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Abstract

This article considers the long-run equilibrium distribution of an economy populated by heterogenous and present biased quasi-hyperbolic discounting agents. In a first configuration with logarithmic utility functions and Cobb-Douglas production technologies, this article establishes the existence and the uniqueness of the equilibrium: only one agent, determined by the highest value of a coefficient building from both the degree of present bias and the rate of discount, will have a positive long-run consumption and a positive long-run wealth. A second configuration with constant elasticities of substitution utilities and linear production technologies is then considered. This article similarly establishes the existence and the uniqueness of the equilibrium. There is generically a unique agent with the highest growth rate for his consumption and his wealth. This agent is determined by both preferences and technology parameters and may change following a technological shock.

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Contents

1. Introduction

The model with heterogenous agents that differ from their valuation of the future and their discount rate has motivated a large literature focused on the features of the associated inter temporal equilibrium. The *Ramsey's Conjecture* states that only the most patient amongst the agents would end up exhibiting positive consumption levels in the long run, all of the other agents being associated with nil consumption levels. It was rigorously proved by Bewley [4] who establishes that the long-run equilibrium assumes the following properties: the most patient amongst the agents holds the whole capital stock of the economy and assumes positive consumption levels. By contrast, all of the remaining agents have asymptotically nil consumption levels. It is also worth mentioning that the level of capital in this model corresponds to a modified golden rule defined from the discount factor of the most patient agent.

The assumption according to which agents would differ in their valuation of the future and their discounting rates has led to numerous further developments. Becker [1] was the first to reconsider this *Ramsey's Conjecture* under the assumption of imperfect financial markets—a non-negativity constraint is imposed on household capital holdings—. The long-run equilibrium is then associated with two classes of agents, namely the most patient *capitalist* agent who owns the whole capital stock of the economy, and the other more impatient *workers* agents who consume their current wages. Becker & Foias [2] and Sorger [12] have later enriched this framework with strategic dimensions.

This article is interested in the status of the aforementioned *Ramsey's Conjecture* when agents have quasi-hyperbolic discounting and simultaneously differ by their discount rates and their degrees of present bias. Dating from the seminal contributions of Phelps & Pollack [9], Laibson [8] and Frederick, Loewenstein & O'Donoghue [6], the benchmark model for analyzing temporal inconsistencies got anchored on the *quasi-hyperbolic discounting hypothesis* and gained popularity through the so-called *$\beta-\delta$ model*, where $\delta \in]0,1[$ features the discount rate and $\beta < 1$ features the present bias. Numerous experiments[1] have indeed consistently illustrated how the *quasi-hyperbolic discounting $\beta - \delta$ model* could provide a more accurate description of actual choices and arbitrages over time than the canonical model as settled by Samuelson [11] and Ramsey [10]. A large number of studies (see Frederick, Loewenstein & O'Donoghue [6]) have also ruled field or laboratory experiments in order to evaluate the β and δ parameters. They all document a strong heterogeneity between the individuals regarding these parameters.

This article considers a simple competitive economy with heterogenous quasi-hyperbolic infinitely-lived agents. Agents preferences being temporally inconsistent, it is assumed that these agents build from a sequence of successive selves, their decisions resulting from a Nash equilibrium between such selves. This assumption is commonly referred as depicting a *sophisticated behaviour*. This is completed by a production side with a standard neoclassical production technology. While no explicit solution would be available in the general case, the argument of this article is specialized to specific parametric formulations that will allow closed forms for the strategies of the agents.

In a first configuration with logarithmic utility functions and Cobb-Douglas production

[1]See Thaler [13], Benzion, Rapoport & Yagil [3], Chapman & Elstein [5] or Frederick, Loewenstein O'Donoghue [6] for a general presentation.

technologies, this article establishes the existence and the uniqueness of an equilibrium with heterogenous quasi-hyperbolic discounting agents. The properties of the long-run steady state remain close from the ones of the classical Ramsey model: only one agent will have a positive long-run consumption and a positive long-run wealth. That agent is however determined by the value of a coefficient where the two parameters β and δ come into play and interact. The impatience of a given agent—a relatively weak value for δ—can be compensated by a relative low present bias—a relatively high value for β.

In a second configuration with constant elasticities of substitution utilities and linear production technologies, this article similarly establishes the existence and the uniqueness of an equilibrium with heterogenous quasi-hyperbolic discounting agents. This competitive equilibrium converges towards a steady growth rate for the capital stock. Interestingly, the consumptions of the agents are going to assume constant but distinct equilibrium growth rates. These individuals rates of growth are not only determined by the two parameters β and δ but also by the constant elasticity of substitution of the utility and the constant productivity of the capital stock. There is generically a unique agent with the highest growth rate for his consumption and his wealth. This agent determines the long-run growth rate for the capital stock of this economy. The consumptions of the other agents become negligible in the long run when compared with the one of this agent. It is further proved that a given perturbation in the value of the constant productivity of the capital stock parameter may end up in modifying the ranking associated with the individual consumption growth rates between two distinct individuals. The case of naïve agents is finally analysed: this is shown to favour (detriment) capital accumulation when the elasticity of substitution is greater (lower) than one

Section 2 introduces a general environment with present biased quasi-hyperbolic discounting agents. An environment with logarithmic utilities and Cobb-Douglas production technologies is considered in Section 3. Another environment with CES utilities and linear production technologies is analyzed in Section 4. All proofs and technicalities are gathered in an final appendix.

2. The Model

Time is discrete. Consider a competitive economy with n infinitely-lived agents indexed by $i \in \{1, 2, \ldots, n\}$. Let $(c_t^i)_{t\in\mathbb{N}}$ denote the consumption sequence of individual i, while $(w_t)_{t\in\mathbb{N}}$ and $(R_t)_{t\in\mathbb{N}}$ respectively stand for the wage and gross interest rates sequences he faces with, for $R_t = 1 + r_t$ and r_t the interest rate.

Letting a_t^i denote the capital stock held at date $t \in \mathbb{N}$ by agent i, his budget constraint for date t is expressed as follows:

$$a_{t+1}^i = R_t a_t^i + w_t - c_t^i.$$

Agents are further subject to an inter-temporal budget constraint defined over their lifespan. Assuming that $\sum_{t=0}^{+\infty}(w_t / \prod_{\tau=0}^{t} R_\tau) < +\infty$, the actualised wealth of agent i, denoted as W_t^i, is defined by:

$$W_t^i = R_t a_t^i + w_t + \sum_{\tau=t+1}^{+\infty} \frac{w_\tau}{\prod_{j=t+1}^{\tau} R_j},$$

while the inter-temporal budget constraint can be expressed, for every $t \in \mathbb{N}$, as:

$$W_{t+1}^i = R_{t+1}(W_t^i - c_t^i), \tag{1}$$
$$\text{with} \quad W_t^i \geqslant 0 \quad \text{for every} \quad t \geqslant 0.$$

The preferences of the agents are associated with quasi-geometric discounting. For a given agent $i \in \{1, 2, \ldots, n\}$, his *self* at time $t \in \mathbb{N}$ ranks consumption sequences according to:

$$u(c_t^i) + \beta_i \Big[\sum_{\tau=1}^{+\infty} (\delta_i)^\tau u(c_{t+\tau}^i) \Big], \; \beta_i > 0, \; \delta_i \in]0, 1[. \tag{2}$$

The decisions of agent $i \in \{1, 2, \ldots, n\}$ stem from the Nash equilibrium of a strategic game between the successive *selves*—a Markov equilibrium, also commonly referred as a *sophisticated behaviour*. Within an infinite horizon setup, it was established by Krusell & Smith [7] that it is possible to build a multiplicity of equilibrium solutions between the different selves. In opposition to this, for a finite horizon environment, the solution obtained by backward induction is unique. This article will retain the infinite horizon solution that is obtained as the limit of the finite horizon one. This enables to select a specific solution amongst the multiple admissible infinite horizon ones.

