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► **To cite this version:**

Jean-François Laslier, Matias Nunez, M Remzi Sanver. A solution to the two-person implementation problem. 2019. halshs-02173504v1

HAL Id: halshs-02173504

<https://shs.hal.science/halshs-02173504v1>

Preprint submitted on 4 Jul 2019 (v1), last revised 5 May 2020 (v3)

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PARIS SCHOOL OF ECONOMICS
ECOLE D'ECONOMIE DE PARIS

WORKING PAPER N° 2019 – 32

A solution to the two-person implementation problem

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**JEL Codes:
Keywords :**

A SOLUTION TO THE TWO-PERSON IMPLEMENTATION PROBLEM^{*}

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JULY 3, 2019

Abstract

We propose a solution to the classical problem of Hurwicz and Schmeidler [1978] and Maskin [1999] according to which no Pareto efficient rule is Nash implementable. To this end, we consider implementation through mechanisms that are deterministic-in-equilibrium while lotteries are allowed off-equilibrium. We show that there are Pareto efficient rules which are implementable and that any such rule is implementable through some simple veto mechanism. Importantly, neither completeness nor transitivity of the preferences over lotteries are required to achieve implementation.

1 Introduction

Can one design some protocol that ensures that two players reach a Pareto efficient agreement in Nash equilibrium? The theorems of Hurwicz and Schmeidler [1978]

^{*}We would like to thank Chris Chambers, Yukio Koriyama, Phil Reny, Larry Samuelson and William Thomson for useful remarks. Financial support from the project IDEX ANR-10-IDEX-0001-02 PSL MIFID is gratefully acknowledged.

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and Maskin [1999], at the outset of implementation theory, provide a negative answer to this question. Assuming that players have complete information, both works showed that no deterministic mechanism guarantees that every Nash equilibrium is Pareto efficient, except dictatorships. The reason is that there is a tension between the conditions for the existence of an equilibrium at every preference profile and those which ensure that each outcome is Pareto efficient. This impossibility, to which we refer as the *two-person implementation problem*, is particularly striking since it is based on a very mild set of assumptions.

This work proposes a solution to this problem based on a modification of the mechanisms used for implementation. More precisely, we examine the consequences of allowing lotteries off-equilibrium, while still ensuring deterministic outcomes in equilibrium. That is, we consider Nash implementation through deterministic-in-equilibrium mechanisms or simply **DE** mechanisms.¹ The **DE** mechanisms use lotteries as endogenous punishments and this triggers the possibility of Nash implementing Pareto efficient rules.

Yet, the notion of Pareto efficiency needs some qualification since allowing for lotteries causes that the definition of Pareto efficiency is more subtle than in the deterministic case (see Bogomolnaia and Moulin [2001] for a discussion). Two classical definitions are ex-ante and ex-post Pareto efficiency. A lottery is ex-ante Pareto efficient if no other lottery Pareto dominates it, whereas it is ex-post Pareto efficient if no pure alternative that can be selected by the lottery is Pareto dominated by some other pure alternative. We first show that the impossibility prevails if one considers ex-ante Pareto efficiency. Yet, we establish that ex-post Pareto efficient implementation is possible under a mild condition that we term “best element bias”: for any set of alternatives, a player prefers the (sure) lottery that consists of his most preferred element in the set to any lottery with support in the same set. Furthermore, we characterize the class of (ex-post) Pareto efficient social choice rules (SCRs) that can be Nash implemented through these mechanisms.

¹To our knowledge, this paper is the first one to consider this idea with two players. See Özkal-Sanver and Sanver [2006], Bochet [2007] and Benoît and Ok [2008] for related ideas. We could replace lotteries with sets of alternatives and the results presented would be identical.

Our main result shows that a SCR is Pareto efficient and Nash implementable if and only if it is among the class of *Pareto-and-veto SCRs* which are determined by some vector $v = (v_1, v_2)$ with $v_1 + v_2 + 1$ being the number of alternatives: any Pareto efficient alternative which is not among the v_i worst alternatives for each player i is selected.²

The sufficiency part, rather than relying on the classical integer games, builds for each implementable SCR a simple veto mechanism that Nash implements it. Importantly, the implementation result just relies on the existence of a best-element bias for both players. Moreover, it does not require neither completeness³ nor transitivity of the preferences over lotteries. Each veto mechanism endows each player i with v_i vetoes to be distributed among the alternatives with, again, $v_1 + v_2 + 1$ being the number of alternatives. The game is simultaneous and the outcome is a full-support lottery over the unvetoes alternatives. The equilibrium reasoning is straightforward: given the vetoes of his opponent, a player can induce any set of unvetoes alternatives as the support of the lottery by adequately casting his vetoes. Thus, due to the best-element bias, his best response is to select his best element among the unvetoes alternatives. These nice features of best responses have three consequences. First, each veto mechanism is **DE** since a unique alternative remains unvetoes in equilibrium, otherwise there is a conflict with best responses. Second, an equilibrium outcome is Pareto efficient since otherwise a player can always, by deviating, select a Pareto dominating alternative. Third, the equilibrium strategies have a natural shape: if x is the implemented alternative and v_i is the number of vetoes, player i vetoes all alternatives preferred to x by his opponent (say k alternatives) and he vetoes also $v_i - k$ among the alternatives less preferred than x by his opponent. If both strategies veto disjoint sets of alternatives, this forces each player to accept his

²It is not the first time that Pareto-and-veto SCRs are found to be of interest in the literature: Abreu and Sen [1991] (pp. 1016-17) present this class of rules as the main example that is virtually implementable but fails to be Nash implementable. In a setting where monetary transfers are allowed, Sanver [2018] designs a direct veto mechanism that implements alternatives which are Pareto efficient and preferable to some disagreement outcome by both players.

³See Schmeidler [1989] who underlines the importance of weakening completeness and writes: "Out of the seven axioms listed here the completeness of the preferences, $(i)a$, seems to me the most restrictive and most imposing assumption".

opponent's strategy, triggering the equilibrium.

The necessity part is more involved. Here, the key concept is the veto power generated by a mechanism: a mechanism μ endows player i with veto power over some set X of alternatives if and only if player i has some strategy that prevents any alternative in X to be selected with positive probability whatever his opponent plays. As we show, any mechanism μ that ensures Pareto efficient outcomes must endow each player i with veto power over every set of alternatives whose cardinality does not exceed some integer v_i^μ with $v_1^\mu + v_2^\mu + 1$ being the number of alternatives. This result almost directly entails that only subcorrespondences of f_v are Nash implementable. Yet, among them, the only Maskin monotonic one is f_v itself, which concludes the proof since the necessity of Maskin monotonicity for Nash implementability prevails for **DE** mechanisms.

The structure of the paper is as follows: Section 2 introduces the basic notions. Section 3 discusses the consequences of imposing ex-ante Pareto efficiency. Section 4 presents the characterization of Pareto efficient and Nash implementable SCRs. The proofs of the necessity and the sufficiency part are respectively presented in Sections 5 and 6. Section 7 presents a review of the literature and Section 8 makes some concluding remarks.

2 Basic notions and notation

A set $N = \{1, 2\}$ of two players face a finite set A of $n + 1 \geq 3$ alternatives. The set of linear orders over A is denoted by \mathcal{L}_A whose generic element \succ_i is the preference of $i \in N$.⁴ The set of preference profiles over A is denoted $\mathcal{L}_A^2 = \mathcal{L}_A \times \mathcal{L}_A$ with $\succ = (\succ_1, \succ_2)$ denoting a generic preference profile. We write $\text{PE}(\succ) = \{x \in A \mid \text{there is no } y \in A \text{ with } y \succ_i x \text{ for all } i \in N\}$ for the set of Pareto efficient alternatives at $\succ \in \mathcal{L}_A^2$. Let $L(x, \succ_i) = \{y \in A : x \succ_i y\}$ be the (strict) lower contour set and $U(x, \succ_i) = \{y \in A : y \succ_i x\}$ be the (strict) upper contour set of $x \in A$ at $\succ_i \in \mathcal{L}_A$.

⁴More precisely, one of $x \succ_i y$ and $y \succ_i x$ holds for any distinct $x, y \in A$ while $x \succ_i x$ fails for all $x \in A$. Moreover, $x \succ_i y$ and $y \succ_i z$ implies $x \succ_i z$ for all $x, y, z \in A$.

