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On the Bitcoin price dynamics: an augmented Markov-Switching model with Lévy jumps*

Julien Chevallier[†], Stéphane Goutte[‡], Khaled Guesmi[§] and Samir Saadi[¶]

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Abstract

This study contributes to the existing literature on the empirical characteristics of virtual currency allowing for a dynamic transition between different economic regimes and considering various crashes and rallies over the business cycle, that is captured by jumps. We combine Markov-switching models with Levy jump-diffusion offer a new model that captures the different sub-period of crises over the business cycle, that is captured by jumps. This method also enables to test the relevance of dynamic measures of regime switching concerning the independent pure-jump process, which are not frequently used in the literature. Bitcoin offers something different than a traditional currency; there is potential value of having a network that helps as a secure repository for the common knowledge of all transactions. Besides, the value of Bitcoin fluctuates so wildly that it may be too risky to serve as a credible store of value.

JEL Codes: G22; C58.

Keywords: Bitcoin; Lévy process; Markov-switching model.

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[†]IPAG Business School Paris and Université Paris 8 (LED)

[‡]Université Paris 8 (LED) and Paris Business School (PSB), Paris. Email: stephane.goutte@univ-paris8.fr

[§]IPAG Business School Paris

[¶]Telfer School of Management, University of Ottawa

1 Introduction

As a peer-to-peer crypto-currency, Bitcoin holds the promise of being free from central banks' and governments' interventions. A decade after its inception following Satoshi Nakamoto's (2008) vision, its price movements are far from being tamed nowadays. This study contributes to the existing literature on the empirical characteristics of virtual currency allowing for a dynamic transition between different economic regimes and considering various crashes and rallies over the business cycle, that is captured by jumps. In our model, we combine Markov-switching models with Lévy jump-diffusion. This strategy allows capturing the different sub-period of crises over the business cycle, that is captured by jumps, whereas the trend is modeled under a Gaussian process. This method also enables to test the relevance of dynamic measures of regime switching concerning the independent pure-jump process, which are not frequently used in the literature.

Our study differs from past ones in that we investigate the Bitcoin price fluctuations. Bitcoin offer something different than a traditional currency; there is potential value of having a network that helps as a secure repository for the common knowledge of all transactions. Besides, the value of Bitcoin fluctuates so wildly that it may be too risky to serve as a credible store of value. Bitcoin hogged the limelight in the cryptocurrency markets in the first week of May 2018 as it fell to one-month lows below \$8,000. Nevertheless, while the world's largest cryptocurrency by market capitalization suffered a 3.6 percent week-on-week drop in prices, it still outclassed other significant names like Bitcoin cash (BCH) and EOS, which both reported double-digit losses.

Furthermore, the US CFTC rules that Bitcoin and other crypto-currencies are commodities subject to the CFTC's jurisdiction. Thus, Bitcoin falls within the definition of "commodity" under the Commodity Exchange Act such that The CFTC regulates Bitcoins derivatives contracts. Consequently, Bitcoin is officially a commodity, according to U.S regulator. Dyhrberg (2015) claim that Bitcoin has liquidity limitations like other commodities. Bitcoin and gold derive most of their value from the fact that they are scarce and costly to extract.

The remainder of the paper is structured as follows. Section 2 details the stochastic model. Section 3 provides an application to the dynamics of the Bitcoin price. Section 4 concludes.

2 Stochastic model

Let (ω, \mathcal{F}, P) be a filtered probability space and T be a fixed terminal time horizon. We propose in this paper to model the dynamics of a sequence of historical values of the price using a regime-switching stochastic jump-diffusion. This model is defined using the class of Lévy processes.

