Circumventing the Hart Puzzle

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Circumventing the Hart Puzzle

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Abstract

The paper demonstrates the existence of sequential equilibria in a pure exchange economy, where asymmetrically informed agents exchange consumption goods and securities of all kinds, on incomplete markets. Standard models rely on Radner's (1972, 1979) rational expectation assumptions, along which agents know the maps between the information signals, the states of nature and the equilibrium prices. As shown by Hart (1975), equilibrium may then fail to exist, even when agents have symmetric information and smooth preferences. In that setting, Duffie-Shafer (1985) shows, from differential topology arguments, that interior equilibria exist generically. The current paper proceeds differently. It drops rational expectations to allow for an infinitesimal uncertainty over future spot prices. This device permits to circumvent Hart's 1975 problem, without using differential topology. Then, the paper shows that a generic condition on payoffs and forecasts guarantees the existence of equilibria. It is consistent with non-transitive preferences, non-interior consumptions, asymmetric information and normalized spot prices at equilibrium. It also serves to prove existence in a more general model, which drops Radner's rational expectations.

Key words: sequential equilibrium, perfect foresight, existence of equilibrium, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

This paper proposes a new proof of existence of equilibrium in incomplete financial markets with differential information. It presents a two-period pure exchange economy, with an ex ante uncertainty over the state of nature to be revealed at the second period. Asymmetric information is represented by private finite subsets of states, which each agent is correctly informed to contain the realizable states. Agents' forecast of the true equilibrium price is correct, but not necessarily perfect. That is, the true price may be expected as one element of a set of possibilities. Agents exchange consumption goods on spot markets, and, unrestrictively, assets of any kind on incomplete financial markets. They have an endowment in each state, and preferences over consumptions, possibly non ordered. Generalizing Cass (1984) to asymmetric information, De Boisdeffre (2007) shows the existence of equilibrium on purely financial markets is characterized, in this setting, by the absence of unlimited arbitrage opportunity. That no-arbitrage condition can be achieved with no price model, along De Boisdeffre (2016), or Cornet-De Boisdeffre (2009), from simply observing asset prices or available transfers on financial markets.

When assets pay off in goods, equilibrium needs not exist in the standard model, as shown by Hart (1975) under symmetric information. His example is based on the collapse of the span of assets' payoffs, that occurs at clearing prices. In our model, an additional problem may arise from differential information. Financial markets may be arbitrage-free for some spot prices, and not for others, in which case equilibrium cannot exist. We solve these two problems jointly, owing to a good property of financial and information structures.

Attempts to resurrect the existence of equilibrium with real assets noticed that the above "*bad*" prices could only occur exceptionally. These attempts include Mc Manus (1984), Repullo (1984), Magill & Shafer (1984, 1985), for potentially complete markets (i.e., complete for at least one price), and Duffie-Shafer (1985, 1986), for incomplete markets. These papers apply to symmetric information, build on differential topology arguments, and demonstrate the generic existence of equilibrium, namely, existence except for a closed set of measure zero of economies, parametrized by the assets' payoffs and agents' endowments. They rely on Radner's (1972) classical assumption that agents have a perfect price foresight, that is, they know the map between the conditional spot price and the state of nature to prevail.

Another way to circumvent Hart's problem is to allow agents for (infinitesimal) uncertainty over the true price to expect. Generically in assets' payoffs and agents' anticipations, this "*tremble*" restores the existence of equilibria on any security market. And it is no pure artifice. After all, the perfect foresight model is an elegant construction, which, stating Radner (1982) himself, "*seems to require of the traders a capacity for imagination and computation far beyond what is realistic*". In the real world, agents are, to some extent, uncertain of future prices, because they lack the required "*structural knowledge*" of how equilibrium prices are determined, in the words of Kurz (1994). The issue of realistic anticipations, consistent with sequential equilibrium, is dealt with in a companion paper [6]. The current paper's results also serve to prove existence in that more general model [6], which discards rational expectations. Hereafter, we simply assume that agents may expect other spot prices than the equilibrium's with positive (arbitrarily small) probabilities to occur.

The current paper's existence theorem applies to both potentially complete and incomplete markets, to ordered and non transitive preferences. It differs from earlier papers dealing with real assets, such as Duffie-Shafer's (1985), in several other ways.