We write $\mathcal{A} = 2^A$ for the power set of A and $\overline{\mathcal{A}} = \mathcal{A} \setminus \{\emptyset, A\}$ for the power set of A without A and the empty set. The set of lotteries over A is denoted $\Delta = \{p : A \rightarrow [0, 1] \mid \sum_{x \in A} p(x) = 1\}$. For each lottery $p \in \Delta$, we let $\text{SUPP}(p) \in \mathcal{A}$ denote the support of lottery p , i.e., the set of alternatives that are assigned a positive probability by p . Slightly abusing notation, we let $\{x\}$ denote both the singleton set consisting of alternative x and the lottery that selects x with probability one.

We extend preferences over alternatives to preferences over lotteries. We let \mathcal{P}_Δ stand for the set of binary relations over Δ . A typical element of \mathcal{P}_Δ is denoted \succeq_i^* with \succ_i^* being its strict part. For any preference profile \succ_i , we let $\kappa(\succ_i)$ denote a set of preference relations over Δ compatible with \succ_i , and, we assume $x \succ_i y \implies \{x\} \succ_i^* \{y\}$, $\forall x, y \in A$, $\forall \succeq_i^* \in \kappa(\succ_i)$. Abusing notation, we write $\kappa(\succ) \subseteq \mathcal{P}_\Delta^2$ for the set of preference profiles over Δ compatible with $\succ \in \mathcal{L}_A^2$.

We now define some mild conditions on the preference extension $\kappa(\cdot)$ to be used throughout:

Best-element bias: The preference extension $\kappa(\cdot)$ satisfies **BEB** iff for any $\succ_i \in \mathcal{L}_A$, any $X \in \mathcal{A}$ with $\#X > 1$ and any $x \in X$, we have $x \succ_i y$ for any $y \in X \setminus \{x\} \implies \{x\} \succ_i^* p$ for any $p \in \Delta \setminus \{\{x\}\}$ with $\text{SUPP}(p) \in 2^X$ and for any $\succeq_i^* \in \kappa(\succ_i)$.

Worst-element bias: The preference extension $\kappa(\cdot)$ satisfies **WEB** iff for any $\succ_i \in \mathcal{L}_A$, any $X \in \mathcal{A}$ with $\#X > 1$ and any $x \in X$, we have $y \succ_i x$ for any $y \in X \setminus \{x\} \implies p \succ_i^* \{x\}$ for any $p \in \Delta \setminus \{\{x\}\}$ with $\text{SUPP}(p) \in 2^X$ and for any $\succeq_i^* \in \kappa(\succ_i)$.

Partition of alternatives: The preference extension $\kappa(\cdot)$ satisfies **PART** iff for any $\succ_i \in \mathcal{L}_A$ and any $X \in \overline{\mathcal{A}}$ with $x \succ_i y$ for any $x \in X$, for any $y \in A \setminus X$, $\exists \succeq_i^* \in \kappa(\succ_i)$ such that $p \succ_i^* q$ for any $p, q \in \Delta$ with $\text{SUPP}(p) \cap X \neq \emptyset$ and $\text{SUPP}(q) \subseteq A \setminus X$.

Monotonicity: The preference extension $\kappa(\cdot)$ satisfies **MON** iff for any $x \in A$, any $q \in \Delta$, and any $\succ_i, \succ'_i \in \mathcal{L}_A$ with $y \in L(x, \succ_i) \implies y \in L(x, \succ'_i)$ for any $y \in \text{SUPP}(q)$, we have $\{x\} \succeq_i^* q$ for some $\succeq_i^* \in \kappa(\succ_i) \implies \{x\} \succeq_i^{**} q$ for some $\succeq_i^{**} \in \kappa(\succ'_i)$.

BEB implies that a player prefers the lottery that picks his best element in any set X of alternatives to any lottery with support in X . **WEB** means that a player prefers

any lottery with support in X to the (sure) one that selects his worst element in X . **BEB** and **WEB** are satisfied by virtually all preference extensions over lotteries.

PART implies that for a player who prefers every element of X to every element of $A \setminus X$, it should be possible to conceive outcomes in X “sufficiently good” and outcomes in $A \setminus X$ “sufficiently bad”, so that there is at least one compatible preference over lotteries where he prefers every lottery with support containing good outcomes from X to every lottery with support consisting only of bad outcomes outside of X . **MON** requires the following: Suppose that the lottery $\{x\}$ is preferred to some lottery q with support Y at some preference \succ_i^* in $\kappa(\succ_i)$. If $\succ'_i \in \mathcal{L}_A$ is such that x improves against elements of Y , then there must be a preference over lotteries compatible with \succ'_i where $\{x\}$ is still preferred to the lottery q .

PART and **MON** are rather weak conditions, that imply some richness of the domain of preferences over lotteries. For example, they both hold if players are expected utility maximizers.⁵

A social choice rule (SCR) is a mapping $f : \mathcal{L}_A^2 \rightarrow \mathcal{A}$. A SCR is Pareto efficient if $f(\succ) \subseteq \text{PE}(\succ)$ for any $\succ \in \mathcal{L}_A$. We say that f is a subcorrespondence of g and write $f \subseteq g$ whenever $f(\succ) \subseteq g(\succ) \forall \succ \in \mathcal{L}_A^2$. A mechanism is a function $\mu : M \rightarrow \Delta$ with $M = M_1 \times M_2$ where $M_i \neq \emptyset$ is the message space of $i \in N$. Let $N^\mu(\succeq^*)$ be the set of Nash equilibria of the normal-form game induced by the mechanism μ when \succeq^* denotes the preferences over lotteries. We say that μ is admissible iff $N^\mu(\succeq^*) \neq \emptyset \forall \succeq^* \in \kappa(\succ), \forall \succ \in \mathcal{L}_A^2$.

A mechanism is deterministic-in-equilibrium (**DE**) iff $\text{supp}(\mu(m)) \in A \forall m \in N^\mu(\succeq^*)$, $\forall \succeq^* \in \kappa(\succ), \forall \succ \in \mathcal{L}_A^2$. A **DE** mechanism μ Nash-implements a SCR $f : \mathcal{L}_A^2 \rightarrow \mathcal{A}$ iff

⁵In order to justify their weakness, one can look at the literature on preference extension axioms over sets of alternatives and deduce preferences over lotteries through preferences over the support of the lotteries. In fact, the previous conditions are satisfied if one considers the well-known preference extension axioms of the literature (such as Gärdenfors [1976] or Kelly [1977]) and deduces preferences over lotteries through the preferences over their supports. If $\kappa(\cdot)$ satisfies **BEB** and **WEB** (which are universally quantified), every sub-correspondence of $\kappa(\cdot)$ satisfies them as well. On the other hand, this need not be the case for (existentially quantified) **PART** and **MON** where violations may occur under additional assumptions that severely restrict the compatible preferences over lottery supports. Nevertheless, these assumptions would be very strong. For example, allowing for each preference over alternatives only one preference over lottery supports determined through the inverse lexicographic rule would entail a violation of **PART**. On the other and, **PART** will be satisfied under the equally restrictive lexicographic rule (see Özyurt and Sanver [2009] for a discussion of the lexicographic and inverse lexicographic rule).

$f(\succ) = \bigcup_{m \in N^\mu(\succeq^*)} \text{SUPP}(\mu(m)) \forall \succeq^* \in \kappa(\succ), \forall \succ \in \mathcal{L}_A^2$. Note that if μ Nash-implements some SCR f , then μ is admissible.

3 Ex-ante Pareto efficiency

A mechanism μ is ex-ante Pareto efficient iff given any $\succ \in \mathcal{L}_A^2$, any \succeq^* in $\kappa(\succ)$, and any $m \in N^\mu(\succeq^*)$, there is no $p \in \Delta$ such that $p \succ_i^* \mu(m)$ for $i \in N$.

A mechanism μ is a dictatorship iff there is some $i \in N$ such that, for each $x \in A$, $\exists m_i \in M_i$ such that $\mu(m_i, m_j) = \{x\}$ for all $m_j \in M_j$.

We show that the notions of ex-ante Pareto efficiency and admissibility clash, hence extending the two-player implementation problem to the setting with lotteries and **DE** mechanisms.

Theorem 1. *Let $\kappa(\cdot)$ satisfy **PART** and **WEB**. If a **DE** mechanism μ is both admissible and ex-ante Pareto efficient, then μ is a dictatorship.*

Proof. Consider a **DE** mechanism μ that is both ex-ante Pareto efficient and admissible. We now characterize the equilibrium outcomes of μ at each preference profile \succ , denoted $f(\succ)$. Take any profile \succ . (In the sequel alternatives a and b stand for the best outcomes at \succ for player 1 and 2, respectively.) Borrowing the vocabulary of Hurwicz and Schmeidler [1978], we think of the mechanism μ as a matrix where player 1 controls rows and player 2 controls columns. We hence write, for every $x \in X$, an $\{x\}$ -row is a row that contains only $\{x\}$ as an outcome and similarly for an $\{x\}$ -column.