2.1 Lévy Process

Definition 1 A Lévy process L_t is a stochastic process such that

1. $L_0 = 0$.
2. For all $s > 0$ and $t > 0$, we have that the property of stationary increments is satisfied. i.e. $L_{t+s} - L_t$ as the same distribution as L_s .
3. The property of independent increments is satisfied. i.e. for all $0 \leq t_0 < t_1 < \dots < t_n$, we have that $L_{t_i} - L_{t_{i-1}}$ are independent for all $i = 1, \dots, n$.
4. L has a Cadlag paths. This means that the sample paths of a Lévy process are right continuous and admit a left limits.

Remark 1 In a Lévy process, the discontinuities occur at random times.

2.2 Markov-switching

Definition 2 Let $(Z_t)_{t \in [0, T]}$ be a continuous time Markov chain on finite space $\mathcal{S} := \{1, 2, \dots, K\}$. Denote $\mathcal{F}_t^Z := \{\sigma(Z_s); 0 \leq s \leq t\}$, the natural filtration generated by the continuous time Markov chain Z . The generator matrix of Z , denoted by Π^Z , is given by

$$\Pi_{ij}^Z \geq 0 \quad \text{if } i \neq j \text{ for all } i, j \in \mathcal{S} \quad \text{and} \quad \Pi_{ii}^Z = - \sum_{j \neq i} \Pi_{ij}^Z \quad \text{otherwise.} \quad (1)$$

Remark 2 The quantity Π_{ij}^Z represents the switch from state i to state j .

2.3 Regime-switching Lévy

Let us define the regime-switching Lévy Model:

Definition 3 For all $t \in [0, T]$, let Z_t be a continuous time Markov chain on finite space $\mathcal{S} := \{1, \dots, K\}$ defined as in Definition 2. A regime-switching model is a stochastic process (X_t) which is solution of the stochastic differential equation given by

$$dX_t = \kappa(Z_t) (\theta(Z_t) - X_t) dt + \sigma(Z_t) dY_t \quad (2)$$

where $\kappa(Z_t)$, $\theta(Z_t)$ and $\sigma(Z_t)$ are functions of the Markov chain Z . Hence, they are constants which take values in $\kappa(\mathcal{S})$, $\theta(\mathcal{S})$ and $\sigma(\mathcal{S})$

$$\kappa(\mathcal{S}) := \{\kappa(1), \dots, \kappa(K)\} \in \mathbb{R}^{K^*}, \quad \theta(\mathcal{S}) := \{\theta(1), \dots, \theta(K)\}, \quad \sigma(\mathcal{S}) := \{\sigma(1), \dots, \sigma(K)\} \in \mathbb{R}^{K^+}.$$

where Y is a stochastic process which could be a Brownian motion or a Lévy process.

Remark 3 *The following classic notations apply:*

- κ denotes the mean-reverting rate;
- θ denotes the long-run mean;
- σ denotes the volatility of X .

Remark 4 • *In this model, there are two sources of randomness: the stochastic process Y appearing in the dynamics of X , and the Markov chain Z . There exists one randomness due to the market information which is the initial continuous filtration \mathcal{F} generated by the stochastic process Y ; and another randomness due to the Markov chain Z , \mathcal{F}^Z .*

- *In our model, the Markov chain Z infers the unobservable state of the economy, i.e. expansion or recession. The processes Y^i estimated in each state, where $i \in \mathcal{S}$, capture: a different level of volatility in the case of Brownian motion (i.e., $Y^i \equiv W^i$), or a different jump intensity level of the distribution (and a possible skewness) in the case of Lévy process (i.e. $Y^i \equiv L^i$).*

Remark 5 *One could propose to use a regime-switching stochastic volatility model, à la Heston, to better capture the flexibility of the volatility changes and levels. Nevertheless, this kind of model increases the computational burden dramatically during the simulations without improving the empirical fit greatly.*

2.4 NIG distribution

We recall the main properties of the Normal Inverse Gaussian (NIG) distribution. Indeed, we assume that a Lévy process L follows a Normal Inverse Gaussian (NIG) distribution. Note that the Variance-Gamma could have been an alternative at this stage (Kaishev and Dimitrova, 2009). The NIG family of distribution was introduced by Barndorff-Nielsen and Halgreen (1977). The NIG density belongs to the family of normal variance-mean mixtures, i.e., one of the most commonly used parametric densities in financial economics.