First, it allows for asymmetric information amongst consumers. Second, financial structures cover any mix of nominal and real assets. Third, it permits to normalize (to arbitrary values) the equilibrium price on every spot market. Finally, but not least, it allows for border consumptions at equilibrium.

In Duffie-Shafer (1985), only the value of one particular consumer's endowment is normalized to one across all states of nature. The relevance and the means of inferring equilibrium prices are no issues, because the perfect foresight assumption, along Radner (1972), is not debated. In the current paper, however, normalizing price anticipations in every state of nature to relevant values is an important step towards forming correct expectations, the topic of the above companion paper [6]. In the latter, agents' beliefs, characteristics and decisions are all private. This privacy typically results in a set of possible equilibrium prices, between which no rational agent is able to choose. And when agents cannot forecast prices with certainty, they need focus on relevant expectations of the possible equilibrium prices to prevail, so that one expectation be self-fulfilling, ex post. The current paper permits this.

Equally important is the issue of border consumptions. So as to apply modulo 2 degree theory - a compulsory step when dealing with real assets, earlier proofs make smoothness assumptions, which result in interior equilibrium consumptions. Yet, given tastes and budget constraints, the assumption that every agent consumes all goods at equilibrium is unrealistic. The current paper overcomes this problem.

As for the technique of proof, the paper drops the classical modulo 2 degree argument of differential topolgy. It even drops the intermediary concept of pseudo-equilibrium. With no fall in rank puzzle, it simply derives the existence of equilibria from the Gale-Mas-Colell (1975-79) fixed-point-like theorem. So that the paper be self-contained, the proof resumes some techniques of De Boisdeffre (2007).

The paper is organized as follows: Section 2 presents the model and equilibrium. Section 3 states and proves the existence Theorem. An Appendix proves Lemmas.

2 The model

Throughout the paper, we consider a pure-exchange economy with two periods, $t \in \{0, 1\}$, and an uncertainty, at $t = 0$, upon which state of nature will randomly prevail, at $t = 1$. Consumers exchange goods, on spots markets, and assets of all kinds, on incomplete financial markets. The sets, I, S, L and J, respectively, of consumers, states of nature, consumption goods and assets are all finite. The state of nature of the first period $(t = 0)$ is denoted by $s = 0$ and we let $\Sigma' := \{0\} \cup \Sigma$, for every subset, Σ, of S. Similarly, $l = 0$ denotes the unit of account and $L' := \{0\} \cup L$.

2.1 Markets and information

Agents consume or exchange the consumption goods, $l \in L$, on both periods' spot markets. At $t = 0$, each agent, $i \in I$, receives privately some correct information signal, $S_i \subset S$ (henceforth given), that tomorrow's true state will be in S_i. We assume costlessly that $S = \cup_{i \in I} S_i$. Thus, the pooled information set, $\underline{\mathbf{S}} := \cap_{i \in I} S_i$, contains the true state, and the relation $\underline{\mathbf{S}} = S$ characterizes symmetric information.

Spot prices are restricted to the unit ball, $\Delta := \{p \in \mathbb{R}^L : \|p\| \leqslant 1\}$. The bound one is chosen for convenience and stands for any positive value on any spot market. Consequently, we restrict w.l.o.g. commodity prices to the set $P := \Delta^{\underline{\mathbf{S}}'}$. At $t = 0$, in addition to the equilibrium price, $p \in P$, which she is assumed to correctly anticipate, each agent, $i \in I$, has an exogenous set, Ω_i, of private forecasts. Such forecasts, $\omega := (s, p_s) \in \Omega_i$, are pairs of a state, $s \in S_i$, and a price, $p_s \in \Delta \cap \mathbb{R}^L_{++}$, that the i^{th}

agent believes to be a possible outcome in the state-s spot market. We let $\Omega := S \times \Delta$ be the forecast set and assume, hereafter, that $\cup_{i \in I} \Omega_i \subset \Omega$ is finite.

Exogenous forecasts may either be wrong or self-fufilling ex post, that is, coincide with equilibrium forecasts. Thus, in the classical equilibrium with symmetric information, agents form exactly one forcast, $\omega := (s, p_s)$, in every state, $s \in \underline{\mathbf{S}} = S$, which is self-fulfilling, ex post. Whether the generic existence of the classical equilibrium may be derived from the current paper's main existence theorem is not our focus. This conjecture, from Remark 1 below, is left for subsequent research.