This article willl consider two separate configurations, *i.e.*, a first one with a logarithmic utility function and a Cobb-Douglas production technology and a second one with an iso-elastic utility function and a linear production technology. As this will be established in the remaining exposition, both of these configurations will be associated with *linear* decision rules for the consumers $i \in \{1, \ldots, n\}$:

$$c_t^i = \lambda_i W_t^i \tag{3}$$

with $\lambda_i \in [0, 1]$. Without loss of generality, agents will henceforward be ranked according to $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$, where it is assumed that $\lambda_1 < \lambda_2$. The solution that is obtained as the limit of the finite horizon game assumes well-behaved regularity properties — they are indeed linear. In opposition to this, the strategies raised by Krusell & Smith [7] were not smooth.

The economy under consideration also integates a competitive representative firm with a production technology $F(K, L)$ that satisfies:

Assumption T. 1. $F(K, L)$ is a function of class $\mathscr{C}^2$, homogeneous of degree one, increases with K as $D_K F(K, 1) > 0$, and is concave.

K_t being the capital stock of period t and L_t the quantity of labour of that same period t, the capital stock being assumed to fully depreciate on use within period t. All individuals inelastically offer $1/n$ units of labour, so that the global labour supply amounts to 1. The competitive wage rate and the competitive interest factor are thus available at date $t \in \mathbb{N}$ as:

$$w_t = \frac{1}{n} D_L F(K_t, 1), \tag{4a}$$
$$R_t = D_K F(K_t, 1). \tag{4b}$$

An equilibrium is then characterised as follows:

Definition 2.1. *A competitive equilibrium of the economy is characterised by a sequence* $(W_t^i, K_t)_{t\in\mathbb{N}}$ *such that, for every* $t \in \mathbb{N}$ *and every* $i \in \{1, 2, \dots, n\}$*:*

$$W_{t+1}^i = R_{t+1}(1-\lambda_i)W_t^i, \tag{5a}$$

$$K_{t+1} = F(K_t, 1) - \sum_{i=1}^{n} \lambda_i W_t^i, \tag{5b}$$

starting from initial conditions $W_0^i = R_0 a_0^i + w_0 + \sum_{t=1}^{+\infty}[w_t/\prod_{\tau=1}^{t} R_\tau]$ *with* a_0^i *given such that* $\sum_{i=1}^{n} a_0^i = K_0$ *and* (w_t, R_t) *given by (4a) and (4b).*

Equation (5*a*) is obtained from the inter-temporal budget constraint (1), respectively replacing R_t by Equation (4*b*) and c_t^i by Equation (3). Similarly, equation (5*b*) derives from the integration of the consumption decision rules (3) into the resource constraint of the economy.

3. An Economy with Logarithmic Preferences and a Cobb-Douglas Production Technology

This section will consider a configuration where $u(c) = \ln(c)$ and $F(K_t, L_t) = (K_t)^\alpha (L_t)^{1-\alpha}$, $\alpha \in [0, 1]$, that in turn implies equilibrium returns of $w_t = (1/n)(1-\alpha)(K_t)^\alpha$ and $R_t = \alpha(K_t)^{\alpha-1}$.

Lemma 3.1. *For* $u(c) = \ln(c)$ *and* $F(K_t, L_t) = (K_t)^\alpha (L_t)^{1-\alpha}$, $\alpha \in [0, 1]$, *the decision rules of the consumers resulting from a Nash equilibrium between the successive selves list as:*

$$c_t^i = \lambda_i W_t^i,$$
$$\lambda_i = \frac{1-\delta_i}{1-\delta_i+\beta_i\delta_i}$$

Lemma 3.2. *Up to its existence, a competitive equilibrium of the economy is characterised, for every* $i \in \{1, 2, \dots, n\}$*, by:*

$$W_t^i = \frac{W_0^i[\alpha(1-\lambda_i)]^t}{\sum_{j=1}^{n}\lambda_j W_0^j[\alpha(1-\lambda_j)]^t/[1-\alpha(1-\lambda_j)]}, \tag{6a}$$

$$K_{t+1} = (K_t)^\alpha(1-\gamma_t), \tag{6b}$$

$$\text{with} \quad \gamma_t := \frac{\sum_{i=1}^{n}\lambda_i W_0^i[\alpha(1-\lambda_i)]^t}{\sum_{i=1}^{n}\lambda_i W_0^i[\alpha(1-\lambda_i)]^t/[1-\alpha(1-\lambda_i)]},$$

starting from initial conditions $W_0^i = R_0 a_0^i + w_0 + \sum_{t=1}^{+\infty}[w_t/\prod_{\tau=1}^{t} R_\tau]$ *with* a_0^i *given such that* $\sum_{i=1}^{n} a_0^i = K_0$ *and* (w_t, R_t) *given by (4a) and (4b).*

Proposition 3.1. — *Consider the above economy:*

(i) *if* $a_0^i \geqslant 0$ *for every* $i \in \{1, \dots, n\}$*, there exists a unique competitive equilibrium;*

(ii) *the competitive equilibrium converges to a steady state characterised by the following features:*

a/ *the capital stock converges to a modified golden rule like steady state with a value* $\bar{K}$ *that corresponds to the only preferences parameters of agent* 1 *through the parameter* λ_1*:*

$$\alpha(\bar{K})^{\alpha-1} = \frac{1}{1-\lambda_1}; \tag{7}$$

b/ *agent 1 is the only agent with a long-run positive inter-temporal wealth* $W^1 = (\bar{K})^{\alpha}[1 - \alpha(1-\lambda_1)]/\lambda_1$. *By contrast with this, the wealth of all the other agents is to converge to* 0*;*

c/ *agent 1 is the only agent with a long-run positive consumption* $\bar{c}^1 = (\bar{K})^{\alpha}[1 - \alpha(1-\lambda_1)]$. *By contrast with this, the consumption of all other agents is to converge to* 0.

This proposition hence yields a range of conclusions which are close to the ones of the classical Ramsey model: only one agent will have a positive long-run consumption and a positive long-run wealth. That agent is however determined by the value of the coefficient λ_i where the two parameters β_i and δ_i come into play and interact. The impatience of a given agent—a relatively weak value for δ_i—can be compensated by a relative low present bias—a relatively high value for β_i.

4. An Economy with C.E.S. Utility and a Linear Production Technology

4.1 The Sophisticated Agent

Preferences are modified to a representation $u(c) = c^{1-1/\sigma}/(1-1/\sigma)$, $\sigma \neq 1$, while the production technology is changed to $F(K_t, L_t) = AK_t$, $A > 0$. It immediately follows that, for every $t \in \mathbb{N}$, $R_t = A$, $w_t = 0$. It is similarly derived that $W_0^i = Aa_0^i$. An extra assumption is needed in order to maintain a bounded utility for the agents:

Assumption PT.1. $\forall\, i \in \{1, 2, \ldots, n\}$, $\delta_i A^{1-1/\sigma} < 1$.

Lemma **4.1**. *For* $u(c) = c^{1-\sigma}/(1-\sigma)$, $\sigma \neq 1$ *and* $F(K_t, L_t) = AK_t$, $A > 0$, *the decision rules of the consumers resulting from a Nash equilibrium between the successive selves list as:*

$$c_t^i = \lambda_i W_t^i$$

where the coefficient λ_i *is defined as the solution of the following equation:*

$$[A(1-\lambda_i)]^{1/\sigma} = \delta_i A[1 - \lambda_i(1-\beta_i)]\cdot \tag{8}$$

Interestingly, in equation (8), the coefficient λ_i not only depends on the preferences parameters σ, δ_i and β_i but also on the technological parameter A.

