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Working on and with division in early China, 3rd century BCE—7th century CE

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The main text refers to line numbers used in the appendix. However, in the main text I put “line” in bold fonts and add *, since the lines might have changed and the references made might become inaccurate. This will be revised after the English polishing.**

1. Introduction

Many hints indicate that division was perceived as both a complex but also a central operation in ancient China and that it has been the object of a mathematical work. These hints testify to a theoretical work with respect to this operation.² They also attest to a work that changed the execution and the mathematical practice of division as well as those of related operations.

We can get a first clue of the difficulty division represented in practitioners' view through the volume devoted to problems linked to the execution of divisions in the earliest extant mathematical writings. Let us take as an example the first mathematical manuscript from early imperial China that archeologists discovered a few decades ago in a tomb sealed at Zhangjiashan (Hubei) in 186 BCE or shortly thereafter, *Writings on mathematical procedures*.³ A rough assessment based on my understanding of division shows that at least 47 slips out of the 190 slips that compose the manuscript, that is to say one fourth of the whole document, are devoted to questions related to the execution of division.⁴ On the

¹ The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement n. 269804 “Mathematical Sciences in the Ancient World (SAW)”. This chapter has been presented, first partly, in the context of the 2013 workshop of the project SAW, on (<http://sawerc.hypotheses.org/workshops/workshop-cultures-of-computation>), and then in draft form during the conference that concluded the workshop, March 2013 (<http://sawerc.hypotheses.org/conferences/conference-cultures-of-computation-and-quantification-in-the-ancient-world> and <https://f-origin.hypotheses.org/wp-content/blogs.dir/946/files/2013/03/Conference-Cultures-of-Computation-abstracts-measures.pdf>). Thanks are due to all participants, and in particular Agathe Keller and Christine Proust.

² I have discussed the use of clues in the conceptual history of mathematics in the ancient world, as well as part of this theoretical work in (Chemla 2014). This chapter returns to this issue from a different perspective.

³ On this manuscript, see (PENG Hao 彭浩 2001), which provides an annotated critical edition, on which, unless otherwise specified, I rely in this chapter.

⁴ In early China, manuscripts were for the most part written on mats of bamboo slips, tied to each other by strings, or on wooden board. More rarely, silk served as the medium of writing. *Writings on mathematical procedures* was written on bamboo slips. I refer to slips in this manuscript according to the numbers Peng Hao (2001) associates to them. The slips that

other hand, the manuscript is composed of 69 sections, two of which are too damaged to be analyzed (and also slips that are too incomplete to be possibly attached to any section, see Morgan and Chemla 2018 (2017): 185-187). Among the sections, there are only 9 sections out of the 67 readable ones, that is, around 13% of them (and also 4 slips out of the 6 damaged ones), in which apparently, division is not in play.⁵ Division is thus as omnipresent as it is difficult. The second known mathematical manuscript, entitled *Mathematics*, which was recently published, does not contradict these claims.⁶

As a result of the work on division that seems to have been carried out between the second century BCE and the first century CE, algorithms of division and related operations appear to have undergone mutations of different types. This chapter addresses features of these mutations that can be perceived, in both theoretical dimensions of division and its actual mathematical practice.⁷ To deal with this issue, I rely on a corpus of texts produced between the end of the 3rd century BCE and the 7th century CE, which includes the earliest extant Chinese mathematical documents available to us and some later texts.

Although my focus is on division and the operations that have been related to it in different contexts, the chapter aims at making a more general point. Its purpose is to show that, far from being only tools, operations have also been objects of inquiry. This holds true for the ancient world as well as for more recent time periods. I will limit myself here to early China. In this case, historians have to capture actors' explorations on operations by identifying and interpreting clues left in the texts. However, as I attempt to show, the clues are numerous and they make meaningful clusters, which allow us to perceive important shifts.

are in my view related to the execution of division include the following ones: 14-15, 23-24, 26-27, 46, 68-69, 74-75, 76-77, 83-84, 86-87, 93-95, 96-97, 159, 160-163, 164-181, 183-184. Some of these slips are translated in the appendix, in which I have gathered the main sources on which I rely in this chapter.

⁵ The percentage is roughly the same if one computes in terms of number of slips. The sections in which division does not feature are: Sections 1, "Multiplying with each other 相乘" (slips 1 to 6); 3, "Multiplying 乘" (slips 8-12); 4, "Increasing or decreasing parts 增減分" (slip 13); 5, "When parts should be halved 分當半者" (slips 14-15); 6, "If parts are halved 分半者" (slip 16); 36, "Regulations for cereals plants 乘禾" (slips 88-90); 41, "Wastage 耗(耗)" (slips 105-108); 42: "Unhusked grain makes husked 粟為米" (slips 109-110); 69, "Making a field into *li* 里田" (slips 188-190). Moreover, Slips 120, 121, 125, and 158, as well as Sections 51 ("Travelling 行" slip 132), and 63 ("A circular timber 【環】(圓)材", slips 156-157) do not have enough readable characters to tell. However, despite their incompleteness, Slips 123 and 124 clearly rest on division. One might consider that since in this context, fractions derive from division, and are handled using division, some of these sections, which deal with fractions ("parts"), should be taken out of this list, which would diminish even further the percentage of the manuscript in which division is not in play.

⁶ XIAO Can (2010) has devoted her Ph. D. dissertation to an annotated edition of this manuscript. It was published recently (XIAO Can 2015). ZHU Hanmin 朱漢民 and CHEN Songchang 陳松長 zhubian 主編 (gen. eds.) 2011) provide the *editio princeps* of the text. For further detail on this manuscript, see below.

⁷ I will mainly dwell on the theoretical dimensions that were not already treated in (Chemla 2014).

2. The documents

My corpus contains two quite distinct types of documents (which I will label, respectively, with the letters A and B). Let me say a few words about each of them, before I turn to division.

As I mentioned previously, I will use Chinese manuscripts (**Items A**) that were recently found in the context of archeological or illegal excavations. The editors of these texts have suggested, for each of these manuscripts, that they had been produced in the 3rd or the 2nd century BCE. These manuscripts include two book-length documents, whose text is now published:

— **Item A1:** *Mathematics* 數 (*Shu*), a manuscript bought in December 2007 on the Hong Kong antiquities market, and whose title was given by the actors themselves. In its editors' view, *Mathematics* has been composed in the time period of Qin kingdom or the Qin dynasty (221 BCE—206 BCE) and probably no later than 212 BCE.⁸

— **Item A2:** *Writings on mathematical procedures* 算數書 (*Suanshushu*), a document found in tomb no. 247 at Zhangjiashan (Hubei). Features of other documents found in the tomb suggest that the tomb was sealed in 186 BCE or slightly later, which gives a *terminus ante quem* for the production of the manuscript. Administrative regulations were found in the same tomb. The editor of *Writings on mathematical procedures*, PENG Hao, has further shown that the manuscript quoted administrative regulations. He put forward the hypothesis that its owner was an official working at the level of a local administration (Peng Hao 2001: 11-12).

Three other manuscripts or groups of manuscripts will be useful for my argument. They are fragmentary in nature, or our knowledge about them is still fragmentary at the present day.

— **Item A3:** I will evoke the writing that seems to be at the present day the oldest extant mathematical document in Chinese and is also the product of illegal excavations. The key fact for me in this chapter is that it consists of a “computation table 算表 (*suan biao*),” which according to its editors was produced in the late Warring States period (475-221 BCE).⁹ I return below to the main features of this table.

⁸ On the date, see (ZHU Hanmin and CHEN Songchang (gen. ed.) 2011, 3), and (XIAO Can 2010: 16), who argue in favor of this *terminus ante quem*. The title is on the verso of slip 0956 (editors' number 1 verso), and its photograph is reproduced in (ZHU Hanmin and CHEN Songchang (gen. ed.) 2011, 3). This book is now kept in the Yuelu Academy 嶽麓書院 (Hunan University) along with other documents that were bought in the same lot (more than 1300 bamboo strips altogether) and they are believed to come from the same site. In 2008, a collector from Hong Kong bought a second lot (more than 30 strips) and gifted it to the Yuelu Academy. On the basis of the shape of the strips, the writing and the content, the two lots are believed to have come from the same tomb.

⁹ Compare (LI Junming and FENG Lisheng 2013: 73). The edition of the table is published and analyzed in (Qinghua daxue chutuwenxian yanjiu yu baohu zhongxin (ed.) and LI Xueqin (gen. ed.) 2013: 12-13, 61-71, 135-143). I am indebted to my colleagues LI Liang and PENG Hao for having helped me access these publications.

— **Item A4:** I will make use of pieces of evidence from a collection of newly acquired mathematical documents that Han Wei (2012, 2013) published. These mathematical documents, which still await complete publication, appear to constitute an important part of a set, which was likewise found in the context of illegal digs and acquired by Beijing University in 2010. Historians also date this set from the Qin period.¹⁰ Han Wei refers to the mathematical writings that it contains and that he quotes as *Mathematical writing. Text A* 算書. 甲種 (*Suan shu. Jia zhong*), *Mathematical writing. Text B* 算書. 乙種 (*Suan shu. Yi pian*) and *Mathematical writing. Text C* 算書. 丙種 (*Suan shu. Bing pian*), since, in contrast to the two manuscripts mentioned above, there is no record of any title given to these writings by practitioners.

— **Item A5:** Finally, the evidence so far available from a newly excavated manuscript, entitled *Mathematical procedures* 算術 (*Suanshu*) (original title, whose photo is however not yet available), will also prove useful. This manuscript has been excavated from tomb M77, sealed before 157 BCE at Shuihudi (Hubei). So far only the photos of ten slips were made available, but they contain useful information for my purpose.¹¹

By contrast to these documents, found during excavations, or in the antiquities market, the other writings (**Items B**) on which I will rely were handed down through the written tradition. They share a common feature: they were all selected, in the 7th century, to become part of an anthology of mathematical “Classics *jing* 經,” or “Canonical texts,” the *Ten Classics of mathematics*, that was presented to the throne in 656. Besides selecting these “Classics,” the project, supervised by Li Chunfeng 李淳風 (602?-670), included selecting ancient commentaries that had been composed on them, preparing an edition for all these documents and composing a subcommentary. The anthology, once ready, was immediately used, with two other auxiliary books, as textbooks in a State-run Mathematics College.¹² These events are essential to understand the nature of the mathematical documents handed down through the written tradition. Two facts will suffice to make this point clear. First, there exists no other mathematical writing composed prior to the 7th century in China that was handed down through the written document but those used as textbooks in this context. Further, there is no extant ancient edition of a Classic that would not include evidence of the commentaries selected, or the subcommentary composed, in the context of the project Li Chunfeng supervised. The books from this anthology used in the present chapter are the following:

— **Item B1:** *The Gnomon of the Zhou [Dynasty]* 周髀 (*Zhoubi*), which is the earliest extant Chinese book that was handed down and is devoted to mathematics (more precisely to mathematical knowledge used, in the context of astral sciences). QIAN Baocong (1963, vol. 1: 3-4) argues that this Classic was composed around 100 BCE. By contrast, Cullen (1996) put forward the view that *The Gnomon of the Zhou [Dynasty]* was a compilation of earlier

¹⁰ HAN Wei 韓巍 (2012:85-89) provides an initial description of these mathematical writings. HAN Wei (2013) deals with knowledge about areas presented in these texts.

¹¹ (Chemla and MA 2011) contains an edition and a translation.

¹² Siu and Volkov (1999) have discussed the State institutions in which mathematics was taught at the time and the evidence we have about the use of these books in the context of learning and taking examinations. (Volkov 2014) discusses the curricula, and in particular the order, in which they were taught.

documents, which was completed in, or after, Wang Mang's reign, in the first century CE.¹³ In what follows, we will see that the study of division provides important clues on this issue. In the context of the project of the anthology, Li Chunfeng's team also prepared an edition of, and a subcommentary on, *The Gnomon of the Zhou [Dynasty]*, together with two earlier commentaries: the commentary completed in the third century by Zhao Shuang 趙爽 and the subcommentary composed by Zhen Luan 甄鸞 active at the Northern Zhou Court (557-581).

— **Item B2:** *The Nine Chapters on Mathematical Procedures* 九章算術 (*Jiuzhang suanshu*), whose title is hereafter abbreviated into *The Nine Chapters*. QIAN Baocong (1963, vol. 1: 83) put forward the view that the compilation of the book was completed in the early Eastern Han dynasty (25-220 CE). I concur with this thesis, for some features of *The Nine Chapters* most probably date to after Wang Mang's reign (9-25).¹⁴ Among the commentaries composed on that Classic prior to the 7th century, only that of Liu Hui 劉徽, completed in 263 and selected by Li Chunfeng's group, survived, together with the subcommentary composed by Li Chunfeng et alii.¹⁵

— **Items B3 and B4:** *Mathematical Classic by Master Sun* 孫子算經 (*Sun zi suanjing*) and *Mathematical Classic by Zhang Qiuqian* 張邱建算經 (*Zhang Qiuqian suanjing*). QIAN Baocong (1963, vol. 2: 275-276) argues that the *Mathematical Classic by Master Sun* was completed and ca. 400, but that the editions available at the present day all bear the mark of later revisions. Moreover, in the case of this Classic, despite the mention of a subcommentary by Li Chunfeng et al. on the first page of each chapter in all the extant ancient editions, its text does not seem to have been handed down in the extant editions. As for the *Mathematical Classic by Zhang Qiuqian (Item B4)*, QIAN Baocong (1963, vol. 2: 325-327) suggests the book was composed between 466 and 485. The text handed down contains "detailed procedures (*cao* 草)" and subcommentary, which the first pages of each chapter in the extant ancient editions attribute, respectively, to the 6th century scholar Liu Xiaosun 劉孝孫 and to Li Chunfeng and his team.¹⁶ These pages also mention a commentary by Zhen Luan that does not seem to have been handed down.

¹³ An English translation of *The Gnomon of the Zhou [Dynasty]* was published in (Cullen 1996). It relies on the critical edition published by QIAN Baocong (1963, vol. 1: 11-80). The (unpaginated) introduction of (GUO and LIU 1998) considers the date of 100 BCE as a *terminus ante quem* for the book. (GUO and LIU 1998) also contains a critical edition of *The Gnomon of the Zhou [dynasty]*. Note, however, that it does not include the subcommentary by Zhen Luan on this Classic (see below).

¹⁴ Compare my introduction to Chapter 6, in (Chemla and GUO 2004: 475-478). Guo Shuchun suggests a much earlier date of composition. See also the pages on the history of *The Nine Chapters* in the (unpaginated) introduction of (GUO and LIU 1998), which present other views on this still contested issue.

¹⁵ See a critical edition and a French translation of *The Nine Chapters* and the two commentaries in (Chemla and GUO 2004).

¹⁶ QIAN Baocong (1963) contains a critical edition of the 7th century *Ten Classics of mathematics*, on which, except for *The Nine Chapters* and unless otherwise specified, I rely. More recently, a new critical edition of the *Ten Classics* was published in (GUO and LIU 1998). The (unpaginated) introduction of (GUO and LIU 1998) adopts the same views as Qian Baocong's on the dates of the *Mathematical Classic by Master Sun* and the *Mathematical Classic by Zhang Qiuqian*.

3. A change in the terminology as an indication of a key turn with respect to division

The first phenomenon that drew my attention to a possible mutation with respect to division in the time period considered relates to terminology. Indeed, the ways of prescribing division testify to an interesting change between the first century BCE and the first century CE. In this respect, the mathematical documents presented above constitute two groups.

On the one hand, *The Gnomon of the Zhou [Dynasty]* (**Item B1**) shows an unexpected continuity with the manuscripts (**Items A**)—at least partly, as explained below (see footnote 25). On the other hand, *The Nine Chapters* (**Item B2**) and the later Classics mentioned above present an essential difference with these earlier documents, whereas, by contrast, they are similar to each other.¹⁷

Since the point will be useful for the argument this chapter makes, let me outline the phenomenon and, to begin with, sketch the similar ways in which division was prescribed in the manuscripts and (parts of) *The Gnomon of the Zhou [Dynasty]*.¹⁸ *Writings on mathematical procedures* (**Item A2**) contains the main modes of prescription used in the early time period that I distinguish. I will thus mainly rely on the evidence this document provides to analyze them.

In *Writings on mathematical procedures*, division can be designated by a first set of expressions that are also found in *The Nine Chapters*. This set is exemplified by the following occurrences:

- “(...), 十而一 (...) /slip 98/ one divides by ten,” literally *shi er yi* “ten then one.” Based on computations with fractions of measurement units of capacity, the result of the division is “為糲十分升之三 *wei bai shi fen sheng zhi san* ... makes three tenths of a *sheng* of finely husked millet.”
- “(...), 令十一而一 (...) /slip 85/ one carries out the division by eleven,” literally *ling shiyi er yi* “making ‘eleven then one’.” The result of the division is given directly as a fraction of a measurement unit of area “十一分步五 *shiyi fen bu wu* five elevenths of a *bu*.”

In fact, in *Writings on mathematical procedures*, this mode of prescription includes variants that are not found in *The Nine Chapters*, such as:

¹⁷ To be more precise, the prescription of division that characterizes the later set of documents can be found in parts of the received text of *The Gnomon of the Zhou [Dynasty]*. Interestingly enough, as we will see below, the passages in which this phenomenon occurs are signalled by the commentator Zhao Shuang as not belonging to the original book, or are introduced in the main text through modern editorial decision.

¹⁸ Guo Shuchun (2002: 525-527) has compared the terminology for division in *Writings on mathematical procedures* and *The Nine Chapters*. Xiao Can 肖燦 (2010: 121ff) has studied the expression of division in *Mathematics*. They both concentrate on whether the prescriptions of division make use of the technical terms for the operands (“dividend 實 *shi*” and “divisor 法 *fa*”) or not. As a result, they do not notice the change in the way of prescribing the operation that appears essential to me.

- “(...) 二尺七寸而一石 (...) /slip 146/ dividing by two *chi* seven *cun* gives the result in *dan*,” literally *er chi qi cun er yi dan* “two *chi* seven *cun* then one *dan*.” Here, a volume is divided by another volume (two *chi* seven *cun*) and the result of the operation is stated to yield measurement units of value *dan*.¹⁹ In this case, the result is the sum of an integral number of *dan* and a fractional part: “為粟卅（四十）六石廿（二十）七分石之八 *wei su xiliu dan nianqi fen dan zhi ba* makes forty-six *dan* eight twenty-sevenths of a *dan* of unhusked millet.”
- “(...) 三成一。 (...) /slip 115/ (...) one divides by three,” literally *san cheng yi* “three produces one”. The division yields, after simplification of the fraction, the result in measurement units of capacity, as “為粟一升七分升三。 *wei su yi sheng qi fen sheng san* ...makes one *sheng* three sevenths of a *sheng* of unhusked millet.”
- “(...) 五而成一。 (...) /slip 113/ (...) one divides by 5,” literally *wu er cheng yi* “five then produces one.” Based on computations with fractions of measurement units of capacity, the result of the division is “為米七分升六。 *wei mi qi fen sheng liu* ... makes six sevenths of a *sheng* of coarsely husked millet.”

Despite differences between them, the above expressions are all characterized by a focus on the transformation, in the quantity divided, of an amount equal to the quantity by which one divides into a unit.²⁰ Given the variety of numerical results produced, these formulations do not seem to refer to a specific procedure, but rather to a general conception of the operation of division.

The second set of expressions shares these basic features. Likewise, the actual results to which they correspond do not seem to imply use of a specific procedure. Moreover, the basic idea underlying all the expressions also insists on the units yielded. However, by contrast to the first set, these other ways of prescribing division in *Writings on mathematical procedures* insist on the comparison between the quantity divided and the one dividing that lies at the basis of the transformation of the dividend into new units. In the manuscripts, they are most often used to prescribe division and present a great number of variants, which I will not all list here. Some of them are found in *The Nine Chapters*. They include:

- “實如法而一。 /slip 67/ one divides the dividend by the divisor,” or “實如法得一。 /slip 35/ dividing the dividend by the divisor yields the result,” or else “實如法得一 (measurement unit)。 /slip 62/ dividing the dividend by the divisor yields the result in (measurement unit).” Literally, these expressions mean, respectively, *shi ru fa er yi* “in the dividend, what is like the divisor then one,” *shi ru fa de yi*, “in the dividend, what is like the divisor yields one ,” *shi ru fa de yi* (measurement unit) “in the dividend, what is like the divisor yields one (measurement unit).” The context of slip 67 illustrates the variety of results yielded by division prescribed in this way, since in fact, the expression simultaneously

¹⁹ On *dan* as measurement units of value, see (Chemla and MA 2015 (2014)).

