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Effectiveness function, core, axiomatization, holdover property, consistency

JEL codes:

C70, C71

Axiomatic Foundations of a Unifying Core*

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July 23, 2018

Abstract

We provide an axiomatic characterization of the core of games in effectiveness form. We point out that the core, whenever it applies to appropriate classes of these games, coincides with a wide variety of prominent stability concepts in social choice and game theory, such as the Condorcet winner, the Nash equilibrium, pairwise stability, and stable matchings, among others. Our characterization of the core invokes the axioms of restricted non-emptiness, coalitional unanimity, and Maskin invariance together with a principle of independence of irrelevant states, and uses in its proof a holdover property echoing the conventional ancestor property. Taking special cases of this general characterization of the core, we derive new characterizations of the previously mentioned stability concepts.

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1 Introduction

Many theorists in economics and political science have been occupied in studying a wide variety of stability concepts in social choice and game theory for a century or more. Generally speaking, these stability concepts are mainly founded on the idea that given some prevailing state, individuals possess some blocking power to oppose that state and exercise it when they have an interest to do so. A stable state is understood to be a state for which no individual or group of individuals has the power to change the status quo by choosing a more desirable situation. This arises, for example, in a general equilibrium of markets where economic agents on both the demand and supply sides do not have any incentive to alter their consumption or production decisions at the given market price. In the same vein, elections in political systems rely on voting

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rules (quorum, majority, etc.) that allow some coalitions of voters to impose their chosen candidate on the entire society. In like manner, equilibrium concepts for non-cooperative games (Nash equilibrium, subgame perfect equilibrium, etc.) recommend a state robust to deviations in strategy in both static and dynamic settings. Likewise, many solution concepts for coalitional games (core, stable set, etc.) stress cooperative agreements on utility allocation that no coalition would contest.

In this article, we consider the general framework of games in effectiveness form (henceforth e-form games), first introduced by Rosenthal (1972), which encompasses a vast range of contexts, including voting problems, normal form games, network problems, and matching models, among others. The canonical e-form game has the following features. A set N of players is equipped with preferences over a set A of states. Players are mutually aware of each other's preferences, can form coalitions, and sign binding agreements to oppose a given state. In addition, the blocking power distribution among coalitions is described by an "effectiveness function"; given a prevailing state a in A , coalition S is effective if it can force all players to move from state a to some state in B . Such a function specifies for every coalition S of players and subset $B \subseteq A$ of states whether or not S is effective for B . Without going into details, this way of defining the effectivity of coalitions is similar to the "inducement correspondence" introduced by Greenberg (1990) and is more general than the notions of "effectivity function" and "effectiveness relation" respectively proposed by Moulin and Peleg (1982) and Chew (1994). The effectiveness function also corresponds to a special case of the "local effectivity function" suggested by Abdou and Keiding (2003).

Since players can behave cooperatively to oppose a given state, the solution concept we consider here is a version of the core of e-form games (see Rosenthal, 1972). A state a is core-stable if there are no coalition S of players and a subset B of states for which S is effective for B at a and in which every player in S strictly prefers every state in B to a . The most remarkable feature of the core is the fact that a wide variety of prominent stability concepts in social choice and game theory, such as the Condorcet winner, the Nash equilibrium, pairwise stability, and stable matchings, among others, coincide with the core applied to some classes of e-form games by means of an appropriate effectiveness function (Propositions 3.1, 3.2, 3.3, and 6.5). More precisely, by fixing what constitutes the blocking power of coalitions, we can express these stability concepts in terms of the core for a suitable class of e-form games.

Despite the diversity of existing stability concepts such as those just mentioned, very little is known about the properties that unify them. To address this issue, we propose to axiomatically characterize the core on a vast range of classes of e-form games. Formally, the core is a correspondence that associates each e-form game with a (possibly empty) subset of core-stable states. Perhaps unexpectedly, the core is characterized on a wide range of classes of e-form games by a set of four axioms which are reasonably weak and intuitive (Theorem 4.4). "Restricted non-emptiness" requires that when the core is non-empty, a solution to contain at least one state. "Coalitional unanimity" establishes that if a state a is selected for an e-form game, then a must belong to any unanimously best set of states B for players in some coalition S effective for B at a . If a state a is selected for an e-form game, then "Maskin invariance" asserts that it is also

selected in an e-form game where a has (weakly) improved in the preference rankings of all players. “Independence of irrelevant states” specifies that if a state is selected for an e-form game, then it is still selected when non-selected states are removed from the game. This latter axiom is in line with other principles of independence widely used in characterizations of game-theoretic solutions (see Nash, 1950a; Arrow, 1950; Chernoff, 1954; Sen, 1969, 1993). For some classes of e-form games such as, for example, those derived from network problems, states cannot be removed without withdrawing players associated with them. Our principle of independence permits withdrawing such players when necessary.

We first show that if a solution is coalitionally unanimous and Maskin invariant, then it is a subsolution of the core (Proposition 4.1). Then, we prove that if a subsolution of the core is non-empty and satisfies independence of irrelevant states, then it is the core (Lemma 4.3), provided the class of e-form games satisfies a new property, called the holdover property, which plays a key role in the proof of this statement. This property echoes the conventional “ancestor property” and specifies that, given a state a in the core of an e-form game, it is always possible to introduce additional states (and their associated players when necessary) in such a way that the core of the new augmented e-form game only contains state a . This methodology constitutes an alternative to the use of the so-called bracing lemma, which is a typical consistency result for many game-theoretic models (see Thomson, 2011). The complementarity of these two approaches is highlighted in Subsection 6.2.

Using the building blocks leading up to our axiomatic characterization of the core (Theorem 4.4), we provide new axiomatic characterizations of the Condorcet winner correspondence (Proposition 5.2), the Nash equilibrium correspondence (Proposition 5.4), and the pairwise stability correspondence (Proposition 5.6). This mainly consists in reformulating our general axioms for specific classes of e-form games underlying these stability concepts and showing that these classes satisfy the holdover property. The Condorcet winner has recently been characterized by Horan et al. (2018) with axioms different from ours. As far as we know, the pairwise stability correspondence has never been characterized axiomatically before. Our characterization of the Nash equilibrium correspondence is compared with the existing ones proposed by Peleg and Tijs (1996) and Ray (2000), allowing two other axiomatic characterizations to be established (Proposition 6.2). By invoking a consistency principle instead of an independence principle and applying a bracing lemma in the framework of e-form games, we provide a second axiomatic characterization of the core (Theorem 6.4) and apply it in the context of the stable matchings (Proposition 6.7).

The rest of the article is organized as follows. Section 2 introduces the framework of e-form games and the related concept of the core. Specific classes of e-form games for which the core coincides with existing stability concepts are constructed in Section 3. Section 4 presents the main axiomatic characterization of the core. Section 5 contains the specific characterizations of the Condorcet winner correspondence, the Nash equilibrium correspondence and the pairwise stability correspondence. The consistency principle for e-form games is discussed in Section 6 and used to characterize the Nash equilibrium correspondence and the stable matchings correspondence. Section 7

concludes. All proofs appear in the Appendix.

2 Preliminaries

2.1 Games in effectiveness form

For a set X , $\mathcal{P}(X)$ denotes the **set of all subsets of X** , and $\mathcal{P}_0(X)$ denotes the **set of all non-empty subsets of X** .

Let \mathcal{N} and \mathcal{A} respectively be fixed and infinite **sets of players and states**. Given a finite subset of players $N \in \mathcal{P}_0(\mathcal{N})$, called a **coalition**, and a subset of states $A \in \mathcal{P}_0(\mathcal{A})$, an **effectiveness function** on (N, A) is a family $E := (E_a, a \in A)$, where for all $a \in A$, $E_a : \mathcal{P}(N) \rightarrow \mathcal{P}(\mathcal{P}_0(A))$ is such that $E_a(\emptyset) = \emptyset$.¹ The statement $B \in E_a(S)$ means that when the current state is $a \in A$, coalition S can force the outcome of the e-form game to be in B . Each player i has a complete and transitive **preference relation** \succeq_i over the set A of states. We sometimes assume that the preference relation is also antisymmetric, depending on the context in which we work in, such as voting problems. We respectively denote \succ_i and \sim_i as the **asymmetric and symmetric parts of \succeq_i** . For $a, b \in A$, $a \succ_i b$ means that player i strictly prefers a to b , and $a \sim_i b$ means that player i is indifferent between a and b . Let $\succeq = (\succeq_i)_{i \in N}$ denote the **preference profile** of all players over A .

A **game in effectiveness form (e-form game)** is a tuple $\Gamma = (N, A, E, \succeq)$ consisting of a coalition $N \in \mathcal{P}_0(\mathcal{N})$ of players, a set $A \in \mathcal{P}_0(\mathcal{A})$ of states, an effectiveness function $E := (E_a, a \in A)$, and a preference profile \succeq over A . We denote a **generic class of e-form games** by \mathcal{K} . Throughout this article, we assume that if $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ for some preference profile \succeq over A then $\Gamma' = (N, A, E, \succeq') \in \mathcal{K}$ for all preference profiles \succeq' over A .

2.2 The solution concept of the core

A **solution** on \mathcal{K} is a correspondence φ that associates each e-form game $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ with a (possibly empty) subset $\varphi(\Gamma) \in \mathcal{P}_0(A)$ of states. A **subsolution** of φ on \mathcal{K} is a correspondence ψ associating each e-form game $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ with a subset $\psi(\Gamma)$ of states in $\varphi(\Gamma)$. A **proper subsolution** ψ of φ on \mathcal{K} is a subsolution of φ on \mathcal{K} such that $\psi \neq \varphi$.

We now introduce the appropriate notion of objection which we use to define the core. For $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, an **objection** against a state $a \in A$ is a pair $(S, B) \in \mathcal{P}_0(N) \times \mathcal{P}_0(A)$ such that:

- (i) $B \in E_a(S)$;
- (ii) for all $b \in B$ and all $i \in S$, $b \succ_i a$ holds.

