Dropping the Cass Trick and Extending Cass' Theorem to Asymmetric Information

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Dropping the Cass Trick and Extending Cass' Theorem to Asymmetric Information

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Abstract

In a celebrated 1984 paper, David Cass provided an existence theorem for financial equilibria in incomplete markets with exogenous yields. The theorem showed that, when agents had symmetric information and ordered preferences, equilibria existed on purely financial markets, supported by any collection of state prices. This theorem built on the so-called "Cass trick", along which one agent had an Arrow-Debreu budget set, with one single constraint, while the other agents were constrained a la Radner (1972), that is, in every state of nature. The current paper extends Cass' theorem to economies with asymmetric information and non-ordered preferences. It refines De Boisdeffre (2007), which characterized the existence of equilibria with asymmetric information by the no-arbitrage condition on purely financial markets. The paper defines no arbitrage prices with asymmetric information. It shows that any collection of state prices, in the agents' commonly expected states, supports an equilibrium. This result is proved without using the Cass trick, in the sense that budget sets are defined symmetrically across all agents. Thus, the paper suggests, in the symmetric information case, an alternative proof to Cass'.

Key words: sequential equilibrium, perfect foresight, existence of equilibrium, rational expectations, incomplete markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

When agents have incomplete or asymmetric information, they seek to infer information from observing markets. A traditional response to that problem is given by the REE (rational expectations equilibrium) model by assuming, quoting Radner (1979), that "*agents have a 'model' or 'expectations' of how equilibrium prices are determined*". Under this assumption, agents know the map between private information signals and equilibrium prices, along a so-called "*forecast function*".

In the simplest setting with two periods, no production and an uncertainty over future states to prevail, Cornet-De Boisdeffre (2002) suggests an alternative approach to the REE, where asymmetric information is represented by private signals, informing each agent that tomorrow's true state will be in a subset of the state space. The latter paper generalizes the classical definitions of equilibrium, no-arbitrage prices and no-arbitrage condition to asymmetric information. In this model, De Boisdeffre (2007) shows that equilibria exist on purely financial markets if they preclude arbitrage. That no-arbitrage condition, which typically holds under asymmetric information, may always be reached by agents observing asset prices or available financial transfers. Along Cornet-De Boisdeffre (2009), or De Boisdeffre (2016), that learning process requires no price model. Such results differ from Radner's (1979) inferences and the related generic existence of fully revealing REE.

In our setting, which drops rational expectations' inferences, the current paper provides new insights on the existence issue. It examines whether the so-called "*Cass trick*" is required, and Cass' theorem holds, under asymmetric information. The Cass trick (1984, 2006) is a device introduced in Radner's (1972) budget sets and equilibria, which consists in replacing the budget constraints of one agent by a single Arrow-Debreu constraint at the first period. The other agents' budget sets are left unchanged. This device enables to define asset prices relative to individual state prices, as the (positively) weighted sum of payoffs across states. It permits to show that any collection of state prices supports an equilibrium.

The current paper extends Cass' results to asymmetric information and non-ordered preferences. Under asymmetric information, it also refines De Boisdeffre (2007), which characterized the existence of one equilibrium by the no-arbitrage condition. Our proof applies, as its main argument, the Gale-Mas-Collel (1975, 1979) fixed-point-like theorem to reaction correspondences, which are formal representations of the market and agents' behaviours. The proof extends Cass' theorem but drops the Cass trick, in the sense that budget sets are defined symmetrically across all agents. Under symmetric information, it thus suggests an alternative proof to Cass'. The paper is organized as follows: Section 2 presents the model. Section 3 states and proves the existence theorem. An Appendix proves Lemmas.

2 The model

We consider a pure-exchange financial economy with two periods, $t \in \{0, 1\}$, and an uncertainty, at $t = 0$, upon which state of nature will randomly prevail at $t = 1$. The economy is finite in the sense that the sets, I, S, L and J, respectively, of consumers, states of nature, consumption goods and assets are all finite. The observed state at $t = 0$ is denoted by $s = 0$ and we let $\Sigma' := \{0\} \cup \Sigma$, whenever $\Sigma \subset S$.

2.1 Markets and information

Agents consume or exchange the consumption goods, $l \in L$, on both periods' spot markets. At $t = 0$, each agent, $i \in I$, receives privately the correct information that tomorrow's true state will be in a subset, S_i, of S. We assume costlessly that $S = \cup_{i \in I} S_i$. Thus, the pooled information set, $\underline{\mathbf{S}} := \cap_{i \in I} S_i$, contains the true state, and the relation $\underline{\mathbf{S}} = S$ characterizes symmetric information.