Proposition 4.1. — *Consider the above economy:*

(i) *if* $a_0^i > 0$ *for every* $i \in \{1, \ldots, n\}$, *there exists a unique competitive equilibrium characterised, for every* $t \in \mathbb{N}$ *and for every* $i \in \{1, \ldots, n\}$, *by:*

$$W_t^i = A^{t+1}(1-\lambda_i)^t a_0^i, \tag{9a}$$
$$c_t^i = \lambda_i A^{t+1}(1-\lambda_i)^t a_0^i, \tag{9b}$$

and by

$$K_{t+1} = AK_t(1-\gamma_t),$$
$$\text{for } \gamma_t = \frac{\sum_{i=1}^n \lambda_i(1-\lambda_i)^t a_0^i}{\sum_{i=1}^n (1-\lambda_i)^t a_0^i};$$

(ii) *the competitive equilibrium converges towards a stationary equilibrium characterised by a growth factor of the capital stock K_{t+1}/K_t that converges towards a limit $A(1-\lambda_1)$.*

Within this environment and as made clear by Proposition 4.1, the consumptions of the individual agents are going to assume constant but distinct equilibrium growth rates. All of the consumptions of the agents $i \geqslant 2$ would hence become negligible in the long run when compared with the one of agent $i = 1$ who is associated with the lowest value for λ_i, the long-run growth rate of the capital stock being further exactly the same as the consumption growth rate of that later agent. From Equation (8), the values of λ_i are to be jointly determined by A, σ, δ_i and β_i for any of the individuals $i \in \{1, \dots, n\}$. The following proposition then establishes how a given perturbation in the value of A may end up in modifying the ranking associated with the coefficients λ_i for two distinct individuals.

PROPOSITION 4.2. — *Let $\sigma > 1$ and assume that two individuals $i \in \{1,2\}$ are characterised by parameters δ_i and β_i such that:*

$$\delta_1 < \delta_2, \quad \beta_1 > \beta_2 \quad \text{and} \quad \beta_1\delta_1 > \beta_2\delta_2.$$

There then exists some critical value for A, labelled $\hat{A}$, such that:

$$\begin{cases} \text{for } A < \hat{A}, & \lambda_1 < \lambda_2; \\ \text{for } A > \hat{A}, & \lambda_1 > \lambda_2. \end{cases}$$

This proposition shows how a technological shock—a change in the parameter A—may result in a change in the identity of the agent who is determining the long-run growth rate of the economy.

4.2 The Naïve Agent

It has up to now been assumed that agents have sophisticated behaviours. An alternative assumption however states as the hypothesis of *naïve agents*. A naïve t-self undertakes its decision at period t *as if* it could also select all of its future decisions. For the logarithmic case of Section 3, naïve and sophisticated behaviours lead to the same decision rules. Oppositely and under the current assumption of CES utility functions, naïve and sophisticated behaviours result in distinct optimal strategies. This section is aimed at comparing these two strategies.

Assume instead that any self determines his behaviour for his decision period *as if* he could equally undertake the whole range of future decisions. This also corresponds to the solution for a self with a perfect commitment capacity. Henceforth labelling as *naïve* such a range of behaviours, it is established :

PROPOSITION 4.3. *For $u(c) = c^{1-\sigma}/(1-\sigma)$, $\sigma \neq 1$ and $F(K_t, L_t) = AK_t$, $A > 0$, consider a naïve agent:*

(i) *His decision rule is given by:*

$$c_t = \lambda_i^n W_t^i,$$
$$\lambda_i^n = \frac{1 - (\delta_i)^\sigma A^{\sigma-1}}{1 - (\delta_i)^\sigma A^{\sigma-1}(1 - (\beta_i)^\sigma)}.$$

(ii) *For* $\beta_i < 1$, $\lambda_i < \lambda_i^n \iff \sigma < 1$ *and* $\lambda_i > \lambda_i^n \iff \sigma > 1$.

(iii) *Consider an economy populated by two agents with the same* (β, δ) *parameters and assume that one is sophisticated while the other one is naïve. The naïve agent accumulates more (less) capital that the sophisticated one when* $\sigma > 1$ *(*$\sigma < 1$*).*

Consider an economy populated with two agents with the same (β, δ) parameters, the first being sophisticated while the second is naïve. For $\sigma > 1$ ($\sigma < 1$), the naïve (sophisticated) agent determines the long-run rate of growth.

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A. Proof of Lemma 3.1

Consider an agent with the preference parameters (β, δ). The game between selves is simplified by introducing the following variables:

$$X_t = \frac{W_t}{\Pi_t}$$
$$x_t = \frac{c_t}{\Pi_t}$$

with Π_t defined as:

$$\Pi_0 = 1$$
$$\text{for } t \geq 1,\ \Pi_t = \prod_{\tau=1}^{t} R_\tau$$

The intertemporal budget constraint (1) becomes: $X_{t+1} = X_t - x_t$ with $x_t \leq X_t$. The objective functions of self t (2) can be written as:

$$\ln x_t + \beta \sum_{\tau=1}^{+\infty} \delta^\tau \ln x_{t+\tau} + H_t$$

with H_t some term that is exogenous for the agent.

Consider the game defined on a T period horizon. The variables X_t and x_t are now denoted by X_t^T and x_t^T. For $t \leq T$, the t-self objective function is:

$$\ln x_t^T + \beta \sum_{\tau=1}^{T-t} \delta^\tau \ln x_{t+\tau}^T + H_t^T$$

The game is solved by backward induction. As the objective function of all selves are homothetic, it is straightforward that the solution will take the following form:

$$x_t^T = \lambda_t^T X_t^T$$

It is first noticed that, for $t = T$, the solution is obvious with $\lambda_T^T = 1$.

Then assume that the game has been solved till period $t+1$. One gets: $\forall \tau$ with $2 \leq \tau \leq T$,

$$x_{t+\tau}^T = \lambda_{t+\tau}^T \left(1 - \lambda_{t+\tau-1}^T\right) \cdots \left(1 - \lambda_{t+1}^T\right) X_{t+1}^T$$

and $x_{t+1}^T = \lambda_{t+1}^T X_{t+1}^T$. Replacing in the t-self objective function, we get:

$$\ln x_t^T + \beta \left(\sum_{\tau=1}^{T} \delta^\tau \right) \ln X_{t+1}^T + G_t^T$$

with G_t^T some term that is exogenous for the self. Finally, the program of self t can be written as:

$$\max_{(x_t^T, X_{t+1}^T)} \left[\ln x_t^T + \beta \left(\sum_{\tau=1}^{T} \delta^\tau \right) \ln X_{t+1}^T \right]$$
$$\text{s.t. } x_t^T + X_{t+1}^T = X_t^T$$