¹Rosenthal (1972) proposed a closed definition of an effectiveness function which, for every coalition $S \in \mathcal{P}(N)$, maps, independent of N , every state $a \in A$ into a collection of subsets of A . Our definition is inspired by that of the effectivity function proposed by Moulin and Peleg (1982), which assigns, independent of the current state, a collection of subsets of A to every coalition $S \in \mathcal{P}(N)$.

Formally, the **core** of $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, denoted by $C(\Gamma)$, is the correspondence that assigns the set of states against which there exists no objection:

$$C(\Gamma) = \{a \in A : \text{there is no objection } (S, B) \text{ against } a\}.$$

An e-form game $\Gamma \in \mathcal{K}$ is **solvable** if it has a non-empty core. The (non-empty) **set of solvable e-form games** over the class \mathcal{K} is denoted by \mathcal{K}_C .

As an example, let us consider $N = \{1, 2, 3\}$, $A = \{a, b, c\}$, and for all $k \in A$, $E_k(S) = \{\{l\}, l \in A\}$ if $|S| \geq 2$, and $E_k(S) = \emptyset$ otherwise. The set of preferences for the players are given by:

\succ_1	\succ_2	\succ_3
a	c	a
b	b	c
c	a	b

The pairs $(\{2, 3\}, \{c\})$ and $(\{1, 3\}, \{a\})$ are objections respectively against states b and c . In this way, the e-form game $\Gamma = (N, A, E, \succeq)$ described above has a unique core-stable state $a \in C(\Gamma)$.

3 Core representations of typical stability concepts

The interpretation of the core may differ from one class of e-form games to another. In this section, we express some prominent stability concepts in terms of the core in a suitable class of e-form games. The proofs of Propositions 3.1, 3.2, and 3.3 are fairly straightforward and thus will be omitted.

3.1 Voting theory, effectivity functions, and the Condorcet winner

The first class of e-form games we consider concerns the selection of a committee in the theory of voting. A **voting problem** is a tuple (N, A, \succeq) , where N is a **set of voters**, A is a **set of candidates**, and $\succeq = (\succeq_i)_{i \in N}$ is a **strict preference profile**, i.e. for each voter i , \succeq_i is a complete, transitive, and antisymmetric preference relation over A . For two different candidates a and b , let $P(a, b)$ denote the number of voters who strictly prefer a to b , so that $P(a, b) + P(b, a) = |N|$. Candidate a is said to **beat** candidate b if $P(a, b) > P(b, a)$, or equivalently $P(a, b) > |N|/2$. For a voting problem (N, A, \succeq) , the **Condorcet winner** is the unique candidate who beats any other candidate in a head-to-head competition whenever it exists. The Condorcet winner is a solution concept that is frequently viewed as a natural desideratum for voting problems.

In the theory of voting, effectivity functions (Moulin and Peleg, 1982) describe the allocation of decision power among various coalitions of voters. In this framework, coalitional power is independent of the chosen candidate, i.e. for all $a, b \in A$ and all $S \in \mathcal{P}(N)$, $E_a(S) = E_b(S)$.

Specifically, the Condorcet winner allows any coalition with a majority to veto any candidate whenever its members agree to do so. As a consequence, independently of

the chosen candidate, the associated effectiveness function $E^{CW} := (E_a^{CW}, a \in A)$ gives veto power to a coalition of size at least $|N|/2$:

$$\forall a \in A, \forall S \in \mathcal{P}_0(N), E_a^{CW}(S) = \begin{cases} \{\{b\}, b \in A\} & \text{if } |S| \geq |N|/2; \\ \emptyset & \text{otherwise.} \end{cases}$$

Let \mathcal{K}^{CW} be the class of e-form games $\Gamma = (N, A, E^{CW}, \succeq)$.

Proposition 3.1. *A candidate a is the Condorcet winner for (N, A, \succeq) if and only if $C(\Gamma) = \{a\}$, where $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}^{CW}$.*

The **Condorcet winner correspondence** associates each e-form game $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}^{CW}$ with the subset $C(\Gamma) \in \mathcal{P}_0(A)$ of candidates. It is worth noting that any effectiveness function gives rise to an equivalent core representation of the Condorcet winner if $|S| \geq |N|/2$, $\{b\} \in E_a(S)$, for all $a, b \in A$, and $E_a(S) = \emptyset$ otherwise.

3.2 Normal form games and Nash equilibrium

As a second class of e-form games, we consider normal form games. A **normal form game** is a tuple (N, Σ_N, \succeq) where N is a finite **set of players**, $(\Sigma_i)_{i \in N}$ are the **sets of strategies**, and $\succeq = (\succeq_i)_{i \in N}$ are complete and transitive **preference relations** over **strategy profiles** $\Sigma_N = \prod_{i \in N} \Sigma_i$. For $S \in \mathcal{P}_0(N)$, we define the cartesian product $\Sigma_S = \prod_{i \in S} \Sigma_i$ and for $\sigma \in \Sigma_N$, we denote the projection of σ on Σ_S by σ_S . We say that $\sigma^* \in \Sigma_N$ is a **Nash equilibrium** (Nash, 1950b) of (N, Σ, \succeq) if:

$$\forall i \in N, \forall \sigma_i \in \Sigma_i, \sigma^* \succeq_i (\sigma_i, \sigma_{N \setminus \{i\}}^*).$$

In the framework of e-form games, the set Σ_N of strategy profiles can be identified with the set A of states. To define the associated effectiveness function $E^{NE} := (E_\sigma^{NE}, \sigma \in \Sigma_N)$, only individual players can oppose a state by a deviation in strategy:

$$\forall \sigma \in \Sigma_N, \forall S \in \mathcal{P}_0(N), E_\sigma^{NE}(S) = \begin{cases} \{ \{(\sigma'_i, \sigma_{N \setminus \{i\}})\} : \sigma'_i \in \Sigma_i \} & \text{if } S = \{i\}; \\ \emptyset & \text{otherwise.} \end{cases}$$

We denote \mathcal{K}^{NE} the class of e-form games $\Gamma = (N, \Sigma_N, E^{NE}, \succeq)$.

Proposition 3.2. *The set of Nash equilibria of (N, Σ_N, \succeq) coincides with $C(\Gamma)$, where $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}$.*

The **Nash equilibrium correspondence** associates each e-form game $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}$ with the subset $C(\Gamma) \in \mathcal{P}_0(\Sigma_N)$ of strategy profiles.

3.3 Network formation and pairwise stability

The third class of e-form games concerns network formation (Jackson and Wolinsky, 1996). Given a finite set N of players, a **network** on N is defined as a **set of edges** g that describes pairwise relations among these players. Two players i and j are **directly connected** in g if and only if $\{i, j\} \in g$. For simplicity, we write $ij \in g$ if nodes i and j are directly connected, and $ij \notin g$ if nodes i and j are not directly connected. The **set of all networks** on N is then $\{g : g \subseteq g^N\}$, where g^N is the complete network. A preference profile is given by $\succeq = (\succeq_i)_{i \in N}$, where for each player $i \in N$, \succeq_i is a complete and transitive preference relation over the set $\{g : g \subseteq g^N\}$. A **network problem** is a tuple $(N, \{g : g \subseteq g^N\}, \succeq)$. We denote $g + ij$ the network obtained by adding link ij (if not already the case) to the existing network g and denote $g - ij$ the network obtained by deleting link ij (if not already the case) from the existing network g (i.e., $g + ij = g \cup \{i, j\}$ and $g - ij = g \setminus \{i, j\}$). A network g is **pairwise stable** (Jackson and Wolinsky, 1996) if the two following conditions hold:

- (i) for all $ij \in g$, $g \succeq_i g - ij$ and $g \succeq_j g - ij$;
- (ii) for all $ij \notin g$, $g + ij \succ_i g$ implies $g \succeq_j g + ij$.²

When considering the framework of e-form games, we identify A with the set $\{g : g \subseteq g^N\}$. The notion of pairwise stability requires that link deletion be one-sided and linked addition be two-sided. As a consequence, the associated effectiveness function $E^{PS} := (E_g^{PS}, g \subseteq g^N)$ only gives blocking power to coalitions of size one or two. First, one-sided link deletion allows any player to delete any of its links:

$$\forall g \subseteq g^N, \forall i \in N, E_g^{PS}(\{i\}) = \{g - ij : ij \in g\}.$$

Second, two-sided link addition allows any pair of players that are currently not linked to form a link:

$$\forall g \subseteq g^N, \forall i, j \in N : i \neq j, E_g^{PS}(\{i, j\}) = \begin{cases} \{g + ij\} & \text{if } ij \notin g; \\ \emptyset & \text{otherwise.} \end{cases}$$

Third, for $S \in \mathcal{P}_0(N)$ such that $|S| \geq 3$, and for all $g \subseteq g^N$, we assume that $E_g^{PS}(S) = \emptyset$. We denote \mathcal{K}^{PS} the class of e-form games $\Gamma = (N, A, E^{PS}, \succeq)$.

Proposition 3.3. *The set of pairwise stable networks of $(N, \{g : g \subseteq g^N\}, \succeq)$ coincides with $C(\Gamma)$, where $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}^{PS}$.*

The **pairwise stability correspondence** associates each e-form game $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}^{PS}$ with the subset $C(\Gamma) \in \mathcal{P}_0(A)$ of networks. Other models of network formation allowing the creation of more than one link at a time by coalitions of arbitrary size

²Jackson and Wolinsky (1996) proposed a stronger version of pairwise stability requiring that adding a link makes one deviating player strictly better off and the other one weakly better off. The notion introduced here was also discussed by Jackson and Wolinsky (1996) and is largely used in the literature.

(see Dutta and Mutuswami, 1997; Jackson and van den Nouweland, 2005) can be represented by means of appropriate effectiveness functions.

4 Axiomatic characterizations of the core

Let \mathcal{K} be a class of e-form games and φ be a solution on \mathcal{K} . The first axiom is the axiom of restricted non-emptiness introduced by Voorneveld and van den Nouweland (1998). It requires that the solution contains at least one state whenever the core is non-empty.

Restricted non-emptiness. For all $\Gamma \in \mathcal{K}_C$, it holds that $\varphi(\Gamma) \neq \emptyset$.

