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# ON MAXIMIN DYNAMIC PROGRAMMING & THE RATE OF DISCOUNT\*

JEAN-PIERRE DRUGEON<sup>†</sup>, THAI HA-HUY<sup>‡</sup> & THI DO HANH NGUYEN<sup>§</sup>

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## ABSTRACT

This article establishes a dynamic programming argument for a maximin optimization problem where the agent completes a minimization over a set of discount rates. Even though the consideration of a maximin criterion results in a program that is not convex and not stationary over time, it is proved that a careful reference to extended dynamic programming principles and a maxmin functional equation however allows for circumventing these difficulties and recovering an optimal sequence that is time consistent. **KEYWORDS:** maximin principle, non-convexities, value function, policy function, supermodularity. **JEL CLASSIFICATION NUMBERS:** C61,D90.

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<sup>†</sup>Paris School of Economics and Centre National de la Recherche Scientifique.

<sup>‡</sup>EPEE, Univ Evry, Université Paris-Saclay; TIMAS, Thang Long University.

<sup>§</sup>Vietnam Maritime University.

# 1. INTRODUCTION

## 1.1 MOTIVATIONS

While it was initiated by Richard Bellman [3] in order to solve multi-stage allocation problems, dynamic programming has swiftly become one of the workhorses of modern dynamic economic analysis. The traditional approach in economics has been set forth by Stokey & Lucas (with Prescott) [17] and is based upon a well known separable criterion where the evaluation of the utility streams is *weighted* through a constant given discount factor  $\delta \in ]0, 1[$ . For every indirect utility  $V$ , a production correspondence  $\Gamma$  and a given  $x_0$ , the value of the payoff is given by  $\sup_{x_{t+1} \in \Gamma(x_t)} \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1})$ . Considering the supremum value  $J(x_0)$  of this program, the dynamic programming approach indeed establishes that it satisfies the *Bellman principle of optimality*, or the satisfaction, for an optimal  $\{x_t^*\}_{t=0}^{\infty}$ , of the *Bellman Equation*  $J(x_t^*) = V(x_t, x_{t+1}^*) + \delta J(x_{t+1}^*)$  for every  $t = 0, 1, 2, \dots$ . Upon the establishment of a solution to this functional equation and the associated determination of the value function, this would, *e.g.*, allow for characterizing complex infinite horizon problems through sequences of stationary two-period problems.

In the recent decades, a large literature has been tracking the limitations of Stokey & Lucas (with Prescott) [17] who, *e.g.*, heavily relied on continuity, boundedness and convexity for establishing the uniqueness of the solution of the Bellman equation through classical fixed point theory. As a matter of illustration and for dealing with the inability of this approach to deal with the unboundedness of the criterion  $V$ , Alvarez & Stokey [1] first provided a direct approach to the logarithmic utility case while Le Van & Morhaim [13] had an unrestricted approach to the optimization problem that was refined by Duran [7] while Rinçon-Zapatero & Rodríguez-Palmero [15] introduced a novel local contraction mapping approach. Following an *order theory* perspective, Kahimigashi [11], [12] has more recently provided an argument that does not impose any topological structure on the optimization problem. Finally, and following a tradition with Streufert [18, 19], Le Van and Vailakis [14], Rinçon-Zapatero & Rodríguez-Palmero [16] or, more recently, Jaśkiewicz, Matkowski and Nowak [10] or Bich, Drugeon & Morhaim [4], some authors have aimed at generalizing dynamic programming approach and techniques to non-separable criteria.

This article will follow a distinct range of concerns and rather draw on the sources of the benchmark optimization problem by drawing the attention to the way the criterion is *weighted* over time. Otherwise stated, what is the relevant way of evaluating these

inter-temporal utility streams? Is there a suitable weighting system that satisfies some *impatience*—decreasing weights over time—, *stability*—time-consistency—properties and that leaves the *sum of the weights equal 1*? A clear cut answer is available: at some date  $t$ , the weighting parameter must be  $(1 - \delta)\delta^t$  while the value of this *normalized payoff function* derives as:

$$\sup_{x_{t+1} \in \Gamma(x_t)} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right].$$

It however goes without saying that, in the case where the decision maker faces a unique discount rate, for the sake of simplicity and with no loss of generality, the term  $(1 - \delta)$  could be overlooked.

Pursue further this line of reasoning and consider now some configuration where the possible discount rate belongs to some set  $\mathcal{D} \subset ]0, 1[$ , *e.g.*, the case where some *disagreements persist about the selected discount rate* for evaluating some utility sequences. Likewise and for every sequence  $\{x_t\}_{t=0}^{\infty}$ , the decision maker ought to consider the augmented set of values

$$\left\{ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right\}_{\delta \in \mathcal{D}},$$

*in lieu* of the standard set.

The above class of concerns relates to the recent decision theory literature that addresses in an axiomatic way the scope for multiple optimal discount rates, *e.g.*, Chambers & Echenique [5] or Dugeon & Ha-Huy [6]. Under some *monotonicity*, *homogeneity*, *constant additivity* and, most importantly, *convexity* and *stability* assumptions on a preference order, the evaluation of the inter-temporal utility generated by the sequence  $\{x_t\}_{t=0}^{\infty}$  assumes a representation that directly echoes the earlier considerations:

$$\inf_{\delta \in \mathcal{D}} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right]. \quad (\text{M})$$

Otherwise stated, the relevance of the aforementioned *augmented* criteria is confirmed but it is to be supplemented by an extra and new *infimum operator* on the rate of discounts. Building on these results from the decision theory literature, this article will consider a maximin optimization problem with an optimally determined rate of discount. Given a compact set of discount factors  $\mathcal{D} = [\underline{\delta}, \bar{\delta}]$ , this leads to the following optimization problem:

$$\sup_{x_{t+1} \in \Gamma(x_t)} \inf_{\delta \in \mathcal{D}} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right]. \quad (\text{MM})$$

## 1.2 MAIN RESULTS & EARLIER LITERATURE

The aim of this article is then to provide a benchmark argument that characterizes the solutions of program (MM) through a *stationary* dynamic programming approach. Interestingly, in spite of the lack of stationarity for the preferences (M) and the absence of convexity in the infimum component of problem (MM), this article sets forth some formal conditions under which the optimal decision rule is time-consistent and stationary while the optimal sequence satisfies a stationary *Bellman-like* equation.

As a result of the absence of convexity in the infimum component of (MM), the argument is somewhat indirect. This builds from first considering a functional equation that is formally related to the optimization problem and then establishing that it assumes a unique solution. In order to prove that this solution corresponds to the value function of the program (MM), the argument proceeds in four steps. (i) Existence and uniqueness results for the optimal policy function relating to this solution of the functional equation. (ii) Under an extra strict supermodularity assumption on the payoff  $V$ , the proof of the monotonicity of the optimal sequence. (iii) A decisive step then proves that, for any  $x_0 > 0$ , there exists a unique admissible  $\delta^* \in \mathcal{D}$  such that the optimal sequence  $\chi^* = \{x_t^*\}_{t=0}^\infty$  is the solution of a classical optimization problem with a fixed discount factor  $\delta^*$ . Upon the determination of the optimal  $\delta^*$ , an extended version of the principle of optimality becomes available: the optimal capital sequence  $\chi^*(\delta^*)$  is considered and proved to satisfy the Bellman-like equation. (iv) The fixed point of that functional equation is eventually proved to be the value function of the maximin problem.

Even though it relies on classical assumptions\*, the article provides a non trivial generalization of the classical results of Stokey & Lucas [17], but also of more recent results in dynamic programming theory, such as Kahimigashi [11], that deal with non-convex structures. A key advance of this article is that an optimally determined rate of discount is compatible with stationary dynamic programming and the principle of optimality. It is finally worth pointing out that, even though both the choice of the discount factor and the optimal policy are stationary, the results are strongly dependent on the initial state.

The closest article from the current contribution is due to Geoffard [8] who introduced a class of optimization programs for which a minimization is to be completed over

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\*One may further notice that the dynamic programming argument of this article could be equally be used with a non-connected set of discount rates for which a time-consistent decision rule would indeed keep on being available.

paths of discount rates. His argument being put in continuous time and motivated by some worst-case scenario analysis, the author considers a *variational utility* model  $\min_{r(t) \in \mathcal{R}} \int_{t=0}^{+\infty} f[c(t), B(t), r(t)] dt$ , for  $\mathcal{R}$  that denotes the set of admissible paths of discount rates, and where  $f$  features current felicity as a function of the current value of consumption  $c(t)$ , the discount factor  $B(t)$  and the discount rate  $r(t)$ . For a production function  $\pi$  that is a quasi-concave function of  $(x, \dot{x})$ , the optimization problem states as  $\max_{x \in X} \min_{B \in \mathcal{B}} \int_{t=0}^{+\infty} f[\pi(x, \dot{x}), B, \dot{B}]$ . In opposition to the structure analyzed through the problem (MM), the whole analysis is however completed under a convexity assumption of the integrand with respect to  $(B, \dot{B})$ , so that the *supinf* and the *infsup* problems naturally coincide and the order can be reversed between the supremum and the infimum operators.

### 1.3 CONTENTS

The article is organized as follows. Section 2 introduces the maximin problem. Section 3 completes a functional approach and a stationary dynamic programming argument for the bounded case. Section 4 characterizes the solution for a special case where the fixed discount problem has a unique steady state and further provides a parametric example. Proofs are gathered in the Appendix.

