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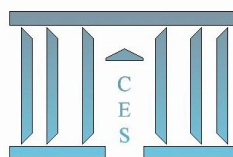
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# Market failures and equilibria in Banach lattices

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## Abstract

In this paper, we consider an economy with infinitely many commodities and market failures such as increasing returns to scale and external effects or other regarding preferences. The commodity space is a Banach lattice possibly without interior points in the positive cone in order to include most of the relevant spaces in economics. We propose a new definition of the marginal pricing rule through a new tangent cone to the production set at a point of its (non-smooth)-boundary. The major contribution is the unification of many previous works with convex or non-convex production sets, smooth or non-smooth, for the competitive equilibria or for the marginal pricing equilibria, with or without external effects, in finite dimensional spaces as well as in infinite dimensional spaces. In order to prove the existence of a marginal pricing equilibria, we also provide a new properness condition on technologies.

**Keywords:** Marginal pricing rule, Banach lattices, Market failures, Properness, General equilibrium.

**J.E.L. codes:** C62, D21, D51, D62

## 1 Introduction

General equilibrium theory in infinite dimensional spaces has been a growing topic during the eighties and nineties and still is a very specialized field which allows to incorporate some non-competitive behavior such as asymmetric information. Since [2] there has been a growing literature whose goal consisted in using very general vector spaces in order to encompass several economic features such as commodity differentiation, uncertainty and infinite time horizon among others. Nevertheless, the above extensions are not straightforward. Indeed, closed and order bounded sets are not compact, equilibrium prices are not necessarily continuous and joint continuity does not hold among others. The first part of that literature was about topological vector spaces mainly Banach

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lattices ([23], [24], [31] and [32] among others). These frameworks cover various economic models: the ones with uncertainty, option pricing and infinite horizon. There are, however, economic questions leading to spaces which are not topological vector lattices. Examples include the models of commodity differentiation ([22] and [20]) and of intertemporal consumption in [19]. Nevertheless their price spaces are lattices. Then, [25], [28] and [27] inaugurated and deepened the use of topological spaces by using vector lattices instead of topological ones. It was shown that it is enough to require that the commodity space is vector lattice endowed with a locally convex topology so that the topological dual space is a lattice too. More recently, [3] and [21] showed that the lattice structure can be left out.

The above literature only deals with convex production sets and the competitive case. The marginal pricing rule behavior comes from [18] in order to treat the case when profit-maximization is no longer suitable. For instance, it happens when there are increasing returns to scale or more general types of non-convexities in production. A proper mathematical treatment in a finite dimensional setting was first provided in [17] and then generalized in [13]<sup>1</sup> through the Clarke's tangent cone<sup>2</sup>. Then [9] proved a marginal pricing equilibrium when several firms do not have convex technologies. This paper, in turn, has been generalized in two ways: by considering externalities and by considering infinitely many commodities. In the first case, the motivation came from the fact that external factors are, very often, source of non-convexities in both production and consumer preferences. [11] have shown by means of an example that the Clarke's normal cone to the production set for a fixed environment is not longer the right concept since discontinuities take place. Consequently, the marginal pricing rule given an external factor must consider not only the close productions for the same level of external effect but also those which are associated to different but close levels of external effect. This means that the producer takes into account that around a given production, the shape of the production set changes when the external effects vary. The new cone they obtain is larger than the Clarke's one which is actually the cost to be paid for extending the model to this situation. Some papers, [30], [6] and [10], have studied the marginal pricing equilibria with infinitely many commodities but only in the space  $L_\infty$ . When externalities are added the same space was used by [15] and [16]. So far, no additional results has been established with non-convexities and/or with externalities in an infinite dimensional setting.

The aim of this paper is first to provide a new definition of the marginal pricing rule based on a new tangent cone in Banach lattices for non-convex production sets with external effects. This definition is extending the previous ones in the literature in the sense that if the production sets are convex, the marginal pricing rule coincides with the competitive behavior, if the production sets are smooth like in [8, 15, 16], we recover the standard unique normal price,

<sup>1</sup>The original working paper was from 1982 and became well known among theorists

<sup>2</sup>Prices are located into the polar of the Clarke's tangent cone, namely, the Clarke's normal cone

in a finite dimensional space without external effects, we recover the Clarke's tangent cone, with external effects, the marginal pricing rule introduced in [11], and if the commodity space is  $L_\infty$ , the concept coincides with the one of [6].

Second, we prove that the marginal pricing rule is compatible with the existence of an equilibrium under assumptions at the same level of generality than the ones used for the competitive equilibrium.

The introduction of  $L_\infty$  (or  $\ell_\infty$ ), although important from an economic point of view, does not allow to consider other situations than those involving infinite horizon analysis. That is why we extend the equilibrium problem to Banach lattices as a way of encompassing many interesting economic models as those cited previously. In order to obtain a marginal pricing equilibrium we have to reformulate both the tangent and the normal cone to a production set  $Y(z)$  at a production  $y$  belonging to it given the external factor  $z$ . It is so not only because the externalities introduces a technical difficulty for getting a suitable set for marginal prices but also because, unlike in  $L_\infty$ , closed and bounded balls are not order bounded.

From a mathematical point of view, the definition of the new tangent cone combines the tools of the Clarke's tangent cone extending by the one in [11] to encompass the externalities and the one of [6] for the infinite dimensional. But to overcome the fact that the open ball may not be order bounded, we have to combine weak\* open neighborhoods and order intervals to get the desired continuity properties. Another contribution of the paper is to provide a so-called local star-shaped assumption on the production sets to avoid an abstract condition (Assumption TC) borrowed from [6]. This assumption is satisfied in finite dimensional spaces thanks to the free disposal assumption or when production sets are convex. It translates the idea that the shape of the production set is not too chaotic like a fractal. It has the advantage of being directly set on the production sets and not as a property of the tangent cone.

As for the existence proof, we adopt the method of [4] by considering the limit of a sequence of equilibria in suitable truncated economies with finite dimensional commodity spaces as in [6]. But we actually introduce a new step to overcome the emptiness of the interior of the positive cone, like in some proofs for competitive equilibrium. Indeed, it consists in restricting the commodity space to the principal ideal generated by a reference vector  $e$ , which could be the total endowments. Then a new topology is generated in the restricted space which is stronger than the relative (norm) topology, so that the interior of the positive cone is non-empty and we can get the existence of an equilibrium in the same way it is done in  $L_\infty$ .

At this stage, a key properness assumption is required to prove that equilibrium in the restricted space is actually an equilibrium in the original economy. More precisely, the question at stake is to extend the price functional from the principal ideal to the whole space. The difficulty in our setting comes from the fact that not all assumptions hold true in the restricted commodity

space. Furthermore, unlike in Walrasian economies, uniform properness is not sufficient to get a common non-null equilibrium price functional satisfying the marginal pricing rule for each producer. Consequently, we have to introduce a new properness condition, which is very close to the spirit of the existing ones in the literature like, for example, [24] and [28], but at the same time, allows us to solve the aforementioned difficulties.

The paper is organized as follows: Section 2 treats with the mathematical structure adopted in the paper. Section 3 presents the model and the Assumptions together with the specification of the new marginal pricing rule. In Section 4 we state an existence result under the particular circumstance that the interior of the positive cone has a non-empty interior. Then, in Section 5 we extend the analysis to more general Banach lattices whose positive cones have an empty interior. Thus, the result of Section 4 is a key-step in this Section. Besides, a new properness condition is stated in order to get the main existence result. Several technical proofs of propositions, lemmas and theorems are given in Appendix.

## 2 Terminology and notation

Let  $L$  be a Banach lattice, i.e., a Riesz space equipped with a complete lattice norm denoted by  $\|\cdot\|$ . The space  $L$  is also endowed with a Hausdorff locally convex-solid topology  $\tau$ , which is weaker than the norm topology and such that all order intervals are  $\tau$ -compact<sup>3</sup>.  $L_+ = \{x \in L \mid x \geq 0\}$  is the positive cone of  $L$  which is  $\tau$ -closed.

Let  $L^*$  be the topological dual of  $L$ . For  $x \in L$  and  $\pi \in L^*$ ,  $\pi(x)$  is the evaluation or the value of the commodity bundle  $x$  for the price  $\pi$ . We denote by  $\sigma^*$  the weak\* topology on  $L^*$  and by  $\|\cdot\|^*$  the dual norm. Let  $L^M$  be the product space given by the cartesian product of  $M$  copies of the space  $L$ . If each space  $L$  is endowed with the topology  $\tau$ , we denote by  $\prod_{L^M} \tau$  the product topology of  $L^M$ . The product space  $L^M$  is also a Banach lattice. For all  $x \in L$ ,  $\mathcal{V}_\tau(x)$  is the set of  $\tau$ -neighborhood of  $x$  and for all  $z \in L^M$ ,  $\mathcal{V}_{\prod_{L^M} \tau}(z)$  is the set of  $\prod_{L^M} \tau$ -neighborhood of  $z$ .

Let  $A : L^M \mapsto L$  be a correspondence. We say that  $A$  has  $\tau$ -closed values if for every  $x \in A$ ,  $A(x)$  is a  $\tau$ -closed subset of  $L$ .

For further details on infinite dimensional spaces and Banach lattices, we refer to [29] and to [1].

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<sup>3</sup>For all  $(x, \xi) \in L^2$ ,  $|x| \leq |\xi|$  implies  $\|x\| \leq \|\xi\|$ .  $B(0, 1) = \{x \in L \mid \|x\| < 1\}$ ,  $B(0, \varepsilon) = \varepsilon B(0, 1)$ ,  $B(x, \varepsilon) = x + \varepsilon B(0, 1)$ ,  $\bar{B}(0, 1) = \{x \in L \mid \|x\| \leq 1\}$ . A subset  $E$  of  $L$  is called solid if  $|y| \leq |x|$  and  $x \in E$  imply  $y \in E$ . For  $x, y \in L$  with  $x \leq y$ ,  $[x, y] = \{z \in L \mid x \leq z \leq y\}$ . The principal ideal generated by  $x \in L$  is  $L(x) = \bigcup_{n \in \mathbb{N}} n[-x, x]$  which is a vector sublattice of  $L$ . An element  $x \in L_+$  is an order unit if  $L(x) = L$ . The  $\|\cdot\|$ -topology is locally solid with a base of neighborhoods of zero which are radial and circled sets.

### 3 The model

There are finite sets of consumers and producers  $I$  and  $J$  respectively. Each element  $z = ((x_i)_{i \in I}, (y_j)_{j \in J})$  belongs to  $L^{I+J}$ , where  $L^{I+J}$  is the product space given by the Cartesian product of  $\#I + \#J$  copies of the space  $L$ . Each consumer  $i$  has a *consumption set* and a other regarding *preference relation*, which depends upon the actions of the other economic agents. Formally, for each  $i \in I$ ,  $X_i : L^{I+J} \rightarrow L_+$  is the consumption correspondence. For the environment  $z \in L^{I+J}$ ,  $X_i(z) \subset L_+$  is the consumption set of the  $i$ -th consumer. We denote by  $\succeq_{i,z}$  the binary preference relation of agent  $i$  on the set  $X_i(z)$ . This relation is assumed to be complete, reflexive and transitive. The relation of strict preference  $x \succ_{i,z} x'$  is then defined by  $x \succeq_{i,z} x'$  and not  $x' \succeq_{i,z} x$ . We do not assume that we can compare two commodity bundles if they do not share the same environment. Let  $\omega_i \in L_+$  be the initial endowment of the  $i$ -th agent such that  $\omega_i \in X_i(z)$  for all  $z \in L^{I+J}$ . Let us denote the total initial endowment of the economy by  $\omega = \sum_{i \in I} \omega_i \neq 0$ .

Each producer  $j$  has a production set which also depends upon the actions of the other agents. For each  $j \in J$ ,  $Y_j : L^{I+J} \rightarrow L$  is the production correspondence. For the environment  $z \in L^{I+J}$ ,  $Y_j(z) \subset L$  is the set of all feasible production plans for the  $j$ -th producer. We denote the  $\|\cdot\|$ -boundary of  $Y_j(z)$  by  $\partial Y_j(z)$ <sup>4</sup>.

The price set is given by  $S = \{\pi \in L_+^* \mid \|\pi\|_{L^*} = 1\}$ .

Let  $r_i : \mathbb{R}^{1+J} \rightarrow \mathbb{R}$  be the *wealth function* of the  $i$ -th consumer. If  $\pi \in S$  and  $(y_j)_{j \in J} \in \prod_{j \in J} Y_j(z)$ , her/his wealth is  $r_i(\pi(\omega_i), (\pi(y_j))_{j \in J})$ . This encompasses the private ownership economy case, i.e., when  $r_i(\pi(\omega_i), (\pi(y_j))_{j \in J}) = \pi(\omega_i) + \sum_{j \in J} \theta_{ij} \pi(y_j)$  for  $\theta_{ij} \geq 0$  and  $\sum_{i \in I} \theta_{ij} = 1$  for all  $j \in J$ .

For a given environment  $z \in L^{I+J}$  and a given initial endowments  $\omega' \in L_+$ , we denote by  $A(\omega', z)$  the set of attainable productions, that is,

$$A(\omega', z) := \{(y'_j) \in \prod_{j \in J} \partial Y_j(z) \mid \sum_{j \in J} y'_j + \omega' \in L_+\}$$

In order to consider only consistent situations, we define the set

$$Z := \left\{ z = ((x_i)_{i \in I}, (y_j)_{j \in J}) \in L^{I+J} \mid \begin{array}{l} \forall i \in I, x_i \in X_i(z) \\ \forall j \in J, y_j \in \partial Y_j(z) \end{array} \right\}$$

The set of *weakly efficient attainable allocations* corresponding to a given total initial endowment  $\omega' \in L_+$  is given by

$$A(\omega') := \{z = ((x_i)_{i \in I}, (y_j)_{j \in J}) \in Z \mid \sum_{i \in I} x_i = \sum_{j \in J} y_j + \omega'\}$$

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<sup>4</sup>We are interested in the boundary of the production set due to the relationship it has with efficiency. Indeed, when  $\text{int} L_+ \neq \emptyset$  free-disposal condition implies that the former equals the set of weakly efficient production vectors given an environment  $z$ , that is,  $\partial Y_j(z) = \{y \in L \mid (\{y\} + \text{int} L_+) \cap Y_j(z) = \emptyset\}$

Finally, we introduce  $e \in L_+$  as a reference commodity bundle satisfying  $\|e\| = 1$ . A natural candidate is  $e = \omega/\|\omega\|$  but this is not always the right choice. Indeed, when  $L$  is the space  $L_\infty$  of essentially bounded measurable functions, it is convenient to choose  $e = \chi$  the constant function equal to 1.