Maskin monotonicity (Maskin, 1999) is one of the key principle in implementation theory but is also desirable in and of itself (see, for example, Kojima and Manea, 2010; Karakaya and Klaus, 2017). Roughly speaking, this axiom, called here Maskin invariance,³ requires that if a state is chosen in an e-form game, then it is also chosen in an e-form game in which the state has (weakly) improved in the preference rankings of all players. We say that a preference profile \succeq' is a **Maskin monotonic transformation of a preference profile \succeq at $a \in A$** if any state that is ranked below or at the same level as a under \succeq_i is also ranked below or at the same level as a under \succeq'_i , i.e. for all $i \in N$, $L_i(\succeq_i, a) \subseteq L_i(\succeq'_i, a)$, where $L_i(\succeq_i, a) = \{b \in A : a \succeq_i b\}$.

Maskin invariance. For all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, all $\Gamma' = (N, A, E, \succeq') \in \mathcal{K}$, and all $a \in \varphi(\Gamma)$ such that \succeq' a Maskin monotonic transformation of \succeq at a , it holds that $a \in \varphi(\Gamma')$.

The next axiom generalizes the well-known unanimity condition that is customarily imposed in many settings (see, for example, Crès et al., 2011). Roughly speaking, this condition means that if everyone prefers a particular state over all other states, then this particular state must be selected. We say that $B \in \mathcal{P}_0(A)$ is **unanimously preferred by $S \in \mathcal{P}_0(N)$ with respect to \succeq** if for all $b \in B$, all $c \in A \setminus B$, and all $i \in S$, it holds that $b \succ_i c$.

Coalitional unanimity. For all $\Gamma \in \mathcal{K}$ and all $a \in \varphi(\Gamma)$, if there exists (S, B) such that $B \in E_a(S)$ and B is unanimously preferred by S with respect to \succeq , then $a \in \varphi(\Gamma) \cap B$.

Coalitional unanimity ensures that a state a in the solution cannot be outside a set B unanimously preferred by a coalition S that is effective for B at a .

Proposition 4.1. *Let \mathcal{K} be a class of e-form games. If φ satisfies coalitional unanimity and Maskin invariance on \mathcal{K} , then φ is a subsolution of the core.*

³We refer to Thomson (2018) for a detailed discussion about the incorrect usage of “monotonicity” and the proposal of “invariance”.

The statement of Proposition 4.1 implies that if an e-form game Γ has an empty core, then it is not possible to find another non-empty solution on a class containing Γ which satisfies both coalitional unanimity and Maskin invariance. The following result can be immediately deduced from Proposition 4.1.

Proposition 4.2. *Let \mathcal{K} be a class of e-form games. If \mathcal{K} is such that for all $\Gamma \in \mathcal{K}_C$, $|C(\Gamma)| = 1$, then the core is the unique solution that satisfies restricted non-emptiness, coalitional unanimity, and Maskin invariance on \mathcal{K} .*

This result is relevant to characterize the core in some specific environments such as normal form games that admit a unique Nash equilibrium (e.g., normal form linear oligopoly games) or the Condorcet winner in voting problems (see Section 5.1).

Loosely speaking, our next axiom establishes that if a state is selected, then this state remains selected even some when non-selected (irrelevant) states are removed. For $N' \in \mathcal{P}_0(N)$ and $A' \in \mathcal{P}_0(A)$, let $E' := \{E'_a, a \in A'\}$ be the effectiveness function on (N', A') defined for all $a \in A'$ and all $S \in \mathcal{P}_0(N')$ by $E'_a(S) = \{B \in E_a(S) : B \subseteq A'\}$. Further, the preference profile $\succeq' = (\succeq'_i)_{i \in N'}$ is defined for all $a, b \in A'$ and all $i \in N'$ by $a \succeq'_i b$ if and only if $a \succeq_i b$. The **e-form subgame of Γ on (N', A')** is denoted by $\Gamma' = (N', A', E', \succeq')$.

Independence of irrelevant states. For all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ and all $a \in \varphi(\Gamma)$, if $\Gamma' = (N', A', E', \succeq')$ is an e-form subgame of Γ such that $\Gamma' \in \mathcal{K}$ and $a \in A'$, then $a \in \varphi(\Gamma')$.

We point out that independence of irrelevant states requires that the e-form subgame Γ' belongs to \mathcal{K} . On the one hand, if N is fixed for all e-form games in \mathcal{K} , independence of irrelevant states closely resembles the seminal axiom of independence of irrelevant alternatives like Nash (1950a).⁴ On the other hand, since it is not always possible to remove some states without withdrawing some players for some classes of e-form games (this is the case, for example, of the class of e-form games representing network problems), the subsets of states and players in the e-form subgame may be closely connected.

Unlike coalitional unanimity and Maskin invariance, we will apply restricted non-emptiness and independence of irrelevant states on classes of e-form games satisfying a new property that echoes the ancestor property defined in Norde et al. (1996).⁵

Holdover property. For all $\Gamma \in \mathcal{K}$ and all $\bar{a} \in C(\Gamma)$, there exists $\bar{\Gamma} \in \mathcal{K}$ such that:

- (i) Γ is an e-form subgame of $\bar{\Gamma}$;
- (ii) $C(\bar{\Gamma}) = \{\bar{a}\}$.

⁴See also Arrow (1950), Chernoff (1954), and Sen (1969, 1993).

⁵See also Peleg et al. (1996) and Peleg and Sudhölter (1997).

When the holdover property holds, it is possible to establish a result that allows independence of irrelevant states to be applied in a way similar to the so-called bracing lemma, a typical result based on a principle of consistency to characterize solution concepts in game-theoretic and economic models (see Thomson, 2011 and Section 6.2 for a detailed discussion).

Lemma 4.3. *Let \mathcal{K} be a class of e-form games satisfying the holdover property. No proper subsolution of the core satisfies restricted non-emptiness and independence of irrelevant states on \mathcal{K} .*

Combining Proposition 4.1 and Lemma 4.3, we obtain our first main characterization of the core of e-form games.

Theorem 4.4. *Let \mathcal{K} be a class of e-form games satisfying the holdover property. The core is the unique solution satisfying restricted non-emptiness, coalitional unanimity, Maskin invariance, and independence of irrelevant states on \mathcal{K} .*

The logical independence of the axioms is discussed in the Appendix.

5 Applications

5.1 Axiomatic characterization of the Condorcet Winner

We propose adapting the axioms of Section 4 to the class of e-form games \mathcal{K}^{CW} defined in Subsection 3.1 representing voting problems. In this context, coalitional unanimity becomes the very standard axiom of majority property stating that if one candidate is preferred by a majority of voters, then that candidate must be the unique chosen candidate.

Majority property. For all $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}^{CW}$, if there exists $b \in A$ and $S \in \mathcal{P}_0(N)$ such that $|S| \geq |N|/2$; and further, for all $c \in A \setminus \{b\}$ and all $i \in S$, $b \succ_i c$, then $\varphi(\Gamma) \subseteq \{b\}$.

Proposition 5.1. *Majority property is equivalent to coalitional unanimity on \mathcal{K}^{CW} .*

Combining Proposition 4.2 with Proposition 5.1, we obtain the following result.

Proposition 5.2. *The Condorcet winner correspondence is the unique solution that satisfies restricted non-emptiness, majority property, and Maskin invariance on \mathcal{K}^{CW} .*

Thus, on the full domain of voting problems there does not exist a nonempty-valued solution that selects the Condorcet winner whenever it exists, and which satisfies both majority property and Maskin invariance.

5.2 Axiomatic characterization of the Nash equilibrium

Following the same line of reasoning, we express the axioms of Section 4 in the context of the class of e-form games \mathcal{K}^{NE} defined in Subsection 3.2 representing normal form games. As a consequence, coalitional unanimity becomes a weak version of the axiom of individual rationality specifying that if the behavior of the players is such that it is possible to reach the favorite strategy profile of a particular player, then this strategy profile must be selected. Let $\mathcal{K} \subseteq \mathcal{K}^{NE}$ be a class of e-form games.

Weak individual rationality. For all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}$, and all $\sigma \in \varphi(\Gamma)$, if there exists $i \in N$ and $\sigma'_i \in \Sigma_i$ such that for all $\sigma'' \in \Sigma_N$ with $\sigma'' \neq (\sigma'_i, \sigma_{-i})$, $(\sigma'_i, \sigma_{-i}) \succ_i \sigma''$ holds, then $\sigma'_i = \sigma_i$.

In Section 6, we establish that weak individual rationality is implied by the two standard axioms of one-person rationality and weak consistency.

Proposition 5.3. *Weak individual rationality is equivalent to coalitional unanimity on $\mathcal{K} \subseteq \mathcal{K}^{NE}$.*

The proof of this result is fairly straightforward and thus omitted. The next axiom was first introduced in the context of normal form games by Peleg and Tijs (1996). Let $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$. For $i \in N$, let $\Sigma'_i \subseteq \Sigma_i$ and $E'^{NE} := \{E'_\sigma{}^{NE}, \sigma \in \Sigma'_N\}$ be the effectiveness function on (N, Σ'_N) defined for all $\sigma \in \Sigma'_N$ and all $S \in \mathcal{P}_0(N)$ by:

$$E'_\sigma{}^{NE}(S) = \begin{cases} \{(\sigma'_i, \sigma_{N \setminus \{i\}}) : \sigma'_i \in \Sigma'_i\} & \text{if } S = \{i\}; \\ \emptyset & \text{otherwise.} \end{cases}$$

Independence of irrelevant strategies. For all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}$, and all $\sigma \in \varphi(\Gamma)$, if $\Gamma' = (N, \Sigma'_N, E'^{NE}, \succeq')$ is an e-form subgame of Γ such that $\Gamma' \in \mathcal{K}$ and $\sigma_i \in \Sigma'_i$ for all $i \in N$, then $\sigma \in \varphi(\Gamma')$.

It is immediate that independence of irrelevant states implies independence of irrelevant strategies over any subclass of \mathcal{K}^{NE} and that for a fixed set of players N , both axioms are equivalent over any subclass of \mathcal{K}^{NE} .