Agents may operate transfers across states in S' by exchanging, at $t = 0$, finitely many assets, $j \in J$ (with $\#J \leqslant \#\underline{\mathbf{S}}$), which pay off, at $t = 1$, conditionally on the realization of forecasts. These conditional payoffs may be nominal or real or a mix of both. The generic payoffs of an asset, $j \in J$, in a state, $s \in S$, are a bundle, $v_j(s) := (v_j^l(s)) \in \mathbb{R}^{L'}$, of the quantities, $v_j^0(s)$, of cash, and $v_j^l(s)$, of each good $l \in L$, which are delivered if state s prevails. All payoffs define a $(S \times L') \times J$ payoff matrix, V, which is identified (with same notation) to a continuous map, $V : \Omega \to \mathbb{R}^J$, relating the forecasts, $\omega := (s, p) \in \Omega$, to the rows, $V(\omega) \in \mathbb{R}^J$, of all assets' payoffs in cash, delivered if state s and price p obtain. At asset price, $q \in \mathbb{R}^J$, agents may thus buy or sell unrestrictively portfolios of assets, $z = (z_j) \in \mathbb{R}^J$, for $q \cdot z$ units of account at $t = 0$, against the promise of delivery of a flow, $V(\omega) \cdot z$, of payoffs across forecasts, $\omega \in \Omega$. Similarly, we restrict w.l.o.g. the asset prices to the set $Q := \{q \in \mathbb{R}^J : \|q\| \leqslant 1\}$.

2.2 The consumer's behaviour and concept of equilibrium

Each agent, $i \in I$, receives an endowment, $e_i := (e_{is})$, granting the commodity bundles, $e_{i0} \in \mathbb{R}^L_+$ at $t = 0$, and $e_{is} \in \mathbb{R}^L_+$, in each expected state, $s \in S_i$, if it prevails. Given the market prices, $p := (p_s) \in P$ and $q \in Q$, and her forecast set, Ω_i, the generic

i^{th} agent's consumption set is $X_i := \mathbb{R}_+^{L\times\underline{\mathbf{S}}'} \times \mathbb{R}_+^{\Omega_i}$ and her budget set is:

$$B_i(p,q) := \{\ (x,z) \in X_i \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z \ \ and \ \ p_s \cdot (x_s - e_{is}) \leqslant V(s,p_s) \cdot z,\ \forall s \in \underline{\mathbf{S}}$$
$$and \ \ \overline{p}_s \cdot (x_\omega - e_{is}) \leqslant V(\omega) \cdot z,\ \forall \omega = (s, \overline{p}_s) \in \Omega_i\ \}.$$

Each consumer, $i \in I$, is endowed with a complete preordering, $\precsim_i$, over her consumption set, representing her preferences. Her strict preferences, $\prec_i$, are represented, for each $x \in X_i$, by the set, $P_i(x) := \{\ y \in X_i : x \prec_i y\ \}$, of consumptions which she strictly prefers to x. The above economy is denoted by $\mathcal{E} = \{(I,S,L,J), V, (S_i)_{i\in I}, (\Omega_i)_{i\in I}, (e_i)_{i\in I}, (\precsim_i)_{i\in I}\}$. Each agent optimizes her consumptions in the budget set. This yields the following concept of equilibrium:

Definition 1 *A collection of prices, $p = (p_s) \in P$, $q \in Q$, and decisions, $(x_i, z_i) \in B_i(p,q)$, for each $i \in I$, is an equilibrium of the economy, $\mathcal{E}$, if the following conditions hold:*

(a) $\forall i \in I,\ (x_i, z_i) \in B_i(p,q)$ *and* $P_i(x_i) \times \mathbb{R}^J \cap B_i(p,q) = \varnothing$*;*

(b) $\sum_{i\in I} (x_{is} - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}}'$*;*

(c) $\sum_{i\in I} z_i = 0$.