The solution is: $x_t^T = \lambda_t^T X_t^T$, $X_{t+1}^T = \left(1 - \lambda_t^T\right) X_t^T$ with

$$\lambda_t^T = \frac{1}{1 + \beta \sum_{\tau=1}^{T} \delta^\tau}$$

For $T \to +\infty$, the limit strategy is derives as:

$$\lim_{T \to +\infty} \lambda_t^T = \lambda = \frac{1}{1 + \beta\delta/(1-\delta)},$$

that completes the statement. QED

B. Proof of Lemma 3.2

The argument of the proof revolves around a careful study of the dynamics of the system as described by Equations (5*a*) and (5*b*). Letting firstly the auxiliary variable χ_t be introduced as:

$$\chi_t := \frac{(K_t)^\alpha}{W_t^1}. \tag{A.1}$$

From Equation (5*a*) considered for $i = 1$, it is obtained that:

$$W_{t+1}^1 (K_{t+1})^{1-\alpha} = \alpha(1-\lambda_1) W_t^1. \tag{A.2}$$

Considering Equation (5*b*) and dividing each member by the corresponding one in Equation (A.2), it is derived that:

$$\frac{(K_{t+1})^\alpha}{W_{t+1}^1} = \frac{(K_t)^\alpha}{\alpha(1-\lambda_1)W_t^1} - \sum_{i=1}^{n} \frac{\lambda_i}{\alpha(1-\lambda_1)} \frac{W_t^i}{W_t^1},$$
$$\iff \chi_{t+1} = \frac{\chi_t}{\alpha(1-\lambda_1)} - \sum_{i=1}^{n} \frac{\lambda_i}{\alpha(1-\lambda_1)} \frac{W_t^i}{W_t^1},$$

where the second equation takes advantage of the definition (A.1). Completing then a further change of variables:

$$\zeta_t := \chi_t [\alpha(1-\lambda_1)]^t, \tag{A.3}$$

it is finally obtained that:

$$\zeta_{t+1} = \zeta_t - \sum_{i=1}^{n} \lambda_i \frac{W_t^i}{W_t^1} [\alpha(1-\lambda_1)]^t. \tag{A.4}$$

Making use of Equation (5*a*) considered for $i = 1$ and $i \in \{2, 3, \ldots, n\}$, it derives that the relative wealth with respect to the most patient agent states as:

$$\frac{W_{t+1}^i}{W_{t+1}^1} = \frac{(1-\lambda_i)W_t^i}{(1-\lambda_1)W_{t+1}^1}$$
$$\implies \frac{W_t^i}{W_t^1} = \left(\frac{1-\lambda_i}{1-\lambda_1}\right)^t \frac{W_0^i}{W_0^1}. \tag{A.5}$$

Replacing then Equation (A.5) in Equation (A.4), it finally derives that:

$$\zeta_{t+1} = \zeta_t - \sum_{i=1}^{n} \lambda_i [\alpha(1-\lambda_i)]^t \frac{W_0^i}{W_0^1}.$$

Developing backward the R.H.S. of the preceding equation:

$$\zeta_t = \zeta_0 - \sum_{i=1}^{n} \lambda_i \frac{W_0^i}{W_0^1} \sum_{\tau=0}^{t-1} [\alpha(1-\lambda_i)]^\tau. \tag{A.6}$$

By definition, ζ_t must be non negative for every $t \in \mathbb{N}$ as $\chi_t \geqslant 0$ prevails for every $t \in \mathbb{N}$. As $\alpha(1-\lambda_i) < 1$ for every $i \in \{1, 2, \ldots, n\}$, $(\zeta_t)_{t\in\mathbb{N}}$ converges to a limit ζ that is determined by Equation (A.6):

$$\zeta = \zeta_0 - \sum_{i=1}^{n} \lambda_i \frac{W_0^i}{W_0^1} \frac{1}{1-\alpha(1-\lambda_i)}.$$

Further, as $\zeta_t \geqslant 0$ for $t \in \mathbb{N}$, it derives that $\zeta \geqslant 0$.

Finally establishing that $\zeta = 0$, assume, *ad absurdum*, that $\zeta > 0$ does prevail. From the respective definitions of χ_t and ζ_t, it derives that:

$$\zeta_t = \frac{(K_t)^\alpha}{W_t^1}\big[\alpha(1-\lambda_1)\big]^t \longrightarrow \zeta > 0 \quad as \quad t \to +\infty,$$
$$\Longleftrightarrow \frac{W_t^1}{(K_t)^\alpha} \sim \frac{1}{\zeta}\big[\alpha(1-\lambda_1)\big]^t \quad as \quad t \to +\infty.$$

It is therefore obtained that:

$$\frac{W_t^1}{(K_t)^\alpha} \to 0 \quad as \quad t \to +\infty,$$
$$and \ \frac{W_t^i}{(K_t)^\alpha} = \frac{W_t^i}{W_t^1}\frac{W_t^1}{(K_t)^\alpha}$$
$$= \left(\frac{1-\lambda_i}{1-\lambda_1}\right)^t \frac{W_0^i}{W_0^1}\frac{W_t^1}{(K_t)^\alpha} \to 0 \quad as \quad t \to +\infty.$$

From Equation (5*b*), it is possible to write:

$$K_{t+1} = (K_t)^\alpha\left[1 - \sum_{i=1}^n \lambda_i \frac{W_t^i}{(K_t)^\alpha}\right].$$

But it has been established that $W_t^i/(K_t)^\alpha \to 0$ as $t \to +\infty$, so that the preceding equation in turn implies that $K_t \to \tilde{K} = 1$ as $t \to +\infty$, for $\tilde{K}$ that is such that $\tilde{K} = (\tilde{K})^\alpha$, which is impossible as this corresponds to an *overaccumulation* configuration. The wealth of the agents would become infinite in conjunction with $\sum_{t=0}^{+\infty}(w_t/\prod_{\tau=0}^t R_\tau) = +\infty$, whence the proof that $\zeta = 0$.

As $\zeta = 0$, it is obtained that:

$$\zeta_0 = \sum_{i=1}^n \lambda_i \frac{W_0^i}{W_0^1}\frac{1}{1-\alpha(1-\lambda_i)}$$
$$= \sum_{i=1}^n \lambda_i \frac{W_0^i}{W_0^1}\sum_{\tau=0}^{+\infty}\big[\alpha(1-\lambda_i)\big]^\tau.$$

Replacing the ensued expression for ζ_0 in Equation (A.6), it is obtained that:

$$\zeta_t = \sum_{i=1}^n \lambda_i \frac{W_0^i}{W_0^1}\sum_{\tau=t}^{+\infty}\big[\alpha(1-\lambda_i)\big]^\tau$$
$$= \sum_{i=1}^n \lambda_i \frac{W_0^i}{W_0^1}\frac{\big[\alpha(1-\lambda_i)\big]^t}{1-\alpha(1-\lambda_i)}.$$

Making use of the respective definitions of χ_t and ζ_t:

$$W_t^1 = \frac{(K_t)^\alpha}{\zeta_t}\big[\alpha(1-\lambda_1)\big]^t. \tag{A.7}$$

Taking advantage of Equation (A.5) of the relative wealths of the agents, it is further inferred that:

$$W_t^i = \frac{W_0^i}{W_0^1}\frac{(K_t)^\alpha}{\zeta_t}\big[\alpha(1-\lambda_i)\big]^t. \tag{A.8}$$

Eventually replacing Equations (A.7) and (A.8) in Equation (5*b*), it is derived that:

$$K_{t+1} = (K_t)^\alpha(1-\gamma_t),$$
$$\gamma_t := \frac{\sum_{i=1}^n \lambda_i W_0^i\big[\alpha(1-\lambda_i)\big]^t}{\sum_{i=1}^n \lambda_i W_0^i\big[\alpha(1-\lambda_i)\big]^t/\big[1-\alpha(1-\lambda_i)\big]}$$

and the details of the statement follow. QED

C. Proof of Proposition 3.1

(i) Existence and uniqueness of the competitive equilibrium. Lemma 3.1 enables to determine the equilibrium dynamics up to the knowledge of the initial wealths

$$W_0^i = Ra_0^i + w_0 + \sum_{t=1}^{+\infty} \frac{w_t}{\prod_{\tau=1}^t R_\tau} \tag{B.1}$$

with a_0^i given such that $\sum_{i=1}^n a_0^i = K_0$. The expression of human wealth $\mu = w_0 + \sum_{t=1}^{+\infty}[w_t / \prod_{\tau=1}^t R_\tau]$ remains unknown. Making use of $(4a)$ and $(4b)$, it is obtained that:

$$\mu = \frac{1-\alpha}{n}\Big[(K_0)^\alpha + \frac{K_1}{\alpha} + \frac{1}{\alpha}\sum_{t=2}^{+\infty} K_t \prod_{\tau=1}^{t-1}\Big(\frac{(K_\tau)^{1-\alpha}}{\alpha}\Big)\Big], \tag{B.2}$$

μ being thus an increasing function of $K_0, K_1, \dots, K_t, \dots$.