In the following, we consider two cases namely when the interior of  $L_+$  is nonempty or when it is empty. In the first case, we assume that  $e$  belongs to the interior of  $L_+$ . In that case, since 0 belongs to the interior of the order interval  $[-e, e]$ , then the norm  $\|\cdot\|$  is equivalent to the lattice norm associated to  $e$  defined by:

$$\|x\|_e = \inf\{t \in \mathbb{R}_+ \mid |x| \leq te\}$$

So, we will then assume that the norm  $\|\cdot\|$  is actually the norm  $\|\cdot\|_e$ , which implies that the closed balls are actually order intervals.

When the interior of  $L_+$  is empty, we assume that  $e$  is a quasi-interior point of  $L_+$  (i.e.  $e$  is strictly positive), which means that the principal ideal  $L(e)$  generated by  $e$  is norm-dense in  $L_+$ .

### 3.1 Marginal pricing equilibrium

To define the marginal pricing rule in presence of externalities with an infinite dimensional commodity space, we combine three ideas: the first one from [11] to deal with the externalities, the second one from [6] to encompass infinite dimensional spaces and, third, we also consider order intervals instead of balls to overcome the possible emptiness of the interior of the positive cone of  $L$ , which is an original contribution of this paper.

For every  $(\bar{y}_j, z) \in \partial Y_j(z) \times Z \subset L^{1+I+J}$  and  $\rho > 0$  we let:

$$\hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j) := \left\{ \nu \in L \mid \begin{array}{l} \exists \eta > 0 \text{ such that } \forall r > 0, \\ \exists V \in \mathcal{V}_{\Pi_{L^{I+J}} \tau(z)}, U \in \mathcal{V}_\tau(\bar{y}_j) \text{ and } \varepsilon > 0 \mid \\ \quad \forall z' \in B(z, \rho) \cap V, \\ \quad \forall \bar{y}'_j \in B(\bar{y}_j, \rho) \cap U \cap Y_j(z') \\ \text{and } \forall t \in (0, \varepsilon), \exists \xi \in r[-e, e] \text{ such that} \\ \quad \bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \xi) \in Y_j(z') \end{array} \right\}$$

Then we define the set

$$\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) := \cap_{\rho > 0} \hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j).$$

By polarity, we define

$$\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) = [\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)]^\circ = \{\pi \in L^* \mid \pi(\nu) \leq 0 \ \forall \nu \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)\}.$$

To comment this definition, we remark that the introduction of the neighborhood  $V$  for the parameter  $z$  is the way to take into account the external



effects on production. Then, the intersection over the parameter  $\rho$  and the perturbation of  $\nu$  by the additional term  $\eta(y_j - y'_j)$  is a way to get the conditional closeness of the graph with respect to the topology  $\tau$ . Finally, choosing  $\xi$  in the order interval  $r[-e, e]$  is the way to narrow the range of  $\xi$ , or, in other words, to say that the vector  $\bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j))$  is close to the set  $Y_j(z')$  for the order relation which is tighter than the norm. Actually, if  $e$  belongs to the interior of  $L_+$ , then we can replace the order interval by a ball, which is closer to the standard definition of a tangent cone in the spirit of Clarke [12]. Finally, note that the set  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$  is not necessarily convex.

The *marginal pricing rule* is now formally defined: given  $(\bar{y}_j, z) \in \partial Y_j(z) \times Z$ , the  $j$ -producer chooses the prices in  $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \cap S$ . In words, to compute the prices according to the marginal pricing rule, the producer takes into account the fact that his production set depends on external effects. Finally, the set of *production equilibria* is

$$PE := \{(\pi, z) \in S \times Z \mid \pi \in \cap_{j \in J} \hat{\mathcal{N}}_{Y_j(z)}(y_j)\}$$

We provide below some properties of the marginal pricing rule. The most important feature are the following: when the production sets  $Y_j(z)$  are convex, then the producers maximize their profit taken the price as given as well as the environment  $z$ ; when there is no externality and the interior of the positive cone is nonempty, it coincides with the concept of [6]; if  $L$  is finite dimensional, then we recover the concept of [11] and if, furthermore, we have no externality, we are back to the definition of the marginal pricing rule based on the Clarke's normal cone as introduced by Cornet in [13].

We are now able to state the definition of a marginal pricing equilibrium.

**Definition 1.** *A marginal pricing equilibrium of the economy  $\mathcal{E}$  is an element  $(z = (x_i)_{i \in I}, (y_j)_{j \in J}, \pi)$  in  $Z \times S$  such that:*

1. *For all  $i \in I$ ,  $\pi(x_i) \leq r_i(\pi(\omega_i), (\pi(y_j))_{j \in J})$  and if  $x'_i \succ_{i,z} x_i$  then  $\pi(x'_i) > r_i(\pi(\omega_i), (\pi(y_j))_{j \in J})$*
2. *For all  $j \in J$ ,  $\pi \in \hat{\mathcal{N}}_{Y_j(z)}(y_j) \cap S$*
3.  $\sum_{i \in I} x_i = \sum_{j \in J} y_j + \omega$

### 3.2 Basic assumptions

We now posit the following assumptions. Some of them became standard in the literature with increasing returns

#### Assumption (C)

For all  $i \in I$

1.  $X_i$  is a convex-valued correspondence with a  $(\prod_{L^{I+J}} \tau, \tau)$  closed graph. Furthermore, for all  $z \in L^{I+J}$ ,  $0 \in X_i(z)$  and  $X_i(z)$  is a solid subset of  $L_+$ .
2. For all  $z \in L^{I+J}$ , the half-line  $\{\delta e \mid \delta > 0\}$  is included in  $X_i(z)$ . For all  $x'_i \in X_i(z)$  and for all  $\delta > 0$  there exists a neighborhood  $V \in \mathcal{V}_{\prod_{L^M} \tau}(z)$  such that  $x'_i + \delta e \in X_i(z')$  for all  $z' \in V$ .
3. For every  $z \in L^{I+J}$ , and all  $\bar{x}_i \in X_i(z)$  both sets  $\{x'_i \in X_i(z) \mid x'_i \succeq_{i,z} \bar{x}_i\}$  and  $\{x'_i \in X_i(z) \mid \bar{x}_i \succeq_{i,z} x'_i\}$  are  $\|\cdot\|$ -closed. For all  $x'_i \in X_i(z)$  such that  $x'_i \succ_{i,z} x_i$ , for all  $t \in (0, 1)$ ,  $tx'_i + (1-t)x_i \succ_{i,z} x_i$ . For every  $z \in A(\omega)$ , there exists  $(x'_i)_{i \in I} \in \prod_{i \in I} (X_i(z) \cap L(e))$  such that  $x'_i \succ_{i,z} x_i$  for all  $i \in I$ .
4. The set  $G_i = \{(x'_i, \bar{x}_i, z) \in L^2 \times L^{I+J} \mid (x'_i, \bar{x}_i) \in X_i(z)^2, x'_i \succeq_{i,z} \bar{x}_i\}$  is a  $(\|\cdot\| \times \tau \times \prod_{L^{I+J}} \tau)$ -closed subset of  $L^2 \times L^{I+J}$ .
5. The wealth function  $r_i : R^{1+n} \rightarrow R$  is continuous and increasing. Furthermore, for all  $((v_i), (v_j)) \in R^{I+J}$ ,  $\sum_{i \in I} r_i(v_i, (v_j)_{j \in J}) = \sum_{i \in I} v_i + \sum_{j \in J} v_j$  and if  $\sum_{i \in I} r_i(v_i, (v_j)_{j \in J}) > 0$  then  $r_i(v_i, (v_j)_{j \in J}) > 0$  for all  $i$ .

**Assumption (P)** For every  $j \in J$

1.  $Y_j : L^{I+J} \rightarrow L$  has a  $(\prod_{L^{I+J}} \tau, \tau)$ -closed graph.
2. For every  $z \in L^{I+J}$ ,  $Y_j(z) \cap L_+ = \{0\}$  and  $Y_j$  satisfies the free-disposal condition, that is,  $Y_j(z) - L_+ = Y_j(z)$ .
3. For all  $z \in L^{I+J}$ , for all  $\bar{y}_j \in \partial Y_j(z)$  and for all  $\delta > 0$ , there exists  $V \in \mathcal{V}_{\prod_{L^M} \tau}(z)$ , such that  $\bar{y}_j - \delta e \in Y_j(z')$  for all  $z' \in V$ .

**Assumption B (Boundedness)** For all  $\omega' \geq \omega$

there exists  $b \in \mathbb{R}_+$  such that, for all  $z \in Z$ ,  $A(\omega', z) \cap L(e)^J \subset [-be, be]^J$ .

**Assumption SA (Survival)** For all  $z \in Z$ ,  $t \in \mathbb{R}_+$ , for all  $(\pi, (\bar{y}_j)) \in S \times A(\omega + te, z)$ , if  $\pi \in \cap_{j \in J} \hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$ , then  $\pi(\sum_{j \in J} \bar{y}_j + \omega + te) > 0$

**Assumption TC (Tangent cone)** For all  $z \in Z$ , for all  $j$ , for all  $\bar{y}_j \in \partial Y_j(z)$ ,  $0 \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ .

*Remark 1.* Note that we assume non-satiation only on the attainable sets. Together with the convexity of preferences, we have local non-satiation. We assume continuity of preference relations with respect to the external factors and to the consumption bundles.

*Remark 2.* Assumption C(1) states that  $X_i(z)$  is solid (i.e., it contains order-intervals). We do so because we are considering consumption sets which are not

equal to the positive cone  $L_+$  (See [31]). We make use of this property in Section 5.

*Remark 3.* By C(1),  $X_i(z)$  is a closed and convex subset of  $L_+$ . Hence, by C(2), one gets that for all  $x \in X_i(z)$ ,  $x + \delta e \in X_i(z)$  for all  $\delta \geq 0$ . We note that Assumption C(2) also implies that if a net  $(z^\gamma) \prod_{L^{I+J}} \tau$ -converges to  $z$ , then for every  $x'_i \in X_i(z)$  there exists a sequence  $(t^\gamma) \in [0, \infty)$  which converges to 0 and such that  $x'_i + t^\gamma e \in X_i(z^\gamma)$  for all  $\gamma$ . This is stronger than the standard lower hemi-continuity condition on  $X_i$ , which will be exploited in Section 4.

From an economic point of view, since  $x'_i + \delta e$  is a possible consumption, Assumption C(2) means that a perturbation of the externalities in a well chosen small neighborhood  $V$  will not produce a so large effect on the consumption possibilities as to  $x'_i + \delta e$  would be no longer possible. Thus a small change in the externalities has a relatively small impact in consumption and can be counterbalanced by a small move along the half-line generated by  $e$ .

*Remark 4.* Assumption C(3) says that feasible consumption vectors are non-satiated in  $L(e)$  rather than in  $L$ . However, as we shall see later,  $L(e)$  includes the relevant commodity bundles in the economy. We refer to [27] who states an analogous assumption for a competitive economy.

*Remark 5.* With regard to the production sets, Assumption P(3) implies that if a net  $(z^\gamma) \prod_{L^{I+J}} \tau$ -converges to  $z$ , then for every  $y'_j \in Y_j(z)$  there exists a sequence  $(t^\gamma) \in [0, \infty)$  which converges to 0 and such that  $y'_j - t^\gamma e \in Y_j(z^\gamma)$  for all  $\gamma$ . As in Remark 3, we point out that this is stronger than the standard lower hemi-continuity condition. Nevertheless, in finite dimensional spaces, P(3) is satisfied if  $z \rightarrow Y_j(z)$  is lower hemi-continuous under the free disposal assumption.

This assumption is economically interpretable similarly to the one on the consumption side. We point out that, in [15], there have been posited less strong assumptions than C(3) and P(3) although they were less economically meaningful. Besides the economic interpretation there is a technical reason for Assumption P(3). We need it in order to prove that profit maximization is the behavior followed by producers when production correspondences are convex-valued (See Proof of Proposition 2 (6)).

*Remark 6.* Boundedness assumption says that feasible production vectors belonging to a principal ideal are order bounded which, in turn, implies that the attainable set in  $L(e)^{I+J}$  is order bounded. In fact, it belongs to  $[-\omega' - \#Jbe, \omega' + \#Jbe]$ .

For spaces whose positive cone has a non-empty interior, Assumption B is automatically satisfied if  $A(\omega')$  is norm-bounded.

*Remark 7* As consequences of Assumption TC, we have  $-L_+ \subset \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$  and  $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \subset L_+^*$ . Later we will show that  $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$  is not reduced to the null vector. This Assumption is also made in [6].

The following assumption is stronger than Assumption TC but it exhibits

two advantages: first, it is stated directly on the production sets, not on the tangent cone, and it has an easier economic interpretation as a reinforcement of Assumption P(3).

**Assumption SP(3)** For all  $z \in L^{I+J}$ , for all  $\bar{y}_j \in \partial Y_j(z)$  and for all  $\delta > 0$  there exists  $V \in \mathcal{V}_{\prod_{L^{I+J}} \tau}(z)$ ,  $U \in \mathcal{V}_\tau(\bar{y}_j)$  such that for all  $z' \in V$ , for all  $\bar{y}'_j \in Y_j(z') \cap U$ , for all  $t \in (0, 1]$ ,  $t(\bar{y}_j - \delta e) + (1-t)\bar{y}'_j \in Y_j(z')$ .

If  $Y_j(\cdot)$  satisfies Assumption SP(3), we say that  $Y_j$  is  $\prod_{L^{I+J}} \tau$ -locally star-shaped with respect to  $\bar{y}_j - \delta e$  whenever  $\bar{y}_j \in Y_j(z)$ . We easily check that Assumption SP(3) is stronger than Assumption P(3). If  $Y_j(z)$  is convex for all  $z$ , Assumption SP(3) is clearly a consequence of Assumption P(3). Now we prove that Assumption TC is satisfied if  $Y_j$  satisfies Assumptions P and SP(3).