## 2. A MAXIMIN PROBLEM

Time is discrete. Consider an economy endowed with a technology defined by a correspondence  $\Gamma : \mathbb{R}_+ \rightarrow \mathcal{P}(\mathbb{R}_+)$  and let  $\text{Gr}(\Gamma)$  denote the graph of  $\Gamma$ . For every  $x_0 \geq 0$ , define  $\Pi(x_0)$  as the set of feasible sequences  $\chi = \{x_t\}_{t=0}^{\infty}$  satisfying  $x_{t+1} \in \Gamma(x_t)$  for any  $t \geq 0$ . Further suppose that the set of eligible discount factors is  $\mathcal{D} = [\underline{\delta}, \bar{\delta}]$ , where  $0 < \underline{\delta} < \bar{\delta} < 1$ . Given  $x_0 > 0$ , consider the maximin problem

$$\sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right], \quad (\text{MM})$$

where  $V : \text{Gr}(\Gamma) \rightarrow \mathbb{R} \cup \{-\infty\}$  and first define  $v(\delta, \chi)$  as the operand in the program (MM):

$$v(\delta, \chi) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}).$$

For every feasible sequence  $\chi \in \Pi(x_0)$ , further let the function  $\hat{v}(\cdot)$  feature its infimum

with respect to the discount factor:

$$\hat{v}(\chi) = \inf_{\delta \in \mathcal{D}} v(\delta, \chi)$$

and let  $\text{dom}(\hat{v}, x_0)$  denote the set of feasible sequences  $\chi \in \Pi(x_0)$  such that  $\hat{v}(\chi) > -\infty$  for a given  $x_0 \geq 0$ .

A range of classical assumptions is first to be introduced on the problem (MM). These assumptions will ensure the concavity of the value function and the compactness of the set of feasible sequences in the product topology.

ASSUMPTION T1.  $\Gamma$  is non-empty, compact-valued and continuous on  $\mathbb{R}_+$ .

ASSUMPTION T2.  $\text{Gr}(\Gamma)$  is convex, *i.e.*, for any  $x, x'$  and  $y \in \Gamma(x)$ ,  $y' \in \Gamma(x')$ , for any  $0 \leq \lambda \leq 1$ ,  $(1 - \lambda)y + \lambda y' \in \Gamma((1 - \lambda)x + \lambda x')$ .

ASSUMPTION V1. The payoff function  $V$  is continuous on  $\text{Gr}(\Gamma)$ .

ASSUMPTION V2. The payoff function  $V$  is strictly concave on  $\text{int}[\text{Gr}(\Gamma)]$ .

The existence and the uniqueness of the solutions of the problem (MM), or equivalently of the ones of the maximization of  $\hat{v}(\chi_t)$  when  $\chi_t \in \Pi(x_0)$ , are then faced with in the following statement:

PROPOSITION 2.1. *Consider the functional  $\hat{v}$  under Assumptions T1 and V1.*

(i) *An optimal  $\chi^*$  does exist.*

(ii) *Further add Assumptions T2 and V2,*

a/  *$\hat{v}$  is strictly concave on  $\text{dom}(\hat{v}, x_0)$ .*

b/ *If  $\sup_{\chi \in \Pi(x_0)} \hat{v}(\chi) > -\infty$ , the maximin problem (MM) has a unique solution  $\chi^* \in \text{dom}(\hat{v}, x_0)$  such that*

$$\begin{aligned} \hat{v}(\chi^*) &= \sup_{\chi \in \Pi(x_0)} \hat{v}(\chi) \\ &= \sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right] \\ &= \max_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right]. \end{aligned}$$

Even though the information brought by Proposition 2.1 may at first sight appear as just recovering fairly classical insights, it is worth emphasizing that it merely provides a

partial answer to the issue at hand. Indeed, and while the supremum component of (MM) and the determination of  $\chi^*$  are faced with and noticeably simplified under the concave structure brought by the retainment of Assumptions T2 and V2, the details of the infimum component of (MM) and the determination of  $\delta^*$  will soon be shown to uncover a range of subtleties that result from the absence of convexity.

### 3. THE FUNCTIONAL EQUATION APPROACH & THE DYNAMIC PROGRAMMING ARGUMENT

#### 3.1 THE VALUE FUNCTION & A BELLMAN-LIKE OPERATOR

The value function of the program (MM) is defined by

$$\begin{aligned} J(x_0) &= \sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right] \\ &= \sup_{\chi \in \Pi(x_0)} \hat{v}(\chi). \end{aligned} \tag{1}$$

It was already pointed out that the details of Proposition 2.1 closely mimic the ones of the standard fixed discount optimization problem considered in Stokey & Lucas [17]. One may thus naturally wonder if the various steps of their associated classical dynamic programming argument would transpose to  $\hat{v}$  on  $\text{dom}(\hat{v}, x_0)$ ? As a first clue in that direction, is it possible to establish that the supremum of  $\hat{v}$ , as it is defined by the value function  $J$  in (1), satisfies a functional Bellman equation?

In order to assess the relevance of this claim and for a feasible sequence  $\chi \in \Pi(x_0)$ , define  $\chi_t = (x_t, x_{t+1}, x_{t+2}, \dots)$  as its truncated formulation that starts at  $t \geq 1$  and observe that the operand  $v(\delta, \chi)$  of the optimisation program (MM) can be split between a component that describes the current payoff and another one that features this discounted operand at a later date:

$$\begin{aligned} v(\delta, \chi_0) &= (1 - \delta)V(x_0, x_1) + \delta v(\delta, \chi_1) \\ &= (1 - \delta)V(x_0, x_1) + (1 - \delta)\delta V(x_1, x_2) + \delta^2 v(\delta, \chi_2) \\ &= \dots \\ &= (1 - \delta) \sum_{t=0}^T \delta^t V(x_t, x_{t+1}) + \delta^{T+1} v(\delta, \chi_{T+1}). \end{aligned}$$



Likewise and from the definition of  $\hat{v}(\cdot)$ ,

$$\hat{v}(\chi_0) = \inf_{\delta \in \mathcal{D}} \{(1 - \delta)V(x_0, x_1) + \delta v(\chi_1, \delta)\}.$$

Note however that the availability of a Bellman-like formulation with an optimal  $\delta$  like

$$\hat{v}(\chi_0) = \{(1 - \delta^*)V(x_0, x_1) + \delta^* \hat{v}(\chi_1)\}$$

would presuppose the hypothetical existence of a  $\delta^* \in \mathcal{D}$  that satisfies:

$$\delta^* \in \operatorname{argmin}_{\delta \in \mathcal{D}} v(\chi_0, \delta) \cap \operatorname{argmin}_{\delta \in \mathcal{D}} v(\chi_1, \delta).$$

The holding of this relationship is nonetheless everything but trivial due to the absence of convexity in the infimum component of the problem (MM).

This direct line of argument being out of reach, it seems more appropriate to follow an indirect approach by wondering whether the value function  $J(\cdot)$  can be recovered as the solution of a functional equation? The following result provides a first clue in this direction by establishing that, under the earlier set of assumptions, some function  $W(\cdot)$  can indeed be recovered as the solution of related maxmin functional equation or, equivalently, as the fixed point of a maxmin operator. In order to simplify matters, one may first notice that the retainment of the following assumption on the payoff will suffice for ensuring that the solution of the problem (MM) is bounded from below and  $J(x_0) > -\infty$  for any  $x_0$  in Proposition 2.1.

ASSUMPTION V3. The payoff function  $V$  is bounded from below.

LEMMA 3.1. [A MAXIMIN BELLMAN-LIKE OPERATOR] *Under Assumptions T1-T2 and V1-V3:*

- (i) *There exists a unique continuous function  $W : \mathbb{R}_+ \rightarrow \mathbb{R}$  which is a solution of the following Bellman-like functional equation<sup>†</sup>:*

$$W(x_0) = \sup_{x_1 \in \Gamma(x_0)} \inf_{\delta \in \mathcal{D}} \{(1 - \delta)V(x_0, x_1) + \delta W(x_1)\}.$$

- (ii) *The function  $W$  is strictly increasing and concave.*

REMARK 3.1. It is worth noticing that the existence argument (i) appears as a special of a recent contribution of Bich, Drugeon & Morhaim [4] that deals with an abstract way

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<sup>†</sup>In [21] and [22], Wakai established an axiomatic basis for such a utility smoothing behavior.

of dynamic programming through *temporal aggregators*. Considering indeed a *discounting mimimizing temporal aggregator* along

$$A(x_0, y, v) = \inf_{\delta \in \mathcal{D}} \{V(x_0, y) + \delta v\},$$

the existence result (i) in Lemma 3.1 appears as a special case of Theorem 4.2 in their article and neither Assumptions **T1-T2**, nor Assumptions **V1-V3** are necessary for the establishment of an existence argument for the function  $W$ . Numerous scholars being however chiefly interested into the numerical aspects of dynamic programming, the authors have currently preferred to let the argument rest upon canonical and well-known assumptions and techniques, the proof of Lemma 3.1 providing, *e.g.*, a useful illustration of contraction mapping techniques for maximin problems.

One may now wonder whether such a solution  $W$  is equivalent to the value function  $J$  and whether an augmented form of the *Principle of Optimality* can be inferred from these results. Otherwised stated, for any  $V, \Gamma, \mathcal{D}$  and  $W$ , does the holding of

$$W(x_0) = \sup_{y \in \Gamma(x_0)} \inf_{\delta \in \mathcal{D}} \{(1 - \delta)V(x_0, y) + \delta W(y)\}$$

imply that  $W(x_0) = \sup_{\chi \in \Pi(x_0)} \hat{v}(\chi)$ , aka the very definition of the value function  $J(x_0)$  in (1)?