Take any $\succ^* \in \kappa(\succ)$ such that for player 1, any lottery whose support contains a is preferred to any lottery whose support does not contain a ; and for player 2, any lottery whose support contains b is preferred to any lottery whose support does not contain b . The existence of \succ^* is ensured by **PART**. Take any alternative $x \neq a, b$. Note that x cannot be in $f(\succ)$ as otherwise μ would have x as an equilibrium outcome while both players prefer the lottery $\{a, b\}$ to x , contradicting ex-ante Pareto efficiency. So $f(\succ) \subseteq \{a, b\}$. Note that this argument applies to any preference profile

\succ . Thus, an ex-ante Pareto optimal and admissible **DE** mechanism gives equilibrium outcomes from the union of tops.

Now consider a preference profile \succ where the players' preferences are completely opposed. Relabel the alternatives as a_1, a_2, \dots, a_m . Take a preference profile where a_1 and a_2 are the respective top and bottom outcomes for player 1 while a_2 and a_1 are the respective top and bottom outcomes for player 2. So the equilibria outcomes of μ belong to $\{a_1, a_2\}$. Note that μ is **DE**, so the lottery with support $\{a_1, a_2\}$ cannot be an equilibrium outcome. Let, without loss of generality, a_1 be an equilibrium outcome. This is the worst element for player 2 and also the worst lottery (due to **WEB**), hence player 1 must have an $\{a_1\}$ -row.

Now, take a preference profile where a_2 and a_3 are the respective top and bottom outcomes for player 1 while a_3 and a_2 are the respective top and bottom outcomes for player 2. So the equilibria outcomes of μ belong to $\{a_2, a_3\}$. We first show that a_3 cannot be an equilibrium outcome. Suppose it is. As a_3 is the worst element and lottery for player 1, player 2 must have an $\{a_3\}$ -column, due to **WEB**, which contradicts player 1 has an $\{a_1\}$ -row. As a result, a_2 is an equilibrium outcome and we argue, mutatis mutandis, player 1 has an a_2 -row.

Iterate by making the arguments for $a_3, a_4, \dots, a_{m-1}, a_m$, proves that for each $a \in A$, player 1 has an $\{a\}$ -row, showing that player 1 is the dictator. Repeating the argument assuming that a_2 is an equilibrium outcome shows that player 2 is the dictator. \square

Theorem 1 shows that ex-ante Pareto efficiency is too restrictive in our setting. As a result, in the sequel, we focus on ex-post Pareto efficient mechanisms. A **DE** mechanism μ is ex-post Pareto efficient iff $\text{SUPP}(\mu(m)) \in \text{PE}(\succ) \forall m \in N^\mu(\geq^*), \forall \geq^* \in \kappa(\succ), \forall \succ \in \mathcal{L}_A^2$. In words, an ex-post Pareto efficient mechanism only selects Pareto efficient alternatives in the support of the outcome. Since our objective is to propose a solution to the classical problem, i.e. the impossibility of implementing Pareto efficient alternatives, exploring the notion of ex-post Pareto efficient in our setting is self-explanatory.

4 Pareto efficient implementable rules

For any $v = (v_1, v_2) \in \{0, 1, \dots, n\}^2$ with $v_1 + v_2 = n$, we define the Pareto-and-veto SCR $f_v : \mathcal{L}_A^2 \rightarrow \mathcal{A}$ as:

$$f_v(\succ) = \overbrace{\text{PE}(\succ)}^{\text{Pareto}} \cap \underbrace{\left\{x \in A \mid \#L(x, \succ_1) \geq v_1\right\} \cap \left\{x \in A \mid \#L(x, \succ_2) \geq v_2\right\}}^{\text{Veto}}.$$

Best $n - v_1$ alternatives for 1 Best $n - v_2$ alternatives for 2

The Pareto-and-veto SCR f_v picks all alternatives that are Pareto efficient and with a lower-contour set larger than v_i for each player i . Note that f_v is well-defined since $v_1 + v_2 = n$.⁶

We design a “strike mechanism”, where for each player $i \in N$, the set $M_i \subseteq A$ represents the set of alternatives he can strike or veto. Given any $v \in \{0, \dots, n\}^2$ with $v_1 + v_2 = n$, we define $\mu_v : M \rightarrow \Delta$ with $M_i = \{X \in \mathcal{A} \mid \#X = v_i\}$ for each $i \in N$ and, for each $m = (m_1, m_2)$, $\mu(m)$ triggers the uniform lottery⁷ over

$$\text{SUPP}(\mu(m)) = A \setminus (m_1 \cup m_2).$$

In other words, an alternative is uniformly drawn from the unvetoes alternatives. As $v_1 + v_2 = n$, $\text{SUPP}(\mu(m))$ is always non-empty.

Theorem 2. *Let $\kappa(\cdot)$ satisfy **BEB**, **WEB**, **MON** and **PART**. If a Pareto efficient SCR f is Nash-implementable by a **DE** mechanism, then $f = f_v$ for some $v \in \{0, \dots, n\}^2$ with $v_1 + v_2 = n$.*

Theorem 3. *Let $\kappa(\cdot)$ satisfy **BEB**. The strike mechanism μ_v Nash implements the SCR f_v for any $v \in \{0, \dots, n\}^2$ with $v_1 + v_2 = n$.*

The proofs of both theorems will be detailed in the next two sections.

⁶The SCR f_v can be empty-valued for $v_1 + v_2 > n$ as it can be observed when the two players have completely opposed preferences while the possibility of ensuring its non-empty-valuedness when $v_1 + v_2 = n$ follows from Theorem 1 of Brams and Kilgour [2001].

⁷Theorem 3 holds replacing the uniform distribution by any probability distribution with full support over the unvetoes alternatives.

These results show the existence of a strong link between implementation through **DE** mechanisms and veto power. Indeed, Theorem 2 shows that under the mild conditions of **BEB**, **WEB**, **PART** and **MON**, a SCR needs to admit some veto structure in order to be both Pareto efficient and implementable. This theorem is very much related to the impossibility result by Hurwicz and Schmeidler [1978] in the following sense. Hurwicz and Schmeidler [1978] show that the only SCRs which are both Pareto efficient and implementable (through a deterministic mechanism) are the dictatorial ones. Note that the dictatorial SCR corresponds to f_v with $v = (n, 0)$ (if player 1 is the dictator) and $v = (0, n)$ (if player 2 is the dictator). Our theorem shows that by allowing lotteries as off-equilibrium punishments, a class of intermediate and, interestingly, non dictatorial SCRs appear: the Pareto-and-veto SCRs.

The counterpart of Theorem 2, Theorem 3, shows that the implementation of non-dictatorial SCRs is indeed possible and that this implementation result can be obtained through simple veto mechanisms. Indeed, it suffices to allow each player i to (simultaneously) veto v_i alternatives and let the outcome be the uniform lottery over the unvetoes alternatives in order to implement the SCR f_v . Note that this possibility result does not rely on the usual techniques of integer/modulo games often present in the implementation literature.

Note that f_v is neutral for any $v \in \{0, \dots, n\}^2$ and f_v is anonymous if and only if $v_1 = v_2$. Thus, under the assumptions of Theorem 2, the following observations trivially follow. With an odd number of alternatives, an anonymous, neutral and Pareto efficient SCR f is Nash-implementable by a **DE** mechanism iff f is a Pareto-and-veto SCR with $v_1 = v_2$. On the contrary, with an even number of alternatives, there exist no anonymous, neutral and Pareto efficient SCR that is Nash-implementable by a **DE** mechanism.

5 Strike mechanisms: sufficiency

As previously mentioned, in each strike mechanism μ_v and each strategy profile $m = (m_1, m_2)$, the outcome is a uniform lottery over the set of unvetoes alternatives

$A \setminus (m_1 \cup m_2)$. Moreover, each strategy m_j vetoes some set of v_j alternatives. Thus, given m_j , the objective for player i is to select the support of the lottery that will determine the outcome. Let $g_v(M_i, m_j) = \{X \in \mathcal{A} \mid \text{SUPP}(\mu_v(m_i, m_j)) = X \text{ for some } m_i \in M_i\}$ be the attainable set of player i at m_j under the strike mechanism μ_v . So the set $g_v(M_i, m_j)$ contains the supports of the uniform lotteries that player i can induce when player j selects m_j under the strike mechanism μ_v .