²⁰ In *Mathematics (Item A1)* and the pieces published from *Mathematical writings A and C (Item A4)* expressions of this type occur, however, most often, in the last two variants (Xiao Can 2010: 121-122; Han Wei 2013).

prescribes two divisions by a divisor equal to 37. The first division yields the result as the fraction $30/37$ *sheng*, whereas the second division has, as its result, 2 *sheng* $7/37$ *sheng*.

- “實如法得...。 /slip 92/ dividing the dividend by the divisor yields (statement of result with measuring units of area),” or “實如法得水、米各一升、掣一斤。 /slip 81/ Dividing the dividend by the divisor yields water and (coarsely) husked grain, respectively, in *sheng* (capacity measurement unit) and mixed (fat) in *jin* (weight measurement unit).” The former statement reads literally: *shi ru fa de...* “in the dividend, what is like divisor yields (...).” The latter, which simultaneously prescribes three divisions (two yielding an amount in capacity, and the other in weight, measuring units), means literally: “In the dividends, what is like the divisor yields, for water and (coarsely) husked grain, respectively one *sheng*, and for the mixed (fat), one *jin*.”

We also find in the manuscripts a great number of variants of this form of prescription that are not evidenced in *The Nine Chapters*. They include:

- “ (...) 為實， (實) 如法而成一。 /slip 124/ (...) makes the dividend, dividing the dividend by the divisor produces the result.” The prescription reads literally: *shi ru fa er cheng yi* “in the dividend, what is like divisor then produces one.” With such expressions, instead of *shi* “dividend,” we can find the statement of an actual quantity, or alternatively, that of the operations yielding this quantity. Moreover, the measurement unit attached to the units produced can be made explicit.
- “ (...) 以為法， (...) 以為實， 如法而一尺。 (...) /slip 56/ (...) is taken as divisor (...) are taken as dividends, dividing gives (the results in the measuring unit) *chi*.” The prescription *ru fa er yi chi* literally means “what is like divisor then one *chi*.” The three results of the divisions to which this expressions refer are an integral amount of the measurement unit *chi*, in two cases increased by a fraction of the same unit.
- “ (...) 以為法， (...) 為實， 實如法得尺。不盈尺者十之， 如法一寸 /slip 42/ (...) is taken as divisor, (...) makes the dividends. In the dividends, what is like the divisor (i.e., Dividing the dividends by the divisor) yields a *chi*. What does not fill up the *chi* (the remainders of the dividends), one ten-folds them. What is like the divisor (i.e., dividing the new values of the dividend by the divisor yields), one *cun*” (see text 2).²¹ The latter type of prescription is quite frequent in *Mathematics*.
- “ (...) 如法得一步 /slip 94/ dividing yields *bu*,” literally, *ru fa de yi bu* “what is like divisor yields one *bu*.” (see text 8)

This long enumeration of modes of prescription and their variants highlights a striking fact. Despite differences among them, all these expressions place emphasis on the production of units, one by one. More often than not, especially in the manuscripts, prescriptions specify the actual measuring unit, or even the sequence of measuring units, with which the result is obtained. By contrast, in *The Nine Chapters* (**Item A2**), clarification

²¹ A set of texts is provided in the appendix. In this appendix, I have numbered the texts and the lines in the translation so as to be able to refer to them more easily.

on the measuring unit attached to the result appears less systematic in the expressions prescribing division.

Moreover, in all these expressions the part played by the divisor in shaping what in the dividend yields the units of the quotient is also stressed. Interestingly, the term chosen to refer to the divisor means “pattern, norm.” In fact, in the context on which we concentrate technical terms referring to the operands are introduced for no other operation but division.

These features are extremely important, as we will see below. They also characterize the third set of expressions by means of which division is referred to in *Writings on mathematical procedures* (**Item A2**). However, this third way of prescribing attests to a specificity, which will precisely make the key difference, in the prescription of division, between, on the one hand, the manuscripts and (part of) *The Gnomon of the Zhou [Dynasty]* (**Item B1**) and, on the other, *The Nine Chapters* (**Item B2**) and other Classics (including another part of *The Gnomon of the Zhou [Dynasty]*). Indeed, as far as I can tell, this additional feature never occurs in *The Nine Chapters* or later mathematical Classics. Let us first illustrate the third type of prescription that occurs in the manuscripts using a few examples, before we clarify this point:

- “ (...) 除十六而得一。 /slip 78/ Dividing by sixteen yields the result," literally *chu shiliu er de yi* “eliminating sixteen then yields one”.
- “除，如法得從 (縱)一步， (...). /slip 167/ Dividing yields the result for the length in *bu*," literally, *chu, ru fa de zong yi bu* “eliminating, what is like the divisor yields, for the length, one *bu*.”

The characteristic feature of these expressions is that they articulate the term *chu*, which I first interpret loosely as “eliminating,” and other expressions of the type described above. Expressions of that family occur not only in *Writings on mathematical procedures* (**Item A2**), but also in *Mathematics* (**Item A1**, see text 7, lines *** 91-92), in *Mathematical writings* (**Item A4**, Han Wei 2012: 87) as well as in *The Gnomon of the Zhou [Dynasty]* (**Item B1**, text 1, lines *** 19-20 and 23). This similarity reveals the historical continuity between these various documents (all the more so that this family of expressions disappears later, as I will show).

I claim that the verb *chu* in the manuscripts, and these parts of *The Gnomon of the Zhou [Dynasty]* in which my third type of expression occurs, *only* refers to *subtraction*. First, it is clear that *chu* refers to subtraction, as can be illustrated, for instance, with the procedure that gives the means to subtract one fraction from another. In this context, after operations that for us correspond to the reduction of the two fractions to the same denominator, the procedure prescribes to *chu*, that is, subtract, the smaller numerator from the larger one. In later texts, like *The Nine Chapters* (**Item B2**), this prescription will be rewritten using the verb *jian* “subtract.” Second, in the context of *Writings on mathematical procedures* and related writings, *chu* also refers to a subtraction that is *repeated* until the moment when the remainder is smaller than the quantity subtracted, and in these cases, the

result of the operation *chu* is precisely the remainder. This is how *chu* is used, for instance, in the context of the simplification of fractions.²²

By contrast to these uses of *chu*, I claim that in the manuscripts, *chu* is *never* used *alone* to refer to a *division*. Another way of asserting this fact is to say that division is always prescribed by a complex expression. Despite variations, the global stability of these expressions nevertheless allows us to consider that division was an operation, and we will discuss the procedures with which this operation was executed.

This terminological analysis leads to a key conclusion: the use of the term *chu* in the expressions of our third set makes explicit that in the context, divisions were carried out by means of subtraction. This feature correlates perfectly well with the emphasis of all these expressions on the production of units in the quotient, one by one (for example, the production of *bu*, one by one, in relation to the prescription “eliminating, what is like the divisor yields, for the length, one *bu*.”) (Chemla 2013). In other words, subtracting the divisor from the quantity divided yields one copy of a given measurement unit in the quotient, and the second half of the expressions, in our third set, makes clear the unit thus yielded—we will analyze these points in greater detail below. These remarks hence suggest that division prescribed in this way corresponds in the manuscripts to an algorithm proceeding by a repeated subtraction, whose iteration yields in succession the units attached to each measurement unit in the quotient (Chemla 2013). The use of *chu* as “subtraction” in these expressions is essential for making this point explicit. I will refer to this way of executing division as “the older execution.”

I further claim that in the first century, we have documents attesting to a major change. It is true that *The Nine Chapters (Item B2)* also contains occurrences of *chu* with the meaning of “subtraction”. However, in these rare cases, it is *disjoint* from the formulation of a division.²³ Moreover, in *The Nine Chapters*, *chu* never refers to a repeated subtraction. The

²² In the algorithm given for the simplification of fractions in *Writings on mathematical procedures* (PENG 2001: 43-45, slips 17-18), repeated subtractions are used, and they are in fact used repeatedly (first the numerator was repeatedly subtracted from the denominator to yield the remainder once subtraction could no longer be carried out, then the remainder was repeatedly subtracted from the denominator, and so on, until one reached the greatest common divisor). The term used in *Writings on mathematical procedures* to refer to each repeated subtraction intending to yield the remainder is precisely *chu*. Similarly, the same algorithm is described in *The Nine Chapters* (Chemla and GUO 2004: 156-157). The term used is no longer *chu*, but *jian* 減 “subtract”. In *The Gnomon of the Zhou [Dynasty]*, we have recurring uses of *chu* referring to a repeated subtraction of this kind, and for each occurrence, the third century commentator Zhao Shuang makes clear what its meaning is, as if Zhao Shuang perceived this use of *chu* as archaic.

²³ This use of *chu* with the meaning of subtraction occurs in the procedure after problem 3.17 (To refer to an expression in *The Nine Chapters*, I designate the problem and procedure in the context of which the expression occurs: the first digit refers to the chapter, the second number to the problem in this chapter). In this case, the fact that the result of the operation is designated as *yu* “remainder” makes clear that *chu* means “subtract.” In the procedure placed after problem 6.16, *chu* also means “subtract,” and the point is made explicit by the commentator. The fact that in this case, the third century commentator insists that in this context *chu* has the meaning of subtracting seems to indicate that (like was the case in the previous footnote with the commentator Zhao Shuang) the third century commentator Liu

key point is that, when, in *The Nine Chapters* and later texts such as the commentaries or the other Classics mentioned above, *chu* is used to prescribe a division, it is *always used alone*, never in association with one of the other expressions described above. In this context, *chu* refers, in and of itself, to a division.

This is the case in the following example from *The Nine Chapters*:

- (procedure after problem 1.2) “以畝法二百四十步除之，即畝數。 **Dividing this by the divisor/norm of the *mu*, 240 *bu*, hence gives the quantity of *mu* (...).**”

In this example, the syntax of the prescription (which I underline using bold characters) is the following *yi x chu zhi* “Dividing this by *x*.” Sometimes *chu x* means “one divides by *x*” (Problems 5.21, 6.21, again from *The Nine Chapters*). One finds variants for these expressions, for instance: the divisor can be omitted from the prescription, the dividend as well, and also both.²⁴ However, the result of the operation *chu* is always a quotient, never a remainder.

Finally, the expression of the division by *chu* can be found in parallel with another mode of prescribing division. This indicates that they are perceived as synonymous. This is the case in the following two parallel procedures from *The Nine Chapters*:

- (9.6) “(...) 餘，倍出水除之，即得水深. (...) The remainder, twice what comes out of the water divides it (*chu zhi*), thus yielding the depth of the water.”
- (9.7) “以去本自乘，令如委數而一。 The distance from the base being multiplied by itself, one makes the division (of this) by the quantity that lies on the ground,” literally *ling ru x er yi* “one carries out ‘what is like *x* then 1’.”

Let me insist: We have *no* similar example of prescription of division by means of *chu* in the manuscripts. The same statement holds true for *The Gnomon of the Zhou [Dynasty]* (**Item B1**), at least *if (and only if)* we accept the third century commentator Zhao Shuang’s

Hui perceives this meaning as an archaism. I interpret as a confirmation of this fact that in all later books in the *Ten Classics of Mathematics*, we find only two restricted contexts in which *chu* refers to a subtraction. In *Mathematical Classic by Master Sun* (**Item B3**, completed ca. 400), only in the very last (and in many respect unusual) problem is *chu* used with the meaning of subtraction. Moreover, *chu* occurs once with the meaning of subtraction in the 6th century *Mathematical Procedures for the Five (Confucian) Canonical Texts*. Zhen Luan, whom we have met above as a subcommentator on *The Gnomon of the Zhou [Dynasty]* (**Item B1**), composed this book on the basis of quotations from ancient canonical texts, or Classics (*jing*), but this time for Confucian scholarship. In the first passage from *Documents*, quoted with its commentary ascribed to Kong Anguo (late second century BCE), *chu* is used with its ancient meaning of subtraction (Qian 1963: 441). In correlation with this, in Zhen Luan’s commentary on this quotation *chu* also occurs with this meaning (Qian 1963: 442). Interestingly, by contrast, nowhere else does this use of *chu* occur in Zhen Luan’s other pieces of commentary nor does it in Li Chunfeng et al.’s subcommentary from the 7th century.

²⁴ From time to time, *chu* means “dividing,” to refer to the fact that one looks for the integral part of the quotient (4.0). On these facts, see the glossary I compiled, (Chemla and GUO Shuchun 2004: 911).

view on the original text.²⁵ This fact points to a crucial phenomenon that again distinguishes division among all other operations that are used in procedures. Interestingly enough, by contrast to operations like adding (*jia* 加), subtracting, multiplying (*cheng* 乘), doubling (*bei* 倍), halving (*ban* 半), *n*-upling (*n zhi n* 之), division in these earlier texts is *never* prescribed by means of a single verb, but always referred to by means of a complex expression. This phenomenon, associated to the fact that only in the case of division are the operands referred to using technical terms, constitutes a clue that in earlier texts, division was perceived as different from the other operations. This remark allows us to perceive the shift in which I am interested: The introduction of *chu* to refer to division in *The Nine Chapters (Item B2)* and the later texts thus gives means to prescribe division using a single verb. Consequently, division is now made closer, similar if you will, to other operations. In addition, prescribing a division with *chu* does not impose that the meaning of the units of the result be specified. This fact is correlated with the progressive neglect, in the

²⁵ Interestingly, for two passages of the received text, for which this statement does not hold true, the third century commentator Zhao Shuang states the passages were not originally in *The Gnomon of the Zhou*. A first occurrence of *chu* with the meaning of “dividing” occurs in (QIAN 1963: 77-79). Zhao Shuang writes about this passage “非周髀本文。蓋人問師之辭。其欲知度之所分，法術之所生。 This is not from the original text of *The Gnomon of the Zhou [Dynasty]*. Probably these are the words of someone asking a teacher. This person wants to know how *du* were cut into parts and how the divisors and procedures were generated.” (I have an article in preparation on this part of the text). A second occurrence related to that use of *chu* is in (QIAN 1963: 27), in the context of the dialogue between Chen zi (Master Chen) and Rong Fang (the section Cullen (1996: 175) calls B. See (QIAN 1963: 23-42.)) It is in fact a prescription of a square root extraction that depends on the new algorithm *chu* (see below). Again, Zhao Shuang writes at the beginning of the section: “非周髀本文… This is not from the original text of *The Gnomon of the Zhou [Dynasty]*…” I have nevertheless to explain why I discard two other passages of *The Gnomon of the Zhou [Dynasty]* in which *chu* might appear to mean “divide.” Indeed, in a third passage, a procedure uses the term *chu* to refer to a division (QIAN 1963: 67). However, there is an editorial problem. As Qian (1963:67, footnote 5) makes clear, in the early 13th century edition, this procedure appears as part of Zhao’s commentary. Qian relied on the other ancient editions to suggest it was once part of the main text. Indeed, this editorial decision makes the whole passage parallel to the following ones. However, we see that there is an editorial problem here, which terminological analysis could help settle. Finally, the fourth passage of *The Gnomon of the Zhou [Dynasty]* to be considered is the procedure computing the lengths of solar shadows, which puts *chu* as “dividing” into play. However, Zhao Shuang’s commentary explicitly explains that the procedure of *The Gnomon of the Zhou [Dynasty]* having problems, he inserted “a new procedure” (Qian 1963: 66). We thus have here a third century text, and not the original text of *The Gnomon of the Zhou [Dynasty]*. In conclusion, it appears from this perspective that Cullen (1996) is right in suggesting that *The Gnomon of the Zhou [Dynasty]* is a compilation of pieces of texts composed at different periods. However, probably the different pieces that he detects call for further analysis (Chemla 2013). Moreover, clearly the history of the technical terminology might be of great help to carry out this work. In this chapter, when I speak of *The Gnomon of the Zhou [Dynasty]*, I refer to the older core of the text as Zhao Shuang referred to it.

prescription itself, of the clarification on the units for the result in the other expressions that are still attested in *The Nine Chapters*.

Let me summarize what the previous discussion has so far established. Against a backdrop of significant similarities in ways of prescribing divisions among all documents, a key difference between the two sets of documents distinguished above emerges: It concerns the *meaning* as well as the *use* of the verb *chu*. This is the essential remark for this chapter.

The meaning of *chu*, I argue, has changed in the prescription of a division, between what we can observe in the documents produced before *The Gnomon of the Zhou [Dynasty] (Item B1)* and what we read in *The Nine Chapters (Item B2)*. In the earlier texts, to designate a division, we never find *chu* alone, but when *chu* is used, it is always inserted in longer expressions, where it refers to subtraction. We have seen more generally that in this context, *all* expressions referring to division are centered on the production of units, and they address the issue of which units are yielded. This remark holds true also for the expressions that involve the term *chu*. Expressions of the first two types (that is, those that do not involve *chu*) are found in *The Nine Chapters*. However, there are hints that there they have lost there their original meaning.²⁶ Moreover, in this later environment, *chu* refers to division and, in that context, it is used alone.

What is more important is that the change of meaning of *chu* that I have evidenced goes along with new developments, related to the division, to which *The Nine Chapters* bears witness. To begin with, I interpret that in the Classic the new meaning of *chu* is correlated with a different way of executing division (I will refer to this as “the new execution”, and it is attached to the division *chu*). Further, in my view, this change is the symptom of a wider turn with respect to operations on at least two intertwined levels. I will distinguish them for the sake of the analysis, before I set out to develop my arguments.

First, the change goes along with a theoretization that develops around the new execution of division. In my view, *The Nine Chapters* testifies to a transformation in the understanding of several operations and of the relationships between them, in which the division *chu* plays a key part, as is clear from the viewpoint of terminology as well as execution.

New algorithms to execute square and cube root extraction seem to have been developed accordingly, and their description like their execution relies on those of the division *chu*. Moreover, the prescription of these operations attests to a change, in which the term *chu* is also pivotal—we return to this point below.

Further, algorithms to solve systems of linear equations also appear (as far as we can tell on the basis of the extant documents). Likewise, the term *chu* and the term for the operands as well as the new execution of division are central in the description and mathematical practice of these algorithms.

These various uses of *chu* and related terms testify to apparently newly established relations between division, root extractions, and the solution of systems of linear equations. A network of operations is thereby built on the basis of the new prescription and execution

²⁶ Among such hints is the fact that one finds such expressions as “Dividing the dividend by the divisor gives the surplus of one over the other 實如法而一，即相多也” (problem 1.14), which can be literally translated as “In the dividend, what is like the divisor then one, hence the surplus of one over the other”. I would argue that such sentences seem to indicate that the expression “Dividing the dividend by the divisor” prescribes the division without bringing into play the literal meaning of the formulation.

of division, since in this context, only the designation of division as *chu* is used, and none of the other modes of prescription.

This collection of facts suggests that these theoretical developments are attached to the change we sketched above in relation to the use of the term *chu* for division.²⁷ I also examine below another dimension of the more global theoretical turn linked to division. We will see that the execution of the older division can be cut into four types of phases. The theoretical features mentioned above relate to Phases 2 and 3 (which one might consider as the execution of a division *stricto sensu*). Phase 1 (the initial phase of a division, which transforms quantities into operands for the execution) is also reconceptualized, and relationships are thereby at the same time established between Phase 1 and other procedures, which places the operation of division in an even wider theoretical context.