Making use of (B.1), the equilibrium law of capital accumulation $(6b)$ becomes:

$$\gamma_t = \frac{\sum_{i=1}^n \lambda_i R_0 a_0^i[\alpha(1-\lambda_i)]^t + \mu\sum_{i=1}^n \lambda_i[\alpha(1-\lambda_i)]^t}{\sum_{i=1}^n \lambda_i R_0 a_0^i[\alpha(1-\lambda_i)]^t/[1-\alpha(1-\lambda_i)] + \mu\sum_{i=1}^n \lambda_i[\alpha(1-\lambda_i)]^t/[1-\alpha(1-\lambda_i)]}, \tag{B.3}$$

$$K_{t+1} = (K_t)^\alpha(1-\gamma_t).$$

Let μ be given. Taking advantage of (B.3), a sequence $(K_t(\mu))_{t\in\mathbb{N}}$ is determined starting from the initial condition K_0. That sequence converges towards a value $\overline{K}$ that is independent of μ and such that $\overline{K} = \alpha(\overline{K})^\alpha(1-\lambda_1)$, whence $\alpha(\overline{K})^{\alpha-1} = 1/(1-\lambda_1)$. Defining then a function $Z(\mu)$ according to:

$$Z(\mu) := \frac{1-\alpha}{n}\Big[(K_0)^\alpha + \frac{K_1(\mu)}{\alpha} + \frac{1}{\alpha}\sum_{t=2}^{+\infty} K_t(\mu) \prod_{\tau=1}^{t-1}\Big(\frac{(K_\tau(\mu))^{1-\alpha}}{\alpha}\Big)\Big].$$

The sequence $(K_t(\mu))_{t\in\mathbb{N}}$ being convergent—the level $\overline{K}$ does indeed correspond to an under-accumulation steady state—, this function is well defined.

The existence of a competitive equilibrium is then available upon the existence of a value μ^* such that $Z(\mu^*) = \mu^*$, uniqueness being reached if and only if the function $Z(\cdot)$ assumes a unique fixed point.

Dealing first with the first argument, consider the two following sequences:

$$\gamma_t^h := \max\Big\{\frac{\sum_{i=1}^n \lambda_i R_0 a_0^i[\alpha(1-\lambda_i)]^t}{\sum_{i=1}^n \lambda_i R_0 a_0^i[\alpha(1-\lambda_i)]^t/[1-\alpha(1-\lambda_i)]}, \frac{\sum_{i=1}^n \lambda_i[\alpha(1-\lambda_i)]^t}{\sum_{i=1}^n \lambda_i[\alpha(1-\lambda_i)]^t/[1-\alpha(1-\lambda_i)]}\Big\};$$

$$\gamma_t^\ell := \min\Big\{\frac{\sum_{i=1}^n \lambda_i R_0 a_0^i[\alpha(1-\lambda_i)]^t}{\sum_{i=1}^n \lambda_i R_0 a_0^i[\alpha(1-\lambda_i)]^t/[1-\alpha(1-\lambda_i)]}, \frac{\sum_{i=1}^n \lambda_i[\alpha(1-\lambda_i)]^t}{\sum_{i=1}^n \lambda_i[\alpha(1-\lambda_i)]^t/[1-\alpha(1-\lambda_i)]}\Big\}.$$

The coefficient γ_t as given by (B.3) emerges as a monotone function of μ, is enclosed between the values associated to $\mu = 0$ and $\mu = +\infty$, whence, and for any $t \in \mathbb{N}$, the satisfaction of:

$$\gamma_t^\ell \leqslant \gamma_t \leqslant \gamma_t^h.$$

Let then the sequences $(\overline{K}_t(\mu))_{t\in\mathbb{N}}$ and $(\underline{K}_t(\mu))_{t\in\mathbb{N}}$ respectively defined by:

$$\overline{K}_0 = K_0; \forall t \in \mathbb{N}, \overline{K}_{t+1} = (\overline{K}_t)^\alpha (1 - \gamma_t^\ell);$$
$$\underline{K}_0 = K_0; \forall t \in \mathbb{N}, \underline{K}_{t+1} = (\underline{K}_t)^\alpha (1 - \gamma_t^h).$$

It is then inferred that,

$$\forall t \in \mathbb{N}, \underline{K}_t \leqslant K_t(\mu) \leqslant \overline{K}_t,$$
$$\lim_{t\to+\infty} \underline{K}_t = \lim_{t\to+\infty} \overline{K}_t = \lim_{t\to+\infty} K_t(\mu) = \overline{K},$$

as $\lim_{t\to+\infty} \gamma_t^\ell = \lim_{t\to+\infty} \gamma_t^h = \lim_{t\to+\infty} = 1 - \alpha(1 - \lambda_1)$. Let then in turn the coefficients $\underline{\mu}$ and $\overline{\mu}$ be defined as:

$$\underline{\mu} = \frac{1-\alpha}{n}\left[(K_0)^\alpha + \frac{\underline{K}_1}{\alpha} + \frac{1}{\alpha}\sum_{t=2}^{+\infty} \underline{K}_t \prod_{\tau=1}^{t-1}\left(\frac{(\underline{K}_\tau)^{1-\alpha}}{\alpha}\right)\right],$$
$$\overline{\mu} = \frac{1-\alpha}{n}\left[(K_0)^\alpha + \frac{\overline{K}_1}{\alpha} + \frac{1}{\alpha}\sum_{t=2}^{+\infty} \overline{K}_t \prod_{\tau=1}^{t-1}\left(\frac{(\overline{K}_\tau)^{1-\alpha}}{\alpha}\right)\right].$$

For every $\mu \geqslant 0$, $\underline{\mu} \leqslant Z(\mu) \leqslant \overline{\mu}$ is hence satisfied. When considered on the interval $[\underline{\mu}, \overline{\mu}]$, the function $Z(\mu)$ corresponds to a continuous function defined over a compact set. Indeed, the convergence of the sum defining $Z(\mu)$ is uniform:

$$\sup_{(\mu)}\left[\frac{1-\alpha}{n}\frac{1}{\alpha}\sum_{t=T}^{+\infty} K_t(\mu)\prod_{\tau=1}^{t-1}\left(\frac{(K_\tau(\mu))^{1-\alpha}}{\alpha}\right)\right] \leqslant \frac{1-\alpha}{n}\frac{1}{\alpha}\sum_{t=T}^{+\infty}\overline{K}_t\prod_{\tau=1}^{t-1}\left(\frac{(\overline{K}_\tau)^{1-\alpha}}{\alpha}\right)$$
$$\text{with} \quad \lim_{T\to+\infty}\frac{1-\alpha}{n}\frac{1}{\alpha}\sum_{t=T}^{+\infty}\overline{K}_t\prod_{\tau=1}^{t-1}\left(\frac{(\overline{K}_\tau)^{1-\alpha}}{\alpha}\right) = 0.$$

The function $Z(\mu)$ being continuous over the compact set $[\underline{\mu}, \overline{\mu}]$, it assumes a fixed point.