Now for all $i \in N$, the set \mathcal{S}_i is defined as an infinite set of possible strategies for player i . For all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}$ and all $i \in N$, we assume that $\Sigma_i \subsetneq \mathcal{S}_i$. We say that a class $\mathcal{K} \subseteq \mathcal{K}^{NE}$ is **extendable** if for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$ and all $i \in N$, there exists $\tilde{\Sigma}_i \subsetneq \mathcal{S}_i$ such that $\Sigma_i \subsetneq \tilde{\Sigma}_i$; and further, if for all preference profiles $\tilde{\succeq}$ over the set $\tilde{\Sigma}_i \times \Sigma_{N \setminus \{i\}}$, $\tilde{\Gamma} = (N, \tilde{\Sigma}_i \times \Sigma_{N \setminus \{i\}}, \tilde{E}^{NE}, \tilde{\succeq}) \in \mathcal{K}$, where \tilde{E}^{NE} is the effectiveness function that extends E^{NE} on $\tilde{\Sigma}_i \times \Sigma_{N \setminus \{i\}}$. A wide variety of classes of e-form games in \mathcal{K}^{NE} are extendable. An example are e-form games representing

normal form games with finite or compact sets of strategies.⁶ We now have the material to provide a new axiomatic characterization of the Nash equilibrium correspondence using Proposition 4.1 and Lemma 4.3.

Proposition 5.4. *Let $\mathcal{K} \subseteq \mathcal{K}^{NE}$ be an extendable class of e-form games such that for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $|N| \geq 2$. The Nash equilibrium correspondence is the unique solution that satisfies restricted non-emptiness, weak individual rationality, Maskin invariance, and independence of irrelevant strategies on \mathcal{K} .*

5.3 Axiomatic characterization of pairwise stable networks

As in the two previous subsections, we write the axioms of Section 4 for the class of e-form games \mathcal{K}^{PS} defined in Subsection 3.3 representing network problems. In this setting, coalitional unanimity can be decomposed into two new axioms.

One-Sided Undesirable Link. For all $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}^{PS}$ and all $g \in \varphi(\Gamma)$, if there exists $i \in N$ such that $g - ij \succ_i g'$ for some $j \in N \setminus \{i\}$ and all $g' \subseteq g^N$ with $g' \neq g - ij$, then $ij \notin g$.

Two-Sided Desirable Link. For all $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}^{PS}$ and all $g \in \varphi(\Gamma)$, if there exists $i, j \in N$ such that $g + ij \succ_k g'$ for all $k \in \{i, j\}$ and all $g' \subseteq g^N$ with $g' \neq g + ij$, then $ij \in g$.

Proposition 5.5. *The conjunction of one-sided undesirable link and two-sided desirable link is equivalent to coalitional unanimity on \mathcal{K}^{PS} .*

The proof of this proposition is omitted since one-sided undesirable link and two-sided desirable link are directly derived from coalitional unanimity. Combining Proposition 4.1 with Lemma 4.3, we obtain the following result.

Proposition 5.6. *The pairwise stability correspondence is the unique solution that satisfies restricted non-emptiness, one-sided undesirable link, two-sided desirable link, Maskin invariance, and independence of irrelevant states on \mathcal{K}^{PS} .*

To the best of our knowledge, this is the first axiomatic characterization of the pairwise stability correspondence.

6 The consistency principle for e-form games

The axiom of consistency we consider in this section is of fundamental interest in axiomatic theory and has been examined from numerous angles.⁷ Throughout this section, we suppose that for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, there exists $(A_i)_{i \in N}$ such that

⁶For example, given $y \in]0, +\infty]$, we can observe that the usual class of games where the set of strategies is the interval $[0, x] \subsetneq [0, y[$ is also extendable.

⁷For comprehensive surveys on consistency and its applications, the reader is referred to Thomson (1990, 2011).

$A = \prod_{i \in N} A_i$. For $S \in \mathcal{P}_0(N)$, we define the cartesian product $A_S = \prod_{i \in S} A_i$ and for $B \in \mathcal{P}_0(A)$ (resp. $a \in A$), we denote B_S (resp. a_S) the projection of B (resp. a) on A_S . Let $\Gamma \in \mathcal{K}$ be an e-form game and $\bar{a} \in A$. The **reduced e-form game of Γ with respect to S and \bar{a}** is the e-form game $\Gamma^{S, \bar{a}} = (S, A_S, E^{S, \bar{a}}, \succeq')$ $\in \mathcal{K}$, where S is a proper coalition of N , and the effectiveness function $E^{S, \bar{a}}$ is defined by setting, for all $a_S \in A_S$ and all $T \in \mathcal{P}_0(S)$,

$$E_{a_S}^{S, \bar{a}}(T) = \{B_S \in \mathcal{P}_0(A_S), \exists B \in E_{(a_S, \bar{a}_{N \setminus S})}(T) \text{ such that } B = B_S \times \{\bar{a}_{N \setminus S}\}\};$$

and further, where the preference profile $\succeq' = (\succeq'_i)_{i \in S}$ over the set A_S is defined by setting, for all $a_S, a'_S \in A_S$ and all $i \in S$, $a_S \succeq'_i a'_S$ if and only if $(a_S, \bar{a}_{N \setminus S}) \succeq_i (a'_S, \bar{a}_{N \setminus S})$. We say that \mathcal{K} is a **reduction-closed class** if for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, all $\bar{a} \in A$, and all $S \subsetneq N$, it holds that $\Gamma^{S, \bar{a}} \in \mathcal{K}$. We now have the material to write consistency in the context of e-form games.

Consistency. For all $\Gamma \in \mathcal{K}$, all $\bar{a} \in \varphi(\Gamma)$, and all $S \subsetneq N$ such that $\Gamma^{S, \bar{a}} \in \mathcal{K}$, it holds that $a_S \in \varphi(\Gamma^{S, \bar{a}})$.

6.1 Consistency and Nash equilibrium

In the literature, consistency is used to characterize the Nash equilibrium correspondence and is generally associated with the controversial axiom of converse consistency (see Thomson, 2011, p. 272). Both axioms pertain to normal form games with variable number of players. Proposition 5.4 proposes a new axiomatic characterization which uses neither consistency nor converse consistency and does not require the variability of the population. In this subsection, we discuss the possible links between Proposition 5.4 and existing axiomatic characterizations of the Nash equilibrium correspondence using consistency. In the context of normal form games, consistency states that if $\sigma \in \Sigma_N$ belongs to the solution of a game, then σ_S belongs to the solution of the game restricted to S , obtained by fixing the strategies of players outside S to $\sigma_{N \setminus S}$. Given an e-form game $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}$ and a strategy profile $\bar{\sigma} \in \Sigma_N$, the **reduced e-form game of Γ with respect to S and $\bar{\sigma}$** is the e-form game $\Gamma^{S, \bar{\sigma}} = (S, \Sigma_S, E^{NE, S, \bar{\sigma}}, \succeq')$ $\in \mathcal{K}^{NE}$, whereby $S \subsetneq N$, the effectiveness function $E^{NE, S, \bar{\sigma}}$ is defined by:

$$\forall \sigma_S \in \Sigma_S, \forall T \in \mathcal{P}_0(S), E_{\sigma_S}^{NE, S, \bar{\sigma}}(T) = \begin{cases} \{ \{(\sigma'_i, \sigma_{S \setminus \{i}\})\} : \sigma'_i \in \Sigma_i \} & \text{if } T = \{i\}; \\ \emptyset & \text{otherwise;} \end{cases}$$

and further, the preference profile $\succeq' = (\succeq'_i)_{i \in S}$ over the set Σ_S is defined for all $\sigma_S, \sigma'_S \in \Sigma_S$ and all $i \in S$ by $\sigma_S \succeq'_i \sigma'_S$ if and only if $(\sigma_S, \bar{\sigma}_{N \setminus S}) \succeq_i (\sigma'_S, \bar{\sigma}_{N \setminus S})$. In the following, we assume that $\mathcal{K} \subseteq \mathcal{K}^{NE}$ is a reduction-closed class. We propose to

weaken Consistency on \mathcal{K} as follows.

Weak consistency. For all $\Gamma \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}$, all $\bar{\sigma} \in \varphi(\Gamma)$, and all $i \in N$ such that $\Gamma^{\{i\}, \bar{\sigma}} \in \mathcal{K}$, it holds that $\sigma_i \in \varphi(\Gamma^{\{i\}, \bar{\sigma}})$.

Since the pionnering work of Peleg and Tijs (1996), several axiomatic characterizations of the Nash equilibrium correspondence or closed solution concepts for normal form games have been proposed in the literature.⁸ Most of them use consistency together with one-person rationality that we adapt for e-form games as follows.

One-person rationality. If $\Gamma = (\{i\}, \Sigma_i, E^{NE}, \succeq_i) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}$, is a one-person e-form game, then $\varphi(\Gamma) = \{\sigma_i \in \Sigma_i : \sigma_i \succeq_i \sigma'_i \text{ for all } \sigma'_i \in \Sigma_i\}$.

The following proposition establishes that one-person rationality together with weak consistency imply weak individual rationality.

Proposition 6.1. *Let $\mathcal{K} \subseteq \mathcal{K}^{NE}$ be a reduction-closed class, and φ be a solution on \mathcal{K} . If φ satisfies one-person rationality and weak consistency on \mathcal{K} , then φ also satisfies weak individual rationality on \mathcal{K} .*

To introduce the next axiom, defined by Peleg and Tijs (1996), we say that player $d \in N$ is **dummy** in $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}$ if $|\Sigma_d| = 1$.

Dummy. For all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}$, if player $d \in N$ is dummy, then $\varphi(\Gamma) = \Sigma_d \times \varphi(\Gamma^{N \setminus \{d\}, \bar{\sigma}})$, where $\bar{\sigma} \in \Sigma_N$.

Ray (2000) proposed a weaker version of the dummy axiom that we can adapt too.

Weak dummy. For all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}$, if player $d \in N$ is dummy, then $\varphi(\Gamma) \subseteq \Sigma_d \times \varphi(\Gamma^{N \setminus \{d\}, \bar{\sigma}})$, where $\bar{\sigma} \in \Sigma_N$.