Second, the turn to which the change in terminology, and accordingly in knowledge, attests also appears to be correlated to a turn in material practices with operations. This facet of the change is elusive. Indeed, whether we consider manuscripts (**Items A**) or Classics (**Items B**), our documents only contain characters, and they all refer to the use of rods and a location on which they are “put” (*zhi* 置) without showing any illustration. In this context, operations were executed in a purely material way. We thus need to grasp this change through restoring practitioners’ material practices that left no traces in our documents. Evidence shows a noticeable amount of continuity in these material practices, since all the mathematical texts mentioned above clearly testify to the fact that rods were used to represent numbers and quantities as material entities to operate on them. Although we have no evidence for that, we may assume that these counting rods were throughout used on a surface, probably an ordinary one, to which I refer loosely as “calculating surface.” Against this backdrop of continuity, I argue that there seems to have occurred a key transformation in the number system on which division *chu* was executed and also in how quantities were dealt with throughout the execution. In my view, our documents attest to a clear-cut change in the way of using rods to represent numbers, which is reflected in how the operations are carried out. Moreover, this seems to have occurred as part of a *wider change* in the way of working with a surface, where positions were granted a key part for all operations linked to the division *chu*. In fact, it is in relation to this new way of working that *The Nine Chapters* allows us to perceive a work on the relationships between operations based on their execution using counting rods and positions.

In brief, in my view, the change in the mathematical practice and the terminology for division is correlated with a theoretical and practical change in several directions. Here, I will highlight some dimensions of this change, whose complete description exceeds the scope of a chapter. For this, I rely on what can be reconstructed of the executions of division in the two contexts brought to light (through an observation of the terminology), from the moment when a dividend and a divisor are specified until the moment when the result is obtained. I emphasize the nuance “what can be reconstructed”, since, as we will soon see, the evidence is difficult to handle. Nevertheless, I argue we can perceive a work carried out

²⁷ With respect to both the terminology and the mode of execution, I have analyzed the relationship between root extractions and division *chu* in (Chemla 1994b). Chemla (2014) is devoted to the shift that brought about this architecture of operations. As for the relationship between the solution of systems of linear equations, as it is carried out in the eighth chapter of *The Nine Chapters*, and division (from the viewpoint again of both the terms used and the algorithm put into play), see Chemla 2000.

on the material culture of operations as well as a reflection on the execution of division that developed in at least two main directions. I suggest that, seen from this perspective, this turn may be the trace of the existence of (at least) two cultures of computation in early China.

4. Division in *The Gnomon of the Zhou [Dynasty]* as a key to the introduction of several types of Phase in the execution of the operation

Division seems to have been perceived as a difficult operation by the authors of *The Gnomon of the Zhou [Dynasty]* (Item B1) as much as by the authors of the earliest Chinese extant mathematical manuscripts. We have emphasized the point above for the manuscripts, showing the space they have devoted to the execution of division. This is also the conclusion that one is tempted to draw from the fact that the only mathematical topic for which we find lengthy detail in *The Gnomon of the Zhou [Dynasty]* is precisely division. The similarities established between the ways of prescribing division in these various documents suggest that they also shared ways of executing division. We will see that there is much evidence in support of this hypothesis.

The divisions discussed in *The Gnomon of the Zhou [Dynasty]* provide an excellent point of departure for our analysis, since they yield evidence to reconstruct an earlier way of executing division. Moreover, they allow us to decompose the execution of division into four types of Phase, which yields analytical tools useful for our discussion. Each of the Phases identified proves to have had a specific history and to have been the focus of a specific effort. In the following sections, we will concentrate on each of them in turn, with a lesser emphasis on Phase 4, which I have addressed elsewhere (Chemla 2013). For the moment, let us introduce the Phases, while at the same time discussing the reconstruction of this early execution for division.

4.1. Reconstructing an older execution of division

To this end, I have translated a text describing a process of division from *The Gnomon of the Zhou [Dynasty]*, together with the commentaries on it, as text 1 in the appendix. Let us examine it.

The procedure of *The Gnomon of the Zhou [Dynasty]* in which we find our document requires dividing the quantity 119000 *li*, expressed with respect to the measuring unit *li* for distances, by 182 days $5/8$ of a day.

After having prescribed “putting” these two quantities presumably on the calculating surface, using rods, the first Phase of the operation amounts to transforming the two values of dividend and divisor into integers. In this case, this is performed using a multiplication by 8. The numbers obtained are, in a sense that will be made more precise in section 6, the smallest integers whose division yields the same result as that between the two quantities with which one started.

Technical terms had been introduced to designate the two quantities to be divided one by the other as “dividend *shi* 實” and “divisor *fa* 法” (lines *** 3-5). The same technical terms are used (dividend and divisor, lines *** 16, 19) to designate the integers thus obtained, respectively, 952000 and 1461. This illustrates the assignment of variables, which the text of the algorithm constantly makes use of. Seen from another perspective, this gives us information on aspects of the material dimensions of the handling of the calculating

surface at the time, since technical terms appear to designate the numerical values that are found in the positions where dividend and divisor were placed at the beginning, at the moment when these technical terms are put into play, and not at the moment when the initial values were placed. 952000 and 1461 thus probably replaced the initial values. Is division the only operation for which operands are modified before the execution starts? Might this explain why, by contrast to other operations, its operands are designated by technical terms? These are open questions.

Note that, by contrast with the quantities with which one started, the two transformed values are stated *without* any measuring unit. Moreover, they are in the form of a decimal expansion, in the sense that they are stated by means of the decimal number system of the Chinese language. However, we do not know how for the author of text 1, these numerical values were represented with rods on the calculating surface. There is no hint indicating that at the time, the statement of this decimal expansion would correspond to a place-value decimal system on the calculating surface (we return to this issue below).

The transformation carried out to yield 952000 and 1461 is designated by the technical term “make communicate” (text 1, line *** 11). In what follows, I refer to this transformation as **Phase 1** in the execution of a division.²⁸ We will see that whereas for this Phase, there seems to be a continuity in *practice* between the earliest extant documents and *The Nine Chapters*, the operation brought into play has undergone *reconceptualization* which is already perceptible in *The Gnomon of the Zhou [Dynasty]* (see section 6).

We will return to the Phase 1. For now, let us focus first on the subsequent two types of Phase. To understand them better, it is useful to recall the relationship between the measuring unit *li* and the smaller unit used in conjunction with it, the *bu*: $1 \text{ li} = 300 \text{ bu}$. The execution of the division between, respectively, 952000 and 1461 described in *The Gnomon of the Zhou [Dynasty]* will yield a result in the form of an integral number of *li*, an integral number of *bu* and a fraction of the *bu*. Note that the operation now operates on “abstract” decimal expansions, written in a way about which we so far know nothing. Also note that, by contrast, it produces a quotient as a measured quantity.

The subsequent part of the execution of the division goes as follows. To begin with, in **lines** *** 19 and 23, a division is prescribed between the “dividend” and the “divisor”, using precisely an expression of the third kind introduced above (“*chu, shi ru fa de yi li* eliminating, in the dividend, what is like the divisor yields one *li*..... 除之,實如法得一里...”). The way in which it is prescribed thus specifies that the units obtained must be interpreted as associated to the measuring unit *li*. In other terms, this division aims at determining the integral part in *li* of the quotient. In fact, this view is not incorrect, but it is partial. In our perception of division, this prescription seems to refer to a division. However, I will suggest that in this context, the operation yielding the integral number of *li* is only one step of the global execution of the division. Indeed, I will argue that in the same way as our division produces a quotient digit by digit, that is, as a sequence of digits, this type of execution produced the quotient as a sequence of components that consist, each, of a number associated with a measurement unit, taken in turn from a decreasing sequence of measurement units (in *The Gnomon of the Zhou [Dynasty]*, as we show shortly the successive components will in fact be a mixture of measurement units and decimal

²⁸ In the chapter entitled “Multiplying Integers: On the diverse practices of medieval Sanskrit authors”, in this book, Keller and Morice-Singh also show that the execution of multiplication involved a Phase of this type. This aspect of computations still awaits systematic research.

expansion). This structure for the executions of division at the time is in fact signaled, in text 1, by the recurrence of expressions of the type “what is like the divisor yields one” This explains why I refer to the production of the integral number of *li* as only part of the process of execution (and in fact the first part). According to this reading, for which I provide evidence below, the overall procedure, whose interpretation we discuss, describes the execution of a single division, and not a sequence of divisions. I call “**Phase 2**” each of these Phases, which corresponds to expressions of the type “in the dividend, what is like the divisor yields one” Such a Phase produces one component of the quotient sought-for in the form of a number attached to a measurement unit. As will soon become clear, it is to be repeated as many times as the quotient has such components.

The part in *li* of the quotient is determined to be 651 *li* and there remains 889 (to which the text refers to, in **line *** 23**, as “what does not fill up the divisor.”) We have no indication on how the 651 *li* was obtained. By contrast, the text is much more revealing about the way in which the part in *bu* of the quotient is obtained. Let us continue reading it. The next sentence (**line *** 24**) prescribes to multiply the remaining 889 by 3, and then to carry out again a Phase 2. The prescription in this Phase 2 makes use of a highly interesting variant of a common prescription of division. It does not read “WHAT IS LIKE THE DIVISOR YIELDS ONE *BU*,” but instead “WHAT IS LIKE THE DIVISOR YIELDS A HUNDRED *BU*.” The same feature holds true in all similar passages. Which clue does this variation provide?

In terms of execution, the third century commentator Zhao Shuang has the same reaction as we have: he suggests that one should have multiplied the remainder by 300 to obtain the dividend, whose division by the divisor 1461 would have produced the (integral) number of *bu* of the quotient (hundreds, tens and units). His interpretation of the procedure, for which by contrast the authors of *The Gnomon of the Zhou [Dynasty]* opted, is that they successively computed a “dividend” for the hundreds of *bu*, and then a “dividend” for the tens of *bu* and finally a “dividend” for the units of *bu*. Zhao Shuang repeats the same interpretation for all the divisions on which the book provides detail.²⁹ In his view, the authors thus decomposed the operation into as many sub-procedures as there are orders of magnitude between the *li* and the *bu*. This corresponds to the three successive prescriptions: “WHAT IS LIKE THE DIVISOR YIELDS A HUNDRED *BU*,” “WHAT IS LIKE THE DIVISOR YIELDS TEN *BU*,” “WHAT IS LIKE THE DIVISOR YIELDS ONE *BU*.” They refer to the repetition of Phase 2 thrice, each operation yielding in this case, not components related to a sequence of measurement units, but rather successive decimal powers of *bu*. Probably, the reason why this decomposition becomes visible is the large gap between the *li* and the *bu*. To decompose the operation in this way, the authors successively multiplied the respective remainders of the successive dividends by 3, by 10 and once again by 10 (**lines *** 24, 30, 36**). I will refer to each of these three Phases as being of the type “**Phase 3**.” As a result, the divisions of these successive dividends by the *same* divisor (**Phase of type 2**) yield, in turn, the number of hundreds of *bu*, the number of tens of *bu*, and the number of units of *bu*.

Probably, the execution of each of these Phases 2 of the division process was carried out by repeated subtractions. We have seen above that *chu* regularly took this meaning in the manuscripts and *The Gnomon of the Zhou [Dynasty]*, that is, precisely in the documents in which we have the prescription of the division under consideration.³⁰ This conclusion is

²⁹ See, for instance, (Qian 1963: 60-61).

³⁰ Footnote 22 provided evidence showing that such repeated subtractions, aiming at yielding the remainder once subtraction could no longer be carried out, were used in

further suggested precisely by how the prescription of these Phases is formulated. Indeed “eliminating *chu*, in the dividend, what is like the divisor yields one (measurement unit)” seems to refer to a combination of repeating a subtraction, and at each repetition, adding a unit of a given sort to the result (the part of quotient) sought-for, while at the same time seeking to determine the remainder of the repeated subtraction. These are precisely the two results that this type of Phase in the execution yields. In fact, in text 1, *chu* is used in the prescription of the division (**line *** 19**), but it is not repeated: like the term “IN THE DIVIDEND,” it is placed before *all* the Phases 2 and 3, stated once and for all. Incidentally, this feature confirms that the text describes the execution of a single operation, whose core part begins on **line *** 19**, with the prescription of *chu*, and ends with the complete production of the quotient, in **line *** 42**.

Perhaps, in the first part of the execution of the division that yielded the number of *li* in the quotient, in addition to repeated subtractions, also tables of multiplication as evidenced in the manuscripts were used. Indeed, tables of multiplication between powers of ten were inserted in *Writings on mathematical procedures (Item A2)* and *Mathematical procedures (Item A5)*,³¹ which echoes the fact, stressed above, that dividend and divisors were transformed into decimal expansions (see Section 4.3). On this point, we can only make conjectures.

It is only for the third and last use of Phase 2 that the prescription of the division makes use of the common expression “WHAT IS LIKE THE DIVISOR YIELDS ONE *BU*.” (**line *** 37**). The fact that before that, the authors used the expressions: “WHAT IS LIKE THE DIVISOR YIELDS A HUNDRED *BU*,” and “WHAT IS LIKE THE DIVISOR YIELDS TEN *BU*” puts the common expression in parallel with the two others. The parallel established between these three formulations (**lines *** 24, 31, 37**) clarifies the meaning of the usual expression: it does emphasize the meaning of the units obtained in the Phase of the division prescribed, that is, the measurement unit to which the integer yielded by repeated subtraction corresponds.

In effect, in this execution of division, the dividend constantly undergoes transformations, whereas the divisor remains invariant. We thus understand that one of the main problems of division might have been to determine the units with respect to which the part of the quotient found should be associated. In relation to the changes in the dividend, this feature might have required a constant control for the successive parts of the quotient.

In this context, the text of *The Gnomon of the Zhou [Dynasty]* somehow shows that the production of the part of the quotient that consists of hundreds or tens of *bu* is of the same type as that of the part corresponding to any measuring unit. This might be related to the promotion of the decimal expansion, to which *The Gnomon of the Zhou [Dynasty]* testifies. We return to this point below, when we see how divisions in the manuscripts likewise yield quotients as quantities unfolding according to a series of measurement units.

Once the integral number of *bu* is obtained, the text prescribing the process of division in *The Gnomon of the Zhou [Dynasty]* turns to the final step, which I will call “**Phase 4**” of the division. In this last step, the final part of the quotient is computed, by means of the introduction of a fraction associated to the measurement unit *bu*, whose numerator is the remainder of the dividend and the denominator, the divisor. In *The Gnomon of the Zhou*

Writings on mathematical procedures and in *The Gnomon of the Zhou [Dynasty]*, and that in these early documents, repeated subtraction was designated using precisely *chu*

³¹ On this point, see (Chemla and MA 2011).

[*Dynasty*], the step is prescribed as follows: “What does not fill up the **divisor**, one **names** it with the divisor.”³²

4.2. Reconstructing the more recent execution of division to which the commentator refers

Several features of Zhao Shuang’s commentary seem to indicate that he contrasts the older process of execution of the division, as described in *The Gnomon of the Zhou [Dynasty]*, (**Item B1**) to another type of execution. Which clues does he give on this other procedure? Zhao Shuang notices several times that the division in *The Gnomon of the Zhou [Dynasty]* is carried out in such a way that the divisor does not change periodically (**line *** 27**).³³ Further, he reads the decimal parts into which the quotient is decomposed in this Classic in terms of “positions” or “digits,” both designated as *wei* 位, a term that does not occur in *The Gnomon of the Zhou [Dynasty]* stricto sensu (**lines *** 27-28, 29, 33, 35, 39-40**). Zhao Shuang also regularly uses the term *chu* alone to refer to a division. This set of hints suggests that most probably he compares the process on which he comments to the one that is described for the division called *chu* in *Mathematical Classic by Master Sun (Item B3)*. This is the one to which I refer as the “new execution” of division *chu*, and one can show this is also the one used by the authors of *The Nine Chapters (Item B2)*,³⁴ and also by later Classics. In fact, all of Zhao Shuang’s remarks on division fit with the hypothesis that this is the execution by reference to which he writes his commentary on the older process of division described here by *The Gnomon of the Zhou [Dynasty]*. Note that it is precisely the algorithm *chu* that Zhen Luan’s commentary on the passage analyzed above puts into play, to carry out the same division as that in *The Gnomon of the Zhou [Dynasty]*, but in a way different from what this Classic describes (**lines *** 45-56**).

Without repeating the argument developed elsewhere, let us evoke this new algorithm to substantiate our claim. In the framework of this more recent algorithm for the execution of division, like in the older procedure, numbers are represented with rods, and both algorithms rely on the fact that values can accordingly be modified. However, now, as is evidenced in *The Nine Chapters*, numerical values are also *shifted*. A set of terms³⁵ occurs in

³² I underline the terms that are specific to Phase 4 in *The Gnomon of the Zhou [Dynasty]* and later Classics using bold characters. (Chemla 2013) shows that this expression differs from the related prescription in the manuscripts, and discusses the consequences of this change. I do not return to this topic here.

³³ Zhao Shuang repeats this comment every time he meets with a division of this type, see another example in (Qian 1963: 60-61).

³⁴ I have developed an argument supporting this thesis in (Chemla 1989), see, in particular, note 10 (Chemla 1989: 69-70, note 10). More recently, I have put forward an argument of a completely different nature, that is, that in the curriculum in which the mathematical Classics were taught, *Mathematical Classic by Master Sun* was taught at the beginning of the elementary curriculum. Accordingly, students learnt first the use of the rods to write numbers and, then, multiplication and division *chu* that the Classic presents in its first pages (Chemla 2016).

³⁵ See my glossary in (Chemla and GUO 2004) for terms like “*bu faire progresser*” (p. 904), “*tui rétrograder*” (pp. 1001-1002), “*zhe rétrograder*” (p. 1031). *Bu* occurs in all Classics and commentaries with this meaning. *Tui* is a later term (with this meaning), for which we have evidence starting from the 3rd century in the commentaries, and also in later Classics. *Zhe* is used in *The Nine Chapters* with this meaning, but not later. The scholar Li Ji, who wrote a

both *The Nine Chapters* and *Mathematical Classic by Master Sun* to refer to shifts leftwards and rightwards, which are used in order to carry out multiplication or division by 10 or powers of 10. None of these terms occur in *The Gnomon of the Zhou [Dynasty]* with this meaning. These clues indicate that the number system at the basis of the new execution of division *chu* is a place-value decimal system. The outline the *Mathematical Classic by Master Sun* gives on the number system used with counting rods, in relation to the division *chu*, supports this claim.

This first feature fits with the fact that Zhao Shuang speaks of “digits.” By comparison, it now makes sense that in *The Gnomon of the Zhou [Dynasty]*, the term “digit” does not occur. In correlation with this clue, the old execution of division, as described in *The Gnomon of the Zhou [Dynasty]*, does not seem to refer to the multiplication by 10 of two successive dividends as carried out by a shift.³⁶ Both hints suggest the hypothesis that the number system on the basis of which the old execution of the division of a dividend by a divisor is carried out (as attested in *The Gnomon of the Zhou [Dynasty]*) is probably different from the one used in China about a century later. It is decimal, but not place-valued.

This conclusion is indirectly supported by the shape of the “Computation table” dated by its editors from the Warring States period (see Section 2, **Item A3**). This table clearly provides resources to multiply by one another numbers from 1 to 9 *and* multiples of ten, from 20 to 90. Had the number system used at the time been place-valued, the table would not include the multiples of tens. Indeed, later multiplication tables only include factors between 1 and 9. In addition, the presence, in *Writings on Mathematical Procedures (Item A2)* and *Mathematical procedures (Item A5)*, of tables of multiplication between powers of 10 also suggests that the number system used in correlation with them was most probably not place-valued. In conclusion, using the term “digit” to comment on the older execution of division, Zhao’s comment appears to be anachronistic.

Let us now illustrate the execution of a division in the framework of the new algorithm on the example treated in *The Gnomon of the Zhou [Dynasty]*. After having multiplied the remainder 889 *li* by 300 (what, Zhao Shuang claims, should be done, **lines ****

Pronunciation and Meaning of The Nine Chapters on Mathematical Procedures (九章算術音義) between the 8th and the 11th century (there is contention about when he was active) rewrote *zhe* into *tui wei* 退位 *tuiwei* (moving backwards by a position). See also the discussion about the early history of the meaning of , in my glossary (Chemla and GUO 2004: 1006-1007).