Because of the number of vetoes at his disposal, player i can induce any subset of alternatives unvetoes by player j in the support of the outcome by adequately casting his vetoes. Thus:

$$g_v(M_i, m_j) = \{X \in \mathcal{A} \mid X \subseteq A \setminus m_j, X \neq \emptyset\}.$$

Building on this key property of the attainable set, the rest of the section proves that the strike mechanism μ_v Nash implements the Pareto-and-veto correspondence f_v . The first consequence is that strike mechanisms are **DE**.

Lemma 1. *Let $\kappa(\cdot)$ satisfy **BEB**. Any strike mechanism μ_v is **DE**.*

Proof. Assume that there is some equilibrium $m = (m_1, m_2)$ with $\#\text{SUPP}(\mu(m)) > 1$. Thus, there is some player i and some alternative $x \in \text{SUPP}(m)$ such that $x \succ_i y$ for any $y \in \text{SUPP}(m) \setminus \{x\}$. Since $x \notin m_j$ and $g_v(M_i, m_j) := \{X \in \mathcal{A} \mid X \subseteq A \setminus m_j\}$, player i has some strategy m'_i that induces $\{x\}$ as an outcome and thus $\mu(m'_i, m_j) = \{x\} \succ_i^* \mu(m)$ due to **BEB**. This contradicts that m is an equilibrium, establishing the proof. \square

Since a strike mechanism is **DE**, it follows that, in equilibrium, no uncertainty remains: both players veto disjoint sets of alternatives and a unique alternative is selected. Equipped with this lemma, we are now ready to prove that the strike mechanism μ with veto vector v Nash implements the Pareto-and-veto SCR with the same veto vector v . The next lines contain the proof of Theorem 3.

Take any $v \in \{0, \dots, n\}^2$ with $v_1 + v_2 = n$, any $\succ \in \mathcal{L}_A^2$, and any $\succeq^* \in \kappa(\succ)$. We first establish $f_v(\succ) \subseteq \bigcup_{m \in N^\mu(\succeq^*)} \mu(m)$.

Take any $x \in f_v(\succ)$. Assume first that we can pick $m_1 \in M_1$ with $m_1 \cup m_2 = A \setminus \{x\}$, $m_1 \cap m_2 = \emptyset$, $x \succ_2 y$ for any $y \in A \setminus m_1$ and $m_2 \in M_2$ with $x \succ_1 y$ for any $y \in A \setminus m_2$.

By construction $\mu(m) = \{x\}$ and $m \in N^\mu(\succeq^*)$ since $g_v(M_i, m_j) := \{X \in \mathcal{A} \mid X \subseteq A \setminus m_j\}$ for each player i and $\kappa(\cdot)$ satisfies **BEB**. The existence of m_1 and m_2 follows from the next observations: Take any pair (m_1, m_2) such that

$$U(x, \succ_2) \subseteq m_1 \subseteq U(x, \succ_2) \cup L(x, \succ_2) \text{ and } U(x, \succ_1) \subseteq m_2 \subseteq U(x, \succ_1) \cup L(x, \succ_1). \quad (1)$$

Note that for any (m_1, m_2) satisfying (1), $x \succ_i y$ for any $y \in A \setminus m_j$ for any $i = 1, 2$ and $j \neq i$ as wanted. We now prove that there is some pair (m'_1, m'_2) satisfying (1) with $m'_1 \cup m'_2 = A \setminus \{x\}$ and $m'_1 \cap m'_2 = \emptyset$. Let $\#U(x, \succ_1) = k$ and $\#U(x, \succ_2) = j$. Since $x \in \text{PE}(\succ)$, it follows that $k + j \leq n$ and $U(x, \succ_1) \cap U(x, \succ_2) \neq \emptyset$.⁸ It follows that $\#(m_1 \cap L(x, \succ_2)) = v_1 - j$ and $\#(m_2 \cap L(x, \succ_1)) = v_2 - k$ for any pair (m_1, m_2) satisfying (1). However, it is always possible to take $v_1 - j$ alternatives from $L(x, \succ_2)$ and $v_2 - k$ different alternatives from $L(x, \succ_1)$ among the $n - (j + k)$ alternatives in $A \setminus (U(x, \succ_1) \cup U(x, \succ_2))$ since $v_1 + v_2 = n$, which proves the existence of (m'_1, m'_2) as wanted.

We now show that $f_v(\succ) \supseteq \bigcup_{m \in N^\mu(\succeq^*)} \mu(m)$. Take some x with $\mu(m) = \{x\}$ for some $m \in N^\mu(\succeq^*)$. We first show that $x \in \text{PE}(\succ)$. Suppose not, i.e., there exists $y \in A$ with $y \succ_i x$ for all $i \in N$. Since m is an equilibrium, we have $m_1 \cap m_2 = \emptyset$ while $\mu(m) = \{x\}$. Thus, $y \in m_i$ for some $i \in N$, say $i = 1$, without loss of generality. Since $g_v(M_1, m_2) := \{X \in \mathcal{A} \mid X \subseteq A \setminus m_2\}$, there is some m'_1 with $\mu(m'_1, m_2) = \{y\} \succ_1^* \mu(m_1, m_2) = \{x\}$ due to **BEB**, contradicting $m \in N^\mu(\succeq^*)$.

We now show $\#L(x, \succ_i) \geq v_i \forall i \in N$. Suppose, without loss of generality, that $\#L(x, \succ_1) < v_1$. As previously argued, the attainable set for any strategy m_2 equals $g_v(M_1, m_2) := \{X \in \mathcal{A} \mid X \subseteq A \setminus m_2\}$. Thus, there is some strategy m'_1 that induces alternative y as an outcome with $\#L(y, \succ_1) = n + 1 - v_2 = v_1 + 1$ and hence $y \succ_1 x$. Therefore, $\mu(m'_1, m_2) = \{y\} \succ_1^* \{x\} = \mu(m)$ due to **BEB**, which contradicts $m \in N^\mu(\succeq^*)$.

⁸To see why, suppose that $k + j \geq n + 1$ so that $U(x, \succ_1) \cap U(x, \succ_2) \neq \emptyset$. Thus, there is some y with $y \succ_i x$ for both $i = 1, 2$, a contradiction.

6 Necessity of veto for implementation

6.1 Maskin monotonicity

We first show that the necessity of Maskin monotonicity for Nash implementability prevails in our framework.

Definition 1. A SCR $f : \mathcal{L}_A^2 \rightarrow \mathcal{A}$ is Maskin Monotonic iff for any $x \in \mathcal{A}$ and any $\succ, \succ' \in \mathcal{L}_A^2$ with $L(x, \succ_i) \subseteq L(x, \succ'_i) \forall i \in N$, $x \in f(\succ) \implies x \in f(\succ')$.

Proposition 1. Let $\kappa(\cdot)$ satisfy **MON**. A SCR $f : \mathcal{L}_A^2 \rightarrow \mathcal{A}$ is Nash implementable by a **DE** mechanism only if f is Maskin monotonic.

Proof. Let f be Nash implementable by some **DE** mechanism μ . Take any $x \in \mathcal{A}$ and any $\succ, \succ' \in \mathcal{L}_A^2$ with $L(x, \succ_i) \subseteq L(x, \succ'_i) \forall i \in N$. Suppose $x \in f(\succ)$. Take any $\succeq^* \in \kappa(\succ)$. As μ Nash implements f , $\exists m = (m_1, m_2) \in N^\mu(\succeq^*)$ with $\mu(m) = \{x\}$. Hence, $\{x\} \succeq_1^* \mu(m'_1, m_2) \forall m'_1 \in M_1$ and $\{x\} \succeq_2^* \mu(m_1, m'_2) \forall m'_2 \in M_2$. By **MON** conjoined with $L(x, \succ_i) \subseteq L(x, \succ'_i) \forall i \in N$, $\exists \succeq^{**} \in \kappa(\succ')$ with $\{x\} \succeq_1^{**} \mu(m'_1, m_2) \forall m'_1 \in M_1$ and $\{x\} \succeq_2^{**} \mu(m_1, m'_2) \forall m'_2 \in M_2$, thus $m \in N^\mu(\succeq^{**})$. As μ Nash implements f , this implies that $x \in f(\succ')$, showing that f is Maskin monotonic. \square

We now ask whether some subcorrespondence of the Pareto-and-veto SCR is Nash implementable. As stated by the following proposition, although f_v is Maskin monotonic, it admits no proper subcorrespondence with the same property.