We establish two other axiomatic characterizations of the Nash equilibrium correspondence which only use standard properties.

Proposition 6.2. *Let $\mathcal{K} \subseteq \mathcal{K}^{NE}$ be a class of e-form games that is both extendable and reduction-closed, and φ be a solution on \mathcal{K} . The following three properties are equivalent:*

- (i) φ is the Nash equilibrium correspondence;
- (ii) φ satisfies restricted non-emptiness, one-person rationality, independence of irrelevant strategies, and weak consistency on \mathcal{K} ;
- (iii) φ satisfies restricted non-emptiness, one-person rationality, independence of irrelevant strategies, and weak dummy on \mathcal{K} .

⁸See, for example, Peleg et al. (1996), Peleg and Sudhölter (1997), Norde et al. (1996) and van Heumen et al. (1996).

6.2 Ancestor Property and axiomatic characterization of stable matchings

In this subsection, we discuss a standard method to axiomatize game-theoretic solutions with the axiom of consistency. This method is called the “bracing lemma” by Thomson (2011). We stem that this method can be adapted to the framework of e-form games and discuss new axiomatizations in matching theory by using the ancestor property together with Proposition 4.1.

6.2.1 Ancestor property and bracing lemma

The following property concerns the nature of the classes of e-form games and adapted the ancestor property defined by Norde et al. (1996) to the framework of e-form games.

Ancestor property. For all $\Gamma = (N, A_N, E, \succeq) \in \mathcal{K}$ and all $a^* \in C(\Gamma)$, there exists $\Gamma' = (N', A'_{N'}, E', \succeq') \in \mathcal{K}$ with $N' \supseteq N$ such that:

- (i) Γ is the reduced e-form game of Γ' with respect to N and a^* ;
- (ii) for all $a' \in C(\Gamma')$, $a'_N = a^*$ holds.

When the ancestor property holds, it is possible to establish a result almost identical to the so-called “bracing lemma” (see Thomson, 2011).

Lemma 6.3. *Let \mathcal{K} be a class of e-form games satisfying the ancestor property. No proper subsolution of the core satisfies restricted non-emptiness and consistency on \mathcal{K} .*

Combining Proposition 4.1 and Lemma 6.3, we obtain our second main characterization of the core.

Theorem 6.4. *Let \mathcal{K} be a class of e-form games satisfying the ancestor property. The core is the unique solution satisfying restricted non-emptiness, coalitional unanimity, Maskin invariance, and consistency on \mathcal{K} .*

6.2.2 A new characterization of the core of matching models

As an application of Theorem 6.4, we propose a new axiomatization of the core of the two-sided one-to-one matching model introduced by Gale and Shapley (1962). The stable matchings correspondence has been axiomatized in the context of one-to-one matching by Sasaki and Toda (1992), Nizamogullari and Özkal-Sanver (2014) and Klaus (2017). We consider a **matching model** consisting of a finite set N of individuals, partitioned in two subgroups with equal numbers of **men** M and **women** W . Each man m (resp. woman w) has a complete, transitive, and antisymmetric strict preference relation P_m (resp. P_w) over the set W (resp. M). A **matching** is a one-to-one mapping $\mu : M \rightarrow W$. A **matching problem** is a tuple $(M, W, (P_m)_{m \in M}, (P_w)_{w \in W})$. A matching μ is **stable** in $(M, W, (P_m)_{m \in M}, (P_w)_{w \in W})$ if there is no pair $(m, w) \in M \times W$ such that $w P_m \mu(m)$ and $m P_w \mu^{-1}(w)$. The deferred

acceptance rule of Gale and Shapley (1962) determines a stable matching that Pareto dominates any other stable matching.⁹

In the setting of e-form games, the set A of states consists of all possible matchings μ . The preferences of the players $(\succeq_i)_{i \in M \cup W}$ over the set A are induced by their preferences $((P_m)_{m \in M}, (P_w)_{w \in W})$ over their partners as follows: for every $m \in M$, $\mu \succ_m \mu'$ if and only if $\mu(m)P_m \mu'(m)$, and for every $w \in W$, $\mu \succ_w \mu'$ if and only if $\mu^{-1}(w)P_w \mu'^{-1}(w)$. Several effectiveness functions lead to a core representation of the stable matchings. We propose the following definition which will turn out to be essential for our axiomatic result. The requirement of stability here is that any man and woman that are currently not partners can make better by creating a link between themselves:

$$\forall \mu \in A, \forall (m, w) \in M \times W, E_\mu^{SM}(\{m, w\}) = \{\{\mu' : \mu'(m) = w\}\},$$

and for $S \in \mathcal{P}_0(N)$ such that $|S| \neq 2$, $E_\mu^{SM}(S) = \emptyset$ for all $\mu \in A$. We denote \mathcal{K}^{SM} the class of e-form games $\Gamma = (N, A, E^{SM}, \succeq)$.

Proposition 6.5. *The set of stable matchings of $(M, W, (P_m)_{m \in M}, (P_w)_{w \in W})$ coincides with $C(\Gamma)$, where $\Gamma = (N, A, E^{SM}, \succeq) \in \mathcal{K}^{SM}$.*

The proof of this result is fairly straightforward and thus omitted. The **stable matchings correspondence** associates each e-form game $\Gamma = (N, A, E^{SM}, \succeq) \in \mathcal{K}^{SM}$ with the subset $C(\Gamma) \in \mathcal{P}_0(A)$ of matchings. The main advantage of this core representation is to have both an adaptation of coalitional unanimity as well as an equivalence between consistency as defined in Section 6 and the one as usually defined in matching theory (see, for example, Sasaki and Toda, 1992). Indeed, for all $\Gamma \in \mathcal{K}^{SM}$, a reduced e-form game $\Gamma^{S, \bar{\mu}}$ belonging to \mathcal{K}^{SM} has necessarily equal numbers of men and women, which means that there exist $M' \subseteq M$ and $W' \subseteq W$ such that $S = M' \cup W'$ and $|M'| = |W'|$. Following the definition of a reduced e-form game, it is immediate that the associated effectiveness function is defined for all matching $\mu|_{M'} : M' \rightarrow W'$ and all $T \in \mathcal{P}_0(M' \cup W')$ by setting $E_{\mu|_{M'}}^{SM, S, \bar{\mu}}(T) = \{\{\mu'|_{M'} : \mu'|_{M'}(m') = w'\}\}$ if $T = \{m', w'\}$, $(m', w') \in M' \times W'$, and $E_{\mu|_{M'}}^{SM, S, \bar{\mu}}(T) = \emptyset$ otherwise.

In the context of matchings, coalitional unanimity becomes a natural axiom that states that soulmates must be married together, where soulmates are a man and a woman who would prefer to be with each other than any other individual (Leo et al., 2017).

Soulmates. For all $\Gamma = (N, A, E^{SM}, \succeq) \in \mathcal{K}^{SM}$ and all $\mu \in \varphi(\Gamma)$, if there exists B such that for all $\mu' \in B$, $\mu'(m) = w$ and B is unanimously preferred by $\{m, w\}$, then $\mu(m) = w$.

Proposition 6.6. *Soulmates is equivalent to coalitional unanimity on \mathcal{K}^{SM} .*

⁹In general assignment problems, the deferred acceptance rule has been characterized by Kojima and Manea (2010).

The proof of this result is fairly straightforward and thus omitted. Applying Proposition 4.1 and Lemma 6.3, we obtain the following result.

Proposition 6.7. *The stable matchings correspondence is the unique solution that satisfies restricted non-emptiness, soulmates, Maskin invariance, and consistency on \mathcal{K}^{SM} .*

7 Conclusion

While a wide variety of solution concepts in social choice and game theory are mainly founded on the same idea of “steady” states, our general axiomatizations of the core provide a comprehensive and unified description of their theoretical contents. The axioms may prove useful in characterizing many other existing stability concepts (e.g., the core of markets games) and potentially others to come.

The non-emptiness of the core of e-form games is also of crucial interest. This question is far beyond the scope of this article and is unlikely to be simple. Therefore, it is left for future research.

Appendix

Proof of Proposition 4.1. Let φ be a solution that satisfies coalitional unanimity and Maskin invariance on \mathcal{K} . Assume for the sake of contradiction that $\varphi(\Gamma) \not\subseteq C(\Gamma)$ for some $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$. Hence, there is a state $a \in \varphi(\Gamma)$ such that $a \notin C(\Gamma)$, i.e. there exists an objection (S, B) against a in Γ . Let \succeq' be the preference profile defined as follows:

- (i) for all $i \in S$, we first move, in any order, the set of states B to the top of player i 's preferences (if not already the case), i.e. for all $b \in B$ and all $c \in A \setminus B$, it holds that $b \succ'_i c$. Second, we move state a just below the worst state in B according to player i 's preferences, i.e. for all $c \in A \setminus (B \cup \{a\})$, $a \succeq'_i c$. Third, for all $c, d \in A \setminus (B \cup \{a\})$, we assume that $c \succeq'_i d$ if and only if $c \succeq_i d$;
- (ii) for all $i \in N \setminus S$, we leave preferences as they are, i.e. for all $i \in N \setminus S$, $\succeq'_i = \succeq_i$.