### 3.2 THE POLICY FUNCTION

This section will establish how, under some extra assumptions, the  $W$ -optimal policy function  $\varphi$  is monotonic. This monotonicity property will reveal as playing a key role in the establishment of the stationarity and time-consistency of the decision rule and, eventually, in the equality between the value function  $J$  and the solution of the functional equation  $W$ .

This line of argumentation is based upon the adoption of the following set of fairly classical assumptions:

ASSUMPTION T3. For  $x$  sufficiently small,  $x \in \text{int}[\Gamma(x)]$ .

ASSUMPTION T4. There exist  $a, b > 0$  such that  $0 < a < 1$  and, for any  $x \geq 0, y \in \Gamma(x)$ ,  $y \leq ax + b$ .

ASSUMPTION V4. The payoff function  $V$  is strictly increasing in its first variable and decreasing in its second one.

Assumptions **T3** and **T4** are traditionally retained in the dynamic programming literature. Assumption **T3** ensures the existence of a non-degenerate feasible sequence and avoids pathological configurations where every feasible sequence converges to zero and the problem becomes trivial. As for Assumption **T4**, it is convenient in ensuring the boundedness of every feasible sequence.

ASSUMPTION T5. The correspondence  $\Gamma$  is *ascending*: for any  $0 \leq x \leq x'$ , for any  $y \in \Gamma(x)$  and for any  $y' \in \Gamma(x')$ , one has  $\min\{y, y'\} \in \Gamma(x)$  and  $\max\{y, y'\} \in \Gamma(x')$ .<sup>‡</sup>

ASSUMPTION V5. The function  $V$  is strictly supermodular.<sup>§</sup>

Assumptions **T5** and **V4** are both more specific and more crucial. These assumptions imply the strict monotonicity of the optimal policy function that is associated with the Bellman-like functional equation. This generalizes to a multiple discount rates configuration the result of Amir [2]. Hence and upon the iteration of this function, the obtained sequence is also strictly monotonic, a key resulting property being that the optimal discount rate will be precluded from any change between successive periods. This property is core to the stationarity and time-consistency of the policy rule and hence decisive in the establishment of the main result.

PROPOSITION 3.1. [THE POLICY FUNCTION] *Consider the previous environment under Assumptions T1-T5 and V1-V5.*

(i) *For every  $x_0 > 0$ , there exists a unique  $x_1^* \in \Gamma(x_0)$  such that*

$$W(x_0) = \min_{\delta \in \mathcal{D}} \{(1 - \delta)V(x_0, x_1^*) + \delta W(x_1^*)\}.$$

(ii) *Let the  $W$ -policy function  $\varphi$  be defined by:*

$$\begin{aligned} \varphi(x_0) &= \operatorname{argmax}_{z \in \Gamma(x_0)} w(x_0, z); \\ w(x_0, z) &= \min_{\delta \in \mathcal{D}} \{(1 - \delta)V(x_0, z) + \delta W(z)\}, \end{aligned}$$

*it is strictly increasing.*

(iii) *Define  $x_t^* = \varphi^t(x_0)$ , or  $x_{t+1}^* = \varphi(x_t^*)$  for any  $t \geq 0$ . The sequence  $\chi^* = \{x_t^*\}_{t=0}^\infty$  is monotonic.*

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<sup>‡</sup>This is a mild assumption that is, *e.g.*, satisfied in the case when, for any  $x$ ,  $0 \in \Gamma(x)$  and  $\Gamma$  is expanding. See Topkis [20].

<sup>§</sup> $V$  is (strictly) supermodular if, for every  $(x, x')$  and  $(y, y')$  that belong to  $\text{Graph}(\Gamma)$ ,  $V(x, y) + V(x', y')(>) \geq V(x', y) + V(x, y')$  prevails whenever  $(x', y')(>) \geq (x, y)$ . When  $V$  is twice differentiable, (strict) supermodularity sums up to (strictly) positive cross derivatives.

### 3.3 AN EXTENDED PRINCIPLE OF OPTIMALITY

The key Proposition 3.2 will now prove that, for every  $x_0 > 0$ , there exists  $\delta^* \in \mathcal{D}$ , with  $\chi^* = \{x_0, \varphi(x_0), \varphi^2(x_0), \dots, \varphi^t(x_0), \dots\}$ , such that the optimal sequence  $\chi^*$  is a solution to an optimization program with a fixed discount factor  $\delta^*$ . For that purpose and for every  $\delta \in \mathcal{D}$ , let  $\mathcal{R}(\delta)$  denote such a benchmark program, henceforward referred to as the corresponding *fixed discount Ramsey problem*:

$$\sup_{\chi \in \Pi(x_0)} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right]. \quad (\mathcal{R}(\delta))$$

The main idea runs as follows. Consider, *e.g.*, some sequence  $\chi^* = \{x_t^*\}_{t=0}^{\infty} = \{\varphi^t(x_0)\}_{t=0}^{\infty}$  that is strictly increasing. For any  $t \geq 0$ , one has  $x_t^* < x_{t+1}^*$ , which implies  $W(x_t^*) < W(x_{t+1}^*)$  and

$$\operatorname{argmin}_{\delta \in \mathcal{D}} \{(1 - \delta)V(x_t^*, x_{t+1}^*) + \delta W(x_{t+1}^*)\} = \underline{\delta}.$$

Hence, for every  $t \geq 0$ :

$$\begin{aligned} W(x_t^*) &= (1 - \underline{\delta})V(x_t^*, x_{t+1}^*) + \underline{\delta}W(x_{t+1}^*) \\ &= (1 - \underline{\delta})V(x_t^*, x_{t+1}^*) + (1 - \underline{\delta})\underline{\delta}V(x_{t+1}^*, x_{t+2}^*) + \underline{\delta}^2 W(x_{t+2}^*) \\ &= \dots \\ &= (1 - \underline{\delta}) \sum_{s=0}^T \underline{\delta}^s V(x_{t+s}^*, x_{t+s+1}^*) + \underline{\delta}^{T+1} W(x_{t+T+1}^*) \\ &= (1 - \underline{\delta}) \sum_{s=0}^{\infty} \underline{\delta}^s V(x_{t+s}^*, x_{t+s+1}^*), \end{aligned}$$

by letting  $T$  converges to infinity. Moreover, and for any  $\delta \in \mathcal{D}$ ,

$$\begin{aligned} W(x_t^*) &= (1 - \underline{\delta}) \sum_{s=0}^{\infty} \underline{\delta}^s V(x_{t+s}^*, x_{t+s+1}^*) \\ &\leq (1 - \delta)V(x_t^*, x_{t+1}^*) + \delta W(x_{t+1}^*) \\ &\leq (1 - \delta)V(x_t^*, x_{t+1}^*) + (1 - \delta)\delta V(x_{t+1}^*, x_{t+2}^*) + \delta^2 W(x_{t+2}^*) \\ &\leq \dots \\ &= (1 - \delta) \sum_{s=0}^{\infty} \delta^s V(x_{t+s}^*, x_{t+s+1}^*), \end{aligned}$$

whence

$$W(x_t^*) = \min_{\delta \in \mathcal{D}} \left[ (1 - \delta) \sum_{s=0}^{\infty} \delta^s V(x_{t+s}^*, x_{t+s+1}^*) \right].$$

The proof that  $\{x_t^*\}_{t=0}^\infty$  is the optimal sequence of a problem  $\mathcal{R}(\underline{\delta})$  then just relies on the concavity of  $V$  and  $W$ , a related class of argument being available for a strictly decreasing sequence  $\{x_t^*\}_{t=0}^\infty$  and a problem  $\mathcal{R}(\bar{\delta})$ .

The detailed results are gathered in the following statement:

**PROPOSITION 3.2.** [AN EXTENDED PRINCIPLE OF OPTIMALITY] *Under assumptions **V1-V4**, **T1-T5**, consider, for a given  $x_0$ , the optimal sequence  $\chi^* = \{x_t^*\}_{t=0}^\infty = \{\varphi^t(x_0)\}_{t=0}^\infty$  as it is defined from Proposition 3.1(iii).*

- (i) *There exists  $\delta^* \in \mathcal{D}$  such that  $\chi^*$  is solution of the fixed discount Ramsey problem  $\mathcal{R}(\delta^*)$ .*
- (ii) *More precisely:*
  - a) *when the sequence  $\{x_t^*\}_{t=0}^\infty$  is strictly increasing, it is a solution to  $\mathcal{R}(\underline{\delta})$ ;*
  - b) *when the sequence  $\{x_t^*\}_{t=0}^\infty$  is strictly decreasing, it is a solution to  $\mathcal{R}(\bar{\delta})$ ;*
  - c) *when the sequence  $\{x_t^*\}_{t=0}^\infty$  is constant, the corresponding value  $\delta^*$  lies between  $\underline{\delta}$  and  $\bar{\delta}$ .*

As this was established by Proposition 3.1, the iteration of the optimal policy function results in a monotonic sequence where the optimal discount rate does not change in the course of the whole iteration. Proposition 3.2 clarifies the regards in which the obtained sequence turns out as being optimal for the fixed discount Ramsey problem with the appropriate value of the discount rate.

Theorem 3.1 is then a direct consequence of Propositions 3.1 and 3.2 and establishes the stationarity of the policy function and an augmented form of the *Principle of Optimality*.