³⁶ This is also a feature that *The Gnomon of the Zhou [Dynasty]* shares with the manuscripts. In text 2 (**line *** 61**), in text 3 (**line *** 66**) and everywhere else, multiplications by 10 are not carried out by a shift of the value. Note that the quotients are yielded as a sequence of components attached to measurement units, even in the context with the measurement system is decimal. Moreover, we have mentioned above the tables of multiplication between powers of 10 that we find in *Writings on mathematical procedures* and *Mathematical procedures*. Additionally, despite the damaged characters of some of these slips, hints indicate that multiplying by 10 or a power of ten was not an obvious operation for the scholars whose activity *Mathematics* reflects—slips 1733, 0791, 0938, 1714, in (Xiao Can 2010: 60, 41), respectively.

25-26), the process yielding the number of *bu* would look like the following succession of configurations on the calculating surface:³⁷

2 6 6 7 0 0 1 4 6 1	Dividend Divisor
2 6 6 7 0 0 1 4 6 1	Dividend Divisor, shifted by 2 positions leftwards to deal with 1 st digit in quotient
1 2 6 6 7 0 0 1 4 6 1	Quotient. Its first digit is placed on top of the last digit of divisor Dividend Divisor, in its shifted position
1 1 2 0 6 0 0 1 4 6 1	Quotient Dividend after subtraction of product of 1 st digit of quotient by divisor Divisor, in its shifted position
1 1 2 0 6 0 0 1 4 6 1	Quotient Dividend Divisor, shifted by one position rightwards to deal with next digit
1 8 1 2 0 6 0 0 1 4 6 1	Quotient. Its 2 nd digit is placed on top of the last digit of divisor Dividend Divisor
1 8 3 7 2 0 1 4 6 1	Quotient Dividend after subtraction of product of 2 nd digit of quotient by divisor Divisor, in its shifted position
1 8 3 7 2 0 1 4 6 1	Quotient Dividend Divisor, shifted by one position rightwards
1 8 2 7 9 8 1 4 6 1	Quotient. Its 3 rd digit is placed on top of the last digit of divisor Dividend after subtraction of product of 3 rd digit of quotient by divisor Divisor in its shifted position

We see, in this new algorithm, a “periodical change” in the divisor. This could very well be what Zhao Shuang refers to using this expression (**line *** 27**), with which he contrasts the older and the new executions. To the contrary, in *The Gnomon of the Zhou*

³⁷ I represent the digits made with rods at the time in China using Arabic numerals. Since the topic exceeds the scope of this chapter, I do not make any attempt here to be more precise with respect to which digits were used and how.

[*Dynasty*], probably in relation to the fact that the number system is not place-valued, the process does not make use of shifts applied to the divisor. What is more, the divisor remains invariant. This is probably the key characteristic feature of the older execution, on which Zhao Shuang insists

In conclusion, the contrast Zhao Shuang outlines indicates the use, in the Classic and the third century commentary, of different types of execution for division. There is no sign indicating that the execution Zhao Shuang had in mind, and to which he referred using the term *chu*, differed from the one used by the authors of *The Nine Chapters*. Further, we have suggested that, given the hints about the number systems that the algorithms provide, the differences between the two executions of division probably also related to a material aspect of the computation: the number system on which procedures operate and its physical properties.

4.3 A hypothetical reconstruction of the older execution of the division as attested to in The Gnomon of the Zhou [Dynasty]

At this point, we can suggest a way of restoring the process of computation that corresponds to what is described in *The Gnomon of the Zhou [Dynasty]* (Item B1). We know rods were used. However, we have no clue about the related tool of computation, except that all texts in our corpus use the same term (“one puts”), to refer to placing numbers written using rods somewhere. Moreover, the terms “dividend” and “divisor” seem to refer to positions where numerical values succeed to one another. Accordingly, we have suggested that this tool is a surface. Although, below, I suggest a hypothesis for a layout of the execution on a surface of that kind, the reader should not take this aspect of the restoring too seriously, since so far we have no evidence about this feature of the computation. I restore the computation, without assuming a place value system. Instead, we use the written number system of the English language, with an exception: to emphasize that in Chinese, the tens are stated in a regular way, we replace “twenty” and the like by “two tens” and the like. With these conventions, I suggest the following execution:

First, one puts dividend and divisor on a surface, probably with rods, but according to a number system we do not know.

		Eight hundred	Eight ten	Nine	Dividend
	One thousand	Four hundred	Six ten	One	Divisor

Multiplying the dividend by 3 (“ONE THREE-FOLDS THIS”, Phase 3), that is, multiplying 889 by 3, which yields 2667, one obtains what Zhao Shuang calls the “dividend of the hundreds” (line *** 24). Note that a “computation table” is used here.

	Two thousand	Six hundred	Six ten	Seven	Dividend
	One thousand	Four hundred	Six ten	One	Divisor

One then iteratively eliminates amounts equal to the divisor (1461) from the dividend (2667), each time incrementing the related units of the quotient (that is, the hundreds) by 1

(This is what corresponds to the prescription: “AND WHAT IS LIKE THE DIVISOR YIELDS A HUNDRED *BU*,” **Phase 2**).

		One hundred			Quotient
	One thousand	Two hundred		Six	Dividend
	One thousand	Four hundred	Six ten	One	Divisor

“WHAT (WHAT OF THE DIVIDEND REMAINS) DOES NOT FILL UP THE DIVISOR, ONE TEN-FOLDS THIS,...” **(Phase 3)**. This likewise gives what Zhao Shuang calls the “dividend of the tens.” Again, a “computation table,” but one dealing only with powers of 10, is used here.

		One hundred			Quotient
Ten thousand	Two thousand		Six ten		Dividend
	One thousand	Four hundred	Six ten	One	Divisor

Now, a repeated subtraction similar to the one carried out above gives units in the quotient corresponding to the tens (here 8), which is what the prescription states: “AND WHAT IS LIKE THE DIVISOR YIELDS TEN *BU*.” **(Phase 2)**

		One hundred	Eight ten		Quotient
		Three hundred	Seven ten	Two	Dividend
	One thousand	Four hundred	Six ten	One	Divisor

The same alternation of Phase 3 and 2 leads one to obtain this time the number of *bu*. In the Classic, it reads: “WHAT (WHAT OF THE DIVIDEND REMAINS) DOES NOT FILL UP THE DIVISOR, ONE TEN-FOLDS THIS,” **(Phase 3)**

		One hundred	Eight ten		Quotient
	Three thousand	Seven hundred	Two ten		Dividend
	One thousand	Four hundred	Six ten	One	Divisor

“AND WHAT IS LIKE THE DIVISOR YIELDS ONE *BU*.” **(Phase 2)**

		One hundred	Eight ten	Two <i>bu</i>	Quotient
		Seven hundred	Nine ten	Eight	Dividend
	One thousand	Four hundred	Six ten	One	Divisor

The execution is completed by the statement of the fraction $798/1461$ *bu* (Phase 4), which completes the integral number of *li* and *bu* determined.

Clearly, by contrast to the other execution, to which Zhao Shuang refers, the divisor here undergoes no change. Moreover, the dividend is not only diminished at each Phase 2, as is the case in the other process of execution, but it is also here multiplied by a given amount at each Phase 3, so as to correspond to the unchanging divisor. This multiplication corresponds to a change of unit in the dividend that makes it possible that using the same divisor yields the next part of the quotient, that is, the component in the quotient

corresponding to the next measuring unit in the sequence.³⁸ We return below to the meaning of these elementary operations.

On the basis of the evidence in *The Gnomon of the Zhou [Dynasty]*, we have thus identified Phases in the old execution of a division. Let us focus in turn on each of them, to examine the evidence we have on them in the manuscripts. To begin with, we will concentrate on the alternation of Phases 2 and 3.

5. Focus on Phases 2 and 3, and their transformation: Determining the core part of the result

We have already seen that the mode of prescribing division was quite close in the manuscripts and in what Zhao Shuang recognizes as the original text of *The Gnomon of the Zhou [Dynasty]* (**Item B1**, see footnote 25). In particular, both documents contain prescriptions of division insisting in general on the production of units and on the nature of the units yielded. They also all prescribe division with *chu* “eliminating” in the same way. As we have shown, *The Gnomon of the Zhou [Dynasty]* provides evidence that puts such expressions in relation to other formulations, and allows us to suggest an interpretation. We can now be more precise about this point.

5.1. Phases 2 and 3 in the manuscripts

In fact, the alternative repetitions of Phases 2 (repeated subtraction yielding units of the quotient) and 3 (adjustment of the dividend by multiplication) identified above, in the execution described in *The Gnomon of the Zhou [Dynasty]* (**Item B1**), are also found explicitly in all the manuscripts that we can read. Moreover, they are expressed and carried out in exactly the same way. For *Mathematics* (**Item A1**), text 3 provides an example in which the result of a division is produced, by means of an alternation of Phases 2 and 3, as a sequence of numbers attached to successive measurement units of capacity, that is, in the context of a decimal system.³⁹ A similar example is found in *Writings on mathematical procedures* (**Item A2**, text 2), where the quotient is produced in the same way, using Phases 2 and 3 in turn. Further, it is produced component by component, following a sequence of measuring units of length, that is, again, in the context of a decimal system. Finally, text 4 shows an example from *Mathematics*, in which the quotient is again produced in the same way, but this time, according to successive measurement units for weight. In the latter example, the sequence of measuring units is thus not decimal. In this case, it thus becomes incontrovertible that the successive components obtained for the quotient are not decimal parts, but components attached to measurement units. Moreover, *Mathematics* (**Item A1**) and *Mathematical*

³⁸ Perhaps, the same ideas are used to determine the integral number of *li* at the beginning of the computation, but as far as I can tell, there is no evidence supporting this idea.

³⁹ Slip 0757 (editors’ number 48), on which Text 3 is written, is incomplete. However, in what remains, we clearly see that the prescription is of the second type that we have introduced. Hence, it does not make use of *chu*. Nevertheless, we have the same phenomenon of alternation of Phases 2 and 3. This testifies to the fact that even with other types of prescription, texts with the same structure recur.

Procedures (Item A5) contain, respectively, a table and procedures yielding mathematical resources to carry out Phases 3 in this context.⁴⁰

In the example of *The Gnomon of the Zhou [Dynasty]*, we have seen that we had no information about the production of the first part of the quotient, 651 *li*. Are the successive digits obtained decimal part by decimal part in the same way? We do not know. The only thing we can observe is the statement of the higher part of the quotient as a decimal expansion. Similarly, when the system of measuring units is not decimal and the distance between successive units is higher than ten (1 *li* = 300 *bu*), we have also seen the relevant parts of quotient obtained and expressed by means of a decimal expansion.

In brief, in all these cases, once dividend and divisor have both been expressed according to a decimal expansion (Phase 1, see section 6), the quotient is produced, sub-procedure by sub-procedure, as a sequence of components, consisting each of a number and a measuring unit, whether the system of units used is decimal or not (taking into account that sometimes the unit is not a real measurement unit, but a decimal order of magnitude). This mathematical practice stands in contrast to what we would do. We would obtain the quotient as a decimal expansion, independently of any measuring system, and then state the result with respect to a measuring unit, or else convert, in the end, the decimal expansion into an expansion along a sequence of measuring units. What the contrast shows is my view, is that here, it seems that the quotient is produced directly *as a measurement value*.

This is what the successive sub-procedures in the execution of this part of the division show. Now, this helps confirm why the prescription of division makes precise what is the unit obtained. When Phases 2 and 3 are repeated, they produce the quotient as a succession of components attached to a sequence of measuring units. The focus of the prescription is on the meaning of the corresponding part of the result in terms of units.

As a result, we have reached a conclusion about the older execution of division: decimal expansions without measurement units are used as *tools* in the execution. More precisely, decimal expansions embody the number system into which one should transform a dividend and a divisor before initiating the production of the quotient. By contrast, the quotient does not seem to be produced in the same number system, but rather *directly* as a measurement value. Moreover, at each Phase 3, the decimal expansion of the dividend is modified, but as decimal expansion, as it seems.⁴¹ The number system used for the operands *in this part of the execution* seems to be different from the system of quantification with which the quotient is produced, whether the system of measuring units is decimal or not.

⁴⁰ For *Mathematics*, see slips with excavation number 0646+0458+0303; and editors' numbering 79-81, in (Zhu and Chen 2011: 12-13, 76-77); see also (Xiao Can 2010: 52), and for *Mathematical Procedures*, see (Chemla and Ma 2011).

⁴¹ Note that here, the procedures given in *Mathematical procedures* to transform a multiplication such as the one in text 4 (multiplication by 24 deriving from the relationship between the measurement units *liang* and *zhu*) into a sequence of multiplications by factors smaller than 10 (here, successively, by 8 and by 3) could be used. Also note that the transformation into decimal expansion is what Ma Biao and I had interpreted was at stake in the available slips of *Mathematical procedures*, and we put forward the hypothesis that "ten *shi* 十" was used verbally to refer to this transformation. For these aspects, see (Chemla and Ma 2011).

These pieces of evidence confirm the assumption that I put forward in Section 4.1, with respect to text 1, from *The Gnomon of the Zhou [Dynasty]*. What all these texts show is *not* a sequence of divisions, but the successive sub-procedures of the execution of a single division that is devoted to the production of a quotient as a measurement value. The iteration we observe is the iteration of the key sub-procedure in the old execution of a division, and not a succession of different divisions, as we would be tempted to believe if we projected our mathematical practice of division onto these texts. We think of a division as an operation producing a number attached to a measuring unit only at the end of the process. Here the *nature* of the result is different, and its *components* are different as well. Misunderstanding this fact might explain why historians have so far failed to recognize that we had in effect here an actual algorithm for the execution of a division.⁴²

In addition, on the basis of the modes of prescription of division, and the evidence provided by the algorithm for simplification of fractions in *Writings on mathematical procedures (Item A2)*, slips 16 and 17) and in *The Nine Chapters (Item B2)*, I have argued above that, probably, in each sub-procedure of such a division, the production of the part of the quotient under construction was obtained by repeated subtraction, including perhaps also multiplication by powers of ten.

Finally, we have noted that in *The Gnomon of the Zhou [Dynasty]*, the multiplications of dividends by ten that these processes used were carried out by actual multiplication and *not* by shifts of the rods representing the number. In fact, the same observation holds true for all the manuscripts. Nowhere in these documents is a multiplication or division of a number by a power of ten carried out by a shift. This phenomenon is correlated with the presence in both *Mathematical procedures (Item A5)* and *Writings on mathematical procedures (Item A2)* of tables of multiplication between powers of ten.⁴³ These facts make it in my view highly improbable that the decimal system with respect to which dividend and divisor were expressed in relation to all these documents be place-valued. Or, if it was, positions were used in a way different from what is evidenced later.

5.2. Phases 2 and 3 vanish from The Nine Chapters

All these features stand in sharp contrast to what we read in *The Nine Chapters (Item B2)*, in particular in relation to the operation *chu* and related operations.

We have already emphasized that in this context, numbers were represented on the calculating surface using rods in such a way that shifting a number leftwards or rightwards by one column carried out, respectively, a multiplication by 10 or a division by 10. We have taken these features as hints that the underlying number system in the execution of the operations as described in *The Nine Chapters* was place-valued. This remark shows how the material properties of the system are reflected in the way in which operations are carried out. This conclusion holds true much more generally.

The main feature to which I now want to call attention is that in *The Nine Chapters* we never find an alternation of Phases 2 and 3. In the first place, this is a textual phenomenon. Never is a prescription of the type “in the dividend, what is like the divisor

⁴² In the historiography of mathematics in ancient China, the only document that has been perceived as describing the execution of a division is *Mathematical Classic by Master Sun*. Otherwise, historians consider we have no earlier evidence about the execution of divisions.

⁴³ The reader is also reminded that in the division from *The Gnomon of the Zhou [Dynasty]* that we observed above, the decimal parts in *bu* of the quotient were treated, and produced, in the same way as components related to a measuring unit are.

yields one (measuring unit)” repeated in the way we have observed above. Such expressions are used once and for all, to prescribe a division. They sometimes specify a measuring unit. Most often, they don’t. As I have suggested above, in *The Nine Chapters*, these expressions seem to have lost their original meaning and to simply prescribe a division.⁴⁴

In my view, these facts confirm the promotion of a new way of dividing, whose presence is made manifest by the shift in terminology I have described above, that is, the new use of the term *chu alone* to prescribe division.

Despite the fact that the process carrying out division *chu* is not made explicit in *The Nine Chapters*, I have indicated above how in my view division was executed at the time. Its main features can be approached on the basis of the fact that the executions of two operations *based* on the process of *chu* are described at length. In the context of *The Nine Chapters*, these operations, that is, square root extraction and cube root extraction, are in a revealing way prescribed as “*divide chu* by extraction of the square root” and “*divide chu* by extraction of the cube root.” The process of their execution (see text 11, and Chemla 1994b) is described as making use of a dividend, a divisor, producing a “quotient.” It can be correlated step by step with the execution of *chu* as it is described in the *Mathematical Classic by Master Sun (Item B3*; for an example, see the reconstruction of the division to which Zhao Shuang alludes in Section 4.2). The correlation between the two processes of execution allows us to capture the execution of the division in parallel to which they are described. It also gives evidence about the layout of the executions, on the surface on which computations were carried with rods. This reveals the key part played by positions (not only the positions for the digits, but also the positions associated with operands) in executing operations and also in establishing relations between them. In other words, our textual evidence indicates a mathematical practice of relating operations, by means of text, layout and processes of execution. To my knowledge, none of these features occurs in the earlier texts. By contrast, they are regularly evidenced in writings written until at least the 13th century (Chemla 1994b), pointing out aspects of the new culture of computation in the framework of which the new way of dividing is to be evidenced.

⁴⁴ In some places in the *Mathematical Classic by Master Sun (Item B3*, Problem 20 of last chapter (Qian 1963: 312-313)), or in the *Mathematical Classic by Zhang Qiujian (Item B4*, Problem 26 of first chapter (Qian 1963: 351-352)), one finds procedures specifying a process of division as an alternation of Phases 2 and 3 in relation to the production of the quotient as a measured quantity. Interestingly enough, they always occur in relation to a quotient produced with respect to the sequence of measuring units for weight. That is to say, they are only used when the gap between successive units of the measuring system is greater than ten. Note that this is precisely the situation dealt with in the passage from *The Gnomon of the Zhou [Dynasty]* analyzed above, and this is the point on which the commentators Zhao Shuang and Zhen Luan either compare the algorithm used to the algorithm *chu*, or offer the algorithm *chu* as an alternative. Moreover, in these cases, the *Mathematical Classic by Master Sun* or in the *Mathematical Classic by Zhang Qiujian* prescribe the divisions producing the successive components of the quotient using the verb *chu* repeatedly (and not a combination of *chu* and other expressions as in *The Gnomon of the Zhou [Dynasty]*.) In other words, the part of the quotient in relation to each of the measuring unit is produced by means of a division *chu*. Archaisms of this type exist in the *Ten Classics of Mathematics*. Another example is given in (Chemla and MA 2011). I will return to them in another publication.

We had emphasized above that the *text* of *The Nine Chapters* did not contain traces of an alternation of Phases 2 and 3. Restoring the *execution* of a division *chu* to which *The Nine Chapters* alludes allows us to see from another angle, and thus confirm, that divisions no longer proceed using the alternation of Phases 2 and 3 to which earlier documents attest. The process corresponding to the division *chu* is still carried out by iteration. However, now, the divisor is constantly shifted and the dividend goes only diminishing (the dividend does not expand and decreases in alternation).

In fact, the main clue highlighting the properties of the number system on which division *chu* operates comes *precisely* from the management of the divisor in the process of the new execution. This shows a correlation between the two main features of the execution, for which we pointed out a change (the number system and the shift of the divisor). Note that, for the shifts of the divisor to be meaningful, the way the other numbers, with which the divisor interacts, are written must also be place-valued.

If we rely on the parallel with root extractions, the execution of division to which the compilers of *The Nine Chapters* (**Item B2**) refer, like those of root extractions, operates on a dividend and a divisor represented with rods as a decimal expansion, using a place value system. The quotient (or the root) is also produced, through an iteration of the core sub-procedure, as a decimal expansion, that is, a succession of decimal components.⁴⁵ The mode of production of the quotient confirms the new role played by the decimal system. It further shows that, by contrast to what we have showed above, the dividend, the divisor and the quotient of a division now appear to be written, operated on, modified and produced in a homogeneous way, more precisely in a same decimal system. We thus see a radical change in the nature of the operands and the quotient, as well as in the technology used to carry out operations. These are other features that also signal in my view a culture of computation different from what is attested to in previous writings.