The economy, $\mathcal{E}$, is called standard if it meets the following conditions:

Assumption A1 *(monotonicity):* $\forall (i,x,y) \in I \times (X_i)^2,\ (x \leqslant y,\ x \neq y) \Rightarrow (x \prec_i y)$;

Assumption A2 *(strong survival):* $\forall i \in I,\ e_i \in \mathbb{R}_{++}^{L\times S_i'}$;

Assumption A3*:* $\forall i \in I$, $\prec_i$ *is lower semicontinuous convex-open-valued and such that* $x \prec_i x + \lambda(y - x)$*, whenever* $(x, y, \lambda) \in X_i \times P_i(x) \times\ ]0,1]$;

Assumption A4 *the system* $\{V(\omega)\}_{\omega \in \cap_{i\in I}\Omega_i}$ *contains* $\#J$ *independent vectors.*

Remark 1 When assets are nominal, Assumption *A4* is not required because no fall in rank occurs. With arbitrary assets, the condition of Assumption *A4* holds generically in forecasts and payoffs. This result is standard from Sard's theorem (see Milnor, 1997, p. 10), and proved in Duffie-Shafer (1985, p. 297-(d)).

3 The existence Theorem and proof

Theorem 1 *A standard economy, $\mathcal{E}$, admits an equilibrium.*

The proof's main argument is the Gale-Mas-Colell (1975, 1979) fixed-point-like theorem. To apply the theorem, we need define, for every agent, $i \in I$, and for markets, lower semi-continuous reaction correspondences over a convex compact set, hence, in a compact economy. The proof proceeds in three steps as follows.

Sub-Section 3.1 presents the auxiliary compact economy, where agents' budget correspondences are slightly modified (to the below B_i', replacing B_i, for each $i \in I$).

Sub-Section 3.2 shows the interiors of the modified budget sets are never empty. This permits to define lower semi-continuous reaction correspondences for every agent in the auxiliary economy. Applying the Gale-Mas-Colell theorem to them yields a so-called (with slight abuse) "*fixed point*" of prices and budgeted decisions.

Sub-Section 3.3 shows the above fixed point is an equilibrium of the economy, $\mathcal{E}$.

3.1 An auxiliary compact economy with modified budget sets

For every $i \in I$, $p = (p_s) \in P$ and $q \in Q$, we consider the augmented budget sets:

$$\begin{aligned}\overline{B}_i(p,q) := \{\ & (x,z) \in X_i \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leqslant 1 - q \cdot z \\ & \textit{and}\ \ p_s \cdot (x_s - e_{is}) \leqslant 1 + V(s,p_s) \cdot z,\ \forall s \in \underline{\mathbf{S}} \\ & \textit{and}\ \ \overline{p}_s \cdot (x_\omega - e_{is}) \leqslant V(\omega) \cdot z,\ \forall \omega = (s, \overline{p}_s) \in \Omega_i\ \};\end{aligned}$$

$$\overline{\mathcal{A}}(p,q) := \{\ [(x_i,z_i)] \in \times_{i \in I}\ \overline{B}_i(p,q) :\ \textstyle\sum_{i \in I}(x_{is} - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}}',\ \ \textit{and}\ \ \sum_{i \in I} z_i = 0\ \}.$$

These sets meet a boundary condition as follows:

Lemma 1 $\exists r > 0 : \forall (p,q) \in P \times Q,\ \forall\ [(x_i,z_i)] \in \overline{\mathcal{A}}(p,q),\ \sum_{i \in I}\ (\|x_i\| + \|z_i\|) < r$

Proof : see the Appendix. □

Remark 2 Lemma 1 is inconsistent with a fall in rank problem a la Hart, which would imply that portfolios from sets $\overline{\mathcal{A}}(p,q)$ (for $(p,q) \in P \times Q$) failed to be bounded.

Lemma 1 permits to reach a compact economy. Thus, we define from Lemma 1, for every $i \in I$ and every $(p,q) \in P \times Q$, the following convex compact sets,

$\overline{X}_i := \{x \in X_i : \|x\| \leqslant r\}$, and $Z := \{z \in \mathbb{R}^J : \|z\| \leqslant r\}$, and

$$\begin{aligned} B_i'(p,q) := \{ \ & (x,z) \in \overline{X}_i \times Z : p_0 \cdot (x_0 - e_{i0}) \leqslant \gamma_{(p_0,q)} - q \cdot z \\ & \textit{and} \ \ p_s \cdot (x_s - e_{is}) \leqslant \gamma_{(s,p_s)} + V(s,p_s) \cdot z, \ \forall s \in \underline{\mathbf{S}} \\ & \textit{and} \ \ \overline{p}_s \cdot (x_\omega - e_{is}) \leqslant V(\omega) \cdot z, \ \forall \omega = (s, \overline{p}_s) \in \Omega_i \ \}. \end{aligned}$$

where $\gamma_{(p_0,q)} := 1 - \min(1, \|(p_0,q)\|)$, $\gamma_{(s,p_s)} := 1 - \|p_s\|$, $\forall s \in \underline{\mathbf{S}}$, so that $B_i'(p,q) \subset \overline{B}_i(p,q)$.