Taking $\delta > 0$, $\alpha \geq 0$, then the density function of a NIG variable $NIG(\alpha, \beta, \delta, \mu)$ is given by

$$f_{NIG}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha}{\pi} \exp(\delta \sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)) \frac{K_1(\alpha \delta \sqrt{1 + (x - \mu)^2 / \delta^2})}{\sqrt{1 + (x - \mu)^2 / \delta^2}}. \quad (3)$$

where K_ν is the third Bessel kind function with index ν . It can be represented with the following integral

$$K_\nu(z) = \frac{1}{2} \int_0^\infty y^{\nu-1} \exp\left(-\frac{1}{2}z(y + y^{-1})\right) dy.$$

For a given real ν , the function K_ν satisfies the differential equation given by

$$x^2 y'' + x y' - (x^2 + \nu^2) y = 0 .$$

This class of distribution is stable by convolution as the classic normal distribution. i.e.

$$NIG(\alpha, \beta, \delta_1, \mu_1) * NIG(\alpha, \beta, \delta_2, \mu_2) = NIG(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2) .$$

Lemma 1 *If $X \sim NIG(\alpha, \beta, \delta, \mu)$ then for any $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$, we have that*

$$Y = aX + b \sim \left(\frac{\alpha}{a}, \frac{\beta}{a}, a\delta, a\mu + b \right) .$$

The log cumulative function of a NIG variable is given by

$$\phi^{NIG}(z) = \mu z + \delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2} \right), \quad \text{for all } |\beta + z| < \alpha , \quad (4)$$

The first moments are given by

$$\mathbb{E}[X] = \mu + \frac{\delta\beta}{\gamma}, \quad \text{Var}[X] = \frac{\delta\alpha^2}{\gamma^3} . \quad (5)$$

with $\gamma = \sqrt{\alpha^2 - \beta^2}$. And finally the Lévy measure of a $NIG(\alpha, \beta, \delta, \mu)$ law is

$$F_{NIG}(dx) = e^{\beta x} \frac{\delta\alpha}{\pi|x|} K_1(\alpha|x|) dx . \quad (6)$$

Remark 6 *Each parameter in $NIG(\alpha, \beta, \delta, \mu)$ distributions can be interpreted as having a different effect on the shape of the distribution:*

- α - tail heaviness of steepness.
- β - skewness.
- δ - scale.
- μ - location.

3 Application on Bitcoin

3.1 Background on Bitcoin price development

The data is retrieved from Coinbase over the period going from July, 10th 2010 to March, 31th 2018 with a daily frequency, totaling 2814 observations. Figure 1 displays the Bitcoin price development.