Proposition 2. Take any $v \in \{0, \dots, n\}^2$ with $v_1 + v_2 = n$. A SCR $f : \mathcal{L}_A^2 \rightarrow \mathcal{A}$ with $f \subseteq f_v$ is Maskin monotonic if and only if $f = f_v$.

The proof of this proposition rather tedious and intricate; it is thus included in the appendix for the sake of clarity. The rest of this section is devoted to show that if a SCR is both implementable and Pareto efficient, then it is a Pareto-and-veto SCR.

6.2 Veto power of mechanisms

For each **DE** mechanism μ and each player i , we let

$$\text{veto}(M_i) = \{X \in \mathcal{A} \mid \exists m_i \in M_i \text{ s.t. } \text{SUPP}(\mu(m_i, m_j)) \cap X = \emptyset \text{ for any } m_j \in M_j\},$$

denote the veto set for player i . When $X \in \text{veto}(M_i)$, we say that player i has veto power over the set X of alternatives, i.e., he has a strategy that ensures that no alternative in this set belongs to the support of the outcome independently of the strategy of his opponent. Note that a player with veto power over X has also veto power over any $Y \subset X$. We first state a result on the structure of the veto power that **DE** mechanisms generate.

Theorem 4. *Let $\kappa(\cdot)$ satisfy **PART** and **WEB**. For any admissible and **DE** mechanism μ and any partition $\{X, \bar{X}\}$ of A with $X, \bar{X} \neq \emptyset$:*

1. *either $\bar{X} \in \text{veto}(M_1)$ or $X \in \text{veto}(M_2)$,*
2. *either $X \in \text{veto}(M_1)$ or $\bar{X} \in \text{veto}(M_2)$.*

Proof. Take any μ that is admissible and **DE**, and any $X \in \mathcal{A} \setminus \{A\}$. Pick some \succ in \mathcal{L}_A^2 such that $x \succ_1 y \forall x \in X, \forall y \in \bar{X}$ and $y \succ_2 x \forall y \in \bar{X}, \forall x \in X$. Take also $\succeq^* \in \kappa(\succ)$ such that $p \succ_1^* q$ for all p with $\text{SUPP}(p) \cap X \neq \emptyset$, for all q with $\text{SUPP}(q) \subseteq \bar{X}$; and $p \succ_2^* q$ for all p with $\text{SUPP}(p) \cap \bar{X} \neq \emptyset$, for all q with $\text{SUPP}(q) \subseteq X$. The existence of \succeq^* is ensured by **PART**. Now suppose, for a contradiction, that player 1 does not have veto power over \bar{X} and player 2 does not have veto power over X . Take any $m = (m_1, m_2) \in N^\mu(\succeq^*)$. As μ is **DE**, $\mu(m) = \{a\}$ for some $a \in A$. Let $a \in X$. As player 1 does not have veto power over \bar{X} , $\exists m'_2 \in M_2$ such that $\text{SUPP}(\mu(m_1, m'_2)) \cap \bar{X} \neq \emptyset$, hence $\mu(m_1, m'_2) \succeq_2^* \{a\}$ due to **WEB**, contradicting $m \in N^\mu(\succeq^*)$. Now, let $a \in \bar{X}$. As player 2 does not have veto power over X , $\exists m'_1 \in M_1$ such that $\text{SUPP}(\mu(m'_1, m_2)) \cap X \neq \emptyset$, hence $\mu(m'_1, m_2) \succeq_1^* \{a\}$, contradicting $m \in N^\mu(\succeq^*)$, again due to **WEB**, which in turn contradicts that μ is admissible. Thus, either $\bar{X} \in \text{veto}(M_1)$ or $X \in \text{veto}(M_2)$.

Now, pick some $\succ \in \mathcal{L}_A^2$ such that $y \succ_1 x \forall y \in \bar{X}, \forall x \in X$ and $x \succ_2 y \forall x \in X, \forall y \in \bar{X}$. Take also $\succeq^* \in \kappa(\succ)$ such that $p \succ_1^* q$ for all p with $\text{SUPP}(p) \cap \bar{X} \neq \emptyset$, for all q with $\text{SUPP}(q) \subseteq X$; and $p \succ_2^* q$ for all p with $\text{SUPP}(p) \cap X \neq \emptyset$, for all q with $\text{SUPP}(q) \subseteq \bar{X}$. Now suppose, for a contradiction, player 1 does not have veto power over X and player 2 does not have veto power over \bar{X} . Take any $m = (m_1, m_2) \in N^\mu(\succeq^*)$. As μ is **DE**, $\mu(m) = \{a\}$ for some $a \in A$. Let $a \in \bar{X}$. As player 1 does not have veto power over X , $\exists m'_2 \in M_2$ such that $\text{SUPP}(\mu(m_1, m'_2)) \cap X \neq \emptyset$, hence $\mu(m_1, m'_2) \succ_2^* \{a\}$ due to **WEB**,

contradicting $m \in N^\mu(\geq^*)$. Now, let $a \in X$. As player 2 does not have veto power over \bar{X} , $\exists m'_1 \in M_1$ such that $\text{SUPP}(\mu(m'_1, m_2)) \cap \bar{X} \neq \emptyset$, hence $\mu(m'_1, m_2) \succ_1^* \{a\}$, again due to **WEB**, contradicting $m \in N^\mu(\geq^*)$, again contradicting that μ is admissible. Therefore, either $X \in \text{veto}(M_1)$ or $\bar{X} \in \text{veto}(M_2)$, which establishes the proof. \square

Definition 2. A mechanism μ is veto neutral for player i iff, for any $X \in \mathcal{A}$ and any permutation $\rho : A \rightarrow A$, $X \in \text{veto}(M_i) \iff \rho(X) \in \text{veto}(M_i)$

Our next result states that if a mechanism is **DE** and ex-post Pareto efficient, then it is veto neutral for both players.

Theorem 5. Let $\kappa(\cdot)$ satisfy **BEB**, **WEB** and **PART**. A mechanism μ that is admissible, **DE** and ex-post Pareto efficient is veto neutral for both players.

Proof. Let μ be an admissible, **DE** and ex-post Pareto efficient mechanism. We first make and prove a claim whose iterative application establishes the proof:

Claim: If some player, say player 1 without loss of generality, has veto power over $X \in \bar{\mathcal{A}}$ under mechanisms μ , then μ gives player 1 veto power over any $X' \in \bar{\mathcal{A}}$ with $\#X' = \#X$ and $\#(X \cap X') = k - 1$.

Proof of the claim: Pick any $k \in \{1, \dots, n\}$ and suppose, for a contradiction, that player 1 has veto power over $X = \{x_1, \dots, x_{k-1}, x_k\}$ but not over $X' = \{x_1, \dots, x_{k-1}, x_{k+1}\}$. Due to **PART** and **WEB**, Theorem 4 holds and implies that there are four possible cases:

- Case 1: $\{X, A \setminus X\} \subseteq \text{veto}(M_1)$ and $A \setminus X' \in \text{veto}(M_1) \cap \text{veto}(M_2)$.
- Case 2: $\{X, A \setminus X\} \subseteq \text{veto}(M_1)$ and $\{X', A \setminus X'\} \subseteq \text{veto}(M_2)$
- Case 3: $\{X, A \setminus X'\} \subseteq \text{veto}(M_1) \cap \text{veto}(M_2)$.
- Case 4: $X \in \text{veto}(M_1) \cap \text{veto}(M_2)$ and $\{X', A \setminus X'\} \subseteq \text{veto}(M_2)$.

Let m_1 be the strategy of player 1 that vetoes X and m_2 be the strategy of player 2 that vetoes $A \setminus X'$. Note that these two strategies exist in all four cases, so the arguments that follow cover all four cases.

Note that $\mu(m_1, m_2) = \{x_{k+1}\}$ for the unique alternative $x_{k+1} \in A \setminus (X \cup A \setminus X')$. Now, consider some $\succ \in \mathcal{L}_A^2$ with $x_{k+1} \succ_1 x \forall x \in \{x_1, \dots, x_{k-1}\}$; $x_{k+1} \succ_2 x \forall x \in \{x_{k+2}, \dots, x_{n+1}\}$; and $x_k \succ_i x_{k+1} \forall i \in N$. This profile exists due to **PART**. Observe that given m_1 , player

2 can only obtain a lottery with support in $A \setminus X$ while at \succ_2 , x_{k+1} is his highest ranked alternative within $A \setminus X$. So he has no incentive to deviate under any $\succeq_2^* \in \kappa(\succ_2)$ due to **BEB**. In a similar vein, given m_2 , player 1 can only obtain a lottery with support in X' while at \succ_1 , x_{k+1} is his highest ranked alternative within X' . So, he has no incentive to deviate under any $\succeq_1^* \in \kappa(\succ_1)$ due to **BEB**. Hence $(m_1, m_2) \in N^\mu(\succeq^*) \forall \succeq^* \in \kappa(\succ)$, contradicting that μ is ex-post Pareto efficient since $\mu(m_1, m_2) = x_{k+1}$ is not ex-post Pareto efficient at \succ , thus establishing our claim.