In addition, there is evidence showing that in this context, it is only *after* the core process of division *chu* has yielded the quotient as a decimal expansion, that, at the *end* of the process, the quotient is turned into a measurement value. Indeed, *The Nine Chapters* contains clues about this final step in a context in which work with measurement values, and specifically weight (non decimal) measuring units, is at stake.⁴⁶ In the context of the procedure given in relation to Problems 2.39 to 2.43, two values have been obtained as a decimal expansion in the dividend position and the divisor position of a division. They are subsequently transformed into measurement values using a prescription that reads as follows:

⁴⁵ Division and root extraction in the *Mathematical Classic by Master Sun*, as well as root extraction in *The Nine Chapters* bear witness to this feature.

⁴⁶ 24 *zhu* = 1 *liang*, 16 *liang* = 1 *jin*, 30 *jin* = 1 *jun*, 4 *jun* = 1 *dan*. See (Chemla and GUO 2004: 252-257) and related footnotes. Analysis is developed in my introduction to chapter 2 (Chemla and Guo 2004: 208-210). In this context, the division is the basis for solving problems in an unusual way, on which I do not dwell here. Later Classics, inserted in the collection *Ten Classics of Mathematics*, also provide similar evidence about the final production of quotients as measurement values. Interestingly, such pieces of evidence always occur when weight measurement units are in play.

“其求石、鈞、斤、兩，以積銖各除法、實，各得其積數，餘各爲銖。 If one looks for (the result) in *dan, jun, jin, liang*, one divides, respectively, divisor and dividend by (successively) the *zhu* that are the products, which respectively yields the corresponding value of the product, and the remainders being respectively taken as *zhu*.”

The technical term *ji zhu* “the *zhu* that are the products” refers to how many *zhu* (that is, the smallest weight measuring unit) each larger measurement unit in the weight system contains.⁴⁷ Dividing the decimal expansion by, successively, “the *zhu* that are the products” for the *dan, jun, jin, liang* transforms the decimal expansions in the dividend and the divisor into a measurement value expressed with respect to the sequence of measurement units, and the remainder in the end is the number of *zhu*. This procedure illustrates how, in *The Nine Chapters*, the quotient, obtained in the form of a decimal expansion, is finally converted, by means of other divisions, into an expansion with respect to a sequence of measuring units. Strikingly, we have no hint that such a procedure was used at the time when the manuscripts were composed.

5.3 Related theoretical transformations evidenced in The Nine Chapters

As alluded to in Section 3, the transformation to which both the new terminology and the new execution for division testifies for Phases 2 and 3 goes hand in hand with other theoretical changes related to division *chu*. Most interestingly for the history of mathematics in my view, a scaffolding of operations is built on the basis of this new division, including in particular square root extraction.

The first element in the shaping of this scaffolding relates to square root extraction. *Writings on mathematical procedures (Item B2)* attests to a procedure used to carry out square root extraction (see text 10). Its name in the context of the manuscript, *Fang tian* “Finding the side of a square cropland” 方田, shows no relation with an operation of division.⁴⁸ Moreover, this procedure is not arithmetic in the sense that it does not rely on the way in which the quantity whose root is sought for is written down to yield the root.⁴⁹ Accordingly, in this form, one cannot speak of “root extraction” as an arithmetical operation. By contrast to this, in the mathematical knowledge to which *The Nine Chapters* attest, square root extraction has become an arithmetic operation, which additionally is tightly related to division (see text 11). It relates to division in the way in which it is prescribed, in the way in which it is executed, and finally in the way positions are used in the execution.

Further, *The Nine Chapters* testifies to emergence of cube root extraction, which is conceived of as a form of division, in exactly the same three respects as square root extraction. Additionally, the solutions of systems of linear equations and quadratic equations

⁴⁷ 24 *zhu* makes 1 *liang*, 384 *zhu* makes 1 *jin*, 11520 *zhu* makes 1 *jun*, 46080 *zhu* makes 1 *dan*.

⁴⁸ *Mathematical Writing. Text A (Item A4)* contains two procedures related to this “square root extraction.” The same name occurs in one of them (Han Wei 2013: 38). This shows the stability of the operation and undermines the hypothesis that *Writings on mathematical procedures (Item A2)* would contain an ad hoc procedure.

⁴⁹ I have developed this definition of “arithmetic” in (Chemla 1994a).

are both addressed in *The Nine Chapters*, and also dealt with as types of division.⁵⁰ It is in all these mathematical contexts (and only in these contexts) that in *The Nine Chapters*, one meets with the most massive use of the term *chu* meaning division and terms related to division, as well as with the massive use of tabular displays and positions (with a mathematical meaning) on the surface on which computations were carried out with counting rods. The former feature relates to knowledge whereas the latter relates to mathematical practice.

Both features are much less prominent in chapters of *The Nine Chapters* that seem to be older, or at least for which the Classic has stronger connections with the manuscripts excavated.⁵¹ This suggests the assumption that some features of the new culture of computation (use of positions, including the place-value number system, and interest in relation between operations) took shape in relation to division and related operations, before some of them became used more widely across mathematical practice. This is what can be assumed from the fact that the commentaries on both *The Nine Chapters* and *The Gnomon of the Zhou [Dynasty]* all use division *chu* and shifts of numbers written down with rods to multiply and divide by powers of 10 much more liberally and in a much wider way than what the Classics attest to.

To summarize, the vanishing of the alternation of Phases 2 and 3 from *The Nine Chapters* appears to go along with several changes. Some of these changes relate to the actual execution of the division and the number system. Others show theoretical developments linked to the operation of division *chu*. Still others relate to elements of mathematical practice of the execution of and work on operations on the calculating surface. In all three, positions on the calculating surface play a part. The apparent conjunction between these changes also suggests that they might have first occurred only in the context of division and related operations, before affecting more widely other domains of mathematics and mathematical practice.

6. Focus on the history of Phase 1 of a division: determining the operands

So far, we have focused on Phases 2 and 3, which we have brought to light through an analysis of a division in *The Gnomon of the Zhou [Dynasty]* (Item B1). We now turn to Phase 1, which in this Classic is prescribed using a single verb, that is, “*tong* 通 make communicate.”⁵² In this context, the operation to which the verb refers takes a dividend and a divisor, expressed using measuring units and a fraction, and transforms them both into a decimal expansion without measurement unit (see Section 4.1). Phases 2 and 3 will then rely on these decimal expansions to proceed.

In Section 4.1, I have claimed that the decimal expansions in question were in a sense “the smallest integers” that could be used to carry out division. What did I mean, and what is the evidence available to support this claim? Right at the outset, let me discard an interpretation that one might be tempted to offer: To my knowledge, at the time when this

⁵⁰ As mentioned above, (Chemla 1994b, 2000) discuss the relationship between these other operations and the division *chu*. (Chemla 2014) concentrates on the conceptual history of arithmetical operations and to the theoretical work required to constitute the scaffolding of operations alluded to here.

⁵¹ I will return to this issue in another publication.

⁵² On this term, see my glossary (Chemla and GUO 2004: 994-998).

passage of *The Gnomon of the Zhou [Dynasty]* was composed (and before), we have no *explicit* evidence that the decimal expansions into which dividend and divisor were first transformed were *simplified* before setting out to produce the quotient.⁵³ The only evidence we have in this Classic indicates that measuring units and denominators were eliminated to yield decimal expansions in place of the original dividend and divisor. By contrast, in this section, we will see that the manuscripts provide evidence that actors aimed at transforming the original dividend and divisor into decimal expansions as short as possible (in a sense that we will now make clear), and we will discuss how this notion was put in practice in actual algorithms.⁵⁴

⁵³ In the last centuries BCE, we have only indirect evidence that this might have been the case. As early as the time when the manuscripts were produced, the way in which operands in rules of three were put into play seems to indicate that simplification was used before dividing. The mathematical practice of the operation “Reducing the width” in all the manuscripts evidences a concern for simplifications that control the size of the finer units, in cases when dissections into smaller units keep accumulating, and thereby keep the decimal expansion of dividend and divisor as small as possible (see, for instance, **Text 7**, and more generally, Peng 2001: 116-123). However, we have no direct evidence on this point. Text 9 presents a procedure from *The Nine Chapters* describing how to carry out a division between quantities that are integers increased by fractions. We return to it below. *The Nine Chapters*, like *The Gnomon of the Zhou [Dynasty]*, prescribes to “make them communicate.” Like Zhao Shuang’s, Liu Hui’s commentary on this text (text 9, **lines** ***121-130) first explains the concrete arithmetical operations to which the prescription refers, and also makes their meaning explicit. The third century commentator Liu Hui, then, appears to address precisely the issue of simplification, when he states dividend and divisor ought be simplified, that is, in the terms he introduces, they must become “相與率 *xiangyulü lü* put in relation with each other” (on this notion, see Li Jimin (1982a and 1982); Guo Shuchun (1984); see also my glossary in (Chemla and Guo 2004: 1012).) This remark occurs in a context in which the commentator accounts for the correctness of the procedure of division as prescribed by *The Nine Chapters*. While developing his interpretation of the meaning of “making communicate,” Liu Hui introduces with full generality the concept of *lü*, that is, a concept referring to values that are only defined up to a factor. The first example of such *lü*’s that he gives relates to the context dealt with: It is provided by the values of a dividend and a divisor brought into relation in an operation of division. The fact that such a pair of dividend and divisor are *lü*’s justifies that they can be both multiplied by a same factor and thereby simultaneously become integers. It also accounts for the fact that the resulting values can be simplified. In Liu Hui’s commentary, this operation of simplifying dividend and divisor, before executing the division, is thus approached theoretically, and leads to the introduction of a concept referring to the “the smallest decimal expansion”, that is, “*lü* put in relation with each other”. The commentary testifies to the fact that at some point in time, simplification before dividing was introduced

⁵⁴ Here, we touch upon an issue awaiting further research, that is, the issue of the actual length of the expansion of numbers dealt with in ancient sources. We have all too often assumed that all numbers were alike, since we practice numbers on the basis of a uniform system. It thus seemed to us that long expansions and short expansions, small quantities and large quantities, were the same. This assumption calls for revision. See C. Proust’s chapter

To begin with, *The Gnomon of the Zhou [Dynasty]* does not make explicit how the operation “make communicate” should be executed. Zhao Shuang’s commentary accounts for the reasons why in his view the operation is necessary, and he translates it into ordinary arithmetical operations: multiplying both dividend and divisor by 8, the “divisor” or denominator. We find the same theoretical term in the text of several algorithms in *The Nine Chapters* (**Item B2**, see, for instance, text 9). If we follow Liu Hui’s interpretation, this term refers to a variety of procedures in relation to the problem dealt with. In the context of a division like the one considered, the two commentators agree with respect to the actual meaning of the prescription, and the values computed.

The essential fact is that the manuscripts contain many procedures that fulfill the same aim (transforming quantities expressed using measurement units and fractions into integers without measurement units), and they do so in different ways, depending on the numbers to which they are applied. Moreover, the actual procedures that the commentators understand the term “making communicate” refers to fit precisely with what the manuscripts evidence. However, in the Classics and in the manuscripts, the *prescription* is made using different textual means, which most importantly have different meanings. This will lead to an important conclusion: Although for Phases 2 and 3, *The Gnomon of the Zhou [Dynasty]* presented similarities with manuscripts, for Phase 1, it is correlated more closely with *The Nine Chapters*. Let us now examine these issues in turn, and in greater detail.

If we look at the manuscripts (**Items A**) from this perspective, we discover that **Phase 1** is the object of a great number of procedures in them. These include the following procedures that give their names to sections in *Writings on Mathematical Procedures*: “Directly sharing *jingfen*” (examples in text 5, text 6), “Reducing the width *shaoguang*” (example in text 7; this procedure represents perhaps an ancient state of the division with fractions), “Erroneously inscribing on a certificate *wuquan* 誤券 (text 8), *danlü* (石率, which I translate for the moment as “lü-ing on the basis of the *dan*,” and to which translation I return below).⁵⁵

At first sight, all these procedures might seem to deal with the execution of the operation of division as a whole. In fact, it appears that their purpose is, more precisely, to determine how to transform dividend and divisor into decimal expansions, depending on which fractions and which measuring units they involve. In the terms I have introduced, they are thus devoted to **Phase 1**. Such an interpretation might help explain why these procedures are not systematically concluded by the prescription of later Phases of division (for example, text 5, or the section “lü-ing on the basis of the *dan*”). Perhaps were they even perceived as independent operations. The fact that dividend and divisor need to be transformed before division *stricto sensu* is carried out might further explain why there are technical terms for the operands of division, and only for them.

on the computation of areas in C. Michel et al. (eds.), *Mathematics and Administration in the ancient world*.

⁵⁵ On the latter procedure, and the theoretical turn to which the related procedures in *The Nine Chapters* attest, see (Chemla 2006). These procedures are formulated in the context of problems that seem to be specific to each of the cases dealt with. I return to these problems in another publication, in the context of a wider inquiry into the use of problems in the manuscripts.

In what follows, we will examine the evidence provided by the manuscripts about **Phase 1**, and we will then show how this part of division seems also to have been later the object of a theoretical work different from what we have observed above.

6.1 Phase 1 according to the manuscripts

The procedure, with which we begin, deals with a topic to which a relatively huge space is devoted in *Writings on mathematical procedures (Item A2)* as well as *Mathematics (Item A1)*. Comparatively, the importance of the mathematical issue addressed using this topic decreases notably in *The Nine Chapters* (in terms of how much space is devoted to it) at the same time as it is reworked from a theoretical viewpoint.

Let us evoke this topic, relying on one of the general procedures attached to it, which is translated in text 8 and entitled “Erroneously inscribing on a certificate.” The specific type of problems covered by this general procedure is clarified by the numerous concrete problems that mention this task. They all focus on the tax to be paid in grain on a cropland having an area A , which is usually expressed in *bu* and sometimes in *mu* (where 1 *mu* equals 240 *bu*). The tax is computed with respect to a norm, which is expressed by the number n of *bu* for which a unit of grain is to be paid. Grain is in this context expressed by measurement units of capacity (that is, a decimal system). So, for instance, problem on slips 96-97 of *Writings on mathematical procedures* mentions a tax of “one *dou* per ten *bu*.” The cropland having an area of one *mu*, the tax that should have been paid was “two *dan* four *dou*.” However, due to an “erroneous inscription on a certificate,” the amount of “two *dan* five *dou*” was written down, and the problem asks to determine the norm to which this wrong inscription corresponds, that is, the number of *bu* for which one *dou* is paid in this context. The procedure solving the problem uses a division, taking the “allotted area” (expressed with measurement units for area) as the dividend and the measurement value wrongly inscribed (expressed with capacity measurement units) as divisor.

I claim that the main point of text 8, which is essential for our analysis of Phase 1, is the transformation of the dividend and the related divisor into decimal expansions without measurement units. This text has been misunderstood so far. This is what can be gathered from the fact that all critical editions seem to have given a wrong text for it. In the appendix, I offer a new edition for it, which relies on the mathematical analysis that I develop here.⁵⁶ It is easy to perceive the importance of this procedure with respect to our topic, if we consider the following statements it contains about measurement units for capacity (remembering that 1 *dan* is 10 *dou*, and 1 *dou*, 10 *sheng*): “taking the *dou* of the certificate as one, taking the *dan* as ten” and, later, “taking the *sheng* of the certificate as one, taking the *dou* as ten.” Clearly, the procedure centers on the transformation of measuring units in quantities into decimal expansions. How it does so is interesting for us.

In the edition I offer for this general procedure, the procedure considers two cases. Either the quantity inscribed on the certificate does not contain the unit of capacity smaller than the *dou*, that is, the *sheng*, or it does.

For the specific problem evoked above, in which the wrong inscription consists of “two *dan* five *dou*,” the quantity has no *sheng*. The division prescribed by both the general and the specific procedures is the following:

⁵⁶ I plan a separate publication for the set of procedures that the manuscripts devote to this topic.

$$\frac{240 \text{ bu}}{2 \text{ dan } 5 \text{ dou}}$$

I claim that in actors' term, it amounts in fact to a rule of three. This interpretation derives from the other general procedure dealing with the same topic: *danlü* "lü-ing on the basis of the *dan*," in which the multiplication by 1 is specified (Chemla 2006). Here, the result sought-for is the amount of area corresponding to "1 *dou*", which can be understood as deriving from the following computation:

$$\frac{240 \text{ bu} \cdot 1 \text{ dou}}{2 \text{ dan } 5 \text{ dou}}$$

As a result of the application of the general procedure "Erroneously inscribing on a certificate," in the "two *dan* five *dou*," "taking the *dou* of the certificate as one, taking the *dan* as ten," the quantity in the divisor is transformed into "twenty-five", whereas the "1 *dou*" yields 1 and the area, in the dividend, is taken as "two hundred and forty." The general procedure stops here (it is limited to Phase 1), whereas the specific procedure prescribes the execution of the division:

$$\frac{240}{25}$$

will yield the number of *bu* for which a tax of "one *dou*" is paid. We thus see how, in this case, the general procedure transforms measurement values into decimal expansions without measurement units.

Another specific problem illustrates the second case of the general procedure "Erroneously inscribing on a certificate," that is, the case in which the wrong inscription has *sheng*. This problem, recorded on slips 68 and 69 of *Writings on mathematical procedures*, also deals with a cropland having an area of "two hundred and forty *bu*." However, instead of providing this area for the "allotted field," the outline mentions the area on which the tax is computed, that is, the "taxed cropland *shuitian*," which is one tenth of the former.⁵⁷ Hence, the area mentioned in the problem is 24 *bu*. Instead of inscribing the amount of tax that would have corresponded to the norm of "eight *bu* pays one *dou*," that is, according to the statement of the problem, "three *dou*," the practitioner recorded "three *dou* one *sheng*." The point is still to compute the norm corresponding to this wrong inscription, that is, for how many *bu* should the tax be 1 *dou*? Solving the problem requires executing the division:

$$\frac{24 \text{ bu}}{3 \text{ dou } 1 \text{ sheng}}$$

or, expressed as a rule of three,

$$\frac{24 \text{ bu} \cdot 1 \text{ dou}}{3 \text{ dou } 1 \text{ sheng}}$$

For these cases, for which the quantity has *sheng*, the general procedure "Erroneously inscribing on a certificate" prescribes: "taking the *sheng* of the certificate as

⁵⁷ See related footnote in the critical edition of **Text 8**.

one, taking the *dou* as ten.” In the rule of three, this corresponds to the following transformation:

$$\frac{24 \text{ bu} \cdot 10}{31}$$

This is precisely what the specific procedure corresponding to the problem prescribes. As for the general procedure under consideration (“Erroneously inscribing on a certificate”), for these cases, it suggests taking the “allotted cropland,” and *not* the “taxed cropland,” as “dividend.” The “allotted cropland” being ten times larger than the “taxed cropland,” this amounts to multiplying by 10. Interestingly, even though, from the viewpoint of the values, this corresponds to taking the same value, from the viewpoint of the operations, the multiplication by 10 is here again avoided. This supports again my hypothesis that at the time, multiplying by 10 was not immediate.

What the general procedure thus reveals is a *contextually determined* transformation of measurement values into decimal expansions without measurement units (precisely what we had observed in *The Gnomon of the Zhou [Dynasty]*). Moreover, the way in which the divisor is transformed satisfies to the principle that the value for the divisor, and accordingly the dividend, is the smallest integer possible that can be obtained by a general procedure. By this I mean the “shorter” decimal expansion. However, we do not have evidence of a general procedure that would enter into the details of specific simplifications.

The other general procedure of *Writings on Mathematical Procedures* that addresses similar questions is precisely *danlü* “lü-ing on the basis of the *dan*.” It deals with the computation of unit prices, that is, a problem similar to the previous one, except for the nature of the quantities and measurement units involved (Chemla 2006). However, the key difference is that this time, the unit prize is sought-for *not* for an *intermediate* measurement unit in a sequence of units (like the *dou*), but for the *highest* measurement unit (the *dan*). Interestingly enough, at the time when the manuscript was produced, *dan* was the highest measurement unit for the system of capacity units as well as weight units (Chemla and Ma 2015 (2014)). For capacity units, the procedure thus complements the one analyzed above.