Figure 1: Bitcoin price

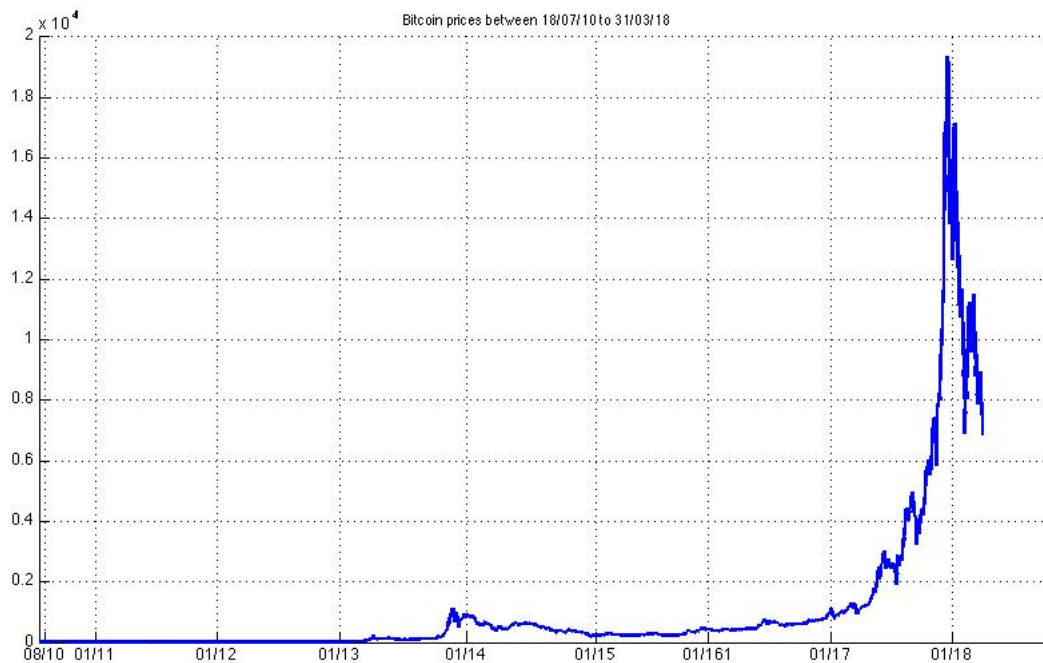
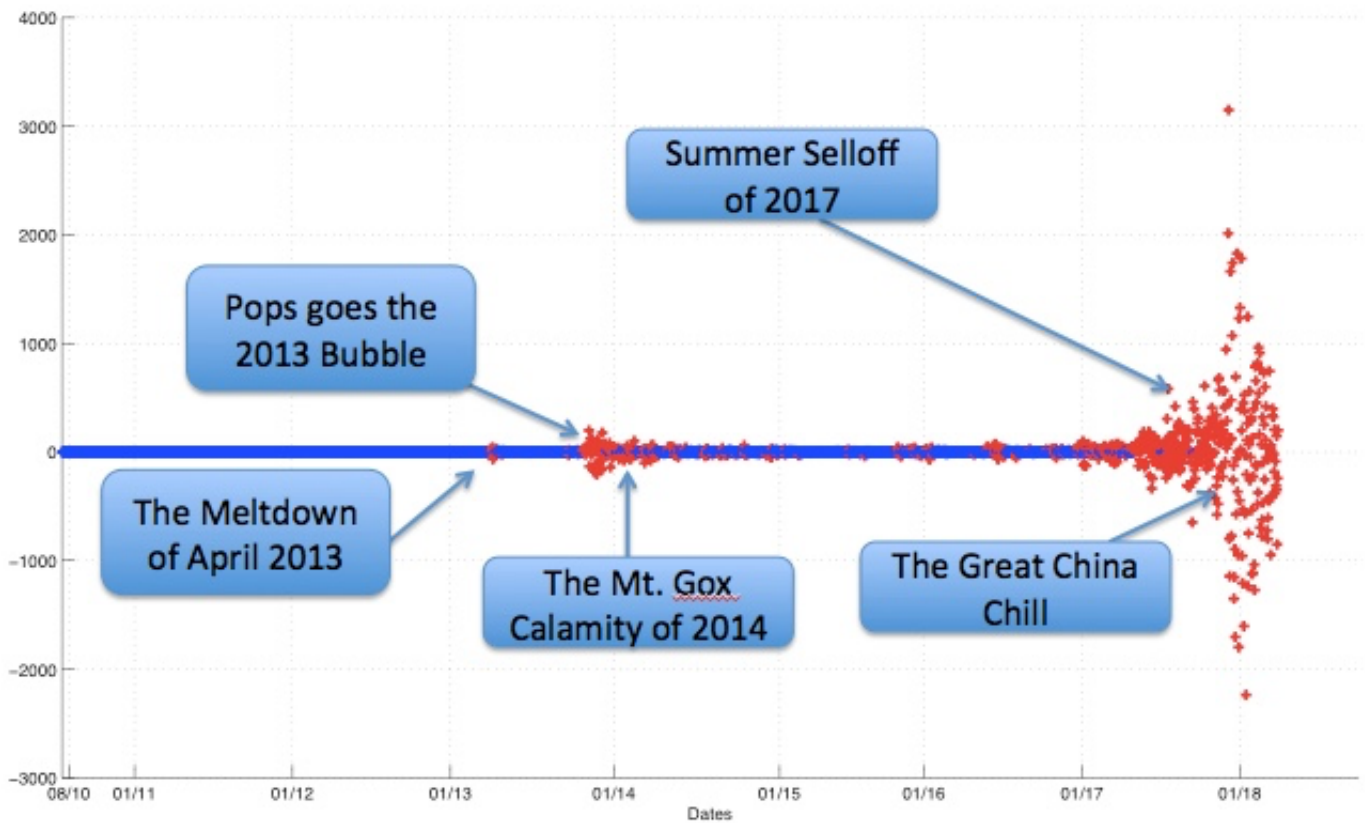