The proof is established by noting that given any $X, Y \in \bar{\mathcal{A}}$ with $\#X = \#Y$, there is a finite sequence of sets X_1, \dots, X_s with $X_1 = X, X_s = Y$ while $\#X_i = \#X$ and $\#(X_i \cap X'_{i+1}) = \#X - 1$ for each $i \in \{1, \dots, s-1\}$. \square

Remark that when μ is veto neutral for some player i , if a set with a given cardinality belongs to $\text{veto}(M_i)$, then any other set with the same cardinality belongs to $\text{veto}(M_i)$ as well. Hence, the veto set for player i can be written as:

$$\text{veto}(M_i) = \{X \in \mathcal{A} \mid \#X \leq v_i^\mu\},$$

where the integer v_i^μ stands for the cardinality of the highest cardinality set over which i has veto power. Note that $v_1^\mu + v_2^\mu \leq n$, as otherwise μ is not well defined.

Theorem 6. *Let $\kappa(\cdot)$ satisfy **BEB**, **WEB** and **PART**. If a mechanism μ is admissible, ex-post Pareto efficient and **DE** then $v_1^\mu + v_2^\mu = n$.*

Proof. Let μ be admissible, ex-post Pareto efficient and **DE**. By Theorem 5, μ is veto neutral for both players so that $v_i^\mu \in \{0, \dots, n\}$ for each $i = 1, 2$. If $v_1^\mu = n$ then $v_1^\mu + v_2^\mu = n$ is trivially satisfied. Now let $v_1^\mu = k \in \{0, \dots, n-1\}$. Take $X \in \bar{\mathcal{A}}$ with $\#X = k+1$. Among the four possible cases stated by Theorem 4, the following two imply $v_1^\mu \geq k+1$, contradicting $v_1^\mu = k$: either player 1 has veto power over both X and $A \setminus X$ or both players have veto power over X . The other two cases are as follows: either both players have veto power over $A \setminus X$; or player 2 has veto power over both X and $A \setminus X$. In the former case, as $\#A \setminus X = n - k$, we have $v_2^\mu \geq n - k$, ensuring $v_1^\mu + v_2^\mu = n$ as wanted. In the latter case, $v_2^\mu \geq \max\{k+1, n-k\}$. If $n-k \geq k+1$, then $v_1^\mu + v_2^\mu = n$. Now, let $k+1 > n-k$, i.e., $k > \frac{n-1}{2}$. Thus $v_1^\mu + v_2^\mu = 2k+1 > n$, giving a contradiction,

concluding the proof. □

6.3 Implementation through DE mechanisms

The next lines summarize how the previous results are enough to prove Theorem 2: namely, if a SCR is Pareto efficient and implementable through a **DE** mechanism, then it is a Pareto-and-veto SCR, provided that $\kappa(\cdot)$ satisfies **BEB**, **WEB**, **MON** and **PART**.

Let f be Pareto efficient and Nash-implementable by a **DE** mechanism μ . So μ is ex-post Pareto efficient as well, thus μ is veto neutral with $v_1^\mu + v_2^\mu = n$ (Theorems 4 and 5 and 6 hold due to **BEB**, **WEB** and **PART**). For each $\succ \in \mathcal{L}_A^2$ and each $i \in N$, let $v_i^f(\succ) = \min_{x \in f(\succ)} \#L(x, \succ_i)$ and $v_i^f = \min_{\succ \in \mathcal{L}_A^2} v_i^f(\succ)$. We first show that $v_i^f \geq v_i^\mu \forall i \in N$. Suppose, for a contradiction, that $v_i^f < v_i^\mu$ for some $i \in N$. So there exists $\succ \in \mathcal{L}_A^2$ and $x \in f(\succ)$ with $\#L(x, \succ_i) < v_i^\mu$. We let, without loss of generality, $i = 1$. Take any $\succeq^* \in \kappa(\succ)$. As μ Nash-implements f , there exists $(m_1, m_2) \in N^\mu(\succeq^*)$ with $\mu(m_1, m_2) = \{x\}$. Now, let $m'_1 \in M_1$ be the strategy where player 1 vetoes his last v_1^μ alternatives with respect to \succ_1 . It follows that any $y \in \text{SUPP}(\mu(m'_1, m_2))$ is such that $y \succ_1 x$. Thus $\mu(m'_1, m_2) \succ_1^* \mu(m_1, m_2)$ due to **WEB**, contradicting $(m_1, m_2) \in N^\mu(\succeq^*)$, which establishes $v_i^f \geq v_i^\mu \forall i \in N$. As $v_1^\mu + v_2^\mu = n$ together with $v_1^f + v_2^f \leq n$, we have $v_1^f + v_2^f = n$, showing that $f \subseteq f_v$. Proposition 2 proves that no proper sub-correspondence of f_v is Maskin monotonic while Proposition 1 proves that Maskin monotonicity is necessary for implementation when **MON** holds. Thus $f = f_v$ for some veto vector v , as desired.

7 Review of the literature

This section summarizes how our work relates to the previous works in the two-player implementation problem.

As previously argued, the pioneering works are those of Hurwicz and Schmeidler [1978] and Maskin [1999]. Following these works, Dutta and Sen [1991] and Moore and Repullo [1990] independently characterize the set of SCRs that can be Nash

implemented with two players and show that suitable domain restrictions (specially in the Euclidean space) can overcome this powerful negative result (see Busetto and Colognato [2009] for a weakening of these conditions). Overall, the conclusion of this literature is rather negative since Pareto efficiency is a pretty mild condition and many applications of interest, as the current one, do not lie in the Euclidean space.

Confronted with this impossibility, the literature has focused on two solutions that modify the mechanism notion⁹: using random mechanisms (see Matsushima [2008] and Abreu and Sen [1991] for the virtual implementation approach) or using dynamic mechanisms (see Anbarci [2006] and De Clippel et al. [2014] among others). Both solutions allow the implementation of Pareto efficient SCRs but impose two respective constraints. Under virtual implementation, as Bochet and Maniquet [2010] puts it, "any alternative can be the outcome of the game as it receives positive probability in the equilibrium lottery". Under dynamic implementation, the order of play matters in determining the outcome which creates first/second mover advantages. Both constraints represent threats to the relevance of these solutions to real-life applications as players may want to avoid ambiguity over the agreed outcome and/or need not agree on the order of play. Thus, our solution focuses on simultaneous mechanisms that ensure deterministic equilibrium outcomes.

Our solution is based on the idea of endowing veto rights to both players. While to our knowledge the literature on virtual implementation did not consider veto-based mechanisms¹⁰, the one on dynamic mechanisms often endows both players with some veto power (see Bloom and Cavanagh [1986a], Bloom and Cavanagh [1986b], Anbarci [2006] and De Clippel et al. [2014] among others). These contributions are then related to the classic literature on sequential voting by veto with many players (see Mueller [1978], Moulin [1981], Felsenthal and Machover [1992] for theoretical works and Yuval [2002] and Yuval and Herne [2005] for an experi-

⁹Other approaches have modified the rationality notion such as partial honesty; see Dutta and Sen [2012] among others.

¹⁰The most closely related papers dealing with virtual implementation are the ones focusing on approval voting with two players (Núñez and Laslier [2015] and Laslier et al. [2017]). The mechanisms are not **DE** and allow for equilibrium lotteries.

mental approach) where each player is assigned a certain number of vetoes to be distributed freely among the alternatives. These dynamic veto mechanisms have been studied by two seminal works in the literature: Moulin [1981] and De Clippel et al. [2014]. Moulin [1981] shows that (weak) dominance solvability leads to unique outcomes. On the other hand, De Clippel et al. [2014] focuses on subgame perfection with $v_1 = v_2$ where, again, there is a unique outcome that coincides with the one obtained under dominance solvability. These are positive results wounded by the fact that the order according to which players cast their vetoes plays a strong role in determining the outcome. As Moulin [1981] puts it, "[v]oting by veto procedures introduce a strong asymmetry among agents: ... the ordering of the agents has a strong influence on the outcome".