The auxiliary economy that we henceforth work with is in anything alike that of Section 2, but agents' portfolio set, consumption sets and budget correspondences, which are now replaced, respectively, by Z, $\overline{X}_i$, B_i', for every $i \in I$, as defined above. Agents' behaviours are replaced by reaction correspondences, presented hereafter. Their budget correspondences meet the following property:

Claim 1 *For every $i \in I$, B_i' is upper semicontinuous.*

Proof Let $i \in I$ be given. The correspondence B_i' is, as standard, upper semicontinuous, for having a closed graph in a compact set. □

3.2 Applying a fixed-point-like argument

Agents' budget sets were modified in the compact economy of sub-Section 3.1, so that their interiors be non-empty. This latter property, proven below, is crucial to demonstrate the lower semi-continuity of the agents' reaction correspondences, serving to apply the Gale-Mas-Colell theorem, along Lemma 2, below. For every $i \in I$ and every $(p,q) \in P{\times}Q$, the interior of $B'_i(p,q)$ is the following set:

$$B''_i(p,q) := \{\ (x,z) \in \overline{X}_i{\times}Z : p_0{\cdot}(x_0 - e_{i0}) < \gamma_{(p_0,q)} - q{\cdot}z$$
$$and\ \ p_s{\cdot}(x_s - e_{is}) < \gamma_{(s,p_s)} + V(s,p_s){\cdot}z,\ \forall s \in \underline{\mathbf{S}}$$
$$and\ \ \overline{p}_s{\cdot}(x_\omega - e_{is}) < V(\omega){\cdot}z,\ \forall \omega = (s,\overline{p}_s) \in \Omega_i\ \}.$$

Claim 2 *The following Assertions hold, for each $i \in I$:*

(i) $\forall (p,q) \in P \times Q,\ B''_i(p,q) \neq \varnothing$*;*

(ii) the correspondence B''_i is lower semicontinuous.

Proof Let $i \in I$ and $(p,q) \in P{\times}Q$ be given.

Assertion *(i)* From Assumption *A2* and the definition of Ω_i, we choose $x \in \overline{X}_i$, such that $(x,0)$ meets all budget constraints with a strict inequality in all state $s \in S'_i$, such that $p_s \neq 0$. Then, if $p_0 \neq 0$, or $(p_0,q) = 0$, the relation $(x,0) \in B''_i(p,q)$ holds. If $p_0 = 0$ and $q \neq 0$, the relation $(x, -q/N) \in B''_i(p,q)$ holds for $N \in \mathbb{N}$ big enough. □

Assertion *(ii)* The convexity of $B''_i(p,q)$ holds and implies, from Assertion *(i)*, $B'_i(p,q) = \overline{B''_i(p,q)}$. From the continuity of the scalar product, the correspondence B''_i is lower semicontinuous at (p,q) for having an open graph in a compact set. □

We introduce an additional fictitious agent for markets and a reaction correspondence for each agent, defined on the convex compact set, $\Theta := P \times Q \times (\times_{i \in I} \overline{X}_i \times Z)$. Thus, we let, for each $i \in I$ and every $\theta := (p,q,(x,z) := [(x_i,z_i)]) \in \Theta$:

$$\Psi_0(\theta) := \{(p',q') \in P \times Q : \textstyle\sum_{s \in \underline{\mathbf{S}}'} (p'_s - p_s){\cdot} \sum_{i \in I}(x_{is} - e_{is}) + (q' - q){\cdot} \sum_{i \in I} z_i > 0\};$$

$$\Psi_i(\theta) := \left\{ \begin{array}{lll} B'_i(p,q) & if & (x_i,z_i) \notin B'_i(p,q) \\ B''_i(p,q) \cap P_i(x_i) \times Z & if & (x_i,z_i) \in B'_i(p,q) \end{array} \right\};$$