Let  $z \in Z$  and  $\bar{y}_j \in \partial Y_j(z)$ . Let  $\rho > 0$  and take  $\eta = 1$ . Let  $r > 0$  and choose  $\delta > 0$  smaller than  $r$ . Let  $V$  and  $U$  be the weak-open sets as given by Assumption SP(3) and  $\varepsilon = 1$ . Then, for all  $z' \in B(z, \rho) \cap V$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U \cap Y_j(z')$ , for all  $t \in (0, \varepsilon)$ ,  $t(\bar{y}_j - \delta e) + (1-t)\bar{y}'_j = \bar{y}'_j + t(\bar{y}_j - \bar{y}'_j - \delta e) \in Y_j(z')$ , which means that  $0 \in \hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j)$ . Since this is true for all  $\rho$ , we conclude that  $0 \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ .

We finally remark that Assumption SP(3) is a consequence of Assumption P(3) when  $L$  is finite dimensional. So, Assumption SP(3) is necessary only to deal with infinite dimensional spaces in the non-convex case.

If Assumption P is satisfied and  $L$  is finite dimensional, let  $z$  and  $\bar{y}_j \in \partial_{Y_j(z)}(\bar{y}_j)$ . Let  $\delta > 0$  and  $V$  be the neighborhood of  $z$  coming from Assumption P(3) that is  $\bar{y}_j - \delta e \in Y_j(z')$  for all  $z' \in V$ . Let  $U = \{\bar{y}_j - \delta e\} + \text{int} L_+$ .  $U$  is a weak open neighborhood of  $\bar{y}_j$  from the free-disposal assumption and since  $L$  is finite dimensional so the norm topology is equal to the weak topology. Thus, for all  $z' \in V$ , for all  $\bar{y}'_j \in Y_j(z') \cap U$  and for all  $t \in (0, 1)$ ,  $t(\bar{y}_j - \delta e) + (1-t)\bar{y}'_j = \bar{y}'_j - t(\bar{y}'_j - \bar{y}_j + \delta e)$  belongs to  $Y_j(z')$  from the free-disposal assumption and the fact that  $\bar{y}'_j - \bar{y}_j + \delta e \in L_+$ .

### 3.3 Comments on the marginal pricing rule

Proposition 2 below allows us to compare the marginal pricing rule with the existing ones in the literature. The proof is given in Appendix.

We first define two correspondences in order to compare with the existing notions in finite dimensional commodity spaces. For every  $(\bar{y}_j, z) \in \partial Y_j(z) \times Z \subset L^{1+I+J}$ , we let

$$\hat{T}_{Y_j(z)}(\bar{y}_j) := \left\{ \nu \in L \left| \begin{array}{l} \forall r > 0 \exists \varepsilon > 0 \mid \forall z' \in B(z, \varepsilon) \\ \forall \bar{y}'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z'), \forall t \in (0, \varepsilon) \\ \exists \xi \in B(0, r) \text{ such that} \\ \bar{y}'_j + t(\nu + \xi) \in Y_j(z') \end{array} \right. \right\}$$

and

$$\hat{N}_{Y_j(z)}(\bar{y}_j) = [\hat{T}_{Y_j(z)}(\bar{y}_j)]^o = \{\pi \in L^* \mid \pi(\nu) \leq 0 \ \forall \nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)\}$$

**Proposition 1.** *Suppose that Assumption P holds, then for every  $(\bar{y}_j, z) \in \partial Y_j(z) \times Z$*

1.  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  is a convex cone and  $-L_+ \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$ .
2.  $\hat{T}_{Y_j(z)}(\bar{y}_j) \subset T_{Y_j(z)}(\bar{y}_j)$  where the former is the Clarke's tangent cone to the set  $\partial Y_j(z)$  at the vector  $\bar{y}_j$ . Without externalities, both notions coincide.

The proof is given in Appendix. The inclusion for the tangent cones implies the reverse inclusion for the normal cones:  $N_{Y_j(z)}(\bar{y}_j) \subset \hat{N}_{Y_j(z)}(\bar{y}_j) \subset L_+$ , where  $N_{Y_j(z)}(\bar{y}_j)$  is the Clarke's normal cone.

**Proposition 2.** *Under Assumptions P and TC, for every  $(\bar{y}_j, z) \in \partial Y_j(z) \times Z$ .*

1.  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$  is a cone,  $-L_+ \subset \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$  and  $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \subset L_+^*$  is a convex and  $\sigma^*$ -closed cone.
2.  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$  and  $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \supset \hat{N}_{Y_j(z)}(\bar{y}_j)$ .
3.  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) \subset \mathcal{T}_{Y_j(z)}(\bar{y}_j)$  where the former is the small tangent cone of [6] to the set  $Y_j(z)$  at  $\bar{y}_j$ .
4. If  $e \in \text{int}L_+$  then  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$  is a convex cone.
5. If  $L$  is finite dimensional, then  $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \cap S = MP(\bar{y}_j, z)$ , where the latter is the marginal pricing rule in finite dimensional economies of [11].
6. If  $Y_j$  is convex-valued

$$\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) = \{\pi \in L^* \mid \pi(\bar{y}_j) \geq \pi(y'_j) \text{ for all } y'_j \in Y_j(z)\}$$

*Remark* Let us note that since for all  $z \in L^{I+J}$  the set  $Y_j(z)$  satisfies free disposal, then  $\nu \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$  if and only if for all  $\rho > 0$  there exists  $\eta > 0$  such that for all  $r > 0$  there are  $V \in \mathcal{V}_{\prod_{L^{I+J}} \tau}(z)$ ,  $U \in \mathcal{V}_\tau(y_j)$  and  $\varepsilon > 0$  such that for all  $z' \in B(z, \rho) \cap V$ , for all  $\bar{y}'_j \in \bar{B}(\bar{y}_j, \rho) \cap U \cap Y_j(z')$  and for all  $t \in (0, \varepsilon)$  it follows that  $\bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) - re) \in Y_j(z')$ .

In order to complete our comparison with the literature, we consider now smooth production sets, i.e., when for all  $z \in Z$  the technology is described by  $Y_j(z) = \{\zeta_j \in L \mid f_j(\zeta_j, z) \leq 0\}$  where  $f_j : L \times L^{I+J} \rightarrow \mathbb{R}$  is a differentiable mapping called the transformation function. Let us consider the following assumption:

**Assumption SB** (Smooth boundary)

1.  $f_j$  is continuous for the  $\tau \times \prod_{L^{I+J}} \tau$  topology on  $L \times L^{I+J}$ ;
2.  $f_j(\cdot, z)$  is Fréchet differentiable and Lipschitz on  $L$ ;
3.  $\nabla_1 f_j(\zeta_j, z)$  belongs to  $L_+^* \setminus \{0\}$  if  $\zeta_j \in \partial Y_j(z)$  where  $\nabla_1 f_j(\zeta_j, z)$  is the gradient of  $f_j$  with respect to  $\zeta_j$ ;
4.  $\nabla_1 f_j$  is continuous for  $\tau \times \prod_{L^{I+J}} \tau$  topology on  $L \times L^{I+J}$  and the norm topology on  $L^*$ .

**Lemma 1.** *If the production set  $Y_j(\cdot)$  is described by a transformation function  $f_j$  satisfying Assumption SB, then  $\hat{N}_{Y_j(z)}(\zeta_j)$  is the half-line generated by  $\nabla_1 f_j(\zeta_j, z)$ , which is also the Clarke's Normal cone to  $Y_j(z)$  at  $\zeta_j$ .*

The proof of this lemma is given in Appendix.

## 4 When $L_+$ has a nonempty interior

In this section, we assume that the positive cone  $L_+$  has a nonempty interior<sup>5</sup> and the vector  $e \in \text{int} L_+$ . Then  $[-e, e]$  is a neighborhood of 0 ([1]). To get the existence of a marginal pricing equilibrium, we follow the approach of Bewley [4]. Thus, we construct a directed set of finite dimensional economies from which we can apply Theorem 3.1 of [11] in each of them in order to get a net of equilibria in the truncated economies, which converges to an equilibrium of the original economy.

Let  $F$  be a finite dimensional subspace of  $L$  containing the vectors  $e$  and  $(\omega_i)_{i \in I}$ . We denote by  $\mathcal{F}$  the family of such subspaces directed by set inclusion. For every  $F \in \mathcal{F}$ , we define its positive cone by  $F_+ = F \cap L_+$  and its interior is  $\text{int} F_+ = F \cap \text{int} L_+$ . We endow each  $F$  with an euclidean structure, the associated norm being denoted  $\|\cdot\|_F$  such that  $\|e\|_F = 1$  and  $\{e^{\perp_F}\} \cap F_+ = \{0\}$ , where  $e^{\perp_F}$  denotes the orthogonal space to  $e$ . Hence, the dual space of  $F$  is  $F$  itself. We denote by  $B^F(x, r)$  (resp.  $\bar{B}^F(x, r)$ ) the open (resp. closed) ball of center  $x$  and radius  $r$  in  $F$  for the euclidean structure.

The truncated consumption correspondence for the commodity space  $F$  is given by  $X_i^F : F^{I+J} \rightarrow F_+$  such that  $X_i^F(z^F) = X_i(z^F) \cap F_+$ . In the same way, the truncated production correspondence  $Y_j^F : F^{I+J} \rightarrow F$ , is defined by  $Y_j^F(z^F) = Y_j(z^F) \cap F$ . We define the set  $Z^F = \{z^F \in F^{I+J} \mid \forall i \in I, x_i^F \in X_i^F(z^F) \text{ and } \forall j \in J, y_j^F \in \partial Y_j^F(z^F)\}$ . Let us note that  $\partial Y_j^F(z^F) \subset \partial Y_j(z^F) \cap F$  thanks to the free-disposal assumption.

Let  $S^F = \{p^F \in F_+^0 \mid \langle p^F, e \rangle_F = 1\}$ , where  $F_+^0$  denotes the positive polar cone of  $F_+$ . The revenue of the  $i$ -th consumer in the truncated economy is given by the same revenue function  $r_i$ . The restricted preference relation on  $X_i^F(z^F)$

<sup>5</sup>Examples of this kind of spaces are  $\mathcal{L}_\infty$  and the space  $C(K)$  of real-valued continuous functionals on a compact Hausdorff space  $K$  endowed with the supremum norm

is  $\succ_{i,z^F}^F$ . Therefore, subeconomies are fully described by  $\mathcal{E}^F = \{(X_i^F, \succ_{i,z^F}^F, r_i, \omega_i)_{i \in I}, (Y_j^F)_{j \in J}\}$  for all  $F \in \mathcal{F}$ .

We point out that for all  $F \in \mathcal{F}$ , for all  $z^F \in F^{I+J}$ , and for all  $i$  and  $j$ ,  $X_i^F(z^F)$  and  $Y_j^F(z^F)$  are non-empty subsets of  $F_+$  and  $F$  respectively. Proposition 2.5 guarantees that for all  $F \in \mathcal{F}$  and all  $(\bar{y}_j, z) \in F^{1+I+J}$ ,  $\hat{\mathcal{N}}_{Y_j^F(z)}(\bar{y}_j) \cap S^F = MP^F(\bar{y}_j, z) =$

$$co \left\{ p \in F \left| \begin{array}{l} \exists (\bar{y}_j^n, z^n) \in L \times L^{I+J} \text{ and } (p^n) \in S^F \\ \text{such that } (\bar{y}_j^n, z^n) \rightarrow (\bar{y}_j, z), (p^n) \rightarrow p \\ \bar{y}_j^n \in \partial Y_j(z^n) \text{ and } p^n \in N_{Y_j(z^n)}(\bar{y}_j^n) \end{array} \right. \right\}$$

Finally we let

$$A^F(\omega') = \left\{ z^F \in Z^F \mid \sum_{i \in I} x_i^F = \sum_{j \in J} y_j^F + \omega' \right\} \subset A(\omega')$$

and

$$PE^F = \left\{ (p^F, z^F) \in S^F \times A^F(\omega) \mid p^F \in \bigcap_{j \in J} MP_j(y_j^F, z^F) \right\}$$

#### 4.1 Existence of marginal pricing equilibria

We can state now the following existence result

**Theorem 1.** *The economy  $\mathcal{E} = \{(X_i, \succ_i, r_i, \omega_i)_{i \in I}, (Y_j)_{j \in J}\}$  has a marginal pricing equilibrium if it satisfies Assumptions (C), (P), (B), (SA) and (TC).*

Within the literature concerning externalities and increasing returns, the above result extends the one of [11] by allowing an infinite dimensional space. It also generalizes [6] since external effects are taken into account. In addition, this model encompasses the one of [16] since we are considering non-smooth technologies. With regard to the papers in which a general pricing rule is considered (for instance, [10] and [15]), we note that even though a larger number of situations can be considered, the existence results require two additional assumptions<sup>6</sup> about boundedness of the losses of the firms and continuity of the pricing rules.

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<sup>6</sup>Known as Assumptions BL and PR respectively

## 4.2 Proof of the Theorem 1

The proof follows the guidelines of [16]. The difference relies on the fact that now we are considering non-smooth technologies.

*Marginal pricing equilibria in the finite dimensional commodity space economy*

We note that neither Survival nor the Local Non-Satiation condition remain true in the subeconomies. Consequently, it seems that Theorem 3.1 of [11] cannot be applied. However, Lemma 2 below shows that whether finite dimensional commodity subspaces are large enough, then weaker versions of those assumptions hold. We first prepare the ground for stating this lemma. Let  $\bar{\vartheta} > 0$  be a real number. By Assumption B, there exists  $a > 2\bar{\vartheta}$ , such that for all  $z \in L^{I+J}$ ,  $A(\omega + \bar{\vartheta}e, z) \subset [-\frac{a}{2}e, \frac{a}{2}e]^J$  and  $A(\omega + \bar{\vartheta}e) \subset [-\frac{a}{2}e, \frac{a}{2}e]^{I+J}$ . Let  $\bar{r} > 2a$  such that  $\{\omega + \bar{\vartheta}e\} + [-\#Jae, \#Jae] \subset [-\bar{r}e, \bar{r}e]$ . Let  $\bar{\lambda}$  be a real number such that  $\bar{\lambda} \geq 2\#J\bar{r} + \|\omega\|_F$ . We remark that  $\bar{\lambda}$  satisfies Lemma 4.2 of [11] in our model.