Focusing then on the uniqueness argument, assume, *ad absurdum*, that there exists two solutions μ_1 and μ_2 to the equation $Z(\mu) = \mu$, with $\mu_1 < \mu_2$. The argument will proceed by showing that the solution sequences would then be such that $K_t(\mu_1) > K_t(\mu_2)$ for any $t \in \mathbb{N}$, that cannot be reconciled with the monotony of $K_t(\mu)$ and will establish uniqueness.

By recurrence, it is first established that:

$$W_0^i = R_0 a_0^i + \mu_1 \qquad \text{and} \qquad W_0^{i'} = R_0 a_0^i + \mu_2,$$

whence and for every $i \in \{1, \ldots, n\}$, the obtention of $W_0^i < W_0^{i'}$.

Letting then $\chi_0^i = W_0^i/(K_0)^\alpha$ and $\chi_0^{i'} = W_0^{i'}/(K_0)^\alpha$, it derives that, for every $i \in \{1, \ldots, n\}$, $\chi_0^i < \chi_0^{i'}$ or, further,

$$\gamma_0 \equiv \frac{\sum_{i=1}^n \lambda_i W_0^i}{(K_0)^\alpha} < \frac{\sum_{i=1}^n \lambda_i W_0^{i'}}{(K_0)^\alpha} \equiv \gamma_0'.$$

It remains to establish that, for every $i \in \{1, \ldots, n\}$, $\chi_1^i < \chi_1^{i'}$ and $\gamma_1 < \gamma_1'$. For that purpose, first consider the ratio of equations $(5a)$ and $(5b)$ when considered at date $t = 0$:

$$\chi_1^i \equiv \frac{W_1^i}{(K_1(\mu))^\alpha} = \frac{\alpha(1-\lambda_i)W_0^i}{(K_0)^\alpha(1-\gamma_0)} = \alpha(1-\lambda_i)\frac{\chi_0^i}{1-\gamma_0},$$
$$\chi_1^{i'} \equiv \frac{W_1^{i'}}{(K_1(\mu))^\alpha} = \frac{\alpha(1-\lambda_i)W_0^{i'}}{(K_0)^\alpha(1-\gamma_0')} = \alpha(1-\lambda_i)\frac{\chi_0^{i'}}{1-\gamma_0'}.$$

But, and as for every $i \in \{1, \dots, n\}$, $\chi_0^i < \chi_0^{i'}$ and $\gamma_0 < \gamma_0'$, it naturally follows that $\chi_1^i < \chi_1^{i'}$ for every $i \in \{1, \dots, n\}$, whence, eventually, the satisfaction of

$$\gamma_1 = \sum_{i=1}^{n} \lambda_i \chi_1^i < \sum_{i=1}^{n} \lambda_i \chi_1^{i'} = \gamma_1'.$$

By recurrence, assume that, for every $i \in \{1, \dots, n\}$, $\chi_t^i < \chi_t^{i'}$ and $\gamma_t < \gamma_t'$: it remains to establish that for every $i \in \{1, \dots, n\}$, $\chi_{t+1}^i < \chi_{t+1}^{i'}$ and $\gamma_{t+1} < \gamma_{t+1}'$. For that intend, considering again the ratio of equations (5*a*) and (5*b*), it is obtained that:

$$\chi_{t+1}^i = \frac{\alpha(1 - \lambda_i)\chi_t^i}{1 - \gamma_t},$$
$$\chi_{t+1}^{i'} = \frac{\alpha(1 - \lambda_i)\chi_t^{i'}}{1 - \gamma_t'}.$$

Whence the holding of $\chi_{t+1}^i < \chi_{t+1}^{i'}$ and

$$\gamma_{t+1} = \sum_{i=1}^{n} \lambda_i \chi_{t+1}^i < \sum_{i=1}^{n} \lambda_i \chi_{t+1}^{i'} = \gamma_{t+1}'.$$

It is thus established that $\gamma_t < \gamma_t'$ for all $t \in \mathbb{N}$. From equation (6*b*), it is inferred that, for every $t \in \mathbb{N}$, $K_t(\mu_1) > K_t(\mu_2)$. The expression of $Z(\mu)$ being however increasing as a function of any of the $K_t(\mu)$, it is eventually obtained that:

$$Z(\mu_1) > Z(\mu_2),$$

a contradiction.

(ii) a/ From Equation (6*b*), the sequence $(K_t)_{t \in \mathbb{N}}$ converges to a steady state $\bar{K}$ such that $\bar{K} = (K)^\alpha \alpha(1 - \lambda_1)$.

b/ From Equation (6*a*), it derives that:

$$\lim_{t \to +\infty} W_t^1 = \frac{[1 - \alpha(1 - \lambda_1)]}{\lambda_1} (\bar{K})^\alpha,$$
$$\lim_{t \to +\infty} W_t^i = 0 \quad \text{for every} \quad i \neq 1.$$

c/ Making use of Equations (6*a*) and of the consumption decision rule (3), it is derived that:

$$\lim_{t \to +\infty} c_t^1 = [1 - \alpha(1 - \lambda_1)](\bar{K})^\alpha,$$
$$\lim_{t \to +\infty} c_t^i = 0 \quad \text{for every} \quad i \neq 1.$$

The details of the statement follow. QED

D. Proof of Lemma 4.1

Consider an agent with the preference parameters (β, δ). As for Lemma 3.1, the following variables are used:

$$X_t = \frac{W_t}{\Pi_t}$$
$$x_t = \frac{c_t}{\Pi_t}$$

with Π_t defined as:

$$\Pi_0 = 1$$

$$\text{for } t \geq 1,\ \Pi_t = \prod_{\tau=1}^{t} R_\tau$$

For a linear technology, $R_\tau = A$ and $\Pi_t = A^t$. The intertemporal budget constraint (1) is $X_{t+1} = X_t - x_t$ with $x_t \leq X_t$.

With a CES utility function, the objective function of self t (2) can be written as :

$$\frac{\left(\delta A^{(\sigma-1)/\sigma}\right)^t}{(\sigma-1)/\sigma}\left\{(x_t)^{(\sigma-1)/\sigma} + \beta\sum_{\tau=1}^{+\infty}\left(\delta A^{(\sigma-1)/\sigma}\right)^\tau (x_{t+\tau})^{(\sigma-1)/\sigma}\right\}$$

Consider the game defined on a T period horizon. The variables X_t and x_t are now denoted by X_t^T and x_t^T. For $t \leq T$, the t-self objective function is proportional to:

$$\frac{1}{(\sigma-1)/\sigma}\left\{\left(x_t^T\right)^{(\sigma-1)/\sigma} + \beta\sum_{\tau=1}^{T-t}\left(\delta A^{(\sigma-1)/\sigma}\right)^\tau \left(x_{t+\tau}^T\right)^{(\sigma-1)/\sigma}\right\}$$

For notational simplicity: $\Delta = \left(\delta A^{(\sigma-1)/\sigma}\right)$. The game is solved by backward induction. As the objective functions of all selves are homothetic, it is straightforward that the solution will take the form:

$$x_t^T = \lambda_t^T X_t^T$$

For $t = T$, the solution is obvious: $\lambda_T^T = 1$. Assuming that the game has been solved till period $t+1$, one obtains, $\forall\tau$ with $2 \leq \tau \leq T$,

$$x_{t+\tau}^T = \lambda_{t+\tau}^T\left(1 - \lambda_{t+\tau-1}^T\right)\ldots\left(1 - \lambda_{t+1}^T\right)X_{t+1}^T$$

and $x_{t+1}^T = \lambda_{t+1}^T X_{t+1}^T$. Replacing in the t-self objective function, we get:

$$\frac{1}{(\sigma-1)/\sigma}\left\{\left(x_t^T\right)^{(\sigma-1)/\sigma} + \beta\Delta(X_{t+1}^T)^{(\sigma-1)/\sigma} Z_t^T\right\},$$

$$\text{with } Z_t^T = \left(\lambda_{t+1}^T\right)^{(\sigma-1)/\sigma} + \Delta\left(\lambda_{t+2}^T\right)^{(\sigma-1)/\sigma}\left(1-\lambda_{t+1}^T\right)^{(\sigma-1)/\sigma}$$
$$+ \cdots + \Delta^{T-t-1}\left(\lambda_T^T\right)^{(\sigma-1)/\sigma}\left(1-\lambda_{T-1}^T\right)^{(\sigma-1)/\sigma}\ldots\left(1-\lambda_{t+1}^T\right)^{(\sigma-1)/\sigma}.$$