Figure 2 explains most of that price action during our study period, namely:

1. The Meltdown of April 2013: In the spring of 2013, a ghastly collapse saw the price of Bitcoin fall from \$233 to \$67 - overnight, i.e., a 71% drop. It would take seven months to recover. The crash of April 2013 came after Bitcoin's first significant brush with the mainstream. The currency had never crossed \$30 before 2013 but a flood of media coverage helped drive it well above \$200. The crash, which followed two smaller jolts in March, rejected in part a correction to speculator exuberance. Some also attribute it to an outage at Mt. Gox, the most popular (at the time) exchange for buying and selling Bitcoin.

Figure 2: Important Dates



2. Pop Goes the 2013 Bubble: Bitcoin spent most of the rest of 2013 around \$120. Then prices jumped ten-fold in the fall: Bitcoin hit a high of \$1,150 in late November, and then the party ended abruptly, and prices tumbled below \$500 by mid-December. It would take more than four years for Bitcoin to reach \$1,000 again. The crazy price run-up of late 2013 appears to have been a classic bubble as amateur investors rushed into Bitcoin for the first time. The frenzy was helped by regulators taking a more positive view of Bitcoin (in the early years, most regarded it as criminal if they had heard of it at all) and by U.S. exchanges like Coinbase that made it easier for average people to buy Bitcoin. When the bubble popped, prices would likely have recovered more quickly but for what happened next.
3. The Mt. Gox Calamity of 2014: The price of Bitcoin had been making significant gains after the bubble pop of 2013 when, in February, the price fell from \$867 to \$439 (a 49% drop). This triggered a doldrums period for Bitcoin that lasted until late 2016. The

February crash came after the operator of Mt. Gox-long the go-to trading place for longtime Bitcoin owners - announced the exchange had been hacked. On February 7, the exchange halted withdrawals and later revealed thieves had made off with 850,000 Bitcoins (which would be worth around \$3.5 billion today). The incident, which created existential doubts about the security of Bitcoin and undercut liquidity in the currency, likely harmed the currency's value for years.

4. Summer Selloff of 2017: In early January, Bitcoin broke \$1,000 for the first time in years and started climbing like crazy. By June, the currency nudged \$3,000- but then lurched back all of a sudden, falling 36% to \$1,869 by mid-July. Why it happened: Even as Bitcoin boomed anew, many worried something was wrong with the code under the hood. Specifically, Bitcoin was slow compared to other crypto-currencies like Litecoin and Ethereum, and its core developers could not agree on how to update the software. This raised the prospect of a "fork" (which would produce two versions of Bitcoin's canonical blockchain) and future schisms, which in turn appeared to give rise to market jitters and the significant fall in price. Ironically, such a fork did materialize in August in the form of rival Bitcoin Cash-but this seems to have done no longterm harm to Bitcoin.
5. The Great China Chill: After fears over the fork subsided, Bitcoin went on another crazy tear: It climbed close to \$5,000 at the start of September before plunging 37% by September 15, shaving off over \$30 billion from Bitcoin's total market cap in the process. Recovery is already underway, though, as prices climbed above \$4,000 three days later. Why it happened: While Bitcoin price moves can be inscrutable, the prime reason for the latest crash can be summed up in one word: China. After it cracked down on so-called "Initial Coin Offerings," there have been widespread rumors the Communist government is going to ban trading crypto-currency altogether. In response, the most prominent exchange, BTCChina, said it would end trading this month. This crackdown, combined with questions about China's de facto monopoly on Bitcoin mining, explains the recent price swoon.