**Lemma 2.** *Under Assumptions (C), (P), (B) (SA) and (TC), there exists a subspace  $\hat{F} \in \mathcal{F}$  such that for all  $F \in \mathcal{F}$ , if  $\hat{F} \subset F$ , then the subeconomy  $\mathcal{E}^F$  satisfies:*

*(SA<sup>F</sup>): For all  $(p^F, z^F, \lambda) \in PE^F \times [0, \bar{\lambda}]$ , if  $(y_j^F)_{j \in J} \in A^F(\omega + \lambda e, z^F)$  then  $\langle p^F, \sum_{j \in J} y_j^F + \omega + \lambda e \rangle_F > 0$*

*(LNS<sup>F</sup>): For all  $((x_i^F)_{i \in I}, (y_j^F)_{j \in J}) \in A^F(\omega)$ , there exists  $(x'_i)_{i \in I} \in \prod_{i \in I} X_i^F(z^F)$  such that  $x'_i \succ_{i, z^F} x_i^F$  for all  $i \in I$ .*

The proof parallels that of [15]. As for (SA<sup>F</sup>) replace Assumption PR in that proof by the result of Proposition 4 below.

**Proposition 3.** *Let  $\hat{F} \in \mathcal{F}$  be the subspace coming from Lemma 2. Under Assumptions (C), (P), (B), (SA) and (TC) if we have  $\hat{F} \subset F$  then the subeconomy  $\mathcal{E}^F$  has an equilibrium  $(z^F, p^F) \in Z^F \times S^F$ .*

*Proof.* We remark that Bonnisseau and Médecin use Assumption (SA) in Lemmas 4.2 (3) and 4.4 and in Claim 4.3. They also fix a parameter  $\bar{t} > 0$  (p. 283). We replace it by the number  $\bar{\vartheta}$  given in Lemma 2. For Lemma 4.2 (3) and Claim 4.3, Survival Assumption is applied only for productions plans which satisfy  $\sum_{j \in J} y_j + \omega + \vartheta e \geq 0$  with  $\vartheta \leq \bar{\vartheta}$ . Given  $\bar{r}$ , it is clear that  $\bar{\vartheta} < \bar{\lambda}$  and then we have that condition (SA<sup>F</sup>) of Lemma 2 is enough to conclude. For Lemma 4.4, we shall prove that (SA<sup>F</sup>) is enough to use the deformation lemma. In doing so, we introduce the following:

Let  $D^F := \bar{B}^F(0, \bar{\lambda})^I \times \bar{B}^F(0, \bar{r})^J$  and  $Z_D^F := Z^F \cap D^F$

$\lambda_j^F : e^{\perp_F} \times F^{I+J} \rightarrow \mathbb{R}$



$$\begin{aligned}
(s_j, z) &\mapsto \lambda_j^F(s_j, z) \\
\Lambda_j^F(s_j, z) &= s_j - \lambda_j^F(s_j, z)e \in \partial Y_j^F(z) \\
X^F(z) &= \sum_{i \in I} X_i^F(z) + F_+ = F_+ \\
Y_0^F(z) &= -X^F(z) \\
\lambda_0^F &: e^{\perp_F} \times F^{I+J} \rightarrow \mathbb{R} \\
(s_j, z) &\mapsto \lambda_0^F(s_j, z) \\
\Lambda_0^F(s_j, z) &= s_j - \lambda_0^F(s_j, z)e \in \partial(-F_+) \\
\theta^F((s_j)_{j \in J}, z) &= \sum_{j \in J} \lambda_j^F(s_j, z) + \lambda_0^F(s_0, z) - \langle \omega, e \rangle_F, \\
\text{with } s_0 &= -\sum_{j \in J} s_j - \text{proj}_{e^{\perp_F}} \omega \\
\Delta^F((s_j)_{j \in J}, z) &= \left\{ (p_j - p)_{j \in J} \mid \begin{array}{l} p_j \in MP_j(\Lambda_j^F(s_j, z), j \in J) \\ p \in N_{-F_+}(\Lambda_0^F(s_0, z)) \cap S^F \end{array} \right\} \\
M_{\bar{\vartheta}}^F(z) &= \left\{ ((s_j)_{j \in J}) \in (e^{\perp_F})^J \mid \sum_{j \in J} \Lambda_j^F(s_j, z) + \omega + \bar{\vartheta}e \in F_+ \right\} \\
\text{for every } z &\in Z_D^F \\
GM_{\bar{\vartheta}}^F &= \left\{ ((s_j)_{j \in J}, z) \in (e^{\perp_F})^J \times Z_D^F \mid \sum_{j \in J} \Lambda_j^F(s_j, z) + \omega + \bar{\vartheta}e \in F_+ \right\} \\
GM_{\bar{\vartheta}, \alpha}^F &= \left\{ ((s_j)_{j \in J}, z) \in (e^{\perp_F})^J \times Z_D^F \mid \bar{\vartheta} \leq \theta^F((s_j)_{j \in J}, z) \leq \alpha \right\} \\
\alpha &= \max \left\{ \theta^F((s_j)_{j \in J}, z) \mid ((s_j)_{j \in J}, z) \in (\bar{B}^F(0, 2a) \cap \{e^{\perp_F}\})^J \times Z_D^F \right\}
\end{aligned}$$

For  $\lambda_j^F$  and  $\lambda_0^F$ , Lemma 4.3 in [11] states that  $\sum_{j \in J} \Lambda_j^F(s_j, z) + \omega + \vartheta e \geq 0$  if and only if  $\theta^F((s_j)_{j \in J}, z) \leq \vartheta$ . Then, the authors apply a deformation lemma for which it must be proved that the conditions of the lemma are satisfied. In particular, it is required that  $0 \notin \Delta^F((s_j)_{j \in J}, z)$  for all  $((s_j)_{j \in J}, z) \in GM_{\bar{\vartheta}, \alpha}^F$ . Otherwise, there exists (see the proof of Lemma 4.4)  $((s_j)_{j \in J}, z) \in (e^{\perp_F})^J \times Z_D^F$  with  $\bar{\vartheta} \leq \theta^F((s_j)_{j \in J}, z) \leq \alpha$  and  $p \in N_{-F_+}(\Lambda_0^F(-\sum_{j \in J} s_j - \text{proj}_{e^{\perp_F}} \omega, z)) \cap S^F$  such that  $p \in \cap_{j \in J} MP_j(\Lambda_j^F(s_j, z), z)$ . By Lemma 4.3 above,  $\sum_{j \in J} \Lambda_j^F(s_j, z) + \omega + \alpha e \geq 0$  and it can be proved that  $p(\sum_{j \in J} \Lambda_j^F(s_j, z) + \omega + \alpha e) = 0$  contradicting the Survival Assumption ( $SA^F$ ) since  $(\Lambda_j^F(s_j, z))_{j \in J} \in A^F(\omega + \alpha e, z)$  and  $\alpha \leq 2\#J\bar{r} + \|\omega\|^F \leq \bar{\lambda}$ . The proof to check this inequality is actually the same as the one provided in [16].

In [11], the Local Non-Satiation Assumption is used in Claim 4.6 where  $z^F \in A^F(\omega)$ . Consequently, Condition (LNS $^F$ ) of Lemma 4 is enough to conclude and the proof of Proposition 3 is complete.  $\square$

### The limit argument

We have a net of finite dimensional equilibria  $\mathcal{E}^F = ((x_i^F)_{i \in I}, (y_j^F)_{j \in J}, p^F)$  for every  $F \in \mathcal{F}$ . Note that  $z^F \in A(\omega)$ . The next lemma shows that we can

extend the price vector  $p^F$  which corresponds to the marginal pricing rule in  $\mathcal{E}^F$  to a continuous linear functional on the entire space  $L$  which is also a marginal pricing rule.

**Lemma 3.** *Let  $Y_j$  be a production correspondence satisfying Assumption P. For every  $F \in \mathcal{F}$ , for every  $(\bar{y}_j, z) \in \partial Y_j^F(z) \times Z^F$  and for every  $p \in MP_j(\bar{y}_j, z)$ , there exists  $\pi \in \hat{N}_{Y_j(z)}(\bar{y}_j) \cap S$  such that  $\pi|_F$  is colinear to  $p$ .*

*Proof.* First of all we show that  $\text{int}\hat{T}_{Y_j(z)}(\bar{y}_j) \cap F \subset \text{int}\hat{T}_{Y_j^F(z)}(\bar{y}_j)$ . Unlike [6], we cannot use hypertangency since  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  is not the Clarke's tangent cone. Let  $\nu \in \text{int}\hat{T}_{Y_j(z)}(\bar{y}_j) \cap F$ , hence there exists  $\delta > 0$  such that  $B(\nu, \delta) \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$ . Take  $\nu' \in B(\nu, \delta) \cap F$ . Since  $\nu' \in \hat{T}_{Y_j(z)}(\bar{y}_j) \cap F$ , for every  $r > 0$  there exists  $\varepsilon > 0$  associated to  $\frac{r}{2}$  such that for all  $z' \in B(z, \varepsilon) \cap F^{I+J}$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \varepsilon) \cap F \cap Y_j(z')$  and for all  $t \in (0, \varepsilon)$  there exists  $\xi \in L$  such that  $\|\xi\| < \frac{r}{2}$  and  $\bar{y}'_j + t(\nu' + \xi) \in Y_j(z')$ . By free-disposal  $\bar{y}'_j + t(\nu' - \frac{r}{2}e) \in Y_j(z')$ . Since  $\nu'$  and  $\frac{r}{2}e$  belong to  $F$ , one has that  $\bar{y}'_j + t(\nu' - \frac{r}{2}e) \in Y_j(z') \cap F$  and thus  $\nu' \in \hat{T}_{Y_j^F(z)}(\bar{y}_j)$ , which implies that  $\nu \in \text{int}\hat{T}_{Y_j^F(z)}(\bar{y}_j)$ .

Let us consider the linear manifold  $N = \{\nu \in F \mid p(\nu) = 0\}$ . It is clear that  $N \cap \text{int}\hat{T}_{Y_j^F(z)}(\bar{y}_j) = \emptyset$ , whence  $N \cap \text{int}\hat{T}_{Y_j(z)}(\bar{y}_j) = \emptyset$ . By Hahn-Banach extension theorem there exists a closed manifold  $H = \{\nu \in L \mid \hat{\pi}(\nu) = 0\}$  containing  $N$  and not intersecting  $\text{int}\hat{T}_{Y_j(z)}(\bar{y}_j)$ . Clearly,  $\hat{\pi}$  is continuous, colinear to  $p$  and it belongs to  $\hat{N}_{Y_j(z)}(\bar{y}_j)$ .  $\square$

The conclusion of this lemma provides a price in  $\hat{N}_{Y_j(z)}(\bar{y}_j)$ , which is smaller than  $\hat{N}_{Y_j(z)}(\bar{y}_j)$ . Nevertheless, the forthcoming limit argument shows that the equilibrium price is actually in  $\hat{N}_{Y_j(z)}(\bar{y}_j)$  since the cone  $\hat{N}_{Y_j(z)}(\bar{y}_j)$  does not satisfy the necessary closedness property.

For all  $j \in J$ , let  $\pi_j^F$  be an extension of  $p^F$  in  $\hat{N}_{Y_j(z)}(y_j^F)$ . The net

$$(z^F, (\pi_j^F)_{j \in J})_{F \in \mathcal{F}} \in A(\omega) \times \prod_{j \in J} \hat{N}_{Y_j(z)}(y_j^F) \cap S$$

has a limit point. Indeed, by Assumption (B),  $(z^F)_{F \in \mathcal{F}}$  belongs to a  $\prod_{L^{I+J}} \tau$ -compact set while  $(\pi_j^F)_{j \in J}$  belongs to a  $\prod_{L^{J,*}} \sigma^*$ -compact set from Alaoglu's Theorem. Hence, there exists a subnet  $(z^{F(\gamma)}, (\pi_j^{F(\gamma)})_{j \in J})_{\gamma \in \Gamma}$  which converges to  $(z, (\pi_j)_{j \in J})$ .

On the other side, since the nets  $(\langle p^{F(\gamma)}, y_j^{F(\gamma)} \rangle)_{\gamma \in \Gamma} = (\pi_j^{F(\gamma)}(y_j^{F(\gamma)}))_{\gamma \in \Gamma}$  and  $(\langle p^{F(\gamma)}, x_i^{F(\gamma)} \rangle)_{\gamma \in \Gamma} = (\pi_j^{F(\gamma)}(x_i^{F(\gamma)}))_{\gamma \in \Gamma}$  are bounded, we can assume without any loss of generality that they converge in  $\mathbb{R}^J$  and  $\mathbb{R}^I$ .

The next result, whose proof is given in Appendix, relates the closure of the graph of  $\hat{N}_{Y_j(\cdot)}(\cdot)$  with  $\hat{\mathcal{N}}_{Y_j(\cdot)}(\cdot)$ .

**Proposition 4.** Let  $(z^\gamma, \pi^\gamma)_{\gamma \in \Gamma}$  be a net in  $A(\omega) \times S$  converging to  $(z, \pi)$  for the product-topology  $\prod_{L_+ \times J} \tau \times \sigma^*$  such that

- a)  $\pi^\gamma(y_j^\gamma)$  converges for all  $j$
- b)  $\pi^\gamma \in \hat{N}_{Y_j(z^\gamma)}(y_j^\gamma)$  for all  $\gamma \in \Gamma$ .