Observe that for all $i \in N$, $L_i(\succeq_i, a) \subseteq L_i(\succeq'_i, a)$. Hence, \succeq' is a Maskin monotonic transformation of \succeq at a . Let $\Gamma' = (N, A, E, \succeq') \in \mathcal{K}$. It follows from Maskin invariance that $a \in \varphi(\Gamma')$. Furthermore, by the definition of \succeq' , it holds that B is unanimously preferred by all members of S . Thus, it follows from coalitional unanimity that $a \in B$, a contradiction. ■

Proof of Lemma 4.3. Let \mathcal{K} be a class satisfying the holdover property and φ be a proper solution of the core, which is defined on \mathcal{K} and satisfies restricted non-emptiness and independence of irrelevant states. Since φ is a proper solution of the core, there exists $\Gamma \in \mathcal{K}$ and $\bar{a} \in C(\Gamma)$ such that $\bar{a} \notin \varphi(\Gamma)$. By the holdover property, there

exists $\bar{\Gamma} \in \mathcal{K}$ such that Γ is an e-form subgame of $\bar{\Gamma}$, where $C(\bar{\Gamma}) = \{\bar{a}\}$. Since φ is a proper subsolution of the core satisfying restricted non-emptiness, we deduce that $\varphi(\bar{\Gamma}) = \{\bar{a}\}$. Since φ satisfies independence of irrelevant states, it follows that $\bar{a} \in \varphi(\Gamma)$, a contradiction. ■

Proof of Theorem 4.4. Let \mathcal{K} be a class of e-form games satisfying the holdover property. From Proposition 4.1 and Lemma 4.3, it suffices to prove that the core defined on \mathcal{K} satisfies restricted non-emptiness, coalitional unanimity, Maskin invariance, and independence of irrelevant states on \mathcal{K} . It is immediate that the core satisfies restricted non-emptiness and coalitional unanimity. Second, we prove that the core satisfies Maskin invariance on \mathcal{K} . Let $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ and $a \in C(\Gamma)$. Let $\Gamma' = (N, A, E, \succeq') \in \mathcal{K}$ be such that \succeq' is a Maskin monotonic transformation of \succeq at a . Assume for the sake of contradiction that $a \notin C(\Gamma')$. Then, there exists an objection (S, B) against a in Γ' . Hence, for all $b \in B$ and all $i \in S$, $b \notin L_i(\succeq'_i, a)$, and so since $L_i(\succeq_i, a) \subseteq L_i(\succeq'_i, a)$, $b \notin L_i(\succeq_i, a)$. This implies that for all $b \in B$ and all $i \in S$, $b \succ_i a$. Thus, (S, B) is also an objection against a in Γ , contradicting $a \in C(\Gamma)$. Third, we show that the core satisfies independence of irrelevant states on \mathcal{K} . Let $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, $a \in C(\Gamma)$ and $\Gamma' = (N', A', E', \succeq')$ be an e-form subgame of Γ such that $\Gamma' \in \mathcal{K}$ and $a \in A'$. Assume for the sake of contradiction that $a \notin C(\Gamma')$. Then, there exists an objection (S, B) against a in Γ' . Since for every $S \in \mathcal{P}_0(N')$, $E'_a(S) \subseteq E_a(S)$, and $\succeq_{N'} = (\succeq_i)_{i \in N'}$, (S, B) is also an objection against a in Γ , contradicting $a \in C(\Gamma)$. It follows that the core also satisfies independence of irrelevant states on \mathcal{K} . ■

Proof of Proposition 5.1. Let φ be a solution on \mathcal{K}^{CW} .

- (i) First, assume that φ satisfies coalitional unanimity on \mathcal{K}^{CW} . Let $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}^{CW}$ and assume that there exists $b \in A$ and $S \in \mathcal{P}_0(N)$ with $|S| \geq |N|/2$, such that for all $c \in A \setminus \{b\}$ and all $i \in S$, $b \succ_i c$. Then $\{b\}$ is unanimously preferred by all members of S and for all $a \in A$, $\{b\} \in E_a^{CW}(S)$. If $\varphi(\Gamma) = \emptyset$, then there is nothing to prove. If $\varphi(\Gamma) \neq \emptyset$, then there exists $a \in \varphi(\Gamma)$, and by coalitional unanimity, $a \in \varphi(\Gamma) \cap \{b\}$. We deduce that $\varphi(\Gamma) = \{b\}$.
- (ii) Second, assume that φ satisfies the majority property on \mathcal{K}^{CW} . Let $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}^{CW}$ and assume that there exist $a \in \varphi(\Gamma)$ and (S, B) such that $B \in E_a^{CW}(S)$ and B is unanimously preferred by all members of S . By the definition of E^{CW} , we deduce that $|S| \geq |N|/2$ and there exists $b \in A$ such that $B = \{b\}$. As a consequence, for all $i \in S$ and all $c \in A \setminus \{b\}$, we have $b \succ_i c$. By the majority property, $\varphi(\Gamma) \subseteq \{b\}$, and we deduce that $a = b$, so $a \in \varphi(\Gamma) \cap \{b\}$.

This concludes the proof. ■

Proof of Proposition 5.4. Let $\mathcal{K} \subseteq \mathcal{K}^{NE}$ be an extendable class such that for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $|N| \geq 2$. By Proposition 4.1 and Lemma 4.3, it is sufficient to prove that \mathcal{K} restricted to games with at least two players satisfies the holdover property. Let $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$ be an e-form game and i and j denote two

distinct players belonging to N . We have seen (see Section 3.2) that the set of Nash equilibria of (N, Σ_N, \succeq) is equal to $C(\Gamma)$. Assume that $|C(\Gamma)| \geq 2$ and let σ^* be a member of $C(\Gamma)$. Since \mathcal{K} is extendable, for all $k \in \{i, j\}$, there exist $s_k, t_k \in \mathcal{S}_k \setminus \Sigma_k$ such that for all weak order $\tilde{\succeq}$ on $\tilde{\Sigma}_N$, $\tilde{\Gamma} = (N, \tilde{\Sigma}_N, \tilde{E}^{NE}, \tilde{\succeq}) \in \mathcal{K}$, whereby:

$$\tilde{\Sigma}_N = \tilde{\Sigma}_{\{ij\}} \times \Sigma_{N \setminus \{i, j\}} \text{ and } \tilde{\Sigma}_k \supseteq \Sigma_k \cup \{s_k, t_k\} \text{ for all } k \in \{i, j\}.$$

Let $(\tilde{\succeq}_\ell)_{\ell \in N}$ be the preference profile on $\tilde{\Sigma}_N$ defined as follows:

Rule 1. $\forall \ell \in N, \forall k, k' \in \{i, j\}, \forall \sigma_k \in \Sigma_k, \forall a_k \in \{s_k, t_k\}, \forall a_{k'} \in \{s_{k'}, t_{k'}\}, \forall \tilde{\sigma} \in \tilde{\Sigma}_N$ such that $\forall k \in \{i, j\}, \tilde{\sigma}_k \notin \{s_k, t_k\} \cup \Sigma_k$, we have:

$$(i) (\sigma_k, \tilde{\sigma}_{N \setminus \{k\}}) \sim_\ell (a_k, \tilde{\sigma}_{N \setminus \{k\}}) \sim_\ell (a_{k'}, \tilde{\sigma}_{N \setminus \{k'\}}) \tilde{\succ}_\ell \tilde{\sigma},$$

$$(ii) (a_k, \sigma_{N \setminus \{k\}}^*) \sim_\ell (a_{k'}, \sigma_{N \setminus \{k'\}}^*) \tilde{\succ}_\ell (a_k, \tilde{\sigma}_{N \setminus \{k\}}).$$

Rule 2. $\forall \ell \in N, \forall k \in \{i, j\}, \forall a_k \in \{s_k, t_k\}, \forall \sigma \in \Sigma_N$, we have $\sigma \tilde{\succ}_\ell (a_k, \sigma_{N \setminus \{k\}}^*)$.

Rule 3. $\forall \ell \in N, \forall \sigma, \sigma' \in \Sigma_N, \sigma \succeq_\ell \sigma' \Leftrightarrow \sigma \tilde{\succeq}_\ell \sigma'$.

Rule 4. $\forall \ell \in N, \forall k \in \{i, j\}, \forall a_k \in \{s_k, t_k\}, \forall \sigma \in \Sigma_N$ such that $\sigma_{N \setminus \{k\}} \neq \sigma_{N \setminus \{k\}}^*$, we have $(a_k, \sigma_{N \setminus \{k\}}) \tilde{\succ}_\ell \sigma$.

Rule 5. $\forall \ell \in N, \forall k \in \{i, j\}, \forall a_k \in \{s_k, t_k\}, \forall a_i \in \{s_i, t_i\}, \forall a_j \in \{s_j, t_j\}, \forall \sigma \in \Sigma_N$ we have $(a_i, a_j, \sigma_{N \setminus \{i, j\}}) \tilde{\succ}_\ell (a_k, \sigma_{N \setminus \{k\}})$.

Rule 6. $\forall \ell \in N \setminus \{i, j\}, \forall \sigma \in \Sigma_N, \forall k \in \{i, j\}, \forall a_k \in \{s_k, t_k\}$, we have:

$$(i) (s_i, s_j, \sigma_{N \setminus \{ij\}}) \sim_i (t_i, t_j, \sigma_{N \setminus \{ij\}}) \tilde{\succ}_i (s_i, t_j, \sigma_{N \setminus \{ij\}}) \sim_i (t_i, s_j, \sigma_{N \setminus \{ij\}}),$$

$$(ii) (s_i, t_j, \sigma_{N \setminus \{ij\}}) \sim_j (t_i, s_j, \sigma_{N \setminus \{ij\}}) \tilde{\succ}_j (s_i, s_j, \sigma_{N \setminus \{ij\}}) \sim_j (t_i, t_j, \sigma_{N \setminus \{ij\}}),$$

$$(iii) (s_i, t_j, \sigma_{N \setminus \{ij\}}) \sim_\ell (t_i, s_j, \sigma_{N \setminus \{ij\}}) \sim_\ell (s_i, s_j, \sigma_{N \setminus \{ij\}}) \sim_\ell (t_i, t_j, \sigma_{N \setminus \{ij\}}).$$

Rule 7. $\forall \ell \in N, \tilde{\succeq}_\ell$ is transitive.

