_N(k,i,i) \\ \begin{cases} \tilde{x}_N(i,i,i) < \tilde{x}_N(i,k,i) \\ \tilde{x}_N(i,i,i) \ge n - \tilde{x}_N(i,k,i) \end{cases} \quad or \quad \begin{cases} \tilde{x}_N(i,i,i) > \tilde{x}_N(i,k,i) \\ \tilde{x}_N(i,i,i) \le n - \tilde{x}_N(i,k,i) \end{cases} \\ \tilde{x}_N(i,i,i) \le \tilde{x}_N(i,i,k) \end{cases}$$

For particular case where $\lambda = \gamma = \mu$ and given Theorem 1, we get:

$$\begin{cases} n \left(d_k c_k^i - d_i c_i^k \right) \ge \lambda \left(2b_i^k - b_k^i \right) \\ \begin{cases} 0 \ge 0 \\ 1 \le 1 \end{cases} \quad or \quad \begin{cases} 0 \le 0 \\ 1 \ge 1 \end{cases} \\ n \left(d_i c_i^k - d_k c_k^i \right) \le \lambda \left(b_k^i - 2b_i^k \right) \end{cases}$$

From equation (b) we can conclude that for the centrist persuader, in case when all persuaders have the equal impact, there is no difference between targeting individual i with other persuaders or choosing a different individual k. We can then omit the systems for the centrist persuader, since they do not play any role. We have:

$$\begin{cases} n \left(d_k c_k^i - d_i c_i^k \right) \ge \lambda \left(2b_i^k - b_k^i \right) \\ n \left(d_i c_i^k - d_k c_k^i \right) \le \lambda \left(b_k^i - 2b_i^k \right) \end{cases}$$

and the final condition is

$$\lambda \left(b_k^i - 2b_i^k \right) \ge n \left(d_i c_i^k - d_k c_k^i \right) \tag{39}$$

7.6   Proof of Proposition 4

We get the limit results by calculating the limits under $\lambda \to 0$ and $\lambda \to +\infty$ in (12), using the results of Theorem 1. Next, we apply the definition of Nash equilibrium and compare the payoffs for (i) and (ii), respectively:

$$\frac{1}{2} \le \frac{3 d_k c_k^i}{2(d_i c_i^k + 2 d_k c_k^i)} \quad \Leftrightarrow \quad d_i c_i^k \le d_k c_k^i$$

$$\frac{1}{2} \le \frac{3 b_k^i}{4n} \quad \Leftrightarrow \quad 2 b_i^k \le b_k^i$$

22

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