We let $P := \{p := (p_s) \in \mathbb{R}^{L \times \underline{\mathbf{S}}'} : \|p\| \leqslant 1\}$ be the set of admissible commodity prices, which each agent is assumed to observe, or anticipate perfectly, a la Radner (1972). Moreover, each agent with an incomplete information forms her private forecasts in the unrealizable states she expects. Such forecasts, (s, p_s^i), are pairs of a state, $s \in S_i \backslash \underline{\mathbf{S}}$, and a price, $p_s^i \in \mathbb{R}_{++}^L$, that the generic i^{th} agent believes to be the conditional spot price in state s. Thus, different agents are allowed to agree or disagree on different forecasts, which are never self-fulfilling. Non restrictively, such idiosyncratic forecasts are exogeneously given, along De Boisdeffre (2007).

Agents may operate financial transfers across states in S' (actually in $\underline{\mathbf{S}}'$) by exchanging, at $t = 0$, finitely many nominal assets, $j \in J$, which pay off, at $t = 1$, conditionally on the realization of the state. We assume that $\#J \leqslant \#\underline{\mathbf{S}}$, so that financial markets be typically incomplete. Assets' payoffs define a $S \times J$ matrix, V, whose generic row in state $s \in S$, denoted by $V(s) \in \mathbb{R}^J$, does not depend on prices. Thus, at asset price, $q \in \mathbb{R}^J$, agents may buy or sell unrestrictively portfolios of assets, $z = (z_j) \in \mathbb{R}^J$, for $q \cdot z$ units of account at $t = 0$, against the promise of delivery of a flow, $V(s) \cdot z$, of conditional payoffs across states, $s \in S$.

2.2 The consumer's behaviour and concept of equilibrium

Each agent, $i \in I$, receives an endowment, $e_i := (e_{is})$, granting the commodity bundles, $e_{i0} \in \mathbb{R}_+^L$ at $t = 0$, and $e_{is} \in \mathbb{R}_+^L$, in each expected state, $s \in S_i$, if it prevails. Given the market prices, $p := (p_s) \in P$ and $q \in \mathbb{R}^J$, and her forecasts, the generic i^{th} agent's consumption set is $X_i := \mathbb{R}_+^{L \times S_i'}$ and her budget set is defined as follows:

$$B_i(p,q) := \{\ (x,z) \in X_i \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z \ \ and \ \ p_s \cdot (x_s - e_{is}) \leqslant V(s) \cdot z,\ \forall s \in \underline{\mathbf{S}}$$
$$and \ \ p_s^i \cdot (x_s - e_{is}) \leqslant V(s) \cdot z,\ \forall s \in S_i \backslash \underline{\mathbf{S}}\ \}.$$

Each consumer, $i \in I$, is endowed with a complete preordering, $\precsim_i$, over her consumption set, representing her preferences. Her strict preferences, $\prec_i$, are represented, for each $x \in X_i$, by the set, $P_i(x) := \{\ y \in X_i : x \prec_i y\ \}$, of consumptions which are strictly preferred to x. In the above economy, denoted by $\mathcal{E} = \{(I,S,L,J), V, (S_i)_{i \in I}, (p_s^i)_{(i,s) \in I \times S_i \backslash \underline{\mathbf{S}}}, (e_i)_{i \in I}, (\prec_i)_{i \in I}\}$, agents optimise their consumptions in the budget sets. This yields the following concept of equilibrium:

Definition 1 *A collection of prices, $p = (p_s) \in P$, $q \in \mathbb{R}^J$, & decisions, $(x_i, z_i) \in B_i(p,q)$, for each $i \in I$, is an equilibrium of the economy, $\mathcal{E}$, if the following conditions hold:*
(a) $\forall i \in I,\ (x_i, z_i) \in B_i(p,q)$ and $P_i(x_i) \times \mathbb{R}^J \cap B_i(p,q) = \varnothing$;
(b) $\sum_{i \in I} (x_{is} - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}}'$;
(c) $\sum_{i \in I} z_i = 0$.

The economy, $\mathcal{E}$, is called standard if it meets the following conditions:

Assumption A1 *(monotonicity): $\forall (i,x,y) \in I \times (X_i)^2,\ (x \leqslant y,\ x \neq y) \Rightarrow (x \prec_i y)$;*
Assumption A2 *(strong survival): $\forall i \in I,\ e_i \in \mathbb{R}_{++}^{L \times S_i'}$;*
Assumption A3*: $\forall i \in I$, $\prec_i$ is lower semicontinuous convex-open-valued and such that $x \prec_i x + \lambda(y-x)$, whenever $(x,y,\lambda) \in X_i \times P_i(x) \times\ ]0,1]$;*
Assumption A4*: $\exists z \in \mathbb{R}^J,\ \forall s \in \underline{\mathbf{S}},\ V(s) \cdot z > 0$.*

2.3 A portfolio decomposition

This sub-Section introduces orthogonal sub-vector spaces of the portfolio set, $\mathbb{R}^J$, in relation to agents' information signals. It defines no-arbitrage prices, and relates them to supporting individual state prices.