Lemma 2 *For each* $i \in I \cup \{0\}$, Ψ_i *is lower semicontinuous.*

Proof See the Appendix. $\square$

We can now apply a fixed-point argument to the above reaction correspondences:

Claim 3 *There exists* $\theta^* := (p^*, q^*, [(x_i^*, z_i^*)]) \in \Theta$, *such that*:

(i) $\forall (p,q) \in P \times Q,\ \sum_{s \in \underline{\mathbf{S}}'} [(p_s^* - p_s) \cdot \sum_{i \in I} (x_{is}^* - e_{is})] + (q^* - q) \cdot \sum_{i \in I} z_i^* \geqslant 0$;

(ii) $\forall i \in I,\ (x_i^*, z_i^*) \in B_i'(p^*, q^*)$ *and* $B_i''(p^*, q^*) \cap P_i(x_i^*) \times Z = \varnothing$.

Proof Quoting Gale-Mas-Colell (1975, 1979): "*Given* $X = \times_{i=1}^m Xi$, *where* X_i *is a non-empty compact convex subset of* $\mathbb{R}^n$, *let* $\varphi_i : X \to X_i$ *be* m *convex (possibly empty) valued correspondences, which are lower semicontinuous. Then, there exists* x *in* X *such that for each* i *either* $x_i \in \varphi_i(x)$ *or* $\varphi_i(x) = \varnothing$". The correspondences Ψ_i, for each $i \in I \cup \{0\}$, meet all conditions of the above theorem and yield Claim 3. $\square$

3.3 An equilibrium of the economy $\mathcal{E}$

The above fixed point, θ^*, meets the following properties, proving Theorem 1:

Claim 4 *Let* $\theta^* := (p^*, q^*, [(x_i^*, z_i^*)]) \in \Theta$, *along Claim 3, be given.*

The following Assertions hold:

(i) $[(x_i^*, z_i^*)] \in \overline{\mathcal{A}}(p^*, q^*)$, *hence,* $\sum_{i \in I} (\|x_i^*\| + \|z_i^*\|) < r$, *from Lemma 1;*

(ii) for every $i \in I$, $(x_i^*, z_i^*) \in B_i'(p^*, q^*)$ *and* $B_i'(p^*, q^*) \cap P_i(x_i^*) \times Z = \varnothing$;

(iii) θ^* *is an equilibrium of the economy* $\mathcal{E}$, *such that* $1 \leqslant \|p_0^*\| + \|q^*\| \leqslant 2$, $p_s^* \in \mathbb{R}^L_{++}$, *for every* $s \in \underline{\mathbf{S}}'$, *and* $\|p_s^*\| = 1$, *for every* $s \in \underline{\mathbf{S}}$.

Proof Assertion (i) Assume, by contraposition, that $\sum_{i \in I} (x_{i0}^* - e_{i0}) \neq 0$ or $\sum_{i=1}^m z_i^* \neq 0$. Then, from Claim 3-$(i)$, the relations $p_0^* \cdot \sum_{i \in I} (x_{i0}^* - e_{i0}) + q^* \cdot \sum_{i=1}^m z_i^* > 0$ and $\gamma_{(p_0^*, q^*)} = 0$ hold. From Claim 3-(ii), the relations $(x_i^*, z_i^*) \in B_i'(p^*, q^*)$ hold, for each

$i \in I$, whose budget constraints at $t = 0$ are written: $p_0^* \cdot (x_{i0}^* - e_{i0}) + q^* \cdot z_i^* \leqslant 0$. Adding them up (for $i \in I$) yields, from above, $0 < p_0^* \cdot \sum_{i \in I} (x_{i0}^* - e_{i0}) + q^* \cdot \sum_{i=1}^m z_i^* \leqslant 0$. This contradiction shows that $\sum_{i \in I} (x_{i0}^* - e_{i0}) = 0$ and $\sum_{i=1}^m z_i^* = 0$.

Assume, now, by contraposition, that $\sum_{i \in I} (x_{is}^* - e_{is}) \neq 0$, for some $s \in \underline{\mathbf{S}}$. Then, from Claim 3-($i$), the relations $p_s^* = \sum_{i \in I} (x_{is}^* - e_{is}) \ / \ \| \sum_{i \in I} (x_{is}^* - e_{is}) \|$ and $\gamma_{(s,p_s)} = 0$ hold. By the same token, summing up budget constraints in state s yields, from Claim 3-(ii) and above: $0 < p_s^* \cdot \sum_{i \in I} (x_{is}^* \text{-} e_{is}) \leqslant V(s, p_s^*) \cdot \sum_{i \in I} \ z_i^* = 0$.