Then,  $\pi(y_j) \leq \lim_\gamma \pi^\gamma(y_j^\gamma)$  and  $\pi \in \hat{N}_{Y_j(z)}(y_j) \cap S$  if  $\pi(y_j) = \lim_\gamma \pi^\gamma(y_j^\gamma)$

We now prove that there is a marginal pricing equilibrium of the economy  $\mathcal{E}$ . Indeed,  $\pi_1 = \pi_2 = \dots = \pi_n > 0$  (See Step 2 in [15]). Now, we denote this price vector simply by  $\pi$ . Next, by C(1) and P(1) together with the fact that  $L_+$  is  $\tau$ -closed,  $((x_i)_{i \in I}, (y_j)_{j \in J}) \in \prod_{i \in I} X_i(z) \times \prod_{j \in J} Y_j(z)$  and  $\sum_{i \in I} x_i = \sum_{j \in J} y_j + \omega$ . We also claim that for all  $i \in I$ , if  $x'_i \succ_{i,z} x_i$ , then  $\pi(x'_i) \geq r_i(\pi(\omega_i), \lim_\gamma (\pi_j^{F(\gamma)}(y_j^{F(\gamma)}))_{j \in J})$ . Indeed, there exists a subspace  $F_0$  such that  $x'_i \in F_0$ . There also exists  $\gamma_0$  and a positive sequence  $\lambda^{F(\gamma)}$  in  $[0, \infty)$  converging to 0 from above such that for all  $\gamma > \gamma_0$ ,  $F_0 \subset F(\gamma)$  and  $x'_i + \delta^{F(\gamma)}e \in X_i(z^{F(\gamma)})$  by Assumption C(2), whence  $x'_i + \delta^{F(\gamma)}e \in X_i(z^{F(\gamma)}) \cap F(\gamma)$ . Since  $(x_i, x'_i, z) \notin G_i$  (see Assumption C(4)), then there exists  $\gamma_1$  such that for all  $\gamma$  larger than  $\gamma_0$  and  $\gamma_1$ ,  $(x_i^{F(\gamma)}, x'_i + \delta^{F(\gamma)}e, z^{F(\gamma)}) \notin G_i$  and thus  $x'_i + \delta^{F(\gamma)}e \succ_{i,z^{F(\gamma)}} x_i^{F(\gamma)}$ . By the equilibrium conditions in  $\mathcal{E}^{F(\gamma)}$  and the fact that  $\pi_{j|F(\gamma)}^{F(\gamma)} = p^{F(\gamma)}$ ,  $\pi_j^{F(\gamma)}(x'_i + \delta^{F(\gamma)}e) > r_i(\pi_j^{F(\gamma)}(\omega), (\pi_j^{F(\gamma)}(y_j^{F(\gamma)}))_{j \in J})$ . Since  $x'_i + \delta^{F(\gamma)}e$  converges to  $x'_i$  in the norm-topology and  $\pi_j = \pi$  for all  $j$ , we have

$$\pi(x'_i) \geq r_i(\pi(\omega), \lim_t (\pi_j^{F(t)}(y_j^{F(t)}))_{j \in J})$$

and the claim is proved.

Next, we point out that by Proposition 4,  $\lim \pi_j^{F(\gamma)}(y_j^{F(\gamma)}) \geq \pi_j(y_j)$  for all  $j \in J$ . By the above result and the local non satiation Assumption C(3), we deduce that  $\pi(x_i) = r_i(\pi(\omega_i), (\pi(y_j))_{j \in J})$  for all  $i \in I$  which, in turn using the market clearing condition, implies that  $\lim \pi_j^{F(\gamma)}(y_j^{F(\gamma)}) = \pi(y_j)$ . Hence again by Proposition 4,  $z \in A(\omega)$  and  $\pi \in \bigcap_{j \in J} \hat{N}_{Y_j(z)}(y_j) \cap S$ . Finally, Condition 1. of Definition 1 is also satisfied thanks to the fact that the income of each consumers is positive, the preferences are continuous and  $0 \in X_i(z)$  for all  $i$ , which allows us to prove that a strictly preferred consumption with respect to  $x_i$  is out of the budget set.

## 5 When $L_+$ has an empty interior

Now we address the case where  $L$  is a Banach lattice whose positive cone has an empty interior. Nevertheless, we recall that we assume that  $e$  is a quasi-interior point of  $L_+$ , that is  $L(e)$ , the principal ideal generated by  $e$  is norm dense in  $L$ . As well known in the literature, a properness condition becomes a key assumption in order to get a competitive equilibrium vector price.

For the consumption side we have:

*Uniform e-properness on preferences ([23])* For all  $i \in I$  and all  $z \in L^{I+J}$ , the preference relation  $\succeq_{i,z}$  is said to be  $e$ -uniformly proper if there is a real number  $\varepsilon_i > 0$  such that for all  $\alpha > 0$  for all  $u \in L$ , if  $\bar{x}_i - \alpha e + \alpha u \in X_i(z)$  and  $\bar{x}_i - \alpha e + \alpha u \succeq_{i,z} \bar{x}_i$  then  $u \notin \alpha B(0, \varepsilon_i)$ .

Actually, the above definition adapts that of Mas-Colell to our case when both the consumption set does not equal the positive cone and externalities are incorporated. Roughly speaking,  $e$  is desirable in the sense that one cannot compensate the loss of  $e$  with a vector which is too small relative to  $e$ . In other words,  $e$  is an extremely desirable bundle ([31])

As for the production sector, Richard introduced uniform  $e$ -properness which is a weakening of that of [24]. We posit and adapt it to the context of our model

*Uniform e-properness on technologies ([28])* For all  $\omega' \geq \omega$ , for all  $z \in A(\omega')$ , the technology  $Y_j(z)$  is  $e$ -uniformly proper if there exists  $\varepsilon_j > 0$  such that for all  $\bar{y}_j \in Y_j(z)$ ,  $(\{\bar{y}_j\} - \Gamma_j) \cap \{\xi \in L : \xi^+ \leq \bar{y}_j^+\} \subset Y_j(z)$ , where  $\Gamma_j = \{\alpha \zeta \mid \alpha > 0 \text{ and } \zeta \in \{e\} + B(0, \varepsilon_j)\}$ .

The above definition implies that if  $y_j^+$  is producible with an amount  $y_j^-$  of inputs, then the output  $y_j^+$  is still feasible if one substitutes the input  $\alpha u$  with  $u \in B(0, \varepsilon_j)$  by  $\alpha e$ . That is to say,  $y_j^+ - (y_j^- + \alpha e - \alpha u) \in Y_j(z)$  for  $u \in B(0, \varepsilon_j)$ .

As for the argument for proving the existence of equilibria, it will consist in finding a vector sublattice for which all the results of Section 4 hold. Then, by an appropriate extension theorem on price functionals, we prove that there is a marginal pricing equilibrium in  $\mathcal{E}$ . Unlike the proofs for competitive equilibria in Banach lattices (or more generally, in vector lattices), some drawbacks appear preventing us to apply this approach directly. First of all, when restricting the original commodity space to an order ideal commodity space, Survival Assumption does not hold. Consequently we would not get an equilibrium in the restricted economy. Second, even if we might prove that Survival Assumption holds true, uniform properness is not sufficient to prove that there exists a positive and continuous extension of prices since the argument as such depends crucially on the profit maximization hypothesis which is not true in non-convex economies. The last difficulty appears even if each producer can set prices according to his marginal pricing rule (i.e. even if we can avoid the second difficulty). When the commodity space is a vector lattice and the price set is a sublattice of the order dual, if  $\{\pi_j\}_{j \in J}$  is the set of prices for which every producer maximizes his profits  $\vee_{j \in J} \{\pi_j\}$  also does for every  $j \in J$ . Hence, uniform properness allows to get an equilibrium price functional (see [28] for a proof). In the present context, since  $L$  is a topological vector lattice, it follows automatically that  $L^*$  has a lattice structure. However, if  $\pi_j \in \tilde{\mathcal{N}}_{Y_j(z)}(y_j)$  for all  $j \in J$  uniform properness is not suitable since it would not guarantee that  $\vee_{j \in J} \{\pi_j\} \in \cap_{j \in J} \tilde{\mathcal{N}}_{Y_j(z)}(y_j)$  as in the Walrasian case. As a remark, uniform

properness is no longer sufficient for proving equilibria.

Consequently, we have to introduce an alternative notion of properness. We shall call it *uniform properness relative to  $L(e)$*

#### Uniform $e$ -properness relative to $L(e)$

We shall say that the technology  $Y_j$  is uniformly proper relative to  $L(e)$  if there exists a real number  $\delta_j > 0$ , which generates the open cone  $\Gamma_j = \{\alpha\xi \mid \alpha > 0, \xi \in \{-e\} + B(0, \delta_j)\}$  such that for all  $z \in Z$  and all  $\bar{y}_j \in Y_j(z)$ ,  $(\{\bar{y}_j\} + \Gamma_j) \cap L(e) \subset Y_j(z)$ . The element  $e$  is the proper vector.

In words, we are saying that if  $y_j$  is producible and we add to it the quantity  $e$  of inputs, then it is still producible if we add a vector which is small enough and the resultant vector is order bounded by some multiple of  $e$ . Thus, marginal rates of substitution with respect to  $e$  are bounded away from zero. In other words, the trace on  $L(e)$  of the extended production set  $Y_j(z) + \Gamma_j$  is included in  $Y_j(z)$ . Our condition is not directly comparable neither to [28] nor to [24]<sup>7</sup>.

We notice that if  $e \in \text{int}L_+$  then uniform properness technology relative to  $L(e)$  follows directly from the free-disposal Assumption since there would be a real number  $\delta_j > 0$ , such that  $-e + B(0, \delta_j) \subset -L_+$  and  $L(e) = L$ . On the other hand, when production correspondences are convex-valued this assumption can be replaced by uniform properness of preferences and technologies given in [23] and [28] respectively.

### 5.1 The main existence theorem

**Theorem 2.** *Let  $\mathcal{E}$  be a Banach lattice economy. There exists  $(\bar{z}, \bar{\pi})$  in  $Z \times \hat{\mathcal{N}}_{Y_j(z)}(y_j) \cap S$  which is a marginal pricing equilibrium if  $\mathcal{E}$  satisfies Assumptions C, P, B, SA, TC and each technology is  $e$ -uniformly proper with respect to  $L(e)$ .*

Theorem 2 extends Theorem 1 since we do not impose any interiority assumption. Apart from the ordered preferences we have assumed, our result is at the same level of generality than the other existence results for competitive equilibrium whereas we encompass two major market imperfections, increasing returns and externalities.

### 5.2 Proof

Let  $L(e)$  be the principal ideal generated by  $e$ . Hence the order interval  $[-e, e]$  is radial in  $L(e)$  and then the gauge of the set  $[-e, e]$  induces a norm topology on  $L(e)$ . We call it the  $\|\cdot\|_e$ -topology. Actually,  $[-e, e]$  is the closed unit ball

<sup>7</sup>Instead of  $L(e)$ , Mas-Colell uses a pre-technology  $Z_j$  which is a closed and convex sublattice of  $L$  satisfying free disposal

on  $L(e)$  while  $B^e(0, 1) = \{x \in L(e) : \|x\|_e < 1\}$  is the open unit ball on  $L(e)$ . Let  $L(e)^*$  denote the  $\|\cdot\|_e$ -dual of  $L(e)$  and let  $\|\cdot\|_e^*$  denote the dual norm on  $L(e)^*$ .

Let  $L(e)_+ = L_+ \cap L(e)$ . Clearly  $L(e)_+$  is  $\|\cdot\|_e$ -closed in  $L(e)$  and has a non-empty  $\|\cdot\|_e$ -interior which contains  $e$ . Obviously,  $\|e\|_e = 1$ . The  $\|\cdot\|_e$ -topology is finer than the topology of  $L(e)$  as a subspace of  $(L, \|\cdot\|)$  with the norm denoted  $\|\cdot\|_{L(e)}$ . Let  $\tau_e$  be the restriction to  $L(e)$  of the topology  $\tau$ . Clearly  $\tau_e$  is included in the  $\|\cdot\|_{L(e)}$ -topology, which itself is included in the  $\|\cdot\|_e$ -topology. Furthermore, it is straightforward to check that  $\tau_e$  is a Hausdorff and locally convex-solid topology such that all order intervals in  $L(e)$  are  $\tau_e$ -compact.

Let  $X_i^e : L(e)^{I+J} \rightarrow L(e)_+$  be the restricted consumption correspondence such that for all  $z \in L(e)^{I+J}$ ,  $X_i^e(z) = X_i(z) \cap L(e)_+$ .  $Y_j^e : L(e)^{I+J} \rightarrow L(e)$  is the restricted production correspondence such that for all  $z \in L(e)^{I+J}$ ,  $Y_j^e(z) = Y_j(z) \cap L(e)$ . We also restrict properly the preference relation by  $\succeq_{i,z}^e$  for all  $z \in L(e)^{I+J}$ . We remark that  $\partial Y_j^e(z) \subset \partial Y_j(z) \cap L(e)$ , whence  $Z^e \subset Z$  and  $A^e(\omega') = A(\omega') \cap Z^e \subset A(\omega')$ . The revenue functions  $(r_i)$  are the same.  $p \in L(e)^*$  is a  $\|\cdot\|_e$ -continuous linear functional on  $L(e)$ . Finally,  $PE^e = \{(p, z) \in L(e)^* \times Z^e \mid p \in \cap_{j \in J} \hat{N}_{Y_j^e(z)}(y_j)\}$ . For all  $z \in L(e)^{I+J}$ ,  $\hat{T}_{Y_j^e(z)}(y_j)$  and  $\hat{N}_{Y_j^e(z)}(y_j)$  are the induced cones in  $L(e)$ .

The economy  $\mathcal{E}^e$  is fully described by  $((X_i^e, \succeq_{i,z}^e, r_i)_{i \in I}, (Y_j^e)_{j \in J}, \omega)$ .

We show that Theorem 1 applies for  $\mathcal{E}^e$ . Clearly, for all  $z \in L(e)^{I+J}$  the  $\|\cdot\|_e$ -interior of  $X_i^e(z)$  is non-empty because of Assumption C(1) on solidity condition and the fact that  $\delta e \in X_i^e(z)$  for all  $\delta > 0$ .  $X_i^e$  satisfies the remaining of Assumption C(1) and Assumption C(2) and since the  $\|\cdot\|_e$ -topology is stronger than the  $\|\cdot\|_{L(e)}$ -topology,  $G_i^e(z)$  is  $\|\cdot\|_e \times \tau_e \times \prod_{L(e)^{I+J}} \tau_e$ -closed in  $L(e)^{2+I+J}$ . As for non satiation, note that it remains true in  $\mathcal{E}^e$  since in Assumption C(3), the improving consumption  $(x'_i)$  is chosen in  $L(e)^I$ . The next lemma whose proof is given in Appendix shows that Assumption SA holds true in the economy  $\mathcal{E}^e$ .

**Lemma 4.** *The economy  $\mathcal{E}^e$  satisfies Assumption SA.*

With respect to the production sector, it is not difficult to observe that Assumption P is fully satisfied on every  $Y_j^e$ . On the other hand, Assumption B is automatically satisfied. Finally one easily checks that Assumption R holds on  $\mathcal{E}^e$ .