Observe that *Rules* 1 through 7 rank strategy profiles in increasing order according to players' preferences. They ensure that for all $\ell \in N, \tilde{\succeq}_\ell$ is a weak order of $\tilde{\Sigma}_N$. The proof that $(N, \Sigma_N, E^{NE}, \succeq)$ is an e-form subgame of $(N, \tilde{\Sigma}_N, \tilde{E}^{NE}, \tilde{\succeq})$ is an immediate consequence of *Rule* 3. Observe that *Rules* 2 and 4 ensure that individual deviations of i and j from σ^* to $\tilde{\Sigma}_N \setminus \Sigma_N$ decrease the situation of all players, while individual deviations of i or j from $\Sigma_N \setminus \{\sigma^*\}$ to $\tilde{\Sigma}_N \setminus \Sigma_N$ improve the situation of all players. We want to prove that $C(\tilde{\Gamma}) = \{\sigma^*\}$. Let $\bar{\sigma} \in \tilde{\Sigma}_N \setminus \{\sigma^*\}$. Several cases arise:

(i) If $\bar{\sigma} \in \Sigma_N \setminus \{\sigma^*\}$, then two subcases can occur:

(a) If $\bar{\sigma}_{N \setminus \{i\}} \neq \sigma_{N \setminus \{i\}}^*$, then by *Rule* 4, we have $(s_i, \bar{\sigma}_{N \setminus \{i\}}) \tilde{\succ}_i \bar{\sigma}$;

- (b) If $\bar{\sigma}_{N \setminus \{i\}} = \sigma_{N \setminus \{i\}}^*$, then $\bar{\sigma}_{N \setminus \{j\}} \neq \sigma_{N \setminus \{j\}}^*$ because $\bar{\sigma} \neq \sigma^*$. Hence, it follows from *Rule 4* that $(s_j, \bar{\sigma}_{N \setminus \{j\}}) \succ_j \bar{\sigma}$.

In both subcases, we conclude that $\bar{\sigma}$ does not belong to $C(\tilde{\Gamma})$.

(ii) If $\bar{\sigma} \in \tilde{\Sigma}_N \setminus \Sigma_N$, then two subcases can occur:

- (a) If there does not exist $k \in \{i, j\}$ such that $\bar{\sigma}_k \in \{s_k, t_k\} \cup \Sigma_k$, then by *Rule 1*, we have $(s_i, \bar{\sigma}_{N \setminus \{i\}}) \succ_i \bar{\sigma}$.
- (b) If there exists $k \in \{i, j\}$ such that $\bar{\sigma}_k \in \{s_k, t_k\} \cup \Sigma_k$, then three cases arise. In each of these cases, we denote by k' the unique element in $\{i, j\} \setminus \{k\}$.
- (b₁) If $\bar{\sigma}_k \in \{s_k, t_k\}$ and $\bar{\sigma}_{k'} \in \{s_{k'}, t_{k'}\}$, then by (ii) and (iii) of *Rule 6*, there exists an improving deviation of i or j .
- (b₂) If $\bar{\sigma}_k \in \{s_k, t_k\}$ and $\bar{\sigma}_{k'} \in \Sigma_{k'}$, then by *Rule 5*, there exists an improving deviation of k' (the case where $\bar{\sigma}_k \in \Sigma_k$ and $\bar{\sigma}_{k'} \in \{s_{k'}, t_{k'}\}$ is similar).
- (b₃) If $\bar{\sigma}_k \in \{s_k, t_k\} \cup \Sigma_k$ and $\bar{\sigma}_{k'} \notin \{s_{k'}, t_{k'}\} \cup \Sigma_{k'}$, then for all $\sigma_{k'} \in \Sigma_{k'}$, $(\sigma_{k'}, \bar{\sigma}_{N \setminus \{k'\}})$ is an improving deviation of k' from $\bar{\sigma}$. On the one hand, if $(\sigma_{k'}, \sigma_{N \setminus \{k, k'\}}) = \sigma_{N \setminus \{k, k'\}}^*$, then it follows from (ii) of *Rule 1* that $(\sigma_{k'}, \bar{\sigma}_{N \setminus \{k'\}})$ is an improving deviation of k' from $\bar{\sigma}$. On the other hand, if $(\sigma_{k'}, \sigma_{N \setminus \{k, k'\}}) \neq \sigma_{N \setminus \{k, k'\}}^*$, then it follows from *Rules 1, 4 and 7* that $(\sigma_{k'}, \bar{\sigma}_{N \setminus \{k'\}})$ is an improving deviation of k' from $\bar{\sigma}$.

In all cases, we conclude that $\bar{\sigma}$ does not belong to $C(\tilde{\Gamma})$.

It remains to prove that σ^* belongs to $C(\tilde{\Gamma})$. First, since σ^* belongs to $C(\Gamma)$, it follows from *Rule 3* that there is no possible deviation from σ^* to Σ_N . Second, it follows from *Rule 2* that there does not exist improving deviation from σ^* to $\tilde{\Sigma}_N \setminus \Sigma_N$. Thus, we conclude that $C(\tilde{\Gamma}) = \{\sigma^*\}$. \blacksquare

Proof of Proposition 5.6. By Proposition 4.1 and Lemma 4.3, it is sufficient to prove that \mathcal{K}^{PS} satisfies the holdover property. Let $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}^{PS}$ and take any $\bar{g} \in C(\Gamma)$. If $|N| = 1$, then there is nothing to prove. Assume that $|N| \geq 2$ and let $\bar{\Gamma} = (N', A', E'^{PS}, \succeq')$ be an e-form game where $N' = N \cup \{\bar{i}\}$. Consider any player $\bar{j} \in N$ and define players' preferences in the following way:

Rule 1. For player $\bar{i} \in N'$, we assume that for all $g, g' \in A' \setminus A$, $g \succ_{\bar{i}}' g'$ iff $|g| > |g'|$; for all $g, g' \in A \setminus \{\bar{g}\}$, $g \sim_{\bar{i}}' g'$; for all $g \in A \setminus \{\bar{g}\}$ and all $g' \in A' \setminus (A \setminus \{\bar{g}\})$, $g' \succ_{\bar{i}}' g$; for all $g \in A' \setminus (A \cup \{\bar{g} + \bar{i}\bar{j}\})$, $g \succeq_{\bar{i}}' \bar{g}$ and $\bar{g} + \bar{i}\bar{j} \sim_{\bar{i}}' \bar{g}$.

Rule 2. For player $\bar{j} \in N$, we assume that for all $g, g' \in A$, $g \succeq_{\bar{j}}' g'$ iff $g \succeq_{\bar{j}} g'$; for all $g \in A$ and all $g' \in A' \setminus A$, $g' \succ_{\bar{j}}' g$; and for all $g, g' \in A' \setminus A$, $g \succ_{\bar{j}}' g'$ iff $|g| < |g'|$.

Rule 3. For all players $k \in N \setminus \{\bar{j}\}$, we assume that for all $g, g' \in A$, $g \succeq_k' g'$ iff $g \succeq_k g'$; for all $g \in A$ and all $g' \in A' \setminus A$, $g \succ_k' g'$; for all $g, g' \in A' \setminus A$ such that $\bar{i}\bar{j} \in g$

and $\bar{i}\bar{j} \in g'$, $g \succ'_k g'$ iff $|g| > |g'|$; for all $g, g' \in A' \setminus A$ such that $\bar{i}\bar{j} \notin g$ and $\bar{i}\bar{j} \notin g'$, $g \succ'_k g'$ iff $|g| < |g'|$; for all $g, g' \in A' \setminus A$ such that $\bar{i}\bar{j} \in g$ and $\bar{i}\bar{j} \notin g'$, $g \succ'_k g'$.

First, observe that Γ is an e-form subgame of $\bar{\Gamma}$. It remains to prove that $C(\bar{\Gamma}) = \{\bar{g}\}$. By construction, it must be clear that \bar{g} is pairwise stable in $\bar{\Gamma}$. Then assume that there exists a pairwise stable network \tilde{g} in $\bar{\Gamma}$ such that $\tilde{g} \neq \bar{g}$. We distinguish two cases:

- (i) First, if $\tilde{g} \in A$, then $\tilde{g} + \bar{i}\bar{j} \succ'_l \tilde{g}$ for all $l \in \{\bar{i}, \bar{j}\}$. So, \tilde{g} is not pairwise stable;
- (ii) Second, if $\tilde{g} \in A' \setminus A$, we distinguish two subcases:
 - (a) If $\bar{i}\bar{j} \in \tilde{g}$ then two cases can occur:
 - i. There exists $k \in N \setminus \{\bar{j}\}$ such that $k\bar{i} \notin \tilde{g}$. In this case, we have $\tilde{g} + k\bar{i} \succ'_l \tilde{g}$ for all $l \in \{k, \bar{i}\}$. So, \tilde{g} is not pairwise stable;
 - ii. For all $k \in N \setminus \{\bar{j}\}$, $k\bar{i} \in \tilde{g}$. In this case, we have $\tilde{g} - \bar{i}\bar{j} \succ'_j \tilde{g}$. So, \tilde{g} is not pairwise stable.
 - (b) If $\bar{i}\bar{j} \notin \tilde{g}$, then there exists $k \in N \setminus \{\bar{j}\}$ such that $k\bar{i} \in \tilde{g}$ since $\tilde{g} \in A' \setminus A$. In this case, we have $\tilde{g} - k\bar{i} \succ'_k \tilde{g}$. So, \tilde{g} is not pairwise stable, a contradiction.