This contradiction proves that $\sum_{i \in I} (x_{is}^* - e_{is}) = 0$ for each $s \in \underline{\mathbf{S}}$ and, from Claim 3 and above, that $[(x_i^*, z_i^*)] \in \overline{\mathcal{A}}(\varpi^*)$. Then, from Lemma 1, $\sum_{i \in I} (\|x_i^*\| + \|z_i^*\|) < r$. □

Assertion (ii) Let $i \in I$ be given. From Claim 3-(ii), we need only show that $B_i'(p^*, q^*) \cap P_i(x_i^*) \times Z = \varnothing$. By contraposition, let $(x_i, z_i) \in B_i'(p^*, q^*) \cap P_i(x_i^*) \times Z$ be given. From Claim 2, there exists $(x_i', z_i') \in B_i''(p^*, q^*) \subset B_i'(p^*, q^*)$. By construction, the relation $(x_i^n, z_i^n) := [\frac{1}{n}(x_i', z_i') + (1 - \frac{1}{n})(x_i, z_i)] \in B_i''(p^*, q^*)$ holds, for every $n \in \mathbb{N}$. From Assumption *A3*, the relation $(x_i^N, z_i^N) \in P_i(x_i^*) \times Z$ also holds, for $N \in \mathbb{N}$ big enough, which implies: $(x_i^N, z_i^N) \in B_i''(p^*, q^*) \cap P_i(x_i^*) \times Z$. The latter contradicts Claim 3-(ii). □

Assertion (iii) The relation $p_s^* \in \mathbb{R}_{++}^L$, for every $s \in \underline{\mathbf{S}}'$, is standard from Assertions (i)-(ii) and Assumption *A1*. From Assertions (i)-(ii), we need only show that $\gamma_{(p_0^*, q^*)} = 0$ and $\gamma_{(s, p_s^*)} = 0$, for every $s \in \underline{\mathbf{S}}$. We show the latter equality for $s = 0$ only (the proof is similar for $s \in \underline{\mathbf{S}}$). Let $i \in I$ be given. Claim 3-(ii) yields: $p_0^* \cdot (x_{i0}^* - e_{i0}) \leqslant q^* \cdot z_i^* + \gamma_{(p_0^*, q^*)}$.

The latter relation holds with equality, from Assertion (i)-(ii) and Assumption *A1*, that is: $p_0^* \cdot (x_{i0}^* \text{-} e_{i0}) - q^* \cdot z_i^* = \gamma_{(p_0^*, q^*)}$. Summing up these relations for $i \in I$ yields, from Assertion (i): $0 = p_0^* \cdot \sum_{i \in I} (x_{i0}^* \text{-} e_{i0}) - q \cdot \sum_{i \in I} z_i^* = \# I \ \gamma_{(p_0^*, q^*)}$. Assertion ($iii$) follows. □

Appendix

Lemma 1 $\exists r > 0 : \forall (p,q) \in P \times Q,\ \forall\ [(x_i, z_i)] \in \overline{\mathcal{A}}(p,q),\ \sum_{i \in I} (\|x_i\| + \|z_i\|) < r$

Proof Let $(p,q) \in P \times Q$, and $[(x_i, z_i)] \in \overline{\mathcal{A}}(p,q)$ be given.