Consequently, all conditions of Theorem 4 are satisfied. Thus, there is an equilibrium  $((x_i)_{i \in I}, (y_j)_{j \in J}, p) \in Z^e \times \hat{N}_{Y_j^e(z)}(y_j) \cap S^{e*}$ . For the next two claims we remark that  $\hat{T}_{Y_j(z)}(y_j) \cap L(e) \subset \hat{T}_{Y_j^e(z)}(y_j)$

**Claim 1.** *There exists a price functional  $\pi \in L^*$  such that  $\pi|_{L(e)} = p$*

*Proof.* Let  $\delta > 0$  be a real number such that  $\delta < \delta_j$  for all  $j \in J$ , where  $\delta_j$  are the parameters coming from uniform  $e$ -properness assumption relative to  $L(e)$

on each production correspondence  $Y_j$ . Since  $L$  is a locally convex topological vector space, we can apply a suitable version of the Hahn-Banach theorem (for instance, see [29], Theorem 4.2, p. 49) to show the existence of a continuous linear functional  $\pi$  which extends  $p$  to the whole space  $L$ . So, we have to show first that  $p$  is  $\|\cdot\|$ -continuous on  $L(e)$ . We stress that accordingly to the above remark  $p$  is  $\|\cdot\|_e$ -continuous. We now prove that the functional  $p$  is bounded on  $B(0, \delta) \cap L(e)$ . Since  $B(0, \delta)$  is circled (then symmetric) it suffices to prove it for any vector in  $B(0, \delta) \cap L(e)_+$ . Let  $\xi \in B(0, \delta) \cap L(e)_+$ . There exists  $n_0 \in \mathbb{N}$  such that  $\bar{\xi} = \frac{1}{n_0}\xi \leq e$ .  $\bar{\xi} \in L(e) \cap B(0, \frac{\delta}{n_0})$ . We show that  $-\frac{1}{n_0}e + \bar{\xi}$  belongs to  $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$ . Indeed, recall that by Assumption TC,  $0 \in \hat{\mathcal{T}}_{Y_j(z)}(y_j)$  and then in  $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$ . Thus, for  $\rho > 0$ , there exists  $\eta > 0$  such that for all  $r > 0$  there exist relative neighborhoods  $V^e \in L(e)^{I+J}$  (for  $V \in \mathcal{V}_{\prod_{L^{I+J}} \tau}(z)$ ) and  $U^e \in L(e)$  (for  $V \in \mathcal{V}_\tau(\bar{y}_j)$ ) together with the real number  $\varepsilon > 0$  such that for all  $z' \in (\{z\} + \rho B^e(0, 1)^{I+J}) \cap V^e$ , for all  $\bar{y}'_j \in (\{y_j\} + \rho B^e(0, 1)) \cap U^e \cap Y_j(z')$  and for all  $t \in (0, \varepsilon)$ ,  $\bar{y}'_j + t(\eta(y_j - \bar{y}'_j) - re) \in Y_j(z')$ . We note that  $-t\frac{1}{n_0}e + t\bar{\xi} \in \Gamma_j$  for  $t > 0$  and then, by the  $e$ -uniform properness relative to  $L(e)$ , it follows that  $\bar{y}'_j + t(\eta(y_j - \bar{y}'_j) - re) - t\frac{1}{n_0}e + t\bar{\xi} = \bar{y}'_j + t(-\frac{e}{n_0} + \bar{\xi} + \eta(y_j - \bar{y}'_j) - re) \in Y(z') \cap L(e)$ . Thus we deduce that  $-\frac{1}{n_0}e + \bar{\xi}$  belongs to  $\hat{\mathcal{T}}_{Y_j^e(z)}^\rho(y_j)$  and since this is true for all  $\rho > 0$ , we get that  $-\frac{1}{n_0}e + \bar{\xi}$  belongs to  $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$ .

Since  $p \in \hat{\mathcal{N}}_{Y_j^e(z)}(y_j)$  and  $p(e) = 1$ , it follows that  $p(\bar{\xi}) \leq \frac{1}{n_0}$ . Hence,  $p(\xi) = n_0 p(\bar{\xi}) \leq 1$ . We conclude that  $p$  is  $\|\cdot\|$ -continuous on  $L(e)$  and thus there exists  $\pi \in L^*$  which extends  $p$ .

□

Because of the free-disposal assumption and the fact that  $\pi(e) = p(e) = 1$ ,  $\pi > 0$ .

**Claim 2.**  $\pi \in \cap_{j \in J} \hat{\mathcal{N}}_{Y_j(z)}(y_j)$ .

*Proof.* Let  $\nu \in \hat{\mathcal{T}}_{Y_j(z)}(y_j) \subset \hat{\mathcal{T}}_{Y_j(z)}^\rho(y_j)$  for any  $j$  and  $\rho > 0$ . If  $\nu \in L(e)$  then  $\pi(\nu) = p(\nu) \leq 0$  and we are done. Let us suppose that  $\nu \notin L(e)$ . From the definition of  $\hat{\mathcal{T}}_{Y_j(z)}^\rho(y_j)$  we know that there exists  $\eta > 0$  such that for  $r > 0$  arbitrary there exist neighborhoods  $V \in \mathcal{V}_{\prod_{L^{I+J}} \tau}(z)$ ,  $U \in \mathcal{V}_\tau(y_j)$  and a real number  $\varepsilon > 0$  such that for all  $z' \in (\{z\} + \rho B^e(0, 1)^{I+J}) \cap V \cap L(e)$ , for all  $\bar{y}'_j \in B^e(y_j, \rho) \cap U \cap L(e) \cap Y_j(z')$  and for all  $t \in (0, \varepsilon)$  there exists  $\xi \in r[-e, e]$  such that  $\bar{y}'_j + t(\nu + \eta(y_j - \bar{y}'_j) + \xi) \in Y(z')$ .

Let us choose  $\beta > 0$ . Let  $\delta_j > 0$  be the parameter coming from uniform  $e$ -properness relative to  $L(e)$ . Since  $\pi$  is  $\|\cdot\|$ -continuous, there exists  $\delta' > 0$  such that  $B(0, \delta') \subset \{x \in L : |\pi(x)| < \beta\}$ . Let  $\delta > 0$  such that  $\delta < \beta\delta_j$  and  $\delta < \delta'$ . Since  $L(e)$  is norm-dense in  $L$  there exists  $u \in B(0, \delta)$  such that  $\nu + u \in L(e)$ . We remark that the vector  $\bar{y}'_j + t(\nu + \eta(y_j - \bar{y}'_j) + \xi) - t\beta e + tu = \bar{y}'_j + t(\nu - \beta e + u + \eta(y_j - \bar{y}'_j) + \xi)$  belongs to  $L(e)$ . Hence, by uniform  $e$ -properness condition, we

deduce that  $\bar{y}'_j + t(\nu - \beta e + u + \eta(y_j - \bar{y}'_j) + \xi) \in Y_j(z') \cap L(e)$ . This in turn implies that  $\nu - \beta e + u$  belongs to  $\hat{T}_{Y_j^e(z)}^\rho(y_j)$  and since this is true for all  $\rho > 0$  we have that  $\nu - \beta e + u$  belongs to  $\hat{T}_{Y_j^e(z)}(y_j)$ . Thus,  $\pi(\nu - \beta e + u) = p(\nu - \beta e + u) \leq 0$ , whence  $\pi(\nu) \leq 2\beta$ . This inequality is true for all  $\beta > 0$  and  $j \in J$  and thus the Claim is proved.  $\square$

The last part of the proof consists in proving that every consumer is in equilibrium according to her/his budget constraint and preferences. Let  $i \in I$  and  $x'_i \in X_i(z)$  such that  $x'_i \succ_{i,z} x_i$  and  $\pi(x'_i) < \pi(x_i)$ . Let us note that  $(x_i, x'_i, z) \notin G_i$ . By Lemma 3 in [27],  $L(e)_+$  is norm-dense in  $L_+$  and thus one can choose a parameter  $\varepsilon > 0$  small enough such that there exists  $u \in B(0, \varepsilon)$ ,  $x'_i + u \in L(e)_+$ ,  $(x_i, x'_i + u, z) \notin G_i$  by Assumption C(4) and  $\pi(x'_i + u) < \pi(x_i)$ . Since there exists  $n_0 \in \mathbb{N}$  for which  $x'_i + u \leq n_0 e$  we deduce by solidity of  $X_i(z)$  (Assumption C(1)) and Assumption C(2) that  $x'_i + u \in X_i(z)$ . Consequently,  $\pi(x'_i + u) = p(x'_i + u) < \pi(x_i) = p(x_i)$ , which contradicts the fact that  $x_i$  is an equilibrium consumption vector for  $i$  in  $\mathcal{E}^e$ .

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## 6 Appendix

### 6.1 Proof of Proposition 1

We first recall the definition of the Clarke’s tangent and normal cones to the set  $Y_j(z)$  at the point  $\bar{y}_j$

$$T_{Y_j(z)}(\bar{y}_j) := \left\{ \nu \in L \mid \begin{array}{l} \forall r > 0 \exists \varepsilon > 0 \mid \forall y'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z), \forall t \in (0, \varepsilon) \\ [y'_j + tB(\nu, r)] \cap Y_j(z) \neq \emptyset \end{array} \right\}$$

$$N_{Y_j(z)}(\bar{y}_j) = [T_{Y_j(z)}(\bar{y}_j)]^o = \{p \in L^* \mid p(\nu) \leq 0 \ \forall \nu \in T_{Y_j(z)}(\bar{y}_j)\}$$

*Proof.*  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  is non-empty since the free-disposal condition implies that  $-L_+ \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$ . Now we show that  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  is a cone. Let  $\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$  and  $\tau > 0$ . Let  $r > 0$  and  $\varepsilon$  be the parameter associated by the definition of  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  to  $\frac{r}{\tau}$ . Hence, for all  $z' \in B(z, \varepsilon)$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z')$  and for all  $t \in (0, \varepsilon)$  there exists  $\xi \in B(0, \frac{r}{\tau})$  such that  $\bar{y}'_j + t(\nu + \xi) \in Y_j(z')$ . Let  $\varepsilon'$  strictly smaller than  $\varepsilon$  and  $\frac{\varepsilon}{\tau}$ . Hence, for every  $z' \in B(z, \varepsilon')$  for every  $\bar{y}'_j \in B(\bar{y}_j, \varepsilon')$  and for every  $t \in (0, \varepsilon')$ , since  $t\tau < \varepsilon$ , there exists  $\xi \in B(0, \frac{r}{\tau})$  such that  $\bar{y}'_j + \tau t(\nu + \xi) = \bar{y}'_j + t(\tau\nu + \tau\xi) \in Y_j(z')$ . As  $\tau\xi \in B(0, r)$ , we have proved that  $\tau\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$  by associating to  $r$  the parameter  $\varepsilon'$  and thus  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  is a cone.

We now show that  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  is convex by proving that

$$\hat{T}_{Y_j(z)}(\bar{y}_j) + \hat{T}_{Y_j(z)}(\bar{y}_j) \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$$

Let  $\nu$  and  $\nu'$  be two vectors in  $\hat{T}_{Y_j(z)}(\bar{y}_j)$ . For  $r > 0$  there exist two non negative real numbers  $\varepsilon$  and  $\varepsilon'$  associated by the definition of  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  to  $\frac{r}{2}$ . Let  $\varepsilon_1 > 0$  smaller than  $\varepsilon$  and  $\frac{\varepsilon'}{1+\|\nu\|+\frac{r}{2}}$ . Hence, for all  $z' \in B(z, \varepsilon_1)$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \varepsilon_1) \cap Y_j(z')$  and for all  $t \in (0, \varepsilon_1)$ , there exists  $\xi \in B(0, \frac{r}{2})$  such that  $\bar{y}''_j = \bar{y}'_j + t(\nu + \xi) \in Y_j(z')$ . We remark that  $\|\bar{y}''_j - \bar{y}_j\| \leq \|\bar{y}'_j - \bar{y}_j\| + t(\|\nu\| + \|\xi\|) < \varepsilon_1 + \varepsilon_1(\|\nu\| + \frac{r}{2}) \leq \varepsilon'$ . So, since  $\varepsilon_1 < \varepsilon'$ , there exists  $\xi' \in B(0, \frac{r}{2})$  such that  $\bar{y}''_j + t(\nu' + \xi') \in Y_j(z')$ . Hence,  $\bar{y}'_j + t(\nu + \nu' + \xi + \xi') \in Y_j(z')$  and  $\xi + \xi' \in B(0, r)$ . So  $\varepsilon_1$  associated to  $r$  satisfies the definition of  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  and we have shown that  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  is closed under addition.

The proof of the second part is trivial given the definition of the Clarke's normal cone. □

## 6.2 Proof of Proposition 2

*Proof. 1.*  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$  is non-empty by Assumption TC. We now prove that  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$  is a cone. Let  $\rho > 0$ ,  $\tau > 0$  and  $\nu \in \cap_{\rho>0} \hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j) \subset \hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j)$ . Then there exists  $\eta > 0$  such that for all  $r > 0$  there exist  $V \in \mathcal{V}_{\prod_{L^I+J} \tau}(z)$ ,  $U \in \mathcal{V}_\tau(\bar{y}_j)$  and  $\varepsilon > 0$  such that for all  $z' \in B(z, \rho) \cap V$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U \cap Y_j(z')$  and for all  $t \in (0, \varepsilon)$ ,  $\bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \xi) \in Y_j(z')$  for some  $\xi \in r[-e, e]$ . Let  $\eta' = \tau\eta$ . Let  $r > 0$ . Let  $V, U$  and  $\varepsilon$  the parameter and the open neighborhoods associated for  $\nu$  to  $\frac{r}{\tau}$ . Let  $\varepsilon' = \frac{\varepsilon}{\tau}$  and note that  $\tau t \in (0, \varepsilon)$  is equivalent to  $t \in (0, \varepsilon')$ . Hence, for all  $z' \in B(z, \rho) \cap V$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U \cap Y_j(z')$  and for all  $\tau t \in (0, \varepsilon)$ , there exists  $\xi \in \frac{r}{\tau}[-e, e]$  such that  $\bar{y}'_j + \tau t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \xi) =$

$\bar{y}'_j + t(\tau\nu + \eta'(\bar{y}_j - \bar{y}'_j) + \tau\xi) \in Y_j(z')$  with  $\tau\xi \in r[-e, e]$ . Hence,  $\tau\nu \in \hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j)$ . Since this holds for all  $\rho > 0$ ,  $\tau\nu \in \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ .

$\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$  is weak\*-closed since it is the intersection of weak\*-closed half spaces  $\{\pi \in L^* \mid \pi(\nu) \leq 0\}$  over the  $\nu$  in  $\hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j)$ .