Read against the background of *The Nine Chapters (Item B2)*, the procedure “lü-ing on the basis of the *dan*” puts into play a rule of three, which relies on the question being formulated as follows: if an amount of cash was exchanged for a given amount of something, what is the price of the *dan*? The structure of the procedure can be represented as follows:

$$\frac{\text{quantity of cash exchanged for an amount of something multiplied by 1 dan}}{\text{amount}}$$

The aim of the procedure, however, remains the production of decimal expansions for the dividend and the divisor, on the basis of which Phases 2 and 3 can proceed. Since cash is not expressed using a whole sequence of measurement units, this task requires mainly transforming the two amounts expressed with respect to weight measurement units (the amount exchanged, and 1 *dan*) into decimal expansions. This is what the procedure does, and it transforms these two amounts in correlation with each other. The key point for us is that the correlative transformation of the unit (1 *dan*) and the divisor is carried out with respect to exactly the same principle: as a result, the decimal expansion obtained for the divisor, and accordingly the dividend, is the smallest possible, in the same sense as above. Interestingly, *Mathematics* contains a complete table of transformation of measurement

units of weight into one another.⁵⁸ Like for Phase 3 (see Section 5.1), this table would be exactly what this procedure would require, in case the quantity in the divisor was expressed in measurement units of weight.

When compared with the other procedures examined above, this other general procedure (“lü-ing on the basis of the *dan*”) presents three specificities that are of interest for us here. First, as we have already emphasized, the structure of the procedure makes the rule of three explicit and, as a result, the correlative transformation of the unit in the dividend and the divisor is systematically brought to light. The correlation is also systematically emphasized using the particle “亦 *yi* likewise.” Second, it makes clear that the meaning of the transformation of the divisor and the unit in the dividend corresponds to a dissection of the units (the term “破 *po* disaggregate” is used to refer to these operations as dissections, see Chemla 2013). Third, the overall procedure encompasses, not only the cases with multiple measurements units, but also the case when the quantity involved contains fractions. Both types of sub-units are dealt with, using *po* “disaggregate”, and their transformations are thereby shown to be of the same nature.

Several other procedures are given in *Writings on mathematical procedures (Item A2)* as well as in *Mathematics (Item A1)* to deal with the case of fractions specifically. Text 6 evidences the fact that the transformation applied to dividend and divisor to transform them into decimal expansions is likewise perceived as a correlative dissection of units into finer units that allow practitioners to eliminate fractions (Chemla 2013). Texts 5 and 7 illustrate how this dissection is carried out, on both dividend and divisor correlatively, using the coarsest unit possible. In particular, text 7 shows how the least common multiple of all denominators is put into play to this end.⁵⁹

In conclusion, whereas, as far as computations are concerned, the procedures put into play differ, depending on the measurement units involved and the fractions in play, *Writings on mathematical procedures* and *Mathematics* abound in procedures whose aim is to carry out Phase 1. They do so differently, but obeying the same general principle with respect to the numbers produced, that is, to keep control of the size of the decimal expansions of the obtained dividend and divisor. The computations in play also follow the same principle from the viewpoint of their meaning, that is, that they are all understood as dissections.

6.2 Theoretical transformations in the conceptualization of the procedures

The procedures just examined were for their main part included in *The Nine Chapters*. Further, as was already mentioned above for some of them, the way of proceeding to fulfill the same task did not change as far as the operations put into play are concerned, if we rely on the testimony of the Classic and the interpretation suggested by the commentators. Let me emphasize that this fact stands in contrast to what we had concluded above about Phases 2 and 3.

However, two key changes are nevertheless attested in the later texts with respect to the procedures used in the earlier texts to carry out Phase 1. These changes are worth attending to. They both testify to the fact that, as we have seen for Phases 2 and 3, Phase 1

⁵⁸ See slips with excavation number 0646+0458+0303; editors’ numbering 79-81, in Zhu and Chen 2011: 12-13, 76-77; Xiao Can 2010: 52.

⁵⁹ On this procedure, see my introduction to chapter 4 (Chemla and GUO 2004: 313-322).

has also been a topic of reflection, through which theoretical innovations appear to have been devised.

To detect the changes, we must observe the *texts* of procedures and the *organization* of mathematical knowledge in the Classic. We thus need to attend to textual phenomena. From this viewpoint, two main types of clues evidence the directions followed in the reflections on Phase 1.

To begin with, the texts through which the procedures were described were reformulated in later texts. When comparing the earlier formulations to the later ones, we see that the ways in which the prescription of operations was rewritten highlighted more general concepts identified in the working of the procedure. Let us, for example, observe text 9, quoted from *The Nine Chapters (Item B2)*.⁶⁰ It corresponds to procedures executing an operation with the same name, “Directly sharing,” in *Writings on mathematical procedures (Item A2)*. Text 5 illustrates the type of text contained in *Writings on mathematical procedures* for the same operation. We see that by contrast, in *The Nine Chapters* the prescription now puts into play prescriptions like “making communicate” and “equalizing” (lines *** 112-113, 127). As was mentioned above, the commentator Liu Hui translates these prescriptions into the same operations as what we read in the procedure of *Writings on mathematical procedures* (lines *** 117-124). The key point here is that, since Liu Hui accounts *at the same time* for the *correctness* of the procedure, he highlights the relationship between the “reasons” of the correctness and the terms used in *The Nine Chapters* to prescribe an execution. For instance, the prescription “making communicate” relates, for him, to the fact that dividend and divisor, being brought into communication through the fact of entering the same operation of division conjointly, must then be transformed conjointly (lines *** 122-124). In Liu Hui’s view, the term not only reflects the reason of their joint transformation in the algorithm, but also prescribes this transformation. This corresponds precisely to the joint multiplication by 8 of dividend and divisor that Zhao Shuang commented upon, in the text 1 of *The Gnomon of the Zhou [Dynasty] (Item B1)*, which we analyzed earlier (lines ***12-15, 17-18, 21-22). It is prescribed in the same way in *The Gnomon of the Zhou [Dynasty]* and *The Nine Chapters*.

We can thus derive two conclusions. First, relying on the commentaries, we see in which respect these new modes of prescription testify to a theoretical work. Phase 1 is now encapsulated in a few theoretical terms. Second, we also note that terms like “making communicate” are already used in *The Gnomon of the Zhou [Dynasty]* and interpreted by the commentator in the same way as Liu Hui does. As far as we can perceive it, this seems to point to an earlier development of theoretical reflections on Phase 1 of division and one that is in continuity with *The Nine Chapters*. As was already emphasized, in this respect *The Gnomon of the Zhou [Dynasty]* is in continuity with the later Classics, and not with the manuscripts.

I have investigated the terms used in the procedure for “Directly sharing” in greater detail elsewhere. I will not develop the analysis any further here. I simply add that, in his

⁶⁰ For a detailed analysis, I refer the reader to (Chemla and GUO 2004: 166-169), and (Chemla 2012: 462-471). Note that in *The Nine Chapters*, the character which I interpret as “directly” is a homophone of that found in *Writings on Mathematical Procedures*, but it is written differently. However, the gloss given by the 7th century commentators corresponds to what we read in the manuscript, which incites me to follow the commentators’ interpretation in this case.

discussion on the *operation* “make communicate,” Liu Hui also introduces a concept referring to *objects* that, “being in a state where they are in relation with each other, communicate” and must thus be transformed conjointly, that is, the concept of *lū* (**lines** *** 122-124). It is precisely with *this* concept that the authors of *The Nine Chapters* rewrite the text of the procedure corresponding to those analyzed in Section 6.1, that is, “Erroneously inscribing on a certificate” and “*lū*-ing on the basis of the *dan*”—this is the point where I return to the translation of the latter expression.

In *The Nine Chapters*, the quantity obtained (whether it is a tax or an amount exchanged) and the unit sought-for are both referred to as *lū*. Again in this context, the commentators establish that the term *lū* is used to designate quantities in order to indicate that they should be transformed conjointly. In effect, the results of the transformations are decimal expansions, and the operations meant are identical to those prescribed in *Writings on mathematical procedures (Item A2)*. Moreover, the commentators also show how the term *lū* relates to the reason for the correctness of the procedure.

The nature of the theoretical work expressed by the formulation of the procedure, as well as the result of this work, are the same as what we have seen for “making communicate.” It is precisely this term *lū* that occurs in the title of the operation “*lū*-ing on the basis of the *dan*.” (Chemla 2006) establishes these procedures from the manuscripts are historically connected with those in the later Classics, showing that the procedure in *The Nine Chapters* was obtained by abstraction on the basis of those found in *Writings on mathematical procedures*. In the latter writing, the character translated as “*lū*-ing” denotes an operation, whereas in *The Nine Chapters* it refers to a property of the quantity thus designated. The theoretical work seems, nevertheless, to have begun at the time when the manuscripts were written and it is in continuity with what *The Nine Chapters* evidences. Exactly the same conclusion derives from the observation of how the procedure “Reducing the width,” quoted from the oldest mathematical manuscript that we can read (text 7), is rewritten in *Writings on mathematical procedures* and then in *The Nine Chapters*.⁶¹

We now see in which respect there is a turn in the conceptualization of the procedures between the manuscripts and the Classics. Further, this turn is related to a reorganization of the architecture of mathematical procedures, a concern that we have already encountered in the previous section, when we were discussing the scaffolding of operations created around the division *chu* (which involved root extractions and other procedures). In correlation with the rewriting of the text of procedures carrying out Phase 1 to which *The Nine Chapters* testifies, their set is restructured into only two procedures, respectively called “Directly sharing” and “Directly *lū*-ing.” We thus see that the names of the operations carried out now reflects a relation between them. The concepts of “making communicate” and *lū*-ing are central in accounting for this relation. Further, the procedure “Directly *lū*-ing” is now itself shown to derive from a more general operation, corresponding to what we call the rule of three (Chemla 2006). Without entering any further into detail here, we see how the mathematical work reveals a reflection on the relationship between operations, similar to what was outlined in Section 5.3. This work is expressed by the way of

⁶¹ The term *jifen* “parts of the product” occurs in the abstracted procedure inserted in *Writings on mathematical procedures*. In this context, it is a verb, whereas in *The Nine Chapters* it is turned into a noun, designating a key concept in the way of proceeding. The Classic also introduces the operation “making communicate” in the description of the procedure. See (Chemla and GUO 2004: 342-345).

organizing chapters in *The Nine Chapters* as well as by the mode of prescribing the procedures.

7. Conclusions

In this chapter, I have relied on a document found in *The Gnomon of the Zhou [Dynasty]* (Item B1) to distinguish between four types of phases in the execution of division. These four type of phases have given us a guideline to inquire into a segment of the history of the execution of division that took place in China, and also the history of related theoretical issues.

A first conclusion derives from my exploration: These four types of phases had distinct histories. However, for each of them, one can perceive that manuscripts, on the one hand, and *The Nine Chapters* (Item B2) and later Classics, on the other, testify to stable differences in knowledge and mathematical practice.

At the end of our analysis, we understand better characteristic features of the older execution of division to which manuscripts testify. To begin with, the same inspiration drives the execution of the four types of phase, that is, they all carry out a change of units in one way or another. This thus emerges as a crucial operation.⁶² Let us review part of our evidence for this.

We have seen that Phase 1 carried out a joint transformation of dividend and divisor given as quantities expressed using a sequence of measurement units into the “smallest” decimal expansion, in a sense that we have discussed. The joint character of the transformation is constantly marked by the use of the particle 亦 “likewise” (e.g., Text 6). The transformation into decimal expansions was evidenced by the use of expressions such as “以石為十 One takes the *dan* as ten” (Text 8). (Chemla and Ma 2011) has claimed that it was marked by a verbal use of “ten 十”. The clearest expressions of the understanding of these transformations as changes of unit, and more precisely a dissection of units into smaller units, are found in Text 6 and also in “lü-ing on the basis of the *dan*.” They are prescribed using expressions like “One is taken as six 以一為六”, or a verb like “disaggregating *po* 破.” In other words, the reason why we perceive practitioners’ understanding of the operation is that the operation is prescribed using expressions that refer to the effect to be obtained, whether it be considered from the viewpoint of the numerical values, or that of their meaning.

Phases 2 and 3 also carry out changes of units, but in a different way. Phases 3 systematically expand the dividend in such a way that the decimal expansion of the divisor can be used to transform portions of the dividend equal to the divisor into a measurement unit. This measurement unit is in fact precisely what the expressions prescribing divisions in the manuscripts all make explicit. In fact, here too, the operations are prescribed using the statement of the reason underlying the operation: “what in the dividend is like the divisor yields one (measurement unit).” Accordingly, the quotient is produced *directly* as a quantity expressed with respect to a sequence of measurement units.

⁶² Evidence about Phase 4 in the manuscripts being also understood as carrying out a change of unit is discussed in (Chemla 2013).

Analysis also reveals how the ideas brought into play in Phase 4, as the manuscripts testify to them, are in continuity with those that lay at the basis of the previous Phases.⁶³

This explains the first characteristic feature that I have underlined: The manuscripts show how throughout the older execution division seems to have been understood as carried out by a key operation that was the change of units. The terminology provides innumerable hints supporting this thesis. Interestingly, changes of unit of this kind are regularly discussed in the commentaries on *The Nine Chapters* in a theoretical way.

By comparison with *The Nine Chapters* and later Classics, the older execution of division as reflected by the manuscripts presents further characteristic features from the joint viewpoint of the knowledge and the mathematical practices brought into play. This execution seems to make use of several types of numerical tables, samples of which are actually found in the manuscripts (tables of multiplications of powers of 10 like those found in *Writings on Mathematical Procedures (Item A2)* and *Mathematical Procedures (Item A5)*; tables of conversion between weight measurement units like those found in *Mathematics (Item A1)* and *Writings on Mathematical Procedures*; table of procedures to transform weight measurement units into smaller ones, like the one in *Mathematical Procedures* (Chemla and Ma 2011), which in fact also carry out transformations into decimal expansions). Moreover, this execution of division does not seem to make a central use of positions, and in particular no hint appears to disclose the use of a decimal place-value number system at the time.

This chapter was inspired by a remark, with which I began, namely a change in the terminology with which divisions were prescribed between the manuscripts and later texts, notably the use of the term *chu*. I have further argued that this change, which in our corpus, is clear starting from *The Nine Chapters* onwards, took place in parallel with a change in the execution of division as well as in related bodies of knowledge and mathematical practice.

Let me review the changes to which the new execution testifies, Phase by Phase, and summarize the key characteristic features of the later texts that we have encountered.

We have seen that in *The Nine Chapters*, Phase 1 is referred to using two key theoretical terms, that is, “make communicate” and *lü*. In other words, the operations to be applied to quantities have become encapsulated in terms referring to them using the reason to carry them out. Moreover, most procedures carrying out Phase 1 have been grouped and reorganized in a new way, under “Directly sharing”, “Directly *lü*-ing”, and “Reducing the width.” In each case, the commentaries make the operations meant explicit, indicating the relationship between these operations and changes of units. Further, they highlight the network of relationships indicated by the terminology. These are the various facets of what I take to be a theoretical shift with respect to Phase 1.

The outcomes of Phase 1 are a dividend and a divisor expressed using a decimal system. Phases 2 and 3 seem to be replaced by the new procedure referred to using the term *chu* alone. The shift in terminology is correlated with a shift in the new execution, characterized by the fact that now the divisor is set in motion. This in turn is correlated with a shift in the number system, which now appears to be not only decimal, but also place-valued. In other words, although like in the earlier periods, rods are still used to represent numbers, they are used in a new way. This is manifested by the key operation that reflects the place-value decimal system and exploits it: the shifts leftwards and rightwards of a

⁶³ (Chemla 2013). This general remark suggests a hypothesis on how in this context fractions were introduced in the way in which they were introduced.

divisor (moving forward and backward) to multiply or divide it by 10, which never occur in the manuscripts and earlier texts, but. In correlation with this, we further note a shift in how the quotient is yielded: it is produced no longer bit of measured quantity by bit of measured quantity, but decimal place by decimal place, and thereafter possibly transformed to be expressed according to a sequence of measuring units.⁶⁴

To recapitulate, the organization of the execution of the division appears as having been similar in the two contexts under consideration, but the operations to be carried out in the different Phases were reconceptualized, for Phase 1, and completely reshaped for Phases 2 and 3. I have also explained how the new execution of the division *chu* took place in parallel with another reshaping, reorganization and development of old and apparently new operations, resulting in the creation of a scaffolding of operations. So in this case too, transformation in execution is reflected in theoretical advancement.

So far, I have addressed the changes identified mainly from the viewpoint of the history of knowledge. However, I also argue that the corpus of texts examined also testify to a *correlative* change in mathematical culture.

The shift in the number system that I have described (and that seems to be closely linked with the new execution of division *chu*) appears to have taken place in the wider context of a more global shift in the use of the calculating surface. In my view, the place-valued number system uses positions (*wei*) in a context in which positions (also *wei*) became more generally central to the practice of mathematics. They were used to execute operations. They were also used with positions being given a mathematical meaning to express mathematical situations, for instance, in the case of systems of linear equations. In addition, positions will play a crucial role in the establishment of relationships between operations (which indicates another link between mathematical practice and theory). Last, but not least, this use of positions will flourish in later traditions that relied on the Classics (Chemla 2016).

In closing this chapter, I will formulate two general conclusions about what is at stake in studies of this kind.

First, the study I have presented shows in which respects, in my view, operations and their execution have been a topic of actors' exploration and reflection, and not only tools, as is often implicitly assumed. This exploration has concentrated on operations *stricto sensu*, on the texture of their execution, their theoretical dimensions, and their relationships to each other. Actors' work on these aspects appears to have also involved a reflection on the material practice of operations. In this context, I have suggested that some of the epistemic techniques that were shaped to deal with operations included elements such as the number system on which to carry them out, and more generally the shaping of practices with positions in the calculating surface. It is in this respect that I have spoken of "different cultures of computation in early China," and that actors appear to have shaped the cultures in which they operate in parallel with their shaping of mathematical knowledge.

Second, this exploration into the history of operations has consequences in how we approach sources of the past. This appears clearly for the case of *The Gnomon of the Zhou [Dynasty]*. We could establish that part of this Classic handed down by the written tradition adhered in several ways to the manuscripts. This indicates that the process of transmission of the Classics has not completely modernized the terminology and the practices.

⁶⁴ About the shift in Phase 4, see (Chemla 2013).

Moreover, we now see that this process has not even uniformized the Classic, since I have insisted that other parts of *The Gnomon of the Zhou [Dynasty]* contained terminology more characteristic of that of *The Nine Chapters* and later Classics than of the manuscripts. The third century commentator indicating, for all these cases, that the sections in which they occur did not belong to the “original” text, this allows us to make several points.

To begin with, the Classic appears to have been produced through compiling parts of text that had been written in different time periods. Even though this conclusion had been formulated long ago, using other tools, we insist that the parts that we now distinguish differ from what other historians have put forward. Moreover, the observation of the terminology for operations enables us to distinguish between parts that are the same as what the commentator claims. This observation invites us to rely more confidently on commentators’ testimony, and it also yields tools to rethink the critical edition at least partly.

Finally, we have seen that the relationship between the manuscripts and *The Gnomon of the Zhou [Dynasty]* differed depending on the Phase of the execution of division dealt with. Phases 1 and 4 present stronger ties with what we find in *The Nine Chapters* than what we read in manuscripts. On the other hand, Phases 2 and 3 are closer to manuscripts. However, a small difference might constitute a major hint: the quotient is produced along a sequence of measurement units and *also* decimal ranks. In this sense, *The Gnomon of the Zhou [Dynasty]* might testify to a shift in the use of decimal expansions that awaits further studies, and, much more importantly, new pieces of evidence.

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Appendix

NB: This appendix contains the main documents on which this chapter relies for its discussion of the history of the execution of division. It presents each document as follows: I first give the Chinese text (sometimes transforming the critical edition mentioned in relation to my own analysis). I then provide an English translation, in which bold and underlined characters insist on the key features of the documents that are discussed in the main text.