**THEOREM 3.1.** [STATIONARY DECISION RULES & STATIONARY DYNAMIC PROGRAMMING] *Under assumptions **T1-T5** and **V1-V5**, for any  $x_0 > 0$ , the following propositions are true:*

- (i) *The decision rule*

$$\varphi(x_0) = \operatorname{argmax}_{z \in \Gamma(x_0)} \left\{ \min_{\delta \in \mathcal{D}} \{(1 - \delta)V(x_0, z) + \delta W(z)\} \right\}$$

*is stationary.*

- (ii) *The value of the Maximin problem is equal to the value of the Minimax problem.*

$$\sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} v(\delta, \chi) = \max_{\chi \in \Pi(x_0)} \min_{\delta \in \mathcal{D}} v(\delta, \chi)$$

$$\begin{aligned}
&= \inf_{\delta \in \mathcal{D}} \sup_{\chi \in \Pi(x_0)} v(\delta, \chi) \\
&= \min_{\delta \in \mathcal{D}} \max_{\chi \in \Pi(x_0)} v(\delta, \chi).
\end{aligned}$$

(iii)  $J(x_0) = W(x_0)$ .

This augmented view of the *principle of optimality* reads as follows: for each initial state  $x_0$ , denote by  $\{x_t^*\}_{t=0}^\infty$  the sequence that is obtained by iterating the optimal policy function  $\varphi$  associated to the Bellman-like functional equation, and by  $\delta^*$  the selected discount rate. One has

$$(1 - \delta^*) \sum_{t=0}^{\infty} (\delta^*)^t V(x_t^*, x_{t+1}^*) = \min_{\delta \in \mathcal{D}} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t^*, x_{t+1}^*) \right],$$

that clarifies the regards in which this functional equation approach provides a clearcut and satisfactory solution to the infimum component of (MM).

## 4. ILLUSTRATIONS

### 4.1 THE ROLE OF INITIAL CONDITIONS

In order to provide some illustration of the main result in Theorem 3.1, consider now a configuration where, for any value  $\delta \in \mathcal{D}$ , the solution of the  $\delta$ -parameterised fixed discount rate problem assumes a unique long-run steady state. Further remark that, under assumptions **T1-T5** and **V1-V5**, there exists a unique *golden rule*  $x^{\mathcal{G}}$  which maximizes  $V(x, x)$ .

ASSUMPTION V6. The function  $\psi(x) = -V_2(x, x)/V_1(x, x)$  is strictly increasing over  $]0, x^{\mathcal{G}}[$ .

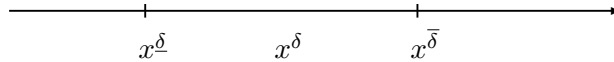


Figure 1: The structure of steady states under Assumption **V6**.

As this is illustrated in Figure 1, under assumption **V6** and for any  $\delta \in \mathcal{D}$ , the optimal solution of the fixed discount Ramsey problem  $\mathcal{R}(\delta)$  assumes a unique non trivial steady state  $x^{\delta}$ , which is further increasing with respect to  $\delta$ , namely:

$$x^{\underline{\delta}} \leq x^{\delta} \leq x^{\bar{\delta}} \quad \text{for every } \delta \in \mathcal{D}.$$

For some initial condition that further satisfies  $x^{\underline{\delta}} \leq x_0 \leq x^{\bar{\delta}}$ , the optimal discount factor  $\delta^*$  emerges as a solution to  $V_2(x_0, x_0) + \delta V_1(x_0, x_0) = 0$  whereas, for  $x_0 > x^{\bar{\delta}}$ , the optimal discount factor is available as  $\delta^* = \bar{\delta}$ . Upon the determination the problem of the optimal  $\delta^*$ , the optimal capital sequence  $\chi^*(\delta^*)$  is considered and proved to satisfy the Bellman-like equation, the solution of that functional equation being equal to the value function.

**COROLLARY 4.1.** [STATIONARY DECISION RULES & LONG-RUN CONVERGENCE] *Under assumptions **T1-T5** and **V1-V6**:*

- (i) for  $x_0 < x^{\underline{\delta}}$ , the optimal sequence is strictly increasing and converges to  $x^{\underline{\delta}}$  and the chosen discount rate is  $\underline{\delta}$ ;
- (ii) for  $x_0 > x^{\bar{\delta}}$ , the optimal sequence is strictly decreasing and converges to  $x^{\bar{\delta}}$  and the chosen discount rate is  $\bar{\delta}$ ;
- (iii) for  $x^{\underline{\delta}} \leq x_0 \leq x^{\bar{\delta}}$ , the optimal sequence is constant and the chosen discount rate lies between  $\underline{\delta}$  and  $\bar{\delta}$ .

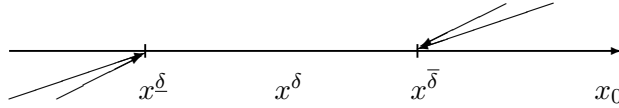


Figure 2: Dependency of the convergence with respect to the initial value  $x_0$ .

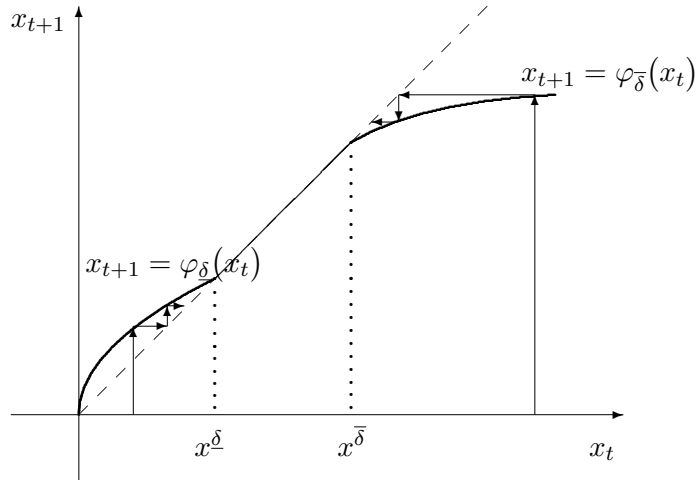


Figure 3: Monotonic convergence and dependency with respect to the initial  $x_0$

With respect to standard optimization programs and as this is illustrated in Figures 2 and

3, the most specific and novel facet of these problems with an optimal rate of discount hence springs from a *strong dependency on the initial condition*  $x_0$ .

## 4.2 AN EXAMPLE

Consider an economy with a logarithmic utility function  $u(c) = \ln c$  and a one good production technology given by  $F(x) = Ax^\alpha$ , where  $0 < \alpha < 1$ . As this shall be argued, even though this characterization more closely corresponds to the unbounded from below case analyzed in Appendix F, the results directly replicate the ones listed in Proposition 3.2. The payoff function has the following form:

$$V(x, y) = \ln(Ax^\alpha - y).$$

and the technology is defined by the correspondence:

$$\Gamma(x) = [0, Ax^\alpha].$$

Given  $x_0 > 0$ , consider the ensuing maximin problem

$$\max_{\chi \in \Pi(x_0)} \min_{\delta \in [\underline{\delta}, \bar{\delta}]} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t \ln(Ax_t^\alpha - x_{t+1}) \right]. \quad (\text{MM-ln})$$

That all of assumptions **V1, V3-V5, T1-T5** are satisfied can readily be checked. Further denoting by  $J(x_0)$  the value function of this problem, it derives that:

$$\begin{aligned} J(x_0) &= \max_{\chi \in \Pi(x_0)} \min_{\delta \in [\underline{\delta}, \bar{\delta}]} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t \ln(Ax_t^\alpha - x_{t+1}) \right] \\ &= \min_{\delta \in [\underline{\delta}, \bar{\delta}]} \max_{\chi \in \Pi(x_0)} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t \ln(Ax_t^\alpha - x_{t+1}) \right]. \end{aligned}$$

For any  $\delta \in [\underline{\delta}, \bar{\delta}]$ :

$$\begin{aligned} v_{\max}(\delta, x_0) &:= \max_{\chi \in \Pi(x_0)} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t \ln(Ax_t^\alpha - x_{t+1}) \right] \\ &= \ln A + \ln(1 - \alpha\delta) + \frac{\alpha\delta}{1 - \alpha\delta} \ln(\alpha\delta) + \frac{\alpha(1 - \delta)}{1 - \alpha\delta} \ln x_0. \end{aligned} \quad (2)$$

and the optimal sequence  $\mathbf{x}_\delta = (x_{\delta,t})_{t=0}^\infty$  of the maximum problem (2) is defined by<sup>¶</sup>

$$x_{\delta,t+1} = \alpha\delta A(x_{\delta,t})^\alpha.$$

It derives that:

$$J(x_0) = \min_{\delta} v_{\max}(\delta, x_0).$$

---

<sup>¶</sup>See section 4.4 in Stokey, Lucas & Prescott [17] for the calculus details.



It can be checked that

$$\frac{\partial v_{\max}}{\partial \delta} = \frac{\alpha[\ln(\alpha\delta) - (1-\alpha)\ln x_0]}{(1-\alpha\delta)^2}.$$

and

$$\frac{\partial v_{\max}}{\partial \delta} = 0 \Leftrightarrow x_0 = (\alpha\delta A)^{\frac{1}{1-\alpha}} \Leftrightarrow \delta = \frac{x_0^{1-\alpha}}{\alpha A}.$$

Three cases are to be considered:

- Case 1.  $x_0 < (\alpha\underline{\delta}A)^{\frac{1}{1-\alpha}}$ .