**2.** From the definition of  $\hat{\mathcal{T}}$ , it is enough to prove that  $\hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j) \subset \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$  for all  $\rho > 0$ . Let  $\rho > 0$  and  $\nu \in \hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j)$ . Consequently, there exists  $\eta > 0$  such that for all  $r > 0$  there exist  $V \in \mathcal{V}_{\Pi_{L^+} \tau}(z)$ ,  $U \in \mathcal{V}_\tau(\bar{y}_j)$  and  $\varepsilon > 0$  associated to  $\frac{r}{2}$ . Let us fix  $\varepsilon' > 0$  strictly smaller than  $\varepsilon, \frac{r}{2\eta}, \rho$  and such that  $B(z, \varepsilon') \subset B(z, \rho) \cap V$  and  $B(\bar{y}_j, \varepsilon') \subset B(\bar{y}_j, \rho) \cap U$ . Thus, for all  $z' \in B(z, \varepsilon')$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \varepsilon') \cap Y_j(z')$  and for all  $t \in (0, \varepsilon')$ , we get the existence of a vector  $\xi \in \frac{r}{2}[-e, e]$  such that  $\bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \xi) \in Y_j(z')$ . Note that  $\|\eta(\bar{y}_j - \bar{y}'_j)\| \leq \frac{r}{2}$  and thus  $\|\xi' = \xi + \eta(\bar{y}_j - \bar{y}'_j)\| \leq r$  and  $\bar{y}'_j + t(\nu + \xi') \in Y_j(z')$ . Consequently,  $\hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j) \subset \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ .

By polarity we obtain that  $\hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j) \subset \hat{\mathcal{N}}_{Y_j(z)}(\bar{y}_j)$ .

**3.**  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) \subset \mathcal{T}_{Y_j(z)}(\bar{y}_j)$  since order intervals are norm-bounded thanks to the fact that the norm is a lattice norm.

**4.** Since  $e \in \text{int}L_+$ , as already mentioned in Section 3, without any loss of generality, we choose the norm as the lattice norm associated to  $e$ . So  $\bar{B}(0, 1) = [-e, e]$  and we can replace it in the definition of  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ ,  $\xi \in r[-e, e]$  by  $\xi \in B(0, r)$ .

We show that  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) + \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j) \subset \hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$  in order to prove the convexity. Let  $\nu$  and  $\nu'$  be in  $\hat{\mathcal{T}}_{Y_j(z)}(\bar{y}_j)$ . Hence they belong to  $\hat{\mathcal{T}}_{Y_j(z)}^\rho(\bar{y}_j)$  for every  $\rho > 0$ . Let us consider the set  $\hat{\mathcal{T}}_{Y_j(z)}^{\rho+1}(\bar{y}_j)$ . There exist  $\eta$  and  $\eta'$  such that for every  $r > 0$  there exist  $\varepsilon, \varepsilon', U, U', V$  and  $V'$  associated to  $\frac{r}{3}$  such that for all  $z' \in B(z, \rho+1) \cap V$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \rho+1) \cap U \cap Y_j(z')$  and for all  $t \in (0, \varepsilon)$ ,  $[\{\bar{y}'_j\} + tB(\nu + \eta(\bar{y}_j - \bar{y}'_j), \frac{r}{3})] \cap Y_j(z') \neq \emptyset$  and for all  $z' \in B(z, \rho+1) \cap V'$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \rho+1) \cap U' \cap Y_j(z')$  and for all  $t \in (0, \varepsilon')$ ,  $[\{\bar{y}'_j\} + tB(\nu + \eta'(\bar{y}_j - \bar{y}'_j), \frac{r}{3})] \cap Y_j(z') \neq \emptyset$ .

There exist  $\alpha \in (0, 1)$ ,  $V''$  and  $U''$  such that  $V'' + B(0, \alpha) \subset V \cap V'$  and  $U'' + B(0, \alpha) \subset U \cap U'$ . Let  $\varepsilon'' > 0$  strictly smaller than  $\varepsilon, \varepsilon', \frac{r}{\eta'(3\|\nu\|+3\rho\eta+r)}$  and  $\frac{3\alpha}{3\|\nu\|+3\rho\eta+r}$ . Hence, for all  $z' \in B(z, \rho) \cap V'' \subset B(z, \rho+1) \cap V$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U'' \cap Y_j(z') \subset B(\bar{y}_j, \rho+1) \cap U \cap Y_j(z')$  and for all  $t \in (0, \varepsilon'') \subset (0, \varepsilon)$ , there exists  $\xi \in B(0, \frac{r}{3})$  such that  $\zeta_j = \bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \xi) \in Y_j(z')$ .

From the definition of  $\varepsilon''$ , one easily checks that  $\|\zeta_j - \bar{y}'_j\| < \alpha < 1$  and  $\eta'\|\zeta_j - \bar{y}'_j\| \leq \frac{r}{3}$ . So  $\zeta_j \in B(\bar{y}_j, \rho+1)$  and  $\|t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \xi)\| < \alpha$ . Consequently, since  $\bar{y}'_j \in U''$  and  $z' \in B(z, \rho+1) \cap V'$ , there exists  $\xi' \in L$  such that  $\|\xi'\| < \frac{r}{3}$  and  $\zeta'_j = \zeta_j + t(\nu' + \eta'(\bar{y}_j - \zeta_j) + \xi') \in Y_j(z')$ . We note that  $\zeta'_j = \bar{y}'_j + t(\nu + \nu' + (\eta + \eta')(\bar{y}_j - \bar{y}'_j) + \xi + \xi' + \eta'(\bar{y}'_j - \zeta_j))$  and  $\|\eta'(\bar{y}'_j - \zeta_j)\| < \frac{r}{3}$ . Consequently,  $[\{\bar{y}'_j\} + tB(\nu + \nu' + (\eta + \eta')(\bar{y}_j - \bar{y}'_j), r)] \cap Y_j(z') \neq \emptyset$  and thus

$\nu + \nu' \in \hat{T}_{Y_j(z)}^\rho(\bar{y}_j)$ . Since this is true for all  $\rho > 0$ , we conclude that  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  is stable under summation. Since  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  is a cone, we get it is convex.

**5.** We first show that when  $L$  is finite dimensional  $\hat{T}_{Y_j(z)}(\bar{y}_j) = \hat{T}_{Y_j(z)}(\bar{y}_j)$ . Let  $r > 0$  and  $\varepsilon$  associated with  $\frac{r}{2}$  such that for all  $z' \in B(z, \varepsilon)$ , for all  $\bar{y}'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z')$  and for all  $t \in (0, \varepsilon)$  there exists  $\xi \in B(0, \frac{r}{2})$  such that  $\bar{y}'_j + t(\nu + \xi) \in Y_j(z')$ . Let  $\rho > 0$  and  $\eta > 0$ . Since in this case all topologies are equivalents and the interior of the positive cone is nonempty, we can choose the closed unit ball equal to the order interval  $[-e, e]$ . We now choose  $\varepsilon' > 0$  smaller than  $\varepsilon, \frac{r}{2\eta}$  and  $\rho$ . Let  $V = B(z, \varepsilon')$  and  $U = B(\bar{y}_j, \varepsilon')$ . Then, for all  $\bar{y}'_j \in U$  one has that  $\|\xi - \eta(\bar{y}_j - \bar{y}'_j)\| < r$  for all  $\xi \in B(\nu, \frac{r}{2})$ . This implies that for all  $z' \in V \cap B(z, \rho)$ , for all  $\bar{y}'_j \in U \cap B(\bar{y}_j, \rho) \cap Y_j(z')$  and all  $t \in (0, \varepsilon')$ , there exists  $\xi \in B(0, \frac{r}{2})$  such that  $\bar{y}'_j + t(\nu + \xi) \in Y_j(z')$ . But  $\bar{y}'_j + t(\nu + \xi) = \bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) - \eta(\bar{y}_j - \bar{y}'_j) + \xi)$ . Since  $\xi - \eta(\bar{y}_j - \bar{y}'_j) \in B(0, r)$ ,  $\nu \in \hat{T}_{Y_j(z)}^\rho(\bar{y}_j)$ , so  $\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$  since this is true for all  $\rho > 0$ .

Let us now consider the following sequential characterisation of  $\hat{T}_{Y_j(z)}(\bar{y}_j)$  when  $L$  is a finite dimensional vector space:

$$\hat{T}_{Y_j(z)}(\bar{y}_j) := \left\{ \nu \in L \mid \begin{array}{l} \forall z^n \rightarrow z, \forall y_j^n \rightarrow \bar{y}_j \text{ such that } y_j^n \in \partial Y_j(z^n) \\ \text{and } \forall t^n \downarrow 0 \exists \nu^n \rightarrow \nu \text{ such that } y_j^n + t^n \nu^n \in Y_j(z^n) \end{array} \right\}$$

Let us recall the marginal pricing rule of [11] when the commodity space is of finite dimension:

$$MP(\bar{y}_j, z) = co \left\{ p \in S \mid \begin{array}{l} \exists z^n \subset \mathbb{R}^{L(I+J)}, z^n \rightarrow z, \\ \exists \bar{y}_j^n \subset \mathbb{R}^L, \bar{y}_j^n \rightarrow \bar{y}_j, \bar{y}_j^n \in \partial Y_j(z^n) \\ \exists p^n \subset \mathbb{R}^L, p^n \rightarrow p, p^n \in N_{Y_j(z^n)}(\bar{y}_j^n) \cap S \end{array} \right\}$$

To prove that  $\hat{N}_{Y_j(z)}(\bar{y}_j) \cap S = MP(\bar{y}_j, z)$ , we first prove the following lemma:

**Lemma 5.** (i) Let  $\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$  and let  $r > 0$  and  $\varepsilon > 0$  as given by the definition of  $\hat{T}_{Y_j(z)}(\bar{y}_j)$ . For all  $z' \in B(z, \varepsilon)$  and all  $\bar{y}'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z')$ ,  $\nu - re \in \hat{T}_{Y_j(z')}(\bar{y}'_j)$ .

(ii) Let  $\nu \in MP(\bar{y}_j, z)^\circ$ , for all  $\delta > 0$ , there exists  $\varepsilon > 0$  such that for all  $z' \in B(z, \varepsilon)$  and all  $\bar{y}'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z')$ ,  $\nu - \delta e \in \text{int} T_{Y_j(z')}(\bar{y}'_j)$ .

*Proof of Lemma 5 (i)* Let  $\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$ . We choose  $\varepsilon' < \varepsilon$  such that  $B(z', \varepsilon') \subset B(z, \varepsilon)$  and  $B(\bar{y}'_j, \varepsilon') \subset B(\bar{y}_j, \varepsilon)$ . Then, for all  $z'' \in B(z', \varepsilon')$  for all  $\bar{y}''_j \in B(\bar{y}'_j, \varepsilon') \cap Y_j(z'')$  and all  $t \in (0, \varepsilon')$  there exists  $\xi \in B(0, r)$  such that  $\bar{y}''_j + t(\nu + \xi) \in Y_j(z'')$ . By the free disposal condition,  $\bar{y}''_j + t(\nu - re) \in Y_j(z'')$ . From the definition of  $\hat{T}_{Y_j(z')}(\bar{y}'_j)$ , this clearly implies that  $\nu - re \in \hat{T}_{Y_j(z')}(\bar{y}'_j)$ .

(ii) Let  $\nu \in MP(\bar{y}_j, z)^\circ$  and  $\delta > 0$ . By contraposition, if for all  $\varepsilon > 0$ , there exists  $z' \in B(z, \varepsilon)$  and  $\bar{y}'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z')$  such that  $\nu - \delta e \notin \text{int} T_{Y_j(z')}(\bar{y}'_j)$ , we can build a sequence  $(z^n, \bar{y}_j^n, p^n)$  converging to  $(z, \bar{y}_j, p)$  such that  $\bar{y}_j^n \in \partial Y_j(z^n)$ ,

$p^n \in N_{Y_j(z^n)}(\bar{y}_j^n) \cap S$  and  $\langle p^n, \nu - \delta e \rangle \geq 0$ . So, at the limit, from the definition of  $MP(\bar{y}_j, z)$ , we get  $p \in MP(\bar{y}_j, z)$  and  $\langle p, \nu - \delta e \rangle \geq 0$ , which implies  $\langle p, \nu \rangle \geq \delta > 0$ , which contradicts  $\nu \in MP(\bar{y}_j, z)^\circ$ .  $\square$

We now proceed with the proof. Let  $\pi \in MP(\bar{y}_j, z)$ , then by Carathéodory's Theorem  $\pi = \sum_{k \in K} \lambda_k p_k$  such that  $\lambda_k \geq 0$ ,  $\sum_{k \in K} \lambda_k = 1$ , and  $(p_k) \in S$  for all  $k$ . Hence, for all  $\delta > 0$ ,  $\langle \pi, \nu - \delta e \rangle = \sum_{k \in L} \lambda_k \langle p_k, \nu - \delta e \rangle$ . By definition, every  $p_k$  is the limit of a sequence  $p_k^n \in N_{Y_j(z^n)}(\bar{y}_j^n) \cap S \subset \hat{N}_{Y_j(z^n)}(\bar{y}_j^n) \cap S$  from Proposition 1. Let  $\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$ . For all  $n \geq n_0$   $\sum_{k \in K} \lambda_k \langle p_k^n, \nu - \delta e \rangle \leq 0$  by the above lemma. Taking limits, one has  $\langle \pi, \nu - \delta e \rangle = \sum_{k \in K} \lambda_k \langle p_k, \nu - \delta e \rangle \leq 0$ . Hence,  $\langle \pi, \nu \rangle \leq \delta$ . Since this is true for all  $\delta > 0$ , we conclude that  $\langle \pi, \nu \rangle \leq 0$  and then  $\pi \in \hat{N}_{Y_j(z)}(\bar{y}_j) \cap S$ .