The program of self t can be written:

$$\max_{(x_t^T, X_{t+1}^T)} \frac{1}{(\sigma-1)/\sigma}\left\{\left(x_t^T\right)^{(\sigma-1)/\sigma} + \beta\Delta(X_{t+1}^T)^{(\sigma-1)/\sigma} Z_t^T\right\}$$
$$\text{s.t. } x_t^T + X_{t+1}^T = X_t^T.$$

The solution is $x_t^T = \lambda_t^T X_t^T$, $X_{t+1}^T = \left(1 - \lambda_t^T\right)X_t^T$, with

$$\lambda_t^T = \frac{1}{1 + (\beta\Delta)^\sigma\left(Z_t^T\right)^\sigma} \tag{6}$$

From the definition of Z_t^T, one gets:

$$Z_t^T = \left(\lambda_{t+1}^T\right)^{(\sigma-1)/\sigma} + \Delta\left(1 - \lambda_{t+1}^T\right)^{(\sigma-1)/\sigma} Z_{t+1}^T \tag{7}$$

Using (6) in t and $t+1$ to eliminate Z_t^T and Z_{t+1}^T in (7), it is obtained that:

$$\frac{1}{\beta\Delta}\left(\frac{1-\lambda_t^T}{\lambda_t^T}\right)^{1/\sigma} = \left(\lambda_{t+1}^T\right)^{(\sigma-1)/\sigma} + \Delta\left(1-\lambda_{t+1}^T\right)^{(\sigma-1)/\sigma}\frac{1}{\beta\Delta}\left(\frac{1-\lambda_{t+1}^T}{\lambda_{t+1}^T}\right)^{1/\sigma}$$

After some simplifications,

$$\lambda_t^T = \frac{1}{1+\Delta^\sigma\left[1-(1-\beta)\lambda_{t+1}^T\right]^\sigma/\lambda_{t+1}^T}$$

Define

$$f(\lambda) = \frac{1}{1+\Delta^\sigma\left[1-(1-\beta)\lambda\right]^\sigma/\lambda}$$

From backward induction with $\lambda_T^T = 1$, all the sequence λ_t^T is defined as: $\lambda_t^T = f^{T-t}(1)$.

The function f has the following properties: $f(0) = 0$, f increasing on $[0,1]$, $f'(0) = 1+(1-\beta)\sigma > 1$. f has two fixed points, 0 and λ^*, which is solution of:

$$\Delta^\sigma\left[1-(1-\beta)\lambda^*\right]^\sigma = 1-\lambda^*$$

From these properties, if $T \to +\infty$, we get for the limit strategy:

$$\lim_{T\to+\infty}\lambda_t^T = \lim_{T\to+\infty} f^{T-t}(1) = \lambda^*,$$

which establishes the statement. QED

E. Proof of Proposition 4.1

(i) The obtention of equations (9a) and (9b) is immediate.

The obtention of the coefficient γ_t proceeds as follows. First, take advantage of (9a):

$$\frac{W_t^i}{W_t^1} = \left(\frac{1-\lambda_i}{1-\lambda_1}\right)^t\frac{a_0^i}{a_0^1}.$$

Parallely make use of (5a) and (5b). It derives that:

$$\begin{aligned}\frac{K_{t+1}}{W_{t+1}^1} &= \frac{AK_t}{A(1-\lambda_1)W_t^1} - \sum_{i=1}^n\frac{\lambda_i W_t^i}{A(1-\lambda_1)W_t^1}\\ \iff \frac{K_{t+1}}{W_{t+1}^1} &= \frac{K_t}{(1-\lambda_1)W_t^1} - \frac{1}{A(1-\lambda_1)}\sum_{i=1}^n\lambda_i\left(\frac{1-\lambda_i}{1-\lambda_1}\right)^t\frac{a_0^i}{a_0^1}.\end{aligned} \tag{E.1}$$

Letting $\zeta_t := (1-\lambda_1)^t K_t/W_t^1$, Equation (E.1) reformulates to:

$$\zeta_{t+1} = \zeta_t - \frac{(1-\lambda_1)^t}{A}\sum_{i=1}^n\lambda_i\left(\frac{1-\lambda_i}{1-\lambda_1}\right)^t\frac{a_0^i}{a_0^1}. \tag{E.2}$$

Forward develop Equation (E.2):

$$\begin{aligned}\zeta_t &= A^{-1}\sum_{i=1}^n\lambda_i\frac{a_0^i}{a_0^1}\sum_{\tau=t}^{+\infty}(1-\lambda_i)^\tau\\ &= A^{-1}\sum_{i=1}^n\lambda_i\frac{a_0^i}{a_0^1}\frac{(1-\lambda_i)^t}{\lambda_i}\\ &= A^{-1}\sum_{i=1}^n\frac{a_0^i}{a_0^1}(1-\lambda_i)^t.\end{aligned}$$

It is eventually obtained that:

$$\begin{aligned}
\frac{W_t^1}{K_t} &= \frac{(1-\lambda_1)^t}{\zeta_t} \\
&= \frac{A(1-\lambda_1)^t}{\sum_{i=1}^n (1-\lambda_i)^t a_0^i / a_0^1}, \\
\frac{W_t^i}{K_t} &= \frac{(1-\lambda_i)^t}{(1-\lambda_1)^t} \frac{a_0^i}{a_0^1} \frac{W_t^1}{K_t} \\
&= \frac{A(1-\lambda_i)^t a_0^i}{\sum_{j=1}^n (1-\lambda_j)^t a_0^j}.
\end{aligned}$$

Whence:

$$\begin{aligned}
\gamma_t &= \frac{\sum_{i=1}^n \lambda_i W_t^i}{AK_t} \\
&= \frac{\sum_{i=1}^n \lambda_i (1-\lambda_i)^t a_0^i}{\sum_{i=1}^n (1-\lambda_i)^t a_0^i}.
\end{aligned}$$

(ii) This is immediate from the expression of γ_t. QED

F. Proof of Proposition 4.2

Let $\chi_i := 1 - \lambda_i$. From Equation (8), it is defined from the following equation:

$$A^{1-1/\sigma}(\chi_i)^{1/\sigma} = \delta_i[\beta_i + \chi_i(1-\beta_i)].$$

Consider then the two ensued straight lines equations $\xi_i(\chi) = \delta_i[\beta_i + \chi(1-\beta_i)]$, $i = 1, 2$. These two lines assume an intersection at $(\hat{\chi}, \hat{\xi})$, for $\hat{\xi} \in]0, 1[$. Its first coordinate is indeed readily computed to:

$$\hat{\chi} = \frac{\delta_1\beta_1 - \delta_2\beta_2}{\delta_2(1-\beta_2) - \delta_1(1-\beta_1)},$$

where $\hat{\chi} > 0$ is ensured under the assumptions of the proposition. Further noticing that:

$$\begin{aligned}
\hat{\chi} < 1 &\iff \delta_1\beta_1 - \delta_2\beta_2 < \delta_2(1-\beta_2) - \delta_1(1-\beta_1) \\
&\iff \delta_1 < \delta_2.
\end{aligned}$$

while the second coordinate is readily computed to:

$$\hat{\xi} = \frac{\delta_1\delta_2(\beta_1 - \beta_2)}{\delta_2(1-\beta_2) - \delta_1(1-\beta_1)}.$$

Consider now the function $\Upsilon(\chi) := A^{1/\sigma - 1}\chi^{1/\sigma}$. It is then noticed that $\Upsilon(0)$ is lower that both of the straight-lines $\xi_i(0) = \delta_i\beta_i$, $i \in \{1, 2\}$. In opposition to this, $\Upsilon(1)$ is greater than $\xi_i(1) = \delta_i$, $i \in \{1, 2\}$. Indeed and under Assumption 1, any of the straight-lines is to assume a unique intersection point with $\Upsilon(\chi)$ within the interval $]0, 1[$.