3.2 MS-Lévy model

In this paper we apply the estimation process of Chevallier and Goutte (2017). This methodology proposes to estimate Eq.(2) in two steps: (i) model parameters in a regime-switching Brownian process, and (ii) the distribution parameters. We fit a regime-switching Lévy model such as Eq.(2) where the stochastic process Y is a Lévy process that follows a Normal Inverse Gaussian (NIG) distribution. Thus the optimal set of parameters to estimate is $\hat{\Theta} := (\hat{\kappa}_i, \hat{\theta}_i, \hat{\sigma}_i, \hat{\alpha}_i, \hat{\beta}_i, \hat{\delta}_i, \hat{\mu}_i, \hat{\Pi})$, for $i \in \mathcal{S}$.

We have the three parameters of the dynamics of X , the four parameters of the density of the Lévy process L , and the transition matrix of the Markov chain Z .

The table ?? reports the results of (i) the set of diffusion parameters, and (ii) the NIG density parameters of the Lévy jump process fitted to each regime. The remaining problem in this work is to specify the number of regimes in the Markov chain. For simplicity, we proceed with two regimes that relate to the ‘boom’ and ‘bust’ phases of the business cycle.

Table 1: Estimated parameters

Process	κ	θ	σ	P_{ii}^Z
State 1	1.0299 (0.1055)	0.3082 (0.0074)	23.8652 (0.7274)	0.9552
State 2	0.9130 (14.4028)	9.9467 (0.0389)	1367.4328 (7529.9928)	0.8550
NIG	α	β	δ	μ
State 1	0.0801 (0.0672)	0.0238 (0.0023)	0.1603 (0.7534)	-0.0499 (10.0061)
State 2	0.0012 (0.0883)	0.0001 (0.0001)	161.0635 (0.5497)	-14.5843 (0.0001)