8 Concluding remarks

We propose a solution to the two-person implementation problem. The answer to the problem are the strike mechanisms that allow simultaneous vetoes by both players. This solution is obtained by altering two key elements of the classic framework: (i) considering mechanisms that allow in-equilibrium pure alternatives and off-equilibrium lotteries and (ii) restricting efficiency to the ex-post Pareto notion.

Our class of **DE** mechanisms is a simultaneous version of the dynamic veto mechanisms (see Moulin [1981]) which, by allowing off-equilibrium set-valued outcomes, resolve the unfairness generated by dynamic mechanisms. To see the difference between our solution and the one proposed by dynamic mechanisms, consider a dynamic game that allows player 1 to veto $n+1-k$ alternatives and player 2 to veto $k-1$ of the remaining k alternatives, where $k \in \{1, \dots, n+1\}$. At each preference profile \succ , the subgame perfect equilibrium outcome of this game is the most preferred alternative of player 1 among $f_v(\succ)$ where $v_1 = n+1-k$ and $v_2 = k-1$. In other words, this dynamic veto mechanism subgame perfect implements a subcorrespondence of f_v by refining it with respect to the true preference of the first mover. One could argue that fairness here could be achieved by randomizing the identity of the first-mover. Yet, this needs qualification since this randomization prevents some alternatives to

arise as the following example shows: when $A = \{a, b, c, d, e\}$, at the preference profile $a \succ_1 b \succ_1 c \succ_1 d \succ_1 e$ and $c \succ_2 b \succ_2 a \succ_2 d \succ_2 e$, the dynamic veto mechanism with $k = 3$ which gives an equal veto power of 2 to both voters implements, by alternating first movers, either a or c but excludes b . However, f_v picks all three of a , b and c . Thus, our simultaneous direct veto mechanisms allow for the implementation of the compromise alternative b whereas their dynamic counterparts fail to do so. This constitutes a strong argument in favor of using simultaneous mechanisms.

We close by noting three limitations of our analysis. First, it is restricted to Nash implementation in pure strategies. Allowing for mixed strategies and exploring the existence of interesting **DE** mechanisms for settings with two or more players seems to be a promising research venue (see Mezzetti and Renou [2012]). Second, the set of implementable SCRs expands if one considers implementation through non-**DE** mechanisms. Indeed, the game-form associated to plurality rule¹¹ Nash implements the union of tops as long as **BEB** holds: i.e., at each preference profile all alternatives that are top-ranked by at least one player. Third, we have considered implementation through ex-post Pareto efficient **DE** mechanisms. Other notions of efficiency (weaker than ex-ante Pareto efficiency) are present in the literature such as stochastic dominance. Whether other SCRs can be Nash implemented through **DE** mechanisms by considering different notions of efficiency remains to be explored.

A Appendix

A.1 Proof of Proposition 2

Checking the “if” part is left to the reader. Now we prove three lemmata whose conjunction establish the “only if” part.

Note that the proof is immediate if either $v_1 = n$ or $v_2 = n$ since, in both cases, f_v is uniquely determined for each preference profile \succ . Thus, $f \subseteq f_v$ directly implies

¹¹In this game form, each player announces a single alternative and one of the announced alternatives is selected randomly. In equilibrium, every player announces his best alternative and thus this mechanism is not **DE**.

that $f = f_v$. For the remaining lemmas, we assume that $0 < v_1, v_2 < n$.

Lemma 1: Take any $f \subseteq f_v$ which is Maskin monotonic, any $\succ \in \mathcal{L}_A^2$ and any $x \in \text{PE}(\succ)$. If $\#L(x, \succ_i) = v_i \forall i \in N$, then $f(\succ) = \{x\}$.

Proof. Take any $f \subseteq f_v$, any $\succ \in \mathcal{L}_A^2$ and any $x \in \text{PE}(\succ)$ with $\#L(x, \succ_i) = v_i \forall i \in N$. Now take any $y \in A \setminus \{x\}$ and suppose $y \in f_v(\succ)$. So $\#L(y, \succ_i) \geq v_i \forall i \in N$, and as $\#L(x, \succ_i) = v_i \forall i \in N$, it must be $\#L(y, \succ_i) > v_i \forall i \in N$, implying $y \succ_i x \forall i \in N$, contradicting $x \in \text{PE}(\succ)$. Hence, $y \notin f_v(\succ)$, thus $y \notin f(\succ)$, implying $f(\succ) = \{x\}$. \square

Lemma 2: Take any $f \subseteq f_v$ which is Maskin monotonic, any $\succ \in \mathcal{L}_A^2$ and any $x \in \text{PE}(\succ)$. If $\#L(x, \succ_1) = v_1$ and $\#L(x, \succ_2) > v_2$ then $x \in f(\succ)$.

Proof. For each \succ , let $V_i(\succ) = \{x \in A : \#L(x, \succ_i) < v_i\}$ be the last v_i alternatives of $i \in N$ at \succ so that $\#V_i(\succ) = v_i$ for both $i = 1, 2$. Since $0 < v_1, v_2 < n$, it follows that $V_1(\succ), V_2(\succ) \neq \emptyset$ for any \succ . Similarly, we let for each \succ , the sets $S(\succ) = L(x, \succ_2) \setminus V_2(\succ)$, $T(\succ) = S(\succ) \setminus f(\succ)$ and $T'(\succ) = T(\succ) \cap V_1(\succ)$.

Consider any $\succ \in \mathcal{L}_A^2$ and any $x \in \text{PE}(\succ)$ with $\#L(x, \succ_1) = v_1$ and $\#L(x, \succ_2) > v_2$. Take any $f \subseteq f_v$ and suppose $x \notin f(\succ)$. Since $\#L(x, \succ_2) > v_2$ by assumption and $\#V_2(\succ) = v_2$, it follows that $S(\succ) \neq \emptyset$. Furthermore, $f(\succ) \subseteq S(\succ)$ by an argument similar to the one used in the proof of Lemma 1.

There are now two cases: either $T(\succ) \neq \emptyset$ or $T(\succ) = \emptyset$.

Assume first that $T(\succ) \neq \emptyset$.

Construct some $\succ' \in \mathcal{L}_A^2$ with $\succ'_1 = \succ_1$ and \succ'_2 such that $L(x, \succ'_2) = L(x, \succ_2) \setminus T'(\succ)$ and $V_2(\succ') = V_2(\succ)$. As $T'(\succ) \subseteq V_1(\succ) = V_1(\succ')$, $x \in \text{PE}(\succ')$ as well. Moreover, as $x \notin f(\succ)$, Maskin monotonicity implies that $x \notin f(\succ')$. Let $S(\succ') = L(x, \succ'_2) \setminus V_2(\succ')$. Note that $S(\succ') = f(\succ) \cup (T(\succ) \setminus T(\succ'))$ and, as a result, $S(\succ') \cap V_1(\succ') = \emptyset$.

It follows that: $f(\succ') \subseteq S(\succ')$.

Now we claim that $\#(V_1(\succ') \cap V_2(\succ')) \geq \#S(\succ')$. If the claim is true then we can

construct¹² $\succ'' \in \mathcal{L}_A^2$ with $\succ_1'' = \succ_1'$ and \succ_2'' differs from \succ_2' only by:

$$V_2(\succ'') = (V_2(\succ') \cup S(\succ')) \setminus X \text{ for some } X \subseteq V_1(\succ') \text{ with } \#X = \#S(\succ').$$

Let $S(\succ'') = L(x, \succ_2'') \setminus V_2(\succ'')$ which implies that $S(\succ'') = X \subseteq V_1(\succ')$. It follows that $f(\succ'') \subseteq S(\succ'') \subseteq V_1(\succ')$, contradicting $f \not\subseteq f_v$.

In order to complete the proof, we now show that the claim is true. To see this, first note that $\#(V_1(\succ') \cap V_2(\succ')) \geq \#S(\succ')$ is equivalent to $\#(V_2(\succ') \setminus V_1(\succ')) \leq v_2 - \#S(\succ')$.