Then  $\partial v_{\max}/\partial \delta > 0$  for all  $\delta \in [\underline{\delta}, \bar{\delta}]$ . Hence

$$J(x_0) = \min_{\delta} v_{\max}(\delta, x_0) = v_{\max}(\underline{\delta}, x_0).$$

In this case, the optimal sequence  $\mathbf{x}^* = (x_t^*)_{t=0}^{\infty}$  is described by

$$x_{t+1}^* = (\alpha\underline{\delta}A)(x_t^*)^{\alpha},$$

The optimal sequence is strictly increasing and converges to  $(\alpha\underline{\delta}A)^{\frac{1}{1-\alpha}}$ .

- Case 2.  $x_0 > (\alpha\bar{\delta}A)^{\frac{1}{1-\alpha}}$ . Then  $\partial v_{\max}/\partial \delta < 0$  for all  $\delta \in [\underline{\delta}, \bar{\delta}]$ . Hence

$$J(x_0) = \min_{\delta} v_{\max}(\delta, x_0) = v_{\max}(\bar{\delta}, x_0).$$

In this case, the optimal sequence  $\mathbf{x}^* = \{x_t^*\}_{t=0}^{\infty}$  is described by

$$x_{t+1}^* = (\alpha\bar{\delta}A)(x_t^*)^{\alpha}.$$

The optimal sequence is strictly decreasing and converges to  $(\alpha\bar{\delta}A)^{\frac{1}{1-\alpha}}$ .

- Case 3.  $x_0 \in [(\alpha\underline{\delta}A)^{\frac{1}{1-\alpha}}, (\alpha\bar{\delta}A)^{\frac{1}{1-\alpha}}]$ . It derives that

$$J(x_0) = v_{\max}(\delta_0, x_0),$$

with  $\delta_0 = \frac{x_0^{1-\alpha}}{\alpha A}$ . In this case  $x_{t+1}^* = \alpha\delta_0 A(x_t^*)^{\alpha} = x_0$  for all  $t \geq 0$ .

## A. PROOF OF PROPOSITION 2.1

*Proof.* (i) This is a straightforward application of the Weierstrass Theorem.

(ii) a/ Fix  $\chi_1, \chi_2 \in \Pi(x_0)$  such that  $\chi_1 \neq \chi_2$  and  $v(\chi_1, \delta), v(\chi_2, \delta) > -\infty$  for every  $\delta \in [\underline{\delta}, \bar{\delta}]$ . Fix  $0 < \lambda < 1$ . Let  $\chi_{\lambda} = (1-\lambda)\chi_1 + \lambda\chi_2$ . From the convexity of  $\Gamma$ , it derives that  $\chi \in \Pi(x_0)$ . Fix  $\delta_{\lambda} \in \mathcal{D}$  such that  $\hat{v}(\chi_{\lambda}) = v(\delta_{\lambda}, \chi_{\lambda})$ .

From the strict concavity of  $V$ , it is obtained that

$$\begin{aligned}
\hat{v}(\chi_\lambda) &= v(\delta_\lambda, (1-\lambda)\chi_1 + \lambda\chi_2) \\
&> (1-\lambda)v(\delta_\lambda, \chi_1) + (1-\lambda)v(\delta_\lambda, \chi_2) \\
&\geq (1-\lambda)\min_{\delta \in \mathcal{D}} v(\delta, \chi_1) + \lambda\min_{\delta \in \mathcal{D}} v(\delta, \chi_2) \\
&= (1-\lambda)\hat{v}(\chi_1) + \lambda\hat{v}(\chi_2).
\end{aligned}$$

(iii) b/ From (1), take the sequence  $\chi^n$  such that  $\lim_{n \rightarrow \infty} \hat{v}(\chi^n) = J(x_0)$ . Since from assumption T2,  $\Pi(x_0)$  is compact in the product topology, it can be supposed that  $\chi^n \rightarrow \chi^*$ . It will first be proved that  $\hat{v}(\chi^*) > -\infty$ .

Fix any  $\epsilon > 0$ , there exists large enough values of  $N$  such that, for  $n \geq N$ ,  $v(\chi^n) > J(x_0) - \epsilon$  prevails. Observe, that from assumption **T2**, there exists  $C > 0$  such that  $0 \leq x_t \leq C$  for any  $\chi \in \Pi(x_0)$ . Hence there exists  $T$  satisfying, for any  $\chi \in \Pi(x_0)$  and for any  $\delta \in \mathcal{D}$ ,

$$(1-\delta) \sum_{t=T+1}^{\infty} \delta^t V(x_t, x_{t+1}) < \epsilon.$$

For any  $n \geq N$ , for any  $\delta \in \mathcal{D}$ :

$$\begin{aligned}
J(x_0) - \epsilon &\leq v(\chi^n) \\
&\leq (1-\delta) \sum_{t=0}^T \delta^t V(x_t^n, x_{t+1}^n) + (1-\delta) \sum_{t=T+1}^{\infty} \delta^t V(x_t^n, x_{t+1}^n),
\end{aligned}$$

which implies

$$\begin{aligned}
(1-\delta) \sum_{t=0}^T \delta^t V(x_t^n, x_{t+1}^n) &\geq J(x_0) - \epsilon - (1-\delta) \sum_{t=T+1}^{\infty} \delta^t V(x_t^n, x_{t+1}^n) \\
&\geq J(x_0) - 2\epsilon.
\end{aligned}$$

Letting  $n$  converge to infinity, for any  $\delta \in \mathcal{D}$ ,

$$(1-\delta) \sum_{t=0}^T \delta^t V(x_t^*, x_{t+1}^*) \geq J(x_0) - 2\epsilon.$$

This inequality prevails for large enough values of  $T$ . Letting  $T$  tend to infinity, for any  $\delta \in \mathcal{D}$ :

$$(1-\delta) \sum_{t=0}^{\infty} \delta^t V(x_t^*, x_{t+1}^*) \geq J(x_0) - 2\epsilon,$$

whence  $\hat{v}(\chi^*) \geq J(x_0) - 2\epsilon$ .

The parameter  $\epsilon$  being chosen arbitrarily,  $\hat{v}(\chi^*) \geq J(x_0)$ . From the strict concavity of  $\hat{v}$ ,  $\chi^*$  is the unique solution of  $\sup_{\chi \in \Pi(x_0)} \hat{v}(\chi)$ .  $\square$

## B. PROOF OF LEMMA 3.1

*Proof.* From the compactness of  $\mathcal{D}$  and  $\Gamma(x_0)$  and for every  $x_0 > 0$ , let  $T$  denote the operator from the space of continuous function into itself:

$$Tg(x_0) = \max_{x_1 \in \Gamma(x_0)} \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta g(x_1)].$$

Supposing that  $g(x) \leq \tilde{g}(x)$  for any  $x$ ,  $Tg(x) \leq T\tilde{g}(x)$  for any  $x$ . Indeed, for each  $x_1 \in \Gamma(x_0)$ , for every  $\delta$ , it is obtained that:

$$(1 - \delta)V(x_0, x_1) + \delta g(x_1) \leq (1 - \delta)V(x_0, x_1) + \delta \tilde{g}(x_1),$$

which implies

$$\min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta g(x_1)] \leq \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta \tilde{g}(x_1)].$$

The inequality being obviously true for every  $x_1$ ,  $Tv(x_0) \leq T\tilde{g}(x_0)$  in turn prevails. By the same arguments, it can be proved that for every constant  $a > 0$ ,  $T(g+a)(x_0) \leq Tg(x_0) + \bar{\delta}a$ .

Since  $T$  satisfies the properties of Blackwell, *i.e.*, monotonicity and discounting, by Theorem 3.3 in Stokey, Lucas & Prescott [17], the operator  $T$  is a contracting map and has a unique fixed point.

In preparation for the concavity of this fixed point function, it will now be proved that the concavity of  $g$  implies the concavity of  $Tg$ . Fix  $x_0$  and  $y_0$  and fix  $(\delta_x, x_1)$  and  $(\delta_y, y_1)$  such that

$$\begin{aligned} Tg(x_0) &= (1 - \delta_x)V(x_0, x_1) + \delta_x g(x_1) \\ &= \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta g(x_1)], \\ Tg(y_0) &= (1 - \delta_y)V(y_0, y_1) + \delta_y g(y_1) \\ &= \min_{\delta \in \mathcal{D}} [(1 - \delta)V(y_0, y_1) + \delta g(y_1)]. \end{aligned}$$

For any  $0 \leq \lambda \leq 1$ , define  $x_0^\lambda = (1 - \lambda)x_0 + \lambda y_0$ ,  $x_1^\lambda = (1 - \lambda)x_1 + \lambda y_1$ . For any  $\delta \in \mathcal{D}$ , it derives that:

$$\begin{aligned} (1 - \delta_x)V(x_0, x_1) + \delta_x g(x_1) &\leq (1 - \delta)V(x_0, x_1) + \delta g(x_1), \\ (1 - \delta_y)V(y_0, y_1) + \delta_y g(y_1) &\leq (1 - \delta)V(y_0, y_1) + \delta g(y_1). \end{aligned}$$

Hence, for any  $\underline{\delta} \leq \delta \leq \bar{\delta}$ :

$$(1 - \lambda)Tg(x_0) + \lambda Tg(y_0) = (1 - \lambda) [(1 - \delta_x)V(x_0, x_1) + \delta_x g(x_1)]$$

$$\begin{aligned}
& + \lambda [(1 - \delta_y)V(y_0, y_1) + \delta_y g(y_1)] \\
\leq & (1 - \lambda) [(1 - \delta)V(x_0, x_1) + \delta g(x_1)] \\
& + \lambda [(1 - \delta)V(y_0, y_1) + \delta g(y_1)] \\
\leq & (1 - \delta)V(x_0^\lambda, x_1^\lambda) + \delta g(x_1^\lambda).
\end{aligned}$$

The inequality being true for every  $\delta$ , it is obtained that:

$$\begin{aligned}
(1 - \lambda)Tg(x_0) + \lambda Tg(y_0) & \leq \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0^\lambda, x_1^\lambda) + \delta g(x_1^\lambda)] \\
& \leq Tg(x_0^\lambda).
\end{aligned}$$

Take  $g^0(x_0) = 0$  for every  $x_0$ , define  $g^{n+1}(x_0) = Tg^n(x_0)$  for any  $n \geq 0$ . By induction,  $g^n(x_0)$  is a concave function for any  $n$ . Taking limits,  $W(x_0) = \lim_{n \rightarrow \infty} g^n(x_0) = \lim_{n \rightarrow \infty} T^n g^0(x_0)$ , the concavity property is available for  $W$ .