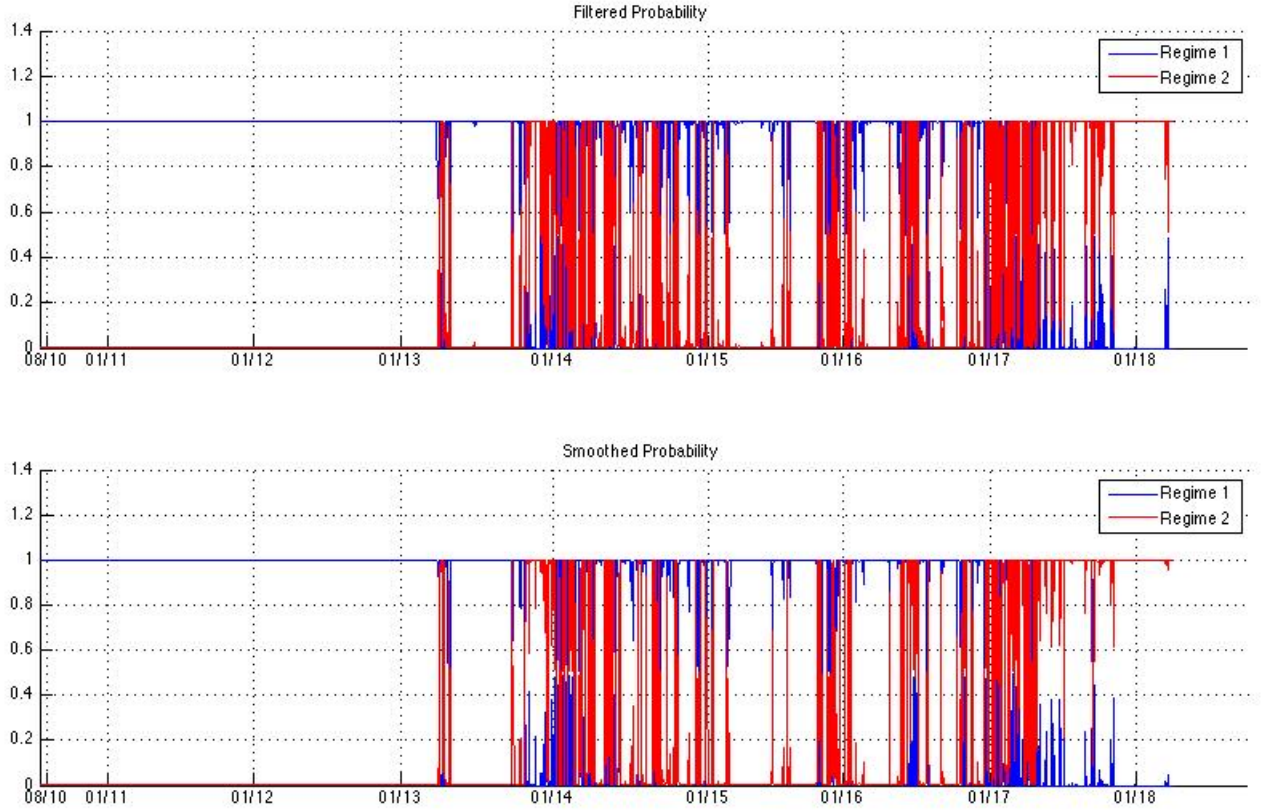
Standard errors are given in parentheses below parameter estimates.

In Table 1, the parameter α represents the jump intensity. The lower the alpha, the higher the jump intensity in a given regime. We observe that $\alpha = 0.0012$ during regime 2, which indicates a high jump intensity. On the contrary, $\alpha = 0.0801$ during regime 1, which points out a low jump intensity.

We also report a plot where each regime is reported with a different color (e.g., blue (red) corresponds to regime 1 (regime 2)). To provide the reader with a clearer picture, we have chosen to plug the regimes identified back into the raw (non-stationary) data. Of course, all the estimates were performed on returns, e.g., stationary data. Below this first plot, the filtered and smoothed probabilities are displayed. They reflect the regime switches at stake.

Inspecting the Figures 1 and 2, we observe that the regime 1 is characterized by a rather stable

Figure 3: Probabilities



price path for the Bitcoin crypto-money (before April 2013 and between all crisis and bubbles whose appear in 2013 and 2014 (i.e. The Meltdown of April 2013, Pops goes the 2013 Bubble and The Mt. Gox Calamity of 2014)). Visually, the model captures well these three main events (by switching in regime two) whose appearance in the standard economic state (i.e., regime one) and then switch definitively in regime two of high volatility and high frequency of jump after 2017. So beyond that date, the data enters the regime two which is characterized by ups and downs, and thus a higher jump activity.

β is the skewness parameter: $\beta < 0$ ($\beta > 0$) implies a density skewed to the left (right). The skewness of the density increases as β increases. In the case where β is equal to 0, the density is symmetric around μ . In both regime the values of β are close to 0; these values reflect a

balance between positive and negative jumps, as illustrated in Figure 2. δ is the scale parameter representing a measure of the spread of the returns. We see that the regime two with high frequency and intensity of jumps exhibits a huge value of delta $\delta = 161.0635$ against a vanishing value for regime one. This fact demonstrates the crisis or bubble situation captured by the regime two of our regime-switching Lévy model.