Let η be a positive integer. Suppose by contradiction that:

$$\#(V_2(\succ') \setminus V_1(\succ')) = v_2 - \#S(\succ') + \eta.$$

Since $S(\succ') \cap V_1(\succ') = \emptyset$ (as proved) and $S(\succ') \cap V_2(\succ') = \emptyset$ (by construction), we have:

$$\#(V_2(\succ') \cup S(\succ')) \setminus V_1(\succ') = v_2 - \#S(\succ') + \eta + \#S(\succ').$$

Again by construction $x \notin V_1(\succ')$ and $x \notin V_2(\succ') \cup S(\succ')$, which implies that:

$$\#((V_2(\succ') \cup S(\succ') \cup \{x\}) \setminus V_1(\succ')) = v_2 - \#S(\succ') + \eta + \#S(\succ') + 1.$$

Finally, $U(x, \succ_2') \subseteq V_1(\succ')$ (by construction¹³) so that $U(x, \succ_2') \setminus V_1 = A \setminus V_1(\succ')$ which implies that:

$$\#(A \setminus V_1(\succ')) = v_2 - \#S(\succ') + \eta + \#S(\succ') + 1 = v_2 + \eta + 1.$$

However, by definition $\#(A \setminus V_1(\succ')) = n + 1 - v_1$ and the previous two equalities imply that

$$v_2 + \eta + 1 = n + 1 - v_1 \implies v_1 + v_2 = n - \eta,$$

¹²Note that $\#V_2(\succ'') = \#V_2(\succ') = v_2$ and therefore if we "add" $\#S(\succ')$ alternatives to $V_2(\succ')$ we need to be able to subtract the same number from $V_1(\succ') \cap V_2(\succ')$ and therefore $\#(V_1(\succ') \cap V_2(\succ')) \geq \#S(\succ')$.

¹³To see why, take some $y \in U(x, \succ_2')$. Since $y \neq x$, then either $y \in U(x, \succ_1')$ or $y \in L(x, \succ_1')$. Assume by contradiction that $y \in U(x, \succ_1')$, then $y \succ_i x$ for both $x = 1, 2$ and thus $x \notin \text{PE}(\succ)$, a contradiction. Hence $y \in L(x, \succ_1')$. However, $L(x, \succ_1') = V_1(\succ')$ since by definition, $\succ_1' = \succ_1$ and $\#L(x, \succ_1) = v_1$.

giving a contradiction in the case in which $T(>') \neq \emptyset$.

Finally, if $T(>') = \emptyset$, then $(S(>) \setminus f(>) \cap V_1(>)) = \emptyset$. However, by definition, $f \subseteq f_v$ and thus $f(>) \cap V_1(>) = \emptyset$. Therefore, $S(>) \cap V_1(>) = \emptyset$. The same argument as in the case with $T(>') \neq \emptyset$ applies replacing $>'$ by $>$. \square

Lemma 3: Take any $f \subseteq f_v$ which is Maskin monotonic, any $> \in \mathcal{L}_A^2$ and any $x \in \text{PE}(>)$. If $\#L(x, >_1) > v_1$ and $\#L(x, >_2) > v_2$ then $x \in f(>)$.

Proof. Take any $f \subseteq f_v$ which is Maskin monotonic, any $> \in \mathcal{L}_A^2$ and any $x \in \text{PE}(>)$ with $\#L(x, >_1) > v_1$ and $\#L(x, >_2) > v_2$. Suppose $x \notin f(>)$. Write $S_i(>) = L(x, >_i) \setminus V_i(>)$ for each $i \in N$. Note that $\#U(x, >_2) \leq n - v_2 - 1$, hence $\#(U(x, >_2) \cap S_1(>)) \leq n - v_2 - 1 < v_1$. Construct some $>' \in \mathcal{L}_A^2$ with $>'_2 = >_2$, $L(x, >'_1) = L(x, >_1)$ and $U(x, >_2) \cap S_1(>) \subseteq V_1(>)$. By construction of $>'$, we have $x \in \text{PE}(>')$ and $U(x, >_2) \cap S_1(>') = \emptyset$. Moreover, $x \notin f(>')$ by Maskin monotonicity of f . Now, construct some $>'' \in \mathcal{L}_A^2$ with $>''_2 = >'_2$ while $>''_1$ differs from $>'_1$ only by $L(x, >''_1) = L(x, >'_1) \setminus S_1(>')$. So $\#L(x, >''_1) = v_1$. As $U(x, >_2) \cap S_1(>') = \emptyset$, we have $x \in \text{PE}(>'')$, thus $x \in f(>'')$ by Lemma 2 which, by Maskin monotonicity, implies $x \in f(>')$, giving a contradiction that establishes Lemma 3. \square

References

- D. Abreu and A. Sen. Virtual implementation in Nash equilibrium. *Econometrica*, 59(4):997–1021, 1991.
- N. Anbarci. Finite alternating-move arbitration schemes and the equal area solution. *Theory and decision*, 61(1):21–50, 2006.
- J.-P. Benoît and E.A. Ok. Nash implementation without no-veto power. *Games and Economic Behavior*, 64(1):51–67, 2008.
- D. Bloom and C. Cavanagh. An analysis of the selection of arbitrators. *American Economic Review*, 76(3):408–22, 1986a.

- D. Bloom and C. Cavanagh. *An analysis of alternative mechanisms for selecting arbitrators*. Number 1224. Harvard Institute of Economic Research, Harvard University, 1986b.
- O. Bochet. Nash implementation with lottery mechanisms. *Social Choice and Welfare*, 28(1):111–125, 2007.
- O. Bochet and F. Maniquet. Virtual Nash implementation with admissible support. *Journal of mathematical economics*, 46(1):99–108, 2010.
- A. Bogomolnaia and H. Moulin. A new solution to the random assignment problem. *Journal of Economic theory*, 100(2):295–328, 2001.
- S.J. Brams and D.M. Kilgour. Fallback Bargaining. *Group Decision and Negotiation*, 10:287–316, 2001.
- J. Busetto and R. Colognato. Reconsidering Two-Agent Nash Implementation. *Social Choice and Welfare*, 32:171–179, 2009.
- G. De Clippel, K. Eliaz, and B. Knight. On the selection of arbitrators. *American Economic Review*, 104:3434–58, 2014.
- B. Dutta and A. Sen. A Necessary and sufficient condition for two-person Nash implementation. *Review of Economic Studies*, 58:121–128, 1991.
- B. Dutta and A. Sen. Nash Implementation with Partially Honest Individuals. *Games and Economic Behavior*, 74:154–169, 2012.
- S.D. Felsenthal and M. Machover. Sequential voting by veto: making the mueller-moulin algorithm more versatile. *Theory and Decision*, 33(3):223–240, 1992.
- P. Gärdenfors. Manipulation of social choice functions. *Journal of Economic Theory*, 13(2):217–228, 1976.
- L. Hurwicz and D. Schmeidler. Construction of outcome functions guaranteeing existence and Pareto optimality of Nash equilibria. *Econometrica*, 46:1447–1474, 1978.

- J.S Kelly. Strategy-proofness and social choice functions without singlevaluedness. *Econometrica: Journal of the Econometric Society*, pages 439–446, 1977.
- J.-F. Laslier, M. Núñez, and C. Pimienta. Reaching consensus through approval bargaining. *Games and Economic Behavior*, 104:241–251, 2017.
- E. Maskin. Nash equilibrium and welfare optimality. *Review of Economic Studies*, 66: 23–38, 1999.
- H. Matsushima. Role of honesty in full implementation. *Journal of Economic Theory*, 139:353–359, 2008.
- C. Mezzetti and L. Renou. Implementation in mixed nash equilibrium. *Journal of Economic Theory*, 147(6):2357–2375, 2012.
- J. Moore and R. Repullo. Nash implementation: A full characterization. *Econometrica*, 58:1083–1099, 1990.
- H. Moulin. Prudence versus sophistication in voting strategy. *Journal of Economic theory*, 24(3):398–412, 1981.
- D.C. Mueller. Voting by veto. *Journal of Public Economics*, 10(1):57–75, 1978.
- M. Núñez and J.-F. Laslier. Bargaining through approval. *Journal of Mathematical Economics*, 60:63–73, 2015.
- I. Özkal-Sanver and M.R. Sanver. Nash implementation via hyperfunctions. *Social Choice and Welfare*, 26(3):607–623, 2006.
- S. Özyurt and M. R. Sanver. A general impossibility result on strategy-proof social choice hyperfunctions. *Games and Economic Behavior*, 66(2):880–892, 2009.
- M. R. Sanver. Implementing Pareto optimal and individually rational outcomes by veto. *Group Decision and Negotiation*, 27(2):223–233, 2018.
- D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, pages 571–587, 1989.

- F. Yuval. Sophisticated voting under the sequential voting by veto. *Theory and Decision*, 53(4):343–369, 2002.
- F. Yuval and K. Herne. Sophisticated behavior under majoritarian and non-majoritarian voting procedures. *Political Behavior*, 27(3):217–237, 2005.