The strict concavity of  $W$  is now going to be established. Consider  $x_0 \neq y_0$ . Define  $(\delta_x, x_1)$ ,  $(\delta_y, y_1)$ ,  $x_0^\lambda$  and  $x_1^\lambda$  as in the first part of this proof. Relying on the same arguments and calculus as well as on the strictly concavity of  $V$ , for any  $\delta \in \mathcal{D}$ , it is obtained that:

$$\begin{aligned}
(1 - \lambda)W(x_0) + \lambda W(x_1) & \leq (1 - \delta) [(1 - \lambda)V(x_0, x_1) + \lambda V(y_0, y_1)] \\
& \quad + \delta [(1 - \lambda)W(x_1) + \lambda W(y_1)] \\
& < (1 - \delta)V(x_0^\lambda, x_1^\lambda) + \delta W(x_1^\lambda).
\end{aligned}$$

From the compacity of the set of discount factors  $\mathcal{D}$ , it derives that:

$$\begin{aligned}
(1 - \lambda)W(x_0) + \lambda W(x_1) & < \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0^\lambda, x_1^\lambda) + \delta W(x_1^\lambda)] \\
& \leq W(x_0^\lambda).
\end{aligned}$$

□

### C. PROOF OF PROPOSITION 3.1

*Proof.* Define  $w(x_0, x_1) = \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta W(x_1)]$ .

(i) Observe that the function  $w(x_0, x_1)$  is strictly concave in  $x_1$ . Indeed, for any  $x_1 \neq y_1$  belonging to  $\Gamma(x_0)$ , and  $0 < \lambda < 1$ , define  $x_1^\lambda = (1 - \lambda)x_1 + \lambda y_1$ . For any  $\delta \in \mathcal{D}$ :

$$\begin{aligned}
(1 - \lambda)w(x_0, x_1) + \lambda w(x_0, y_1) & \leq (1 - \lambda) [(1 - \delta)V(x_0, x_1) + \delta W(x_1)] \\
& \quad + \lambda [(1 - \delta)V(x_0, y_1) + \delta W(y_1)]
\end{aligned}$$

$$< (1 - \delta)V(x_0, x_1^\lambda) + \delta W(x_1^\lambda).$$

The definition set  $\mathcal{D}$  for the discount factors  $\delta$  being compact, it is obtained that:

$$\begin{aligned} (1 - \lambda)w(x_0, x_1) + \lambda w(x_0, y_1) &< \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1^\lambda) + \delta W(x_1^\lambda)] \\ &= w(x_0, x_1^\lambda). \end{aligned}$$

The strict concavity of  $w(x_0, \cdot)$  implies that this function assumes a unique maximizer  $x_1$ .

(ii) The monotonicity of the policy function  $\varphi$  is now going to be established. For that purpose, it must first be checked that there cannot exist a couple  $(x_0, z_0)$  such that  $x_0 < z_0$  and  $\varphi(x_0) > \varphi(z_0)$ . Suppose the opposite and let  $y_0 = \operatorname{argmin}_{[x_0, z_0]} \varphi(y)$ .

(a) Consider the first case with  $y_0 < \varphi(y_0)$ . This implies  $W(y_0) < W(\varphi(y_0))$ . Take

$$\delta_{y_0} \in \operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(y_0, \varphi(y_0)) + \delta W(\varphi(y_0))].$$

It derives that:

$$(1 - \delta_{y_0})V(y_0, \varphi(y_0)) + \delta_{y_0}W(\varphi(y_0)) < W(\varphi(y_0)),$$

that implies  $V(y_0, \varphi(y_0)) < W(\varphi(y_0))$ . There further exists  $\epsilon > 0$  such that

$$V(y, \varphi(y)) < W(\varphi(y)) \text{ for any } y_0 - \epsilon < y < y_0 + \epsilon.$$

For every  $y$  belonging to this interval, the following is satisfied:

$$\operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(y, \varphi(y)) + \delta W(\varphi(y))] = \{\underline{\delta}\}.$$

Recall that, in this case, for any  $y_0 - \epsilon < y < y_0 + \epsilon$

$$\varphi(y) = \operatorname{argmax}_{y' \in \Gamma(y)} [(1 - \underline{\delta})V(y, y') + \underline{\delta}W(y')].$$

From the super modularity of  $V$ , using the Topkis's theorem quoted in Amir [2],  $\varphi$  is strictly increasing (ascending) in  $]y_0 - \epsilon, y_0 + \epsilon[$ , hence for  $y_0 - \epsilon < y < y_0$ , the inequality  $\varphi(y) < \varphi(y_0)$ , a contradiction with the choice of  $y_0$ .

(b) The case where  $y_0 > \varphi(y_0)$  is faced with by applying the same arguments and a contradiction is similarly obtained.

- (c) Consider the third case.  $x_0 < y_0$  and  $\varphi(x_0) > \varphi(y_0) = y_0$  are simultaneously satisfied. But, and from Assumption **V4**,  $V$  is increasing in its first argument and decreasing in its second one,

$$\begin{aligned} V(x_0, \varphi(x_0)) &< V(y_0, y_0) \\ &= W(y_0) \\ &< W(\varphi(x_0)). \end{aligned}$$

Hence

$$\operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, \varphi(x_0)) + \delta W(\varphi(x_0))] = \{\underline{\delta}\}.$$

Since  $\varphi(x_0) > x_0$ ,  $\varphi(x) > x$  prevails in a neighborhood of  $x_0$ . Denoting by  $I_{x_0}$  the set of  $z \in ]x_0, y_0[$  such that for any  $x_0 < x < z$ , it derives that:

$$V(x, \varphi(x)) < W(\varphi(x)) \text{ which is equivalent to } x < \varphi(x).$$

It is further obvious that for any  $x_0 < x < z$  with  $z \in I_{x_0}$ , one has:

$$\operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(x, \varphi(x)) + \delta W(\varphi(x))] = \{\underline{\delta}\},$$

which implies for any  $x_0 < x < z$ :

$$\varphi(x) = \operatorname{argmax}_{x' \in \Gamma(x)} [(1 - \underline{\delta})V(x, x') + \underline{\delta}W(x')].$$

Hence, on the interval  $I_{x_0}$ , from the super modularity of  $V$ , the function  $\varphi$  is strictly increasing. Take now  $\bar{z} = \sup(I_{x_0})$ . If  $z_0 < \varphi(z_0)$ , then and by the continuity of  $V$  and  $W$ ,  $\varphi(x) > x$  prevails for any  $x_0 < x < \bar{z} + \epsilon$ , for  $\epsilon$  that is sufficiently small: a contradiction. Hence  $\varphi(\bar{z}) \leq \bar{z}$ . From the continuity of  $\varphi$ ,  $\varphi(\bar{z}) = \bar{z}$ . From the increasing property of  $\varphi$  on  $I_{x_0}$ , it is obtained that:

$$\begin{aligned} \varphi(x_0) &< \varphi(\bar{z}) \\ &\leq z \\ &\leq y_0 \\ &= \varphi(y_0) \\ &< \varphi(x_0), \end{aligned}$$

a contradiction.

It has been proved that, for any  $x_0 < z_0$ ,  $\varphi(x_0) \leq \varphi(z_0)$ . Suppose that there exists a couple  $(x_0, z_0)$  such that  $x_0 < z_0$  and  $\varphi(x_0) = \varphi(z_0)$ . This implies for any  $x_0 \leq y \leq z_0$ ,  $\varphi(y) = \varphi(x_0) = \varphi(z_0)$ . There hence exists  $y_0 \in [x_0, z_0]$  such that  $\varphi(y_0) \neq y_0$ . Using the same arguments as in the first part of the proof,  $\varphi$  is to be strictly increasing in a neighborhood of  $y_0$ , that contradicts with the result that  $\varphi$  is constant on  $[x_0, z_0]$ .

The monotonicity of  $\chi^* = \{x_t^*\}_{t=0}^\infty$  is a direct consequence of the increasingness property for  $\varphi$ .  $\square$

## D. PROOF OF PROPOSITION 3.2

*Proof.* First consider the case of a sequence  $\chi^* = \{x_t^*\}_{t=0}^\infty = \{\varphi^t(x_0)\}_{t=0}^\infty$  that is strictly increasing. The objective will be to prove that  $\chi^*$  is solution of a fixed discount Ramsey problem  $\mathcal{R}(\underline{\delta})$ .