Thus, we conclude that $C(\bar{\Gamma}) = \{\bar{g}\}$. ■

Proof of Proposition 6.1. Let φ be a solution on $\mathcal{K} \subseteq \mathcal{K}^{NE}$ where \mathcal{K} is a reduction-closed class. Assume that φ satisfies one-person rationality and weak consistency on \mathcal{K} . Let σ be an element of $\varphi(\Gamma)$. Assume that there exists $i \in N$ and $\sigma'_i \in \Sigma_i$ such that for all $\sigma'' \neq (\sigma'_i, \sigma_{-i})$, $(\sigma'_i, \sigma_{-i}) \succ_i \sigma''$. Since $\sigma \in \varphi(\Gamma)$ and φ satisfies weak consistency, we have $\sigma_i \in \varphi(\Gamma^{\{i\}, \sigma})$. Since φ satisfies one-person rationality, we have $\{\sigma'_i\} = \varphi(\Gamma^{\{i\}, \sigma})$. Hence, we obtain $\sigma_i = \sigma'_i$, which means that φ satisfies weak individual rationality. ■

Proof of Proposition 6.2. Let $\mathcal{K} \subseteq \mathcal{K}^{NE}$ be a class of e-form games, which is both extendable and reduction-closed, and φ be a solution on \mathcal{K} . We omit the proof that the Nash equilibrium correspondence satisfies the axioms. Assume that φ satisfies restricted non-emptiness, one-person rationality, independence of irrelevant strategies, and weak consistency on \mathcal{K} . Since φ satisfies one-person rationality and weak consistency, then by a similar argument to Peleg and Tijs (1996)[Proposition 2.8], we obtain that φ is a subsolution of the Nash equilibrium correspondence. We conclude with Lemma 4.3 that φ is the Nash equilibrium correspondence. Now, assume that φ satisfies restricted non-emptiness, one-person rationality, independence of irrelevant strategies, and weak dummy. Since φ satisfies weak dummy and independence of irrelevant strategies, then by Ray (2000)[Proposition 2], we obtain that φ satisfies weak consistency and conclude with the previous argument that φ is the Nash equilibrium correspondence. ■

Proof of Lemma 6.3. Let \mathcal{K} be a class satisfying the ancestor property and φ be a proper solution of the core, defined on \mathcal{K} and which satisfies restricted non-emptiness and consistency. Since φ is a proper solution of the core, there exist

$\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ and $a^* \in C(\Gamma)$ such that $a^* \notin \varphi(\Gamma)$. By the ancestor property, there exists $\Gamma' = (N', A'_{N'}, E', \succeq') \in \mathcal{K}$ with $N' \supseteq N$ such that Γ is the reduced e-form game of Γ' with respect to a^* and N , and for all $a' \in C(\Gamma')$, $a'_{N'} = a^*$. Since φ is a proper subsolution of the core satisfying restricted non-emptiness, there exists $a' \in \varphi(\Gamma') \subseteq C(\Gamma')$. Since φ satisfies consistency, it follows that $a'_{N'} = a^* \in \varphi(\Gamma)$, which is a contradiction. ■

Proof of Theorem 6.4. Let \mathcal{K} be a class of e-form games satisfying the ancestor property. From Proposition 4.1 and Lemma 6.3, it suffices to prove that the core defined on \mathcal{K} satisfies restricted non-emptiness, coalitional unanimity, Maskin invariance, and consistency on \mathcal{K} . By Theorem 4.4, it remains to prove that the core satisfies consistency on \mathcal{K} . Let $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ and $\bar{a} \in C(\Gamma)$. Let $S \subsetneq N$ be a proper subcoalition of N and $\Gamma^{S, \bar{a}} = (S, A_S, E^{S, \bar{a}}, \succeq') \in \mathcal{K}$ be the reduced e-form game of Γ with respect to S and \bar{a} . Assume for the sake of contradiction that $\bar{a}_S \notin C(\Gamma^{S, \bar{a}})$. Then, there exists an objection (T, B_S) , with $T \in \mathcal{P}_0(S)$ and $B_S \in E_{\bar{a}_S}^{S, \bar{a}}(T)$, against \bar{a}_S in $\Gamma^{S, \bar{a}}$. It follows that there exists $B \in E_{\bar{a}}(T)$ with $B = B_S \times \{\bar{a}_{N \setminus S}\}$, and for all $b \in B$ and all $i \in T$, $b \succ_i \bar{a}$, contradicting $\bar{a} \in C(\Gamma)$. ■

Proof of Proposition 6.7. To prove that any solution φ satisfying soulmates and Maskin invariance on \mathcal{K}^{SM} is a subsolution of the stable matchings correspondence, it suffices to adapt the proof of Proposition 4.1. The proof that no proper subsolution of the stable matchings correspondence satisfies restricted non-emptiness and consistency on \mathcal{K}^{SM} is a direct consequence of Sasaki and Toda (1992)[Lemma 1, p. 222] and Lemma 6.3. ■

Independence of the axioms

Now, we comment on the logical independence of the axioms in Theorem 4.4. The following two examples respectively show that restricted non-emptiness and coalitional unanimity are independent. First, consider the solution φ_1 on \mathcal{K} such that $\varphi_1(\Gamma) = \emptyset$ for all $\Gamma \in \mathcal{K}$. Then φ_1 violates restricted non-emptiness but vacuously satisfies coalitional unanimity, Maskin invariance, and independence of irrelevant states on \mathcal{K} . Then, take the solution φ_2 on \mathcal{K} such that $\varphi_2(\Gamma) = A$ for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$. Then φ_2 satisfies restricted non-emptiness, Maskin invariance, and independence of irrelevant states but not coalitional unanimity on \mathcal{K} .

Due to the general framework of e-form games, the independence of Maskin invariance and independence of irrelevant states cannot be established for every class \mathcal{K} of e-form games. First, observe that Maskin invariance can sometimes be replaced by independence of irrelevant states in the proof of Proposition 4.1. In this proof, we assume by contradiction that $a \in \varphi(\Gamma)$ and $a \notin C(\Gamma)$, i.e. there exists an objection (S, B) against a in Γ . At this step, we must assume that the e-form subgame $\Gamma' = (N, B \cup \{a\}, E', \succeq')$ of Γ belongs to \mathcal{K} . Then, it follows from independence of irrelevant states that $a \in \varphi(\Gamma')$. However, B is unanimously preferred by S with respect to

\succeq' and $B \in E'_a(S)$. By coalitional unanimity, it holds that $a \in B$, a contradiction. As a consequence, the axioms of restricted non-emptiness, coalitional unanimity and independence of irrelevant states imply Maskin invariance in specific environments. Moreover, when the core contains at most one state (this is the case, for example, with the class \mathcal{K}^{CW}), we can easily be convinced that independence of irrelevant states is implied by the three other axioms.

References

- Abdou, J. and H. Keiding (2003). On necessary and sufficient conditions for solvability of game forms. *Mathematical Social Sciences* 46(3), 243–260.
- Arrow, K. J. (1950). A difficulty in the concept of social welfare. *Journal of Political Economy* 58(4), 328–346.
- Chernoff, H. (1954). Rational selection of decision functions. *Econometrica* 22(4), 422–443.
- Chew, M. S.-Y. (1994). Farsighted coalitional stability. *Journal of Economic Theory* 63(2), 299–325.
- Crès, H., I. Gilboa, and N. Vieille (2011). Aggregation of multiple prior opinions. *Journal of Economic Theory* 146(6), 2563–2582.
- Dutta, B. and S. Mutuswami (1997). Stable networks. *Journal of Economic Theory* 76(2), 322–344.
- Gale, D. and L. S. Shapley (1962). College admissions and the stability of marriage. *The American Mathematical Monthly* 69(1), 9–15.
- Greenberg, J. (1990). *The Theory of Social Situations: An Alternative Game-Theoretic Approach*. Cambridge University Press.
- van Heumen, R., B. Peleg, S. Tijs, and P. Borm (1996). Axiomatic characterizations of solutions for bayesian games. *Theory and Decision* 40(2), 103–129.
- Horan, S., M. J. Osborne, and M. R. Sanver (2018). Positively responsive collective choice rules and majority rule: a generalization of May’s theorem to many alternatives. *Working paper*.
- Jackson, M. O. and A. van den Nouweland (2005). Strongly stable networks. *Games and Economic Behavior* 51(2), 420–444.
- Jackson, M. O. and A. Wolinsky (1996). A strategic model of social and economic networks. *Journal of Economic Theory* 71(1), 44–74.

- Karakaya, M. and B. Klaus (2017). Hedonic coalition formation games with variable populations: core characterizations and (im)possibilities. *International Journal of Game Theory* 46(2), 435–455.
- Klaus, B. (2017). Consistency and its converse for roommate markets. *Games and Economic Behavior* 104, 43–58.
- Kojima, F. and M. Manea (2010). Axioms for deferred acceptance. *Econometrica* 78(2), 633–653.
- Leo, G., J. Lou, M. Van der Linden, Y. Vorobeychik, and M. H. Wooders (2017). Matching soulmates. *Working paper*.
- Maskin, E. (1999). Nash equilibrium and welfare optimality. *The Review of Economic Studies* 66(1), 23–38.
- Moulin, H. and B. Peleg (1982). Cores of effectivity functions and implementation theory. *Journal of Mathematical Economics* 10(1), 115–145.
- Nash, J. F. (1950a). The bargaining problem. *Econometrica* 18(2), 155–162.
- Nash, J. F. (1950b). Equilibrium points in n-person games. *Proceedings of the national academy of sciences* 36(1), 48–49.
- Nizamogullari, D. and İ. Özkal-Sanver (2014). Characterization of the core in full domain marriage problems. *Mathematical Social Sciences* 69, 34–42.
- Norde, H., J. Potters, H. Reijniere, and D. Vermeulen (1996). Equilibrium selection and consistency. *Games and Economic Behavior* 12(2), 219–225.
- Peleg, B., J. Potters, and S. Tijs (1996). Minimality of consistent solutions for strategic games, in particular for potential games. *Economic Theory* 7(1), 81–93.
- Peleg, B. and P. Sudhölter (1997). An Axiomatization of Nash Equilibria in Economic Situations. *Games and Economic Behavior* 18(2), 277–285.
- Peleg, B. and S. Tijs (1996). The consistency principle for games in strategic form. *International Journal of Game Theory* 25(1), 13–34.
- Ray, I. (2000). Consistency, dummy and independence of irrelevant strategies for strategic games. *Social Choice and Welfare* 17(2), 217–222.
- Rosenthal, R. W. (1972). Cooperative Games in Effectiveness Form. *Journal of Economic Theory* 5(1), 88–101.
- Sasaki, H. and M. Toda (1992). Consistency and characterization of the core of two-sided matching problems. *Journal of Economic Theory* 56(1), 218–227.
- Sen, A. (1969). Quasi-transitivity, rational choice and collective decisions. *The Review of Economic Studies* 36(3), 381–393.

- Sen, A. (1993). Internal consistency of choice. *Econometrica* 61(3), 495–521.
- Thomson, W. (1990). The consistency principle. *Game theory and applications* 187.
- Thomson, W. (2011). Consistency and its converse: an introduction. *Review of Economic Design* 15(4), 257–291.
- Thomson, W. (2018). On the terminology of economic design: a critical assessment and some proposals. *Review of Economic Design* 22, 67–99.
- Voorneveld, M. and A. van den Nouweland (1998). A new axiomatization of the core of games with transferable utility. *Economics letters* 60(2), 151–155.