Finally define $\hat{A}$ from the holding of $(\hat{A})^{1/\sigma-1}(\hat{\chi})^{1/\sigma} = \hat{\xi}$.

– For $A < \hat{A}$, one obtains

$$A^{1/\sigma-1}(\hat{\chi})^{1/\sigma} > (\hat{A})^{1/\sigma-1}(\hat{\chi})^{1/\sigma},$$

whence the holding of $\chi_1, \chi_2 \in]0, \hat{\chi}[$ and the satisfaction of $\chi_1 > \chi_2$, that implies $\lambda_1 < \lambda_2$.

– For $A > \hat{A}$, one oppositely obtains

$$A^{1/\sigma-1}(\hat{\chi})^{1/\sigma} < (\hat{A})^{1/\sigma-1}(\hat{\chi})^{1/\sigma},$$

whence the holding of $\chi_1, \chi_2 \in]\hat{\chi}, 1[$ and the satisfaction of $\chi_1 < \chi_2$, that implies $\lambda_1 > \lambda_2$.

The statement follows. QED

G. Proof of Proposition 4.3

Consider a naïve agent with parameters (β, δ). For a naïve self deciding at period 0, all of the decisions starting from date 0 do correspond to the ones taken by an agent with a discount factor of δ, *i.e.*, without any biais and for $\beta = 1$. The corresponding value for the coefficient λ is hence available as:

$$\lambda = 1 - \delta^{\sigma} A^{\sigma-1}.$$

For every $t \geqslant 1$, the law of evolution of his wealth and the level of his consumption would be available as:

$$W_{t+1} = A(1-\lambda)W_t,$$
$$c_t = \lambda W_t.$$

The associated value of the objective by date $t = 0$ hence derives as:

$$\frac{(c_0)^{1-1/\sigma}}{1-1/\sigma} + \frac{\beta}{1-1/\sigma}(\lambda A)^{1-1/\sigma}(W_0 - c_0)^{1-1/\sigma}\left\{\sum_{\tau=1}^{+\infty} \delta^{\tau}\left[A(1-\lambda)\right]^{(\tau-1)(1-1/\sigma)}\right\}.$$

The first-order condition of the optimisation problem is available as:

$$(c_0)^{-1/\sigma} = \beta(\lambda A)^{1-1/\sigma}\frac{(W_0 - c_0)^{-1/\sigma}\delta}{1-\delta\left[A(1-\lambda)\right]^{1-1/\sigma}}.$$

From this equation, the c_0 solution assumes a form $c_0 = \lambda$ and

$$(\lambda^n)^{-1/\sigma} = \frac{\beta\delta\left[(1-(\delta)^{\sigma}A^{\sigma-1})A\right](1-\lambda^n)^{-1/\sigma}}{1-\delta\left[\delta^{\sigma}A^{\sigma}\right]^{1-1/\sigma}}$$
$$\lambda^n = \frac{1-\delta^{\sigma}A^{\sigma-1}}{1-\delta^{\sigma}A^{\sigma-1}(1-\beta^{\sigma})}.$$

In order to compare λ and λ^n, let $\chi := 1-\lambda$ and $\chi^n := 1-\lambda^n$. The coefficient χ is defined as the solution of:

$$\chi = \delta^{\sigma}A^{\sigma-1}[\beta + (1-\beta)\chi]^{\sigma},$$

that is to be compared with:

$$\chi^n = \frac{\delta^{\sigma}A^{\sigma-1}\beta^{\sigma}}{1-\delta^{\sigma}A^{\sigma-1}(1-\beta^{\sigma})}.$$

This leads to:

$$\chi^n > \chi \Longleftrightarrow \chi^n > \delta^{\sigma}A^{\sigma-1}[\beta + (1-\beta)\chi^n]^{\sigma}.$$

Having completed a range of simplifications, it is eventually obtained that the satisfaction of this inequation reformulates as the one of:

$$\left[1-\delta^{\sigma}A^{\sigma-1}\left(1-\beta^{\sigma}\right)\right]^{\sigma-1} > \left[1-\delta^{\sigma}A^{\sigma-1}\left(1-\beta^{\sigma-1}\right)\right]^{\sigma}.$$

Consider the function $G(\cdot)$ defined by:

$$G(\beta) = (\sigma-1)\ln\left[1-\delta^{\sigma}A^{\sigma-1}\left(1-\beta^{\sigma}\right)\right] - \sigma\ln\left[1-\delta^{\sigma}A^{\sigma-1}\left(1-\beta^{\sigma-1}\right)\right].$$

From the previous results, $\chi > \chi^{n} \Leftrightarrow G(\beta) > 0$. Further noticing that $G(1) = 0$, the derivative of $G(\cdot)$ is available as:

$$\begin{aligned}
DG(\beta) &= \sigma(\sigma-1)\left[\frac{\delta^{\sigma}A^{\sigma-1}\beta^{\sigma-1}}{1-\delta^{\sigma}A^{\sigma-1}\left(1-\beta^{\sigma}\right)} - \frac{\delta^{\sigma}A^{\sigma-1}\beta^{\sigma-2}}{1-\delta^{\sigma}A^{\sigma-1}\left(1-\beta^{\sigma-1}\right)}\right] \\
&= \sigma(\sigma-1)\delta^{\sigma}A^{\sigma-1}\beta^{\sigma-2}\left[\frac{\beta}{1-\delta^{\sigma}A^{\sigma-1}\left(1-\beta^{\sigma}\right)} - \frac{1}{1-\delta^{\sigma}A^{\sigma-1}\left(1-\beta^{\sigma-1}\right)}\right] \\
&= \sigma(\sigma-1)\delta^{\sigma}A^{\sigma-1}\beta^{\sigma-2}\left[\frac{(\beta-1)\left(1-\delta^{\sigma}A^{\sigma-1}\right)}{\left[1-\delta^{\sigma}A^{\sigma-1}\left(1-\beta^{\sigma}\right)\right]\left[1-\delta^{\sigma}A^{\sigma-1}\left(1-\beta^{\sigma-1}\right)\right]}\right]
\end{aligned}$$

Then, and for $\beta < 1$, $DG(\beta) < 0$ if $\sigma > 1$ while $DG(\beta) > 0$ if $\sigma < 1$. As $G(1) = 0$, it is obtained that, for $\beta < 1$:

$$\begin{cases} \chi^{n} > \chi & \text{if } \sigma > 1 \\ \chi^{n} < \chi & \text{if } \sigma < 1. \end{cases}$$

As a consequence:

$$\begin{cases} \lambda^{n} < \lambda & \text{if } \sigma > 1 \\ \lambda^{n} > \lambda & \text{if } \sigma < 1. \end{cases}$$

The statement follows. QED