The texts are arranged following the organization of the article. Moreover, the first nine texts relate to division, and whereas texts 10 and 11 give evidence about root extraction.

Text 1, Division in *The Gnomon of the Zhou [Dynasty]* (Item B1). The introduction of the four types of phases in the execution. Edition used: (QIAN Baocong 1963, vol. 1: 52). In my view, this part of the text dates to *ca.* 100 BCE. Measurement units for distance used in the text: 1 *li* 里 = 300 *bu* 步. Note that the term *chu* (marked in bold characters) is used with different meanings and in expressions having a different syntax, in *The Gnomon of the Zhou [Dynasty]*, on the one hand, and in the commentaries, on the other.

(…) 其南北游，日六百五十一里一百八十二步、一千四百六十一分步之七百九十八。

術曰：置十一萬九千里爲實。以半歲一百八十二日八分日之五爲法，

(Zhao Shuang)半歲者，從外衡去內衡以爲法。除相去之數得一日所行也。

而通之，

(Zhao Shuang)通之者，數不合齊，常⁶⁵以法等，得相通入，以八乘也。

得九十五萬二千爲實，

(Zhao Shuang)通十一萬九千里。⁶⁶

所得一千四百六十一爲法，除之

(Zhao Shuang)通百八十二日、八分日之五也。

實如法得一里。不滿法者，三之，如法得百步。

⁶⁵ GUO Shuchun and LIU Dun (1998: 41), note 132, remark that all ancient editions have *chang* here, except the Ming edition. QIAN Baocong (1963, vol. 1: 52) does not have this character. Nor does he include a note. GUO Shuchun and LIU Dun (1998: 41), note 132, suggest this is the correct reading. They accordingly suggest another punctuation, which I follow here.

⁶⁶ Here, as in many other places, in the editions linked to the *Siku quanshu* 四庫全書 (Complete Library of the Four Treasuries) enterprise at the end of the 18th century, Dai Zhen 戴震 includes a slightly different way of expressing the same amount: “一十一萬九千里。” The Song reading probably derives from a more ancient document. However, note that this feature is not systematic in Dai Zhen’s edition.

(Zhao Shuang)一里三百步。當以三百乘，而言三之者，不欲轉法，便以一位爲百實。故從一位命爲百。

不滿法者，十之，如法得十步

(Zhao Shuang)上不用三百乘，故此十之，便以一⁶⁷位爲十實。故從一位命爲十。

不滿法者，十之，如法得一步

(Zhao Shuang)復十之者，但一位爲實。故從一位，命爲一。

不滿法者，以法命之。

(Zhao Shuang)位盡於一步，故以法命其餘分爲殘步。

臣鸞曰：求南北游法：置冬至十一萬九千里，以半歲日分母八乘之，得九十五萬二千爲實。通半歲一百八十二日、八分日之五，得一千四百六十一。以除得六百五十一里。不盡八百八十九，以三百乘之，得二十六萬六千七百。復以法除之，得一百八十二步，不盡七百九十八。卽得日南北遊日六百五十一里一百八十二步、一千四百六十一分步之七百九十八。

⁶⁷ GUO Shuchun and LIU Dun (1998: 41), note 134, indicate that this “一 yi one” was added by Dai Zhen in the editions carried out in relation to the *Siku quanshu* enterprise (see Dai Zhen’s note). They adopt this suggestion.

1 ITS (I.E., THE SUN'S) NORTH-SOUTH DISPLACEMENT PER DAY IS SIX HUNDRED AND FIFTY-ONE *LI* ONE HUNDRED
2 AND EIGHTY-TWO *BU* SEVEN HUNDRED NINETY-EIGHT ONE THOUSAND FOUR HUNDRED SIXTY-FIRSTS OF A *BU*.
3 PROCEDURE : ONE PUTS (ON THE CALCULATING SURFACE) ONE HUNDRED AND NINETEEN THOUSAND *LI* AS
4 DIVIDEND.⁶⁸ ONE TAKES HALF A YEAR, ONE HUNDRED AND EIGHTY-TWO DAYS FIVE EIGHTHS OF A DAY,⁶⁹ AS A
5 DIVISOR,
6 (Zhao Shuang's commentary) The reason why one takes "half a year" is that one takes the
7 (time required for the sun to move) from the outermost *heng* to the innermost *heng* as
8 divisor.⁷⁰ **Dividing (*chu*, by this, i.e., the time elapsed between the moments when the sun**
9 **was on one and then on the other) the quantity of the distance between them yields what is**
10 **travelled in one day.**
11 AND ONE MAKES THEM (I.E., THE VALUES IN THE DIVIDEND AND THE DIVISOR) COMMUNICATE,
12 (Zhao Shuang's commentary) The reason why "one makes them communicate" is that these
13 values are not homogeneous with each other, constantly making the divisor (i.e., the
14 denominator) equal results in making them communicate with each other, (hence) one
15 multiplies by eight.
16 WHICH YIELDS NINE HUNDRED FIFTY-TWO THOUSAND AS DIVIDEND,
17 (Zhao Shuang's commentary) this is making one hundred and nineteen thousand *li*
18 communicate.
19 THE ONE THOUSAND FOUR HUNDRED SIXTY-ONE WHICH IS OBTAINED MAKES THE DIVISOR, **ELIMINATING (*chu*)**
20 **THIS,**
21 (Zhao Shuang's commentary) this is making one hundred and eighty-two days five eighths of
22 a day communicate.
23 WHAT IN THE DIVIDEND IS LIKE THE DIVISOR YIELDS ONE *LI*. WHAT DOES NOT FILL UP THE DIVISOR (I.E., THE
24 REMAINDER OF THE DIVIDEND), ONE THREE-FOLDS THIS, AND WHAT IS LIKE THE DIVISOR YIELDS A HUNDRED *BU*.
25 (Zhao Shuang's commentary) One *li* being three hundred *bu*, one should multiply (the
26 remainder) by three hundred, but the reason why one says "one three-folds this" is that one
27 does not want to change periodically the divisor, one then takes a single position (i.e., a
28 single digit of the quotient) as corresponding to the dividend of the hundreds. Therefore the
29 single following position (i.e., a single digit of the quotient) is named as hundred.⁷¹
30 WHAT DOES NOT FILL UP THE DIVISOR (I.E., THE REMAINDER OF THE DIVIDEND), ONE TEN-FOLDS THIS,⁷² AND
31 WHAT IS LIKE THE DIVISOR YIELDS TEN *BU*.

⁶⁸ It has been made clear in the preceding lines that this amount corresponded to the north-south displacement of the sun between the winter and the summer solstice. The passage of *The Gnomon of the Zhou [Dynasty]* (without the commentaries) is translated in (Cullen 1996: 187-188). Here I modify the translation in several places.

⁶⁹ In this context, the year has 365 + 1/4 days.

⁷⁰ I follow (Cullen 1996: 183-185), in leaving the term *heng* untranslated. In the context of *The Gnomon of the Zhou [Dynasty]*, *heng* designates the subsequent orbits of the sun around the celestial pole at precise moments of the year. The Sun goes from the outermost to the innermost *heng* and back during a year. Here one is interested in the displacement per day of the orbit along a north-south axis.

⁷¹ One names the digit in the same way as the prescription of the division formulated it: hundreds of *bu*. The term used for "naming" is the same as the one used subsequently, when naming the remainder with the divisor and giving the final part of the quotient as a fraction.

⁷² Note that this multiplication by ten is not said to be performed by a shift of rods.

32 (Zhao Shuang's commentary) Above, one has not used the multiplication by three hundred,
33 therefore here one "ten-folds this." One then takes a single position (i.e., a single digit of the
34 quotient) as corresponding to the dividend of the tens. Therefore the single following
35 position (i.e., a single digit of the quotient) is named as ten.

36 WHAT DOES NOT FILL UP THE DIVISOR (I.E., THE REMAINDER OF THE DIVIDEND), ONE TEN-FOLDS THIS,⁷³ AND
37 WHAT IS LIKE THE DIVISOR YIELDS ONE *BU*.

38 (Zhao Shuang's commentary) The reason why again "one ten-folds this" is that one only
39 takes a single position as corresponding to the dividend.⁷⁴ Therefore the single following
40 position (i.e., a single digit of the quotient) is named as one (i.e., as making the units).

41 WHAT DOES NOT FILL UP THE DIVISOR (I.E., THE REMAINDER OF THE DIVIDEND), ONE NAMES THIS WITH THE
42 DIVISOR.

43 (Zhao Shuang's commentary) The positions (digits) have been exhausted up to the units of
44 *bu*, therefore taking the divisor to name the remaining parts corresponds to breaking *bu*.

45 **Zhen Luan says:** Method for looking for the north-south displacement: One puts (on the
46 calculating surface) the one hundred and nineteen thousand *li* (from) the winter solstice and
47 one multiplies this by the denominator eight of the (number of) days of the half year,
48 yielding nine hundred fifty-two thousand as dividend. One makes one hundred and eighty-
49 two days five eighths of a day (corresponding to) the half-year communicate, which yields
50 one thousand four hundred sixty-one. With (this) **dividing (*chu*)** yields six hundred fifty-one *li*.
51 The eight hundred eighty-nine which are not exhausted, one multiplies them by three
52 hundred, yielding two hundred sixty-six thousand seven hundred. Again, one **divides (*chu*)**
53 this by the divisor, yielding one hundred and eighty-two *bu*. What is not exhausted is seven
54 hundred ninety-eight, hence yielding for the displacement of the sun from north to south
55 daily six hundred fifty-one *li* one hundred and eighty-two *bu* seven hundred ninety-eight one
56 thousand four hundred sixty-firsts of a *bu*.

Text 2, Execution of division in *Writings on mathematical procedures* (Item A2). Example of Phases 2, 3, 4. Edition used: (PENG, 2001: 56). The document was found in a tomb sealed in 186 BCE or shortly thereafter. Measurement units for length used in the text:

1 *chi* 尺 = 10 *cun* 寸.

"(...) 术 (術) 曰: 直(置)二_L、直(置)四_L、直(置)八_L、直(置)十六_L、直(置), /41/ 卅
(三十) 二, 并以為法_L, 以五尺偏(遍)乘之各自為實₌, (實) 如法得尺。不盈尺者
十之, 如法一寸_L。不盈寸者, 以法命分。 (…), /42/

57 "(...) Procedure: one puts (on the calculating surface) two_L, one puts four_L, one puts eight_L,
58 one puts sixteen_L, one puts thirty-two. The sum is taken as divisor. _L Multiplying them all by
59 five *chi* makes, for each respectively, a dividend. In the dividends, what is like the divisor (i.e.,
60 Dividing the dividends by the divisor) yields a *chi*. What does not fill up the *chi* (i.e., the
61 remainders of the dividends), one ten-folds them. What (i.e., the new values for the
62 dividends, respectively) is like the divisor (i.e., dividing by the divisor yields), one *cun*. What
63 does not fill up the *cun*, one names their parts with the divisor."

⁷³ Again, note that this multiplication by ten is not said to be performed by a shift of rods.

⁷⁴ Note that the term "dividend" refers throughout the text to what is placed in the position of the dividend at the moment of the computation when the text refers to it.

Text 3, Execution of division in *Mathematics* (Item A1). Example of Phases 2 and 3. Edition used: (ZHU Hanmin and CHEN Songchang 2011: 9). The slip is incomplete and was perhaps originally placed after slip 0842 (editors' number 47, see (ZHU Hanmin and Chen Songchang 2011: 8)). The editors consider the manuscript was completed no later than 212 BCE.

Measurement units for capacity used in the text: 1 *dou* 斗 = 10 *sheng* 升.

0757 (editors' number 48) (incomplete slip, only the end of the procedure)

(...) 法，以租乘分田，如法一斗，不盈斗者，十之，如法得一升。

64 (...) divisor, one multiplies the cropland shared (?) by the tax. What (i.e., in the dividend) is
65 like the divisor (yields) one *dou*. What does not fill up the *dou* (i.e., the remainder of the
66 dividend), one ten-folds this. What (in the latter number as dividend) is like the divisor
67 yields one *sheng*.

Text 4, Execution of division in *Mathematics* (Item A1). Example of Phases 2, 3, and 4, with specific evidence on Phase 2. According to the editors, the text was produced no later than 212 BCE. Edition used: (ZHU Hanmin and CHEN Songchang 2011: 5). Measurement units for weight used in the text: 1 *liang* 兩 makes 24 *zhu* 銖.

(...) 為 [尹+貝]⁷⁵ (實)， (...) /1743 (editors' number 17)/

(...) 為法，如法一兩，不盈兩者，以一為廿四，乘之，如法一朱（銖），不盈⁷⁶朱（銖）者，以法命分。 /1835+1744 (editors' number 18)/

68 (...) makes the dividend (...), makes the divisor. What (i.e., in the dividend) is **like the divisor**
69 **(yields) one *liang***. What does not fill up the *liang* (i.e., the remainder of the dividend), **taking**
70 **one as twenty-four, one multiplies this** (i.e., the remainder). What (in the latter number as
71 dividend) is like the divisor (yields) one *zhu*. What does not fill up the *zhu* (i.e., the
72 remainder of the dividend), one names its parts with the divisor.

Text 5. Evidence about Phase 1 of division, in the case when one divides between quantities of **cash and persons** having **integers and fractions**. The procedure is identical to the one described in a section of *Writings on mathematical procedures* (Item A2, found in a tomb sealed in 186 BCE or shortly thereafter). The heading of the section, which is the name of the procedure, reads: 徑分 *jingfen* "Directly sharing". The example comes from another section of *Writings on mathematical procedures*. Edition used: (Peng 2001: 45-47). See also text 9.

⁷⁵ In the manuscript, these two characters are combined on top of one another to write the character interpreted as a variant for the character for "dividend."

⁷⁶ Xiao Can (2010: 26) gives the last four characters as hardly readable, since they occur where the slip was broken, and hence damaged. She features the characters she restores relying on the remaining parts between brackets, as uncertain. ZHU Hanmin 朱漢民 and CHEN Songchang 陳松長 zhubian 主編 (gen. eds.) 2011: 15, 42 (footnote 2) give them as certain. I follow them.

“(….)五人分七錢少半、(半)錢，人得一錢卅(三十) /23/ 分錢十七。術⁷⁷曰：下⁷⁸三分，以一為六，即因而六⁷⁹人⁸⁰以為法，亦六錢⁸¹以為實。 (...) /24/

73 Five persons **sharing/dividing** (*fen*) seven cash one third and half cash, each person
74 obtains one cash seventeen thirtieths cash. Procedure: (Since) **below** (there is) a third, **one**
75 **takes one as six. Hence six-folding** by the *yin* (method of multiplication) the persons, one
76 takes (the result) as the divisor. **Likewise** six-folding the cash, one takes (the result) as the
77 dividend.”

Text 6, Phase 1 of division evidenced for a similar case, but with more complex measuring units, as part of a procedure in *Mathematics* (Item A1, document produced no later than 212 BCE). Edition used: (ZHU Hanmin and CHEN Songchang 2011: 6, 48), slips 0788+0775+0984 (editors' numbers 29, 30, 31). Measurement units for weight used in the text: 16 *liang* 兩 make 1 *jin* 斤. The *bu* is used as a measurement unit for areas.

Context: The norm being that the tax on a cropland of 5 *bu* is 1 bundle, on 16 *bu* of taxed cropland one is required to give a total weight of 5 *jin* in bundles, that is, 80 *liang*. From this, one can determine that a bundle weighs 25 *liang* (a numerical value that will be needed in the inverse computation from which I quote below). The statement of the problem claims that instead of writing 5 *jin* on the certificate, one made a mistake, inscribing wrongly the amount as 5 *jin* 1 *liang* (that is, 81 *liang*; see similar procedures below). It is asked to diminish the quantity of *bu* that defines the norm (that is, the amount of *bu* for which one owes a bundle) so that what was written on the certificate becomes conform to the new norm. One thus asks: for how many *bu* should the tax be one bundle (that is, 25 *liang*)? The result is given as $4 + \frac{76}{81}$ *bu*, which was clearly obtained as the result of a rule of three: 16 *bu* yielding 81 *liang*, what are the *bu* yielding 25 *liang*? So, in the direct computation, one had multiplied 16 *bu* by 25 *liang* and divided the result by 81 *liang* to get the result. This is what can be gathered on the basis of the inverse computation, which the slips record thereafter. To achieve the aim, stated at the beginning, of “restoring” the 81 *liang* (namely, the wrong

⁷⁷ Peng (2001: 46) gives the character as 术, while it reads 術, see (Cullen 2004; Chōka zan kankan Sansūsho kenkyū kai 2006, plate 21).

⁷⁸ PENG Hao (2001: 47), note 11, suggests that 有 *you* “there is” was omitted after 下 *xia* “below.” GUO Shirong (2001) agrees on this remark, arguing on the basis of the occurrence of the same expression in the following section.

⁷⁹ Between the characters 六 *liu* “six” and 人 *ren* “persons,” Chōka zan kankan Sansūsho kenkyū kai (2006: 138) note that the manuscript has a hook punctuation, which was not reproduced in the edition (PENG 2001). They punctuate: “以一為六、即因而六。人以為法，亦六錢以為實。” On the basis of the whole sentence as well as the parallel procedure below, I tend to consider this mark as an interpolation.

⁸⁰ PENG Hao (2001: 47), note 12, suggests that 數 *shu* “number” was omitted after 人 *ren* “person.” I am not sure that the technical language displayed in the document as well as in *The Nine Chapters* requires to consider this as an omission.

⁸¹ Peng Hao (2001: 47), note 13, again suggests that 數 *shu* “number” was omitted after 錢 *qian* “cash.”

amount written on the certificate, from which one started, and by which one divided), the inverse computation prescribes to divide the product of 16 *bu* and 25 *liang* by the norm found as a result and obtain the amount written on the certificate sought-for. The inverse computation can be represented as follows:

$$\frac{25 \text{ liang} \cdot 16 \text{ bu}}{4 + \frac{76}{81} \text{ bu}}$$

The concern of “restoring” always occurs in contexts in which one has divisions or root extractions (see another example in Text 10, below). The text referring to the inverse computation reads as follows:

“(…) 欲復之，置一束兩數以乘兌(稅)田，而令以一為八十一為[尹+貝](實)，亦【令所奕步一為八十一，不分者，從之以為】⁸²/slip 0775 (editors’ number 30) 法，[尹+貝](實)如法一兩。(blank space) /slip 0984 (editors’ number 31)/

78 “(…) If one wants to restore this (the eighty-one *liang*), one puts the quantity of *liang* of one
79 bundle and multiplies with (this) the cropland taxed, and **makes one to be taken as eighty-**
80 **one** to make the dividend.⁸³ One **likewise** takes **one as eighty-one in the *bu*** that were
81 diminished (weakened).⁸⁴ Those that are not divided into parts/shared⁸⁵ join these (i.e., 4
82 times 81) which is taken as divisor. Dividing the dividend by the divisor yields the result in
83 *liang*.”

Text 7, Evidence for Phase 1 for a **division** between an **area** and a **length having multiple fractions**. The procedure quoted below comes from *Mathematics (Item A1)*, no later than 212 BCE), and is one among several similar procedures, introduced by the expression “Reducing the width.” This name for the procedure, and similar texts of procedures, are found in other manuscripts and in *The Nine Chapters (Item B2)*. Edition used: The whole section is in (ZHU Hanmin and CHEN Songchang 2011: 22-24, 119-125). I only use here part of the section (slips 0958+0789+0855, editors’ numbers 169-170-171). Measurement units for area used in the text: the *bu*, which is also used as a measurement unit for distances. As a measurement unit for area, 1 *bu* is a square whose side is 1 *bu*.

下有十分，以[一]為二千五百廿（二十）_レ，半為千二百六十_レ，三分為八百卅（四十）_レ，四分為六百卅（三十）_レ，五分為五百四，六分為四百廿（二十），七分為三百六

⁸² The right side of the lower section of the slip is damaged and only the left side of its characters can be read.

⁸³ The inversion of the rule of three has the consequence that the “unit of the product” must be transformed into 81. Note this “unit”, which is that of the product.