To prove the converse inclusion, we use the duality between closed convex cones and we actually prove that  $MP(\bar{y}_j, z)^\circ \subset \hat{T}_{Y_j(z)}(\bar{y}_j)$ . Let  $\nu \in MP(\bar{y}_j, z)^\circ$ . To prove that  $\nu \in \hat{T}_{Y_j(z)}(\bar{y}_j)$ , it suffices to show that  $\nu - \delta e \in \hat{T}_{Y_j(z)}(\bar{y}_j)$  for all  $\delta > 0$ . From the above lemma, for all  $\delta > 0$ , there exists  $\varepsilon > 0$  such that for all  $z' \in B(z, \varepsilon)$  and all  $\bar{y}'_j \in B(\bar{y}_j, \varepsilon) \cap Y_j(z')$ ,  $\nu - \delta e \in \text{int} T_{Y_j(z')}(\bar{y}'_j)$ . So, from the characterisation of the interior of the Clarke' tangent cone, there exists  $\tau(\bar{y}'_j, z') > 0$  such that for all  $t \in [0, \tau(\bar{y}'_j, z')]$ ,  $\bar{y}'_j + t(\nu - \delta e) \in Y_j(z')$ . Let  $z^n \rightarrow z$  and  $\bar{y}_j^n \rightarrow \bar{y}_j$  such that  $\bar{y}_j^n \in \partial Y_j(z^n)$ . For  $n$  large enough,  $z^n \in B(z, \varepsilon)$  and  $\bar{y}_j^n \in B(\bar{y}_j, \varepsilon) \cap Y_j(z')$ . So, we can build a sequence  $t^n \downarrow 0$  such that  $t^n < \tau(\bar{y}_j^n, z^n)$ . Hence,  $\bar{y}_j^n + t^n(\nu - \delta e) \in Y_j(z^n)$  for all  $n$  large enough, which implies that  $\nu - \delta e \in \hat{T}_{Y_j(z)}(\bar{y}_j)$ .  $\square$

6. Let  $Y_j$  be a convex valued correspondence. Let

$$PM(\bar{y}_j, z) = \{\pi \in L^* \mid \pi(\bar{y}_j) \geq \pi(y'_j) \forall y'_j \in Y_j(z)\}$$

be the profit maximization behaviour. Let  $\zeta_j - \bar{y}_j \in (Y_j(z) - \{\bar{y}_j\})$ . Let  $\rho > 0$ ,  $\eta = 1$ ,  $r > 0$ ,  $0 < \delta < r$ ,  $\varepsilon = 1$  and  $U = L$ . By Assumption P(3), there exists  $V \in \mathcal{V}_{\prod_{L^M} \tau}(z)$  such that  $\zeta_j - \delta e \in Y_j(z')$  for all  $z' \in V$ . Let  $z' \in B(z, \rho) \cap V$  and  $\zeta_j - \delta e \in Y_j(z')$ . Then, for  $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap Y_j(z')$  and  $t \in (0, 1)$ ,  $t(\zeta_j - \delta e) + (1 - t)\bar{y}'_j \in Y_j(z')$  since  $Y_j(z')$  is convex. But this means that  $\bar{y}'_j + t(\zeta_j - \delta e - \bar{y}_j + (\bar{y}_j - \bar{y}'_j)) \in Y_j(z')$ . Since  $-\delta e \in r[-e, e]$  we have that  $\zeta_j - \bar{y}_j \in \hat{T}_{Y_j(z)}^\rho(\bar{y}_j)$ . Since this is true for all  $\rho > 0$ ,  $\zeta_j - \bar{y}_j \in \hat{T}_{Y_j(z)}(\bar{y}_j)$  and thus  $\hat{N}_{Y_j(z)}(\bar{y}_j) \subset (Y_j(z) - \{\bar{y}_j\})^\circ = PM_j(\bar{y}_j, z)$ . The converse is immediate since  $PM(\bar{y}_j, z) = N_{Y_j(z)}(\bar{y}_j) \subset \hat{N}_{Y_j(z)}(\bar{y}_j) \subset \hat{N}_{Y_j(z)}(\bar{y}_j)$ .

### 6.3 Proof of Lemma 1

*Proof.* Let us note that Assumption SB (2) implies that for all  $\zeta_j \in \partial Y_j(z)$ ,  $T_{Y_j(z)}(\zeta_j) = \{\nu \in L \mid \nabla_1 f_j(\zeta_j, z)(\nu) \leq 0\}$  is the Clarke's tangent cone to  $Y_j(z)$  at  $\zeta_j$  ([12], Theorem 2.4.7, Corollary 2, p. 57).

Since  $e$  is in the quasi-interior of  $L_+$  and  $\nabla_1 f_j(\zeta_j, z) \in L_+^* \setminus \{0\}$ ,  $\nabla_1 f_j(\zeta_j, z)(e)$  is strictly positive. Let  $\nu \in L$  such that  $\nabla_1 f_j(\zeta_j, z)(\nu) \leq 0$ , let  $\rho > 0$ ,  $\eta = 1$  and  $r > 0$ . Let  $\alpha > 0$  such that  $\alpha < \frac{r\beta}{2(2\|\nu\|+4\rho+r)}$  where  $\beta = \nabla_1 f_j(\zeta_j, z)(e)$ . Since  $\nabla_1$  is continuous, there exist neighborhoods  $U \in \mathcal{V}_\tau(\zeta_j)$  and  $V \in \mathcal{V}_{\prod_{L^I+J} \tau}(z)$  such that  $\|\nabla_1 f_j(\zeta_j, z) - \nabla_1 f_j(\zeta'_j, z')\| < \alpha$  for all  $(\zeta'_j, z') \in U \times V$ .

Let  $U' = \{\zeta'_j \in L \mid \nabla_1 f_j(\zeta_j, z)(\zeta'_j - \zeta_j) < \frac{r\beta}{4}\}$  be a weak neighborhood of  $\zeta_j$ . There exists another convex neighborhood  $U''$  of  $\zeta_j$  and  $\delta > 0$  such that  $U'' + B(0, \delta) \subset U' \cap U$ .

Let  $\varepsilon > 0$  such that  $\varepsilon < \frac{2\delta}{2(2\|\nu\|+4\rho+r)}$ . From the Mean Value Theorem, for all  $z' \in V \cap B(z, \rho)$ , for all  $\zeta'_j \in U'' \cap B(\zeta_j, \rho) \cap Y_j(z')$  and all  $t \in (0, \varepsilon)$ , there exists  $\zeta''_j$  in the segment  $[\zeta'_j, \zeta'_j + t(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e)]$  such that

$$f(\zeta'_j + t(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e), z') = f(\zeta'_j, z') + t\nabla_1 f_j(\zeta''_j, z')(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e)$$

From our choice of  $\varepsilon$ , one deduces that  $\zeta''_j \in U$ , so  $\|\nabla_1 f_j(\zeta''_j, z') - \nabla_1 f_j(\zeta_j, z)\| \leq \alpha$ . Since  $\nabla_1 f_j(\zeta''_j, z')(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e) = (\nabla_1 f_j(\zeta''_j, z') - \nabla_1 f_j(\zeta_j, z))(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e) + \nabla_1 f_j(\zeta_j, z)(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e)$ , one deduces from the previous definitions and inequalities that  $\nabla_1 f_j(\zeta''_j, z')(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e) < 0$ . Since  $f(\zeta'_j, z') \leq 0$ , we get  $f(\zeta'_j + t(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e), z') \leq 0$ , that is  $\zeta'_j + t(\nu + \zeta_j - \zeta'_j - \frac{r}{2}e) \in Y_j(z')$ . Since  $-\frac{r}{2}e \in [-re, re]$ , we obtain that  $\nu \in \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\zeta_j)$ . Since this is true for all  $\rho > 0$ , we have that  $\nu \in \hat{\mathcal{T}}_{Y_j(z)}(\zeta_j)$   $\square$

## 6.4 Proof of Proposition 4

In this subsection,  $e$  belongs to the interior of  $L_+$  and the norm is the norm associated to  $e$ , that is the closed unit ball is the interval  $[-e, e]$ .

We first state and prove the following lemma.

**Lemma 6.** *Let  $\rho > 0$  and  $\nu \in \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$ . From the definition of  $\hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$ , there exists  $\eta > 0$  such that for  $\bar{r} > 0$ , there exist  $\varepsilon_{\bar{r}} > 0$ ,  $V_{\bar{r}}$  and  $U_{\bar{r}}$  such that for all  $z' \in B(z, \rho) \cap V_{\bar{r}}$  and all  $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U_{\bar{r}} \cap Y_j(z')$ ,  $t \in (0, \varepsilon_{\bar{r}})$ ,  $\bar{y}'_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) - \bar{r}e) \in Y_j(z')$ . Then, for all  $z' \in B(z, \rho) \cap V_{\bar{r}}$  and all  $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U_{\bar{r}} \cap Y_j(z')$ ,  $\nu + \eta(\bar{y}_j - \bar{y}'_j) - 2\bar{r}e \in \hat{\mathcal{T}}_{Y_j(z')}(\bar{y}'_j)$ .*

*Proof.* Let  $\rho > 0$  and  $\nu \in \hat{\mathcal{T}}^{\rho}_{Y_j(z)}(\bar{y}_j)$ . Let  $z' \in B(z, \rho) \cap V_{\bar{r}}$  and  $\bar{y}'_j \in B(\bar{y}_j, \rho) \cap U_{\bar{r}} \cap Y_j(z')$ . Let  $\varepsilon' > 0$  smaller than  $\varepsilon_{\bar{r}}$ ,  $\frac{\bar{r}}{\eta}$  and such that  $B(z', \varepsilon') \subset B(z, \rho) \cap V_{\bar{r}}$  and  $B(\bar{y}'_j, \varepsilon') \subset B(\bar{y}_j, \rho) \cap U_{\bar{r}}$ . For all  $z'' \in B(z', \varepsilon')$ , for all  $\bar{y}''_j \in B(\bar{y}'_j, \varepsilon') \cap Y_j(z'')$  and for all  $t \in (0, \varepsilon')$ ,  $\bar{y}''_j + t(\nu + \eta(\bar{y}_j - \bar{y}''_j) - \bar{r}e) = \bar{y}''_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) + \eta(\bar{y}'_j - \bar{y}''_j) - \bar{r}e) \in Y_j(z'')$ . Since  $\bar{y}''_j \in B(\bar{y}'_j, \varepsilon')$ ,  $\eta(\bar{y}'_j - \bar{y}''_j) \geq -\eta\varepsilon'e \geq -\bar{r}e$ . Hence, by the free-disposal property,  $\bar{y}''_j + t(\nu + \eta(\bar{y}_j - \bar{y}'_j) - 2\bar{r}e) \in Y_j(z'')$ . So, for all  $r > 0$ , the vector  $\nu + \eta(\bar{y}_j - \bar{y}'_j) - 2\bar{r}e$  satisfies the definition of  $\hat{\mathcal{T}}_{Y_j(z')}(\bar{y}'_j)$  with  $\varepsilon'$  has above and  $\xi = 0 \in B(0, r)$ .  $\square$

Now we proceed with the proof of Proposition 4:

*Proof.* From Assumption B,  $A(\omega)$  is bounded, so, there exists  $\rho > 0$  large enough such that for all  $(z', z'') \in A(\omega)^2$ ,  $z'' \in B(z', \rho)$  and  $y_j'' \in B(y_j', \rho)$ .

Let  $\nu \in \hat{\mathcal{T}}_{Y_j(z)}^\rho(y_j)$ . From the previous Lemma, there exists  $\eta > 0$  such that for  $\bar{r} > 0$ , there exist  $V_{\bar{r}}$  and  $U_{\bar{r}}$  such that for all  $z' \in B(z, \rho) \cap V_{\bar{r}}$  and all  $\bar{y}_j' \in B(y_j, \rho) \cap U_{\bar{r}} \cap Y_j(z')$ ,  $\nu + \eta(y_j - \bar{y}_j') - 2\bar{r}e \in \hat{\mathcal{T}}_{Y_j(z')}(\bar{y}_j')$ .

Since  $(z^\gamma, \pi^\gamma)_{\gamma \in \Gamma}$  converges to  $(z, \pi)$  for the product-topology  $\prod_{L^{I+J}} \tau \times \sigma^*$  and  $\pi^\gamma(y_j^\gamma)$  converges for all  $j$  and  $\pi^\gamma \in \hat{N}_{Y_j(z^\gamma)}(y_j^\gamma)$  for all  $\gamma \in \Gamma$ , there exists  $\gamma_0$  such that for all  $\gamma \geq \gamma_0$ ,  $z^\gamma \in V_{\bar{r}} \cap A(\omega) \subset V_{\bar{r}} \cap B(z, \rho)$  and  $y_j^\gamma \in U_{\bar{r}} \cap B(y_j, \rho) \cap Y_j(z^\gamma)$ . From the above lemma,  $\nu + \eta(y_j - y_j^\gamma) - 2\bar{r}e \in \hat{\mathcal{T}}_{Y_j(z^\gamma)}(y_j^\gamma)$ . Since  $\pi^\gamma \in \hat{N}_{Y_j(z^\gamma)}(y_j^\gamma)$ ,  $\pi^\gamma(\nu + \eta(y_j - y_j^\gamma) - 2\bar{r}e) \leq 0$ . Taking limits, we obtain  $\pi(\nu) + \eta\pi(y_j) - \eta \lim_\gamma \pi^\gamma(y_j^\gamma) \leq 2\bar{r}$ . Since this is true for all  $\bar{r} > 0$  we deduce that  $\pi(\nu) + \pi(y_j) \leq \lim_\gamma \pi^\gamma(y_j^\gamma)$ .

From Assumption TC,  $0 \in \hat{\mathcal{T}}_{Y_j(z)}(y_j) \subset \hat{\mathcal{T}}_{Y_j(z)}^\rho(y_j)$ , so the previous inequality leads to  $\pi(y_j) \leq \lim_\gamma \pi^\gamma(y_j^\gamma)$ .

If  $\pi(y_j) = \lim_\gamma \pi^\gamma(y_j^\gamma)$ , then we get  $\pi(\nu) \leq 0$  for all  $\nu \in \hat{\mathcal{T}}_{Y_j(z)}^\rho(y_j)$ , which is larger than  $\hat{\mathcal{T}}_{Y_j(z)}(y_j)$  and we conclude that  $\pi \in \hat{N}_{Y_j(z)}(y_j)$ .  $\square$

## 6.5 Proof of Lemma 4

*Proof.* We want to prove that for all  $(p, z, t) \in S^e \times Z^e \times \mathbb{R}_+$ , if  $(y_j) \in A^e(\omega + te, z)$  then  $p(\sum_{j \in J} y_j + \omega + te) > 0$ . By Claims 1 and 2, there exists a continuous, positive, linear functional  $\pi$  which extends  $p$  to the whole space  $L$  and  $\pi \in \cap_{j \in J} \hat{N}_{Y_j(z)}(y_j)$ . Hence,  $(\pi, z, t) \in S \times Z \times \mathbb{R}_+$  and  $(y_j) \in A(\omega + te, z)$ . Since  $\sum_{j \in J} y_j + \omega + te$  belongs to  $L(e)_+$  we get by Assumption SA,  $0 < \pi(\sum_{j \in J} y_j + \omega + te) = p(\sum_{j \in J} y_j + \omega + te)$  and the Lemma is proved.  $\square$