Selecting any  $\delta^* \in \operatorname{argmin}_{\delta \in \mathcal{D}}[(1 - \delta)V(x_0, x_1^*) + \delta W(x_1^*)]$ , it derives that  $W(x_0) < W(x_1^*)$ , whence

$$(1 - \delta^*)V(x_0) + \delta^*W(x_1^*) < W(x_1^*).$$

This implies that  $V(x_0, x_1^*) < W(x_1^*)$ . Hence

$$\operatorname{argmin}_{\delta \in \mathcal{D}}[(1 - \delta)V(x_0, x_1^*) + \delta W(x_1^*)] = \{\underline{\delta}\}.$$

Since  $x_0 < x_1^*$ ,  $\varphi(x_0) < \varphi(x_1^*)$ , or  $x_1^* < x_2^*$ . By induction, it is obtained that  $x_t^* < x_{t+1}^*$  and

$$\operatorname{argmin}_{\delta \in \mathcal{D}}[(1 - \delta)V(x_t^*, x_{t+1}^*) + \delta W(x_{t+1}^*)] = \{\underline{\delta}\} \text{ for any } t \geq 0.$$

This implies, for any finite  $T$

$$\begin{aligned} W(x_0) &= (1 - \underline{\delta})V(x_0, x_1^*) + \underline{\delta}W(x_1^*) \\ &= (1 - \underline{\delta})V(x_0, x_1^*) + (1 - \underline{\delta})\underline{\delta}V(x_1^*, x_2^*) + \underline{\delta}^2W(x_{t+2}^*) \\ &\dots \\ &= (1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(x_t^*, x_{t+1}^*) + \underline{\delta}^{T+1}W(x_{T+1}^*). \end{aligned}$$

Observe that from assumptions **T3** and **T4**, there exists a  $C > 0$  such that  $x_0 \leq x_t^* < C$ .

This implies

$$\lim_{T \rightarrow \infty} \underline{\delta}^T W(x_T^*) = 0.$$

Hence

$$\begin{aligned} W(x_0) &= (1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(x_t^*, x_{t+1}^*) + \underline{\delta}^{T+1} W(x_{T+1}^*) \\ &= (1 - \underline{\delta}) \sum_{t=0}^{\infty} \underline{\delta}^t V(x_t^*, x_{t+1}^*), \end{aligned}$$

by letting  $T$  tends to infinity.

The aim is now to prove that  $\chi^*$  is a solution of the fixed discount problem  $\mathcal{R}(\underline{\delta})$ . Suppose the opposite and denote by  $\hat{\chi} = \{\hat{x}_t\}_{t=0}^{\infty}$  the unique solution of  $\mathcal{R}(\underline{\delta})$ . The sequence  $\{W(\hat{x}_t)\}$  being uniformly bounded, the date  $T$  can be selected as being large enough in order to satisfy:

$$(1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(\hat{x}_t, \hat{x}_{t+1}) + \underline{\delta}^{T+1} W(\hat{x}_{T+1}) > (1 - \underline{\delta}) \sum_{t=0}^{\infty} \underline{\delta}^t V(x_t^*, x_{t+1}^*).$$

For  $0 \leq \lambda \leq 1$ , define  $x_t^\lambda = (1 - \lambda)\hat{x}_t + \lambda x_t^*$ . Recall that

$$V(x_t^*, x_{t+1}^*) < W(x_{t+1}^*) \text{ for every } t \geq 0.$$

Take  $\lambda$  sufficiently close from 1 so that, for every  $0 \leq t \leq T$ , the inequality  $V(x_t^\lambda, x_{t+1}^\lambda) < W(x_{t+1}^\lambda)$  is satisfied: for any  $0 \leq t \leq T$ ,

$$\operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(x_t^\lambda, x_{t+1}^\lambda) + \delta W(x_{t+1}^\lambda)] = \{\underline{\delta}\}.$$

Hence

$$\begin{aligned} W(x_0) &\geq (1 - \underline{\delta})V(x_0, x_1^\lambda) + \underline{\delta}W(x_1^\lambda) \\ &\geq (1 - \underline{\delta})V(x_0, x_1^\lambda) + (1 - \underline{\delta})\underline{\delta}V(x_1^\lambda, x_2^\lambda) + \underline{\delta}^2W(x_2^\lambda) \\ &\dots \\ &\geq (1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(x_t^\lambda, x_{t+1}^\lambda) + \underline{\delta}^{T+1} W(x_{T+1}^\lambda). \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{t=0}^T \underline{\delta}^t V(x_t^\lambda, x_{t+1}^\lambda) &\geq (1 - \lambda) \sum_{t=0}^T \underline{\delta}^t V(\hat{x}_t, \hat{x}_{t+1}) + \lambda \sum_{t=0}^T \underline{\delta}^t V(x_t^*, x_{t+1}^*), \\ W(x_{T+1}^\lambda) &\geq (1 - \lambda)W(\hat{x}_{T+1}) + \lambda W(x_{T+1}^*). \end{aligned}$$

Hence, and from the definition of  $T$

$$W(x_0) \geq (1 - \underline{\delta}) \left[ (1 - \lambda) \sum_{t=0}^T \underline{\delta}^t V(\hat{x}_t, \hat{x}_{t+1}) + \lambda \sum_{t=0}^T \underline{\delta}^t V(x_t^*, x_{t+1}^*) \right]$$



$$\begin{aligned}
& + \underline{\delta}^{T+1} [(1 - \lambda)W(\hat{x}_{T+1}) + \lambda W(x_{T+1}^*)] \\
= & (1 - \lambda) \left[ (1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(\hat{x}_t, \hat{x}_{t+1}) + \underline{\delta}^{T+1} W(\hat{x}_{T+1}) \right] \\
& + \lambda \left[ (1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(x_t^*, x_{t+1}^*) + \underline{\delta}^{T+1} W(x_{T+1}^*) \right] \\
> & (1 - \underline{\delta}) \sum_{t=0}^{\infty} \underline{\delta}^t V(x_t^*, x_{t+1}^*) \\
= & W(x_0),
\end{aligned}$$

a contradiction.

Observe that since  $\chi^* = \operatorname{argmax} \mathcal{R}(\underline{\delta})$ , the monotonic sequence  $\{x_t^*\}_{t=0}^{\infty}$  converges to a steady state of  $\mathcal{R}(\underline{\delta})$ . Supposing that  $x_0 > x_1^*$ , by using the same arguments, it can be proved that  $\chi$  is strictly decreasing, is a solution of a fixed discount problem  $\mathcal{R}(\bar{\delta})$  and converges to a steady state of that problem  $\mathcal{R}(\bar{\delta})$ .

Consider now the case  $\varphi(x_0) = x_0$  and hence  $x_t^* = x_0$  for any  $t$ . Two cases are to be considered:

- (i) For any  $\epsilon > 0$ , there exists  $x_0 - \epsilon < x < x_0 + \epsilon$  satisfying  $x \neq \varphi(x)$ .
- (ii) There exists  $\epsilon > 0$  such that for any  $x_0 - \epsilon < x < x_0 + \epsilon$ ,  $x = \varphi(x)$ .

First consider case (i). Take a sequence  $x_0^n$  that converges to  $x_0$  and such that  $\varphi(x_0^n) \neq x_0^n$  for any  $n$ . Since, in this case, the sequence is either solution of  $\mathcal{R}(\underline{\delta})$ , or of  $\mathcal{R}(\bar{\delta})$ , without loss of generality, it can be assumed that, for any  $n$ , the sequence  $\{\varphi^t(x_0^n)\}_{t=0}^{\infty}$  is a solution of the fixed discount Ramsey problem  $\mathcal{R}(\delta)$ , with  $\delta \in \{\underline{\delta}, \bar{\delta}\}$ .

But  $\lim_{n \rightarrow \infty} \varphi(x_0^n) = \varphi(x_0) = x_0$ , which implies  $\lim_{n \rightarrow \infty} \varphi^2(x_0^n) = x_0$ . From the Euler equation

$$V_2(x_0^n, \varphi(x_0^n)) + \delta V_1(\varphi(x_0^n), \varphi^2(x_0^n)) = 0,$$

it is derived that:

$$V_2(x_0, x_0) + \delta V_1(x_0, x_0) = 0.$$

Hence the sequence  $x_t = x_0$  for any  $t$  is solution of  $\mathcal{R}(\delta)$ . Suppose now that for  $x_0 - \epsilon < x < x_0 + \epsilon$ ,  $\varphi(x) = x$ . This implies that  $W(x) = V(x, x)$  for  $x \in ]x_0 - \epsilon, x_0 + \epsilon[$ , that in its turn implies

$$W'(x_0) = V_1(x_0, x_0) + V_2(x_0, x_0).$$

Observe that, the function  $(1 - \delta)V(x_0, x_1) + \delta W(x_1)$  is linear in  $\delta$  and concave in  $x_1$ , the following also prevails:

$$\max_{x_1 \in \Gamma(x_0)} \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta W(x_1)] = \min_{\delta \in \mathcal{D}} \max_{x_1 \in \Gamma(x_0)} [(1 - \delta)V(x_0, x_1) + \delta W(x_1)].$$

From  $\varphi(x_0) = x_0$ , there thus exists  $\delta \in \mathcal{D}$  such that

$$(1 - \delta)V_2(x_0, x_0) + \delta W'(x_0) = 0.$$

This implies

$$(1 - \delta)V_2(x_0, x_0) + \delta V_1(x_0, x_0) + \delta V_2(x_0, x_0) = 0,$$

that is equivalent to

$$V_2(x_0, x_0) + \delta V_1(x_0, x_0) = 0,$$

For any  $t$ , the sequence  $x_t = x_0$  is thus a solution of  $\mathcal{R}(\delta)$ .