In Table 1, of particular interest is the volatility parameter σ : it is more than sixty times higher during regime 2 than during regime 1. The relatively higher jump intensity during regime 2, therefore, translates into higher volatility levels. The mean-reverting parameters κ are close to zero in both regimes.

When inspecting the filtered and smoothed probabilities in the middle and bottom panels of Figure 3, we notice that the probability of staying in the current regime is very high (e.g., close to unity). This information is also visible in the last column of Table 1: the probability to stay in regime 1 (in regime 2) is equal to 95.52% (85.50%). If there is some general form of persistence in the chain (e.g., high probability of staying in a given regime), then this could have important implications for the computation of the Value-at-Risk and dynamic portfolio allocation, because the benefits of portfolio diversification would be less volatile.

What we learn mainly from these probability graphs is that the stochastic process fitted to each regime does not have the same jumps characteristics during the sample period. Indeed, there are periods with a prominent presence of jumps (recorded during regime 2) in the Bitcoin price, and others without. Hence, this first set of results illustrates the interest of resorting to the regime-switching Lévy model to model the Bitcoin dynamics.

3.3 Regime-switching classification

An ideal model is one that classifies the regimes sharply, with smoothed probabilities close to either zero or one. In order to measure the quality of regime classification, we propose the following two measures:

- (1) The **regime classification measure (RCM)** introduced by Ang and Bekaert (2002) and generalized for multiple states by Baele (2005).

$$\text{RCM}(K) = 100 \cdot \left(1 - \frac{K}{K-1} \frac{1}{T} \sum_{k=1}^N \sum_{Z_{t_k}} \left(P \left(Z_{t_k} = i | \mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)} \right) - \frac{1}{K} \right)^2 \right), \quad (7)$$

where the quantity $P \left(Z_{t_k} = i | \mathcal{F}_{t_M}^X; \hat{\Theta}_1^{(n)} \right)$ is the smoothed probability and $\hat{\Theta}_1^{(n)}$ is the vector of parameters estimated. The constant serves to normalize the statistic to be between 0 and 100. Good regime classification is associated with low RCM statistic value: a value of 0 means perfect regime classification, and a value of 100 implies that no information about the regimes is revealed. Consequently, even if a model has the highest log-likelihood value, its RCM needs to be close to zero.

- (2) The **smoothed probability indicator**, introduced by Goutte and Zou (2013) and developed by Goutte (2014). With this indicator, a proper classification for the data can be achieved when the smoothed probability is less than 0.1 or greater than 0.9. This means that the data at time $t \in [0, T]$ is in one of the regimes at the 10% error level.

Table 2: Regime Classification Measure (RCM) and Smoothed Probability Indicator

	RCM	$P^{10\%}$
<i>Financial markets</i>		
Bitcoin	4.96	94.88%

For the Bitcoin, we observe in Table 2 that the RCM statistic is close to zero (4.96). In that case, the regime-switching Lévy model can discriminate correctly between the two regimes. In the second column of 2, we notice that the smoothed probability indicator is equal to 94.88%. Again, we conclude that the discrimination between regime is accurately performed by the model, as it is very close to the upper bound of 100%. With this battery of diagnostic tests, we have established the robustness of the results obtained with the Lévy regime-switching model.

4 Conclusion

Most of the Bitcoin price downward movements seem to coincide with speculative run-ups coupled with exogenous shocks, such as an exchange hack or government regulation forbidding the use of cryptocurrencies. In most cases, Bitcoin has bounced back from the downward movements within months or even weeks - suggesting Bitcoin buyers hodling for the long run.

Based on this mean-reverting behavior, we investigate a new way to model the dynamics of the Bitcoin price. This paper studies the volatility of the Bitcoin price using a Markov-switching model augmented with jumps under the Lévy form.

Distinct and significant dynamics are captured during crisis and bubbles periods. We identify the persistence of periods of high volatility and jumps. Policy implications unfold in terms of portfolio/risk management for hedging and pricing future derivatives on cryptocurrencies.

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