⁸⁴ $4 + \frac{76}{81} \text{ bu}$, 4 becoming 81, this operation does not affect the fractional part: 步 *bu* here

designates only the completed *bu*

⁸⁵ “Those that are not divided into parts/shared” refers here to the 76 which are expressed with respect to a unit (the eighty-first part of a *bu*) that is not cut.

十，八分 /0958/ 為三百一十五_レ，九分為二百八十_レ，十⁸⁶為二百五十二，同之七千三百八十一，以為法，直（置）二百卅（四十）步，亦以一為二千五百廿（二十），凡六十萬四千八百，除 /0789/之，如法得一步，為從（縱）八十一步有（又）七千三百八十一分步之六千九百卅（三十）九，成田一畝。 /0855/

84 “(Since) below there is a tenth, one takes [one] as two thousand five hundred twenty _レ, a
85 half as one thousand two hundred sixty _レ, a third as eight hundred forty _レ, a fourth as six
86 hundred thirty _レ, a fifth as five hundred and four, a sixth as four hundred twenty, a seventh
87 as three hundred sixty, an eighth /0958/ as three hundred fifteen _レ, a ninth as two hundred
88 eighty _レ, [a] ten[th] as two hundred fifty-two; summing these (gives) seven thousand three
89 hundred and eighty-one, which is taken as divisor. One puts two hundred and forty *bu* and
90 likewise one takes one as two thousand five hundred twenty, (which makes) altogether six
91 hundred and four thousand eight hundred. **Eliminating (*chu*) /0789/** from this, what is like
92 the divisor yields one *bu*, (i.e., dividing by the divisor yields the result in *bu*), which makes,
93 for the length, eighty-one *bu* and six thousand nine hundred and thirty-nine seven thousand
94 three hundred and eighty-firsts *bu*, which generates one *mu*. /0855/

Text 8, Evidence of Phase 1, for a division between an area and a quantity expressed using several measuring units of the system of capacity units. The procedure comes from *Writings on mathematical procedures (Item A2)*, a document found in a tomb sealed in 186 BCE or shortly thereafter. Edition on which I relied to offer the following edition: (PENG Hao 2001: 82-83). Measurement units for capacity used in the text: 1 *dan* 石 = 10 *dou* 斗 = 100 *sheng* 升. The *bu* is used as a measurement unit for areas. Full explanations for understanding this text are given in Section 6.1.

NB: I translate the text as it occurs on the slips. However, I consider that the passage on which I added grey color on slip 93 is superfluous, since it was written again on slip 94. See the related footnotes.

“誤券 租禾誤券者，術(術)曰：⁸⁷毋升者直(置)稅田⁸⁸數以為實，而以券⁸⁹斗為一，以石為十，并以為法，如法得一步。其券有⁹⁰者，直(置)與 (chord, space) ●⁹¹/93 /

⁸⁶ Here it seems that the character 分 *fen* “part” was omitted. Neither Xiao Can (2010: 83), nor ZHU Hanmin 朱漢民 and CHEN Songchang 陳松長 *zhubian* 主編 (gen. eds.) (2011: 124) mention it.

⁸⁷ GUO Shirong (2001: 280) considers that the scribe omitted *qi quan* 其券. In my view, this is not impossible. However, what follows suggests that this was not perceived as such by the actors.

⁸⁸ I think that the scribe omitted here the character “步 *bu*” (measurement unit). This is clear from the passage below (on slip 94), which in my view, repeats what is on this slip 93, starting from one character before this first omission (see footnote 91). Indeed, there is a second omission below (see footnote 90). Writing again the expression, this time correctly, the scribe writes “田步數 *tian bu shu* the quantity of *bu* of the cropland”. This should be understood with the complete expression, in which this segment occurs here, as “稅田步數

shuitian bu shu the quantity of *bu* of the taxed cropland." GUO Shirong (2001: 280) also suggests this omission of "步 *bu*", without, however, putting forward the same reason. By contrast to the second omission, this omission does not seem to me to raise mathematical problems, or problems of understanding. The second one is much worse, and indeed none of the colleagues who wrote on this passage so far seems to have restored the omitted character correctly.

⁸⁹ PENG Hao (2001: 83), notes 2 and 5, considers that here like below, the character "之 *zhi*" was omitted. GUO Shirong (2001: 280) suggests the same idea. In the context of the hypothesis that the whole passage has been written twice (the second time, correctly), I think the character was not perceived as required in the original document.

⁹⁰ For the same reason as in the preceding footnotes, I think that the scribe omitted "升 *sheng*" here. (Peng 2001: 83), note 3, suggests rather that "斗 *dou*" was omitted. All the other scholars adopted this idea, or put the same idea forward independently (GUO Shirong 2001: 280; GUO Shuchun 2001: 211; Cullen 2004: 131, note 60; Chôka zan kankan Sansûsho kenkyûkai 2006: 78-79). However, the procedure thereby restored is not correct, these authors paying no attention to the opposition *shui tian* 税田/*yu tian* 與田, which is essential for the meaning of the procedure. See the following footnote. In a context of the research that D.P. Morgan and I have begun on the mistakes in *Writings on mathematical procedures*, I showed this mistake that everybody had overlooked to Morgan, who immediately observed a dot placed on the slip, on the left hand side of the character before the omission. Most of the critical editions and translations of *Writings on mathematical procedures* had also overlooked this dot, except (Chôka zan kankan Sansûsho kenkyûkai 2006: 78-79, and Plate 12). Morgan and I formed the hypothesis that the dot pointed out the mistake, as did the dot placed at the bottom of slip 93 (and the dot placed where a character is missing on slip 95). We also put forward the hypothesis that the scribe perceived the dot at the bottom of slip 93 as an injunction to write again the passage in which two omissions had occurred, and copied again from one character before the first omission (Morgan and Chemla 2018 (2017)).

⁹¹ On the next slip, starting on a new slip, the scribe wrote once more the same passage which I put in grey characters, but restoring the two characters omitted. This seems to have been imposed on him, after someone checking (whose presence is manifested by dots) signaled key omissions and mistakes. The scribe's rewriting starts from "田步数 *tian bu shu* the quantity of *bu* of the (taxed) cropland" and goes until the occurrence, below, of "與田步数 *yu tian bu shu* the quantity of *bu* of the allotted cropland". The reason why it must be so is that the opposition between "與田 *yu tian* allotted cropland" and "稅田 *shui tian* taxed cropland" is essential for the procedure to be correct. The area of latter is one tenth of that of the former (see *Mathematics*, slip 0939, (XIAO Can 2010: 23-24, ZHU Hanmin and CHEN Songchang 2011: 38).) The piece of information is what justifies the correctness of the final part of the procedure. This point is explained mathematically in Section 6.1, where its relevance for my purpose is discussed. To the contrary, this quantitative information does not fit with restoring the missing character as "斗 *dou*" (see previous footnote). Hence my conclusion is that the middle part of the section was copied twice. (Cullen 2004: 69-70), noticing the similarity, concludes that it derives from the compilation of two different sources. I don't think this makes sense. To support the conclusion that the opposition on which the procedure relies is simply that between the cases with and those without *sheng*, we must rule out another possibility, that is, that the measuring unit omitted in the

田步数以為實，而以券斗為一，以石為十，并以為法，如法得一步。其券有升者，直
(置) 與田步数以為實，而以 /94/
券之升為一，以斗為十，并為法，如⁹²●得一步。 /95/

95 **Erroneously inscribing on a certificate:** In case, when collecting tax in millet, one
96 erroneously inscribed (an amount) on a certificate, procedure: **if there is no sheng** (in the
97 amount described), one puts (on the calculating surface) the number of the taxed cropland,
98 taking this as dividend, and then taking each *dou* of the certificate as one; taking each *dan*
99 as ten, one sums (the results) and takes (the result) as divisor. Dividing by the divisor yields
100 the result in *bu*. In case on the certificate there are[•], one puts (on the calculating surface)
101 the allotted (*mark of the lower chord binding slips, then space, and then*) ●/93/ the number
102 of *bu* of the (taxed) cropland, taking this as dividend, and then **taking each dou of the**
103 **certificate as one, taking each dan as ten**, one sums (the results) and takes (the result) as
104 divisor. Dividing by the divisor yields the result in *bu*. In case on the certificate **there are**
105 **sheng**, one puts (on the calculating surface) the number of *bu* of the allotted cropland,⁹³
106 taking it as dividend, and **then taking** (*mark of the lower chord binding slips*) /94/ **each**
107 **sheng of the certificate as one, taking each dou as ten**, one sums (the results) to make the
108 divisor. Dividing by the ●[divisor] yields the result in *bu*. /95/”

expression “其券有者…” in case on the certificate there are…” might have been the *dan* 石. The middle sentence would then read “其券有石者” and it would have to be understood as meaning “if in this certificate there are only *dan*” (which is already a problem). However, for the computation to be correct if this were the case, one would have to consider that the character “與” (allotted) in the expression “與田步数 *yu tian bu shu* the quantity of *bu* of the allotted cropland” is a mistake for “稅” (taxed) “稅田步数 *shui tian bu shu* the quantity of *bu* of the taxed cropland”. This seems highly improbable, since it would not make sense of many material features of the section. Moreover, the case of having *dan* is dealt with by the previous general procedure presented. So I consider that this rules out the last possibility, and this led me to the conclusion that the key opposition presented in the procedure is that between cases with and cases without *sheng*, and as a result ,that the middle part was written twice (the second time correctly). It was on the basis of this analysis that (Morgan and Chemla 2018 (2017) could interpret the meaning of the dots. (PENG Hao 2001: 83), note 4, considers that since the dot at the bottom of the slip is not a separating mark, either the “copyist” interpolated it, or he drew it as a mark.

⁹² PENG (2001: 83), note 5, considered there should have been the character “法 *fa* divisor”, instead of the dot, and suggested that the “copyist” left a mark for the missing character. GUO Shirong (2001: 280), GUO Shuchun (2001: 211), Cullen (2004: 131, note 61), also consider the character *fa* was omitted. I agree with this conclusion. Guo Shirong (2001: 280) suggests that the punctuation mark indicated the beginning of a slip. (Morgan and Chemla 2018 (2017) suggests that like the dots on slip 93, this dot also signalled an omission.

⁹³ The “allotted cropland” being ten times larger in area than the “taxed cropland,” the dividend is thereby immediately multiplied by 10, which is essential for the procedure to be correct. Note moreover that it would be normal that the tax be computed on the basis of the “taxed cropland”, and that it is only for the computation to be correct that one is invited to compute the tax with the allotted cropland. Note that this reveals that multiplying by 10 was not immediate, which reinforces my hypothesis that the number system contemporary with *Writings on mathematical procedures* was not the place-value number system.

Text 9. Evidence for Phase 1, in *The Nine Chapters* (Item B2). Procedure to divide between quantities of **cash and persons** having **integers and fractions**. The procedure is referred to as 徑 (經) 分 *jingfen* “Directly sharing” in *The Nine Chapters* (a book completed in my view in the 1st century CE). Edition used: (Chemla and GUO 2004: 167-169). For explanations, see Section 6.2 and (Chemla 2012: 462-471). See also text 5.

又有三人三分人之一，分六錢三分錢之一、四分錢之三。
問人得幾何。

荅曰：人得二錢八分錢之一。

經分 臣淳風等謹按：經分者，自合分已下，皆與諸分相齊，此乃直求一人之分。以人數分所分，故曰經分也。

術曰：以人數爲法，錢數爲實，實如法而一。有分者通之；母互乘子知，齊其子。母相乘者，同其母。以母通之者，分母乘全內子。乘，散全則爲積分，積分則與分子相通，故可令相從。凡數相與者謂之率。率知，自相與通。有分則可散，分重疊則約也。等除法實，相與率也。故散分者，必令兩分母相乘法實也。重有分者同而通之。又以法分母乘實，實分母乘法。此謂法、實俱有分，故令分母各乘全分內子，又令分母互乘上下。

109 (1. 18) “SUPPOSE AGAIN ONE HAS THREE PERSONS AND A THIRD OF A PERSON SHARING SIX CASH, ONE THIRD
110 AND THREE FOURTHS CASH. ONE ASKS HOW MUCH A PERSON GETS.

111 “ANSWER: A PERSON GETS TWO CASH ONE EIGHTH CASH.

112 “DIRECTLY SHARING/MAKING PARTS. **Your servant, Chunfeng, and the others comment respectfully:** As for
113 “Directly sharing/making parts”, from (the procedure) “Gathering parts” onwards, (the procedures) all made
114 the (quantity of) parts homogeneous with each other, only here does one directly seeks the part of one person.
115 One shares that which is shared by the number of persons, this is why one says “Directly sharing”.⁹⁴

116 PROCEDURE: ONE TAKES THE QUANTITY OF PERSONS AS **DIVISOR**, THE QUANTITY OF CASH AS **DIVIDEND** AND ONE
117 DIVIDES THE DIVIDEND BY THE DIVISOR. IF THERE IS ONE TYPE OF PART, ONE **MAKES THEM COMMUNICATE**.

118 “If “the denominators multiply the numerators that do not correspond to them”, it is to homogenize the
119 corresponding numerators. If the “denominators are multiplied with each other”, it is to equalize the
120 corresponding denominators.⁹⁵

121 “With the help of the denominator “**making them communicate**” is **multiplying** by the denominator of the
122 parts the integers (or: integral parts of the quantities) and incorporating these (the results) into the numerators.
123 **By multiplying, one disaggregates the integers, thus making the parts of the product/accumulated parts**
124 **(jifen)**. The parts of the product and the numerators hence **communicate** with each other, this is why one can
125 make them join each other.

126 “**Whenever quantities are given/put in relation with each other, one calls them *lü*. *Lü*'s**, being by nature in
127 relation to each other, **communicate**. If there are parts, one can disaggregate; if parts are reiterated
128 superpositions, one simplifies.

⁹⁴ The explanation given by the commentary is conform to the way in which the name of the procedure is written in *Writings on mathematical procedures*, namely, as the homophone *jingfen* 徑分. I thus interpret the name in *The Nine Chapters* accordingly.

⁹⁵ These two statements quote the commentary after the procedure to add up fractions. They repeat the context in which “equalizing” and “homogenizing” are introduced. Here, they echo the “equalizing” in the text of the procedure as formulated in *The Nine Chapters*.

129 “**Divisor and dividend**, divided by the equal [number (that is, their g.c.d.)], are *lǚ*’s put in relation with each
 130 **other**. Therefore, if one disaggregates the parts, one necessarily makes the two denominators both multiply
 131 divisor and dividend.”
 132 “IF THERE ARE SEVERAL TYPES OF PARTS, ONE **EQUALIZES** THEM AND THEN **MAKES THEM COMMUNICATE**.
 133 Otherwise, one multiplies the dividend by the denominator of the divisor and multiplies the divisor by the
 134 denominator of the dividend. This means that, if divisor and dividend both have parts, then one makes the
 135 denominators respectively multiply the (corresponding) integral parts and (the result) be incorporated into the
 136 (corresponding) numerators and then one makes the denominators multiply the (quantity) above or below that
 137 does not correspond to them.”

Text 10, Evidence about an older execution of square root extraction in *Writings on mathematical procedures* (Item A2, found in a tomb sealed in 186 BCE or shortly thereafter).
 The translation is followed by an attempt at reconstructing what the computation might have looked like when carried out with rods. However, the layout is wholly tentative. Moreover, in the absence of any information on the number system, present-day place-value decimal system is used. Edition used: (PENG Hao 2001: 124-125). Measurement units for area used in the text: 1 *mu* 畝 = 240 *bu* 步. The *bu* is also used as a measurement unit for distances. As a measurement unit for area, 1 *bu* is a square whose side is 1 *bu*.

方田 田一畝方幾何步_レ? 曰: 方十五步卅(三十)一分步十五。術(術)曰: 方十五步不足十五步, 方十六步有徐(餘)十六步_レ⁹⁶。曰: 并贏(盈)、不足以為法_レ, 不足, /185/子乘贏(盈)母, 贏(盈)子乘不足母, 并以為實。復之, 如啓廣之術(術)。_レ, 186,

138 Finding the side of the square cropland⁹⁷ How many *bu* has the side of a square of
 139 the cropland of one *mu*? Answer: the side of the square is fifteen *bu* fifteen thirty-firsts of a
 140 *bu* (i.e., fifteen parts of the *bu* cut into thirty-one parts). Procedure: if the side of the square
 141 were fifteen *bu*, it would fail (to fill up one *mu*, i.e., 240 *bu* by) fifteen *bu*; if the side of the
 142 square were sixteen *bu*, it would exceed (one *mu*, i.e., 240 *bu* by) sixteen *bu*. It is said: one
 143 adds the excess and the deficit to make the divisor. The numerator of the deficit /slip 185/
 144 multiplies the denominator of the excess, the numerator of the excess multiplies the
 145 denominator of the deficit,⁹⁸ one adds (the results) to make the dividend. One restores this
 146 (original value) like for the procedure for detaching the width.⁹⁹ /slip 186/

⁹⁶ In (Chôka zan kankan Sansûsho kenkyûkai 2006), Ohkawa et al. read here a hook punctuation mark which neither PENG (2001) nor the others authors mentioned above signal.

⁹⁷ An operation with the same name and a similar procedure occurs in the manuscript dated from the Qin dynasty and kept at Beijing University (Han Wei 2013: 38-39).

⁹⁸ The use of the terms “excess” and “deficit” suggests that the procedure carrying out the operation later called “Excess and deficit” in *The Nine Chapters* is used here (Chemla and GUO 2004: 562-565). This fact seems confirmed by the operations prescribed by the text of the procedure. However, it is unclear why the terms “numerator” and “denominator” are used here. In (Chôka zan kankan Sansûsho kenkyûkai 2006 : 16), Ohkawa et al. suggest a layout placing the deficit and the excess on a top row, with, respectively, the corresponding assumed value below. If this layout proved correct, this would mean that the layout explicitly described in *The Nine Chapters* is not the same. This hint about the transformations

15	15	Deficit Excess (left : result, right : supposition)		15	
16	16		16	15 * 16	16 * 15 + 15 * 16
15	+	16	15	+	16

Text 11, Evidence about the new execution of a square root extraction in *The Nine Chapters* (Item B2, completed in my view in the 1st century CE).

I only give my attempt at reconstructing what the computation may have looked like when carried out with rods. For a translation, see (Chemla 1994b: 253-254).

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	(steps)
			2	2	2	upper
55225	55225	55225	55225	55225	15225	middle
1	1	1	1	2	2	lower

Step 6	Step 7	Step 9	Step 11	Step 13	Step 14	Step 15	(steps)
2	2	23	23	23	23	23	upper
15225	15225	15225	15225	15225	2325	2325	middle
2	4	4	4	43	43	46	lower
			1	3	3		Below

of calculating practices still awaits systematic study. Before such research has been carried out, I attempt another way of restoring the layout of the computations in the following table, using the layout for the placement of numerator and denominator explicitly attested to later in computations with fractions, that is, numerator on the left and denominator on the right. My assumption is thus that the terms numerator and denominator could refer to positions.

⁹⁹ The concern for “restoring” (*fu* 復) is attested in several other sections of *Writings on mathematical procedures* (see, for another example, Text 6, above) and in Liu Hui’s commentary on *The Nine Chapters*, see my glossary of technical terms (Chemla and GUO 2004: 924-925). The term *fu* appears in contexts in which divisions and root extractions are in play, and refers to the requirement that by applying the inverse operation to the result of these operations, one should restore the original value. In *Writings on mathematical procedures*, the procedure “detaching the width” prescribes how to compute the width of a rectangle, when one knows its area and its length. It ends on the remark that the procedure is similar to “detaching the length.” This corresponds to another section in the manuscript, in which a procedure determines the length, when knowing the area of a rectangle and its width. This section is composed of two paragraphs, in which the procedure is dealt with, first, in general terms and, then, in more specific terms. At the end of both, the inverse operation is addressed. In “Finding the side of the square cropland”, the side of a square is found, and multiplying it by itself should restore the area in the same way as in “detaching the length,” (or “detaching the width”) multiplying the length computed by the given width (or swapping the two) should restore the area.