□

## E. PROOF OF THEOREM 3.1

*Proof.* (i) This follows as an immediate corollary of Proposition 3.2.

(ii) For each  $x_0 > 0$ , define  $\chi^* = \{x_t^*\}_{t=0}^\infty = \{\varphi^t(x_0)\}_{t=0}^\infty$ . Using Propositions 3.2, there exists  $\delta^* \in \mathcal{D}$  such that  $\chi^*$  is a solution of  $\mathcal{R}(\delta^*)$  and

$$W(x_0) = (1 - \delta^*) \sum_{t=0}^{\infty} (\delta^*)^t V(x_t^*, x_{t+1}^*).$$

For any  $\delta \in \mathcal{D}$ , it is obtained that:

$$\begin{aligned} W(x_0) &\leq (1 - \delta)V(x_0, x_1^*) + \delta W(x_1^*) \\ &\leq (1 - \delta)V(x_0, x_1^*) + \delta((1 - \delta)V(x_1^*, x_2^*) + \delta W(x_2^*)) \\ &\dots \\ &\leq (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t^*, x_{t+1}^*). \end{aligned}$$

Hence

$$\begin{aligned} W(x_0) &\leq \min_{\delta \in \mathcal{D}} \left[ (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t^*, x_{t+1}^*) \right] \\ &\leq J(x_0). \end{aligned}$$

Suppose now that  $(\tilde{\delta}, \tilde{\chi})$  satisfies

$$v(\tilde{\delta}, \tilde{\chi}) = \max_{\chi \in \Pi(x_0)} \min_{\delta \in \mathcal{D}} v(\delta, \chi).$$

The value  $\chi^* = \{x_t^*\}_{t=0}^\infty$  being an optimal solution of the problem  $\mathcal{R}(\delta^*)$ , it is obtained that:

$$\begin{aligned} J(x_0) &= v(\tilde{\delta}, \tilde{\chi}) \\ &\leq v(\delta^*, \tilde{\chi}) \\ &\leq v(\delta^*, \chi^*) \\ &= W(x_0). \end{aligned}$$

Hence:

$$\begin{aligned} J(x_0) &= \max_{\chi \in \Pi(x_0)} \min_{\delta \in \mathcal{D}} v(\delta, \chi) \\ &= \min_{\delta \in \mathcal{D}} \max_{\chi \in \Pi(x_0)} v(\delta, \chi) \\ &= W(x_0), \end{aligned}$$

that establishes the details of the statement.  $\square$

## F. PAYOFFS UNBOUNDED FROM BELOW

Assumption **V3** is relaxed and the payoff becomes unbounded from below. As a result of this unboundedness, contracting map techniques cannot any longer be used to prove the existence of a solution to the functional Bellman-like equation. However, and thanks to the supplementary condition **V6**, the existence argument can be established through some simple *guess and verify* methods.<sup>‡</sup>

Under assumptions **T1-T5** and **V1-V2**, **V4-V6** and for every  $\delta$  there exists a unique long run steady state  $x^\delta$  for the fixed discount Ramsey problem  $\mathcal{R}(\delta)$ , and  $x^\delta \leq x^\delta \leq x^{\bar{\delta}}$ , for every  $\delta \in \mathcal{D}$ . Define then  $J_\delta(x_0)$  as the value function of  $\mathcal{R}(\delta)$ :

$$J_\delta(x_0) = \sup_{\chi \in \Pi(x_0)} (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}).$$

Consider now the function  $W(\cdot)$  as defined by:

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<sup>‡</sup>Observe however that it would be more difficult to apply these methods were one to relax Assumption **V6**, for there would then exist multiple steady solutions for any  $\mathcal{R}(\delta)$ .

- (i) For  $0 \leq x_0 \leq x^\delta$ ,  $W(x_0) = J_{\underline{\delta}}(x_0)$ ,
- (ii) For  $x^\delta \leq x_0 \leq x^{\bar{\delta}}$ , take  $\delta$  satisfying  $V_2(x_0, x_0) + \delta V_1(x_0, x_0) = 0$ ,  $W(x_0) = J_\delta(x_0) = V(x_0, x_0)$ .
- (iii) For  $x \geq x^{\bar{\delta}}$ ,  $W(x_0) = J_{\bar{\delta}}(x_0)$ .

Making use of the monotonicity of  $x^\delta$  with  $\delta \in \mathcal{D}$ , the following preparation lemma can first be established,

LEMMA F.1. *Assume **T1-T5** and **V1-V2, V4-V6**. The function  $W$  is strictly concave.*

*Proof.* Obviously, it is well known in the literature on fixed discount Ramsey problems that  $W$  is strictly concave in  $]0, x^\delta]$ ,  $[x^\delta, x^{\bar{\delta}}]$  and  $[x^{\bar{\delta}}, +\infty[$ . It just remains to prove that  $W$  is differentiable at  $x^\delta$  and  $x^{\bar{\delta}}$ . The left derivative of  $W$  at  $x^\delta$  is first equal to the derivative of  $J_{\underline{\delta}}$ :

$$W'_-(x^\delta) = J'_{\underline{\delta}}(x^\delta).$$

It is also well known that

$$\begin{aligned} J'_{\underline{\delta}}(x^\delta) &= (1 - \delta)V_1(x^\delta, x^\delta) \\ &= V_1(x^\delta, x^\delta) + V_2(x^\delta, x^\delta) \\ &= W'_+(x_0), \end{aligned}$$

whence  $W'_-(x^\delta) = W'_+(x^\delta)$ , or  $W$  is differentiable at  $x^\delta$ . The proof of the differentiability at  $x^{\bar{\delta}}$  relies on the same arguments.  $\square$

PROPOSITION F.1. *Assume **T1-T5** and **V1-V2, V4-V6**. The function  $W$  is solution of the functional equation*

$$W(x_0) = \max_{x_1 \in \Gamma(x_0)} \min_{\delta \in \mathcal{D}} \{(1 - \delta)V(x_0, x_1) + \delta W(x_1)\}.$$

*Proof.* Denote by  $\varphi_\delta$  the optimal policy function of  $\mathcal{R}(\delta)$ . Consider the case  $x_0 < x^\delta$ . The system of inequalities  $x_0 < \varphi_{\underline{\delta}}(x_0) < x^\delta$  is available, hence

$$\begin{aligned} W(x_0) &= J_{\underline{\delta}}(x_0) \\ &= (1 - \underline{\delta})V(x_0, \varphi_{\underline{\delta}}(x_0)) + \underline{\delta}J(\varphi_{\underline{\delta}}(x_0)) \\ &= (1 - \underline{\delta})V(x_0, \varphi_{\underline{\delta}}(x_0)) + \underline{\delta}W(\varphi_{\underline{\delta}}(x_0)) \\ &= \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, \varphi_{\underline{\delta}}(x_0)) + \delta W(\varphi_{\underline{\delta}}(x_0))]. \end{aligned}$$

It remains to prove that

$$W(x_0) \geq \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta W(x_1)]$$

for any  $x_1 \in \Gamma(x_0)$ . But, and for every  $x_1 \in \Gamma(x_0)$ ,

$$\min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta W(x_1)] \leq (1 - \underline{\delta})V(x_0, x_1) + \underline{\delta}W(x_1).$$

The function  $(1 - \underline{\delta})V(x_0, x_1) + \underline{\delta}W(x_1)$  being strictly concave in  $x_1$ , it hence attains its maximum at  $x_1 = \varphi_{\underline{\delta}}(x_0)$ . Recall indeed that  $x_0 < \varphi_{\underline{\delta}}(x_0) < x^{\underline{\delta}}$ , and  $W(\varphi_{\underline{\delta}}(x_0)) = J_{\underline{\delta}}(\varphi_{\underline{\delta}}(x_0))$  and  $W'(\varphi_{\underline{\delta}}(x_0)) = J'_{\underline{\delta}}(\varphi_{\underline{\delta}}(x_0))$ . This implies that:

$$(1 - \underline{\delta})V_2(x_0, \varphi_{\underline{\delta}}(x_0)) + \underline{\delta}W'(\varphi_{\underline{\delta}}(x_0)) = 0.$$

From the strict concavity of  $(1 - \underline{\delta})V(x_0, x_1) + \underline{\delta}W(x_1)$  and for any  $x_1 \in \Gamma(x_0)$ :

$$(1 - \underline{\delta})V(x_0, x_1) + \underline{\delta}W(x_1) \leq (1 - \underline{\delta})V(x_0, \varphi_{\underline{\delta}}(x_0)) + \underline{\delta}W(\varphi_{\underline{\delta}}(x_0)).$$

The same arguments can be used for the remaining cases  $x^{\underline{\delta}} \leq x_0 \leq x^{\bar{\delta}}$  and  $x^{\bar{\delta}} \leq x_0$  and the argument of the proof is complete. □

**THEOREM F.1.** *Assume **T1-T5** and **V1-V2, V4-V6**.*

(i) *The value of the Maximin problem is equal to the value of the Minimax problem*

$$\begin{aligned} \sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} v(\delta, \chi) &= \max_{\chi \in \Pi(x_0)} \min_{\delta \in \mathcal{D}} v(\delta, \chi) \\ &= \inf_{\delta \in \mathcal{D}} \sup_{\chi \in \Pi(x_0)} v(\delta, \chi) \\ &= \min_{\delta \in \mathcal{D}} \max_{\chi \in \Pi(x_0)} v(\delta, \chi). \end{aligned}$$

(ii)  $J(x_0) = W(x_0)$ .

This characterization of payoffs unbounded from below will prove useful for the characterization of a benchmark example in Section 4.

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