- As standard, the relations, $x_{is} \in [0,e]^L$, where $e := \sum_{i \in I} \|e_i\|$, hold, for every $(i,s) \in I \times \underline{\mathbf{S}}'$, from the non-negativity and market clearing conditions on $\overline{\mathcal{A}}(p,q)$.
- From above and the positive price expectations on all forecasts, $\omega \in \cup_{i \in I} \Omega_i \subset S \times \mathbb{R}^L_{++}$, Lemma 1 will be proved if the portfolios, (z_i), are bounded uniformly for $(p,q) \in P \times Q$ and $[(x_i, z_i)] \in \overline{\mathcal{A}}(p,q)$. We now prove this latter property.
- Let $\delta = 1 + \sum_{i \in I} \|e_i\|$. Assume, by contraposition, that, for every $k \in \mathbb{N}$, there exist $(p^k, q^k) \in P \times Q$ and $[(x_i^k, z_i^k)] \in \overline{\mathcal{A}}(p^k, q^k)$, such that $\|z^k\| := \sum_{i \in I}\ \|z_i^k\| > k$. For every $k \in \mathbb{N}$, we let $(z_i'^k) := (\frac{z_i^k}{\|z^k\|}) \in \mathbb{R}^{J \times I}$, whose sequence admits a cluster point, $(z_i) \in \mathbb{R}^{J \times I}$, such that $\|(z_i)\| = 1$, while the relations $[(x_i^k, z_i^k)] \in \overline{\mathcal{A}}(p^k, q^k)$ imply:

  $\sum_{i \in I}\ z_i'^k = 0\ ,\ V(\omega) \cdot z_i'^k \geqslant -\delta/k, \forall (i, \omega, k) \in I \times \cap_{i \in I} \Omega_i \times \mathbb{N}$, and, passing to the limit,
  $\sum_{i \in I}\ z_i = 0,\ V(\omega) \cdot z_i \geqslant 0, \forall (i, \omega) \in I \times \cap_{i \in I} \Omega_i$,

The latter relations imply $V(\omega) \cdot z_i = 0$, for every $(i, \omega) \in I \times \cap_{i \in I} \Omega_i$ and, from Assumption *A4*, $(z_i) = 0$. This contradicts the above relation $\|(z_i)\| = 1$. This contradiction proves that portfolios from the sets $\overline{\mathcal{A}}(p,q)$ are bounded uniformly in $(p,q) \in P \times Q$. From above, this suffices to prove Lemma 1. □

Lemma 2 *For each* $i \in I \cup \{0\}$*,* Ψ_i *is lower semicontinuous.*

Proof The correspondences Ψ_0 is lower semicontinuous for having an open graph.

- Assume that $(x_i, z_i) \notin B_i'(p,q)$. Then, $\Psi_i(\theta) = B_i'(p,q)$.

Let V be an open set in $\overline{X}_i \times Z$, such that $V \cap B_i'(p,q) \neq \varnothing$. It follows from the convexity of $B_i'(p,q)$ and the non-emptyness of the open set $B_i''(p,q)$ that $V \cap B_i''(p,q) \neq \varnothing$. From Claim 2, there exists a neighborhood U of (p,q), such that $V \cap B_i'(p',q') \supset V \cap B_i''(p',q') \neq \varnothing$, for every $(p',q') \in U$.

Since $B_i'(p,q)$ is nonempty, closed, convex in the compact set $\overline{X}_i \times Z$, there exist two open sets V_1 and V_2 in $\overline{X}_i \times Z$, such that $(x_i, z_i) \in V_1$, $B_i'(p,q) \subset V_2$ and $V_1 \cap V_2 = \varnothing$. From Claim 1, there exists a neighborhood $U_1 \subset U$ of (p,q), such that $B_i'(p',q') \subset V_2$, for every $(p',q') \in U_1$. Let $W = U_1 \times (\times_{j \in I} W_j)$, where $W_i := V_1$ and $W_j := \overline{X}_j \times Z$, for every $j \in I \backslash \{i\}$. Then, W is a neighborhood of θ, such that $\Psi_i(\theta') = B_i'(p',q')$, and $V \cap \Psi_i(\theta') \neq \varnothing$, for all $\theta' := (p',q',(x',z')) \in W$. Thus, Ψ_i is lower semicontinuous at θ.□

- Assume that $(x_i, z_i) \in B_i'(p,q)$, i.e., $\Psi_i(\theta) = B_i''(p,q) \cap P_i(x) \times Z$.

Lower semicontinuity is immediate if $\Psi_i(\theta) = \varnothing$. Assume $\Psi_i(\theta) \neq \varnothing$. We recall that P_i (from Assumption *A3*) is lower semicontinuous with open values and that B_i'' has an open graph. As corollary and from Lemma 1-(i), the correspondence $(p',q',(x',z')) \in \Theta \mapsto B_i''(p',q') \cap P_i(x_i') \times Z \subset B_i'(p',q')$ is lower semicontinuous at θ. Then, from the latter inclusions, Ψ_i is lower semicontinuous at θ. □

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