For each $i \in I$, we let $Z_i := \sum_{s \in S_i} \mathbb{R}V(s) \subset \mathbb{R}^J$ be the span of the payoff matrix' rows in the range of the information set, S_i. We let $Z_i^o := \{z \in \mathbb{R}^J : V(s) \cdot z = 0, \forall s \in S_i\}$, its orthogonal complement, and $\underline{Z}^o := \{ z \in \mathbb{R}^J : V(s) \cdot z = 0, \forall s \in \underline{\mathbf{S}} \}$, be the spaces of useless portfolios in the eyes of, respectively, the i^{th} agent and a (possibly missing) fully informed agent.

We also define $Z^o := \sum_{i \in I} Z_i^o$ and $Z^{o\perp} := \cap_{i \in I} Z_i$, its orthogonal complement. Whereas the relation $Z^o \subset \underline{Z}^o$ holds for any information structure, (S_i), from the definition, the converse inclusion may fail, as shown on the heuristic example below. We therefore let $Z^* := \underline{Z}^o \cap Z^{o\perp} \subset \mathbb{R}^J$ be reduced to $\{0\}$, if and only if $Z^o = \underline{Z}^o$.

For all portfolio, $z \in \mathbb{R}^J$, we henceforth denote $z = z^1 + z^* + z^2$ the so-called (with slight abuse) "*orthogonal decomposition*" of z on $Z^o \oplus Z^* \oplus \underline{Z}^{o\perp}$. We use this notation throughout and may assume, from Assumption *A4*, that the last $J^2 > 0$ assets belong to $\underline{Z}^{o\perp}$, and all other assets (if any) belong to $\underline{Z}^o$. Thus, $V(s) = V(s)^2$ holds for every $s \in \underline{\mathbf{S}}$. We now define no-arbitrage prices and their supporting state prices.

Definition 2 *A no-arbitrage price is an asset price, $q \in \mathbb{R}^J$, which meets one of the following equivalent conditions, and we let $\mathcal{NA}$ be their set:*

(a) $\nexists (i, z) \in I \times \mathbb{R}^J$, $-q \cdot z \geqslant 0$ *and* $V(s) \cdot z \geqslant 0$, $\forall s \in S_i$, *with one strict inequality;*

(b) $\forall i \in I$, $\exists \lambda_i := (\lambda_{is}) \in \mathbb{R}^{S_i}_{++}$, $q = \sum_{s \in S_i} \lambda_{is} V(s)$.

Scalars, $(\lambda_{is}) \in \times_{i \in I} \mathbb{R}^{S_i}_{++}$, *which meet the above condition* (b), *are said to support the no-arbitrage price,* $q \in \mathcal{NA}$, *and called (individual) state prices. We denote* $\mathcal{NA}(\lambda) := \{q \in \mathcal{NA} : q^2 = \sum_{s \in \underline{\mathbf{S}}} \lambda_s V(s)\}$, *for every* $\lambda := (\lambda_s) \in \mathbb{R}^{\underline{\mathbf{S}}}_{++}$, *and* $\mathcal{NAC} = \cup_{\lambda \in \mathbb{R}^{\underline{\mathbf{S}}}_{++}} \mathcal{NA}(\lambda)$.

Proof The equivalence between the Assertions (i) and (ii) of Definition 2 is standard and proved in Cornet-De Boisdeffre (2002, Lemma 1, p. 398). □

We henceforth assume, at no cost from Cornet-De Boisdeffre (2009), that the information structure, (S_i), is arbitrage-free, i.e., admits a n-a price. Indeed, the latter paper shows that agents, starting from any information structure, (S_i), may always infer, with no price model, a refined information structure, which is arbitrage-free.

The following heuristic example and Claim 1 show that Z^* may not be reduced to $\{0\}$ and that the sets $\mathcal{NA}$ and $\mathcal{NAC}$ do not coincide, in general, but that any collection of symmetric state prices, $\lambda := (\lambda_s) \in \mathbb{R}^{\mathbf{S}}_{++}$, supports a no-arbitrage price, $q \in \mathcal{NA}(\lambda)$, and, from Theorem 1 below, supports an equilibrium.

Example Consider an economy, $\mathcal{E}$, with two agents, $i \in \{1,2\}$, three states, $s \in \{1,2,3\}$, an information stucture, $S_1 := \{1,2\}$ and $S_2 := \{2,3\}$, and one asset, whose price is $q = 1$. If the payoff matrix is $V = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, Assumption *A4* fails and the relations $Z^* = \mathbb{R}$ and $q \in \mathcal{NA} \cap {}^c\mathcal{NAC}$ hold. If $V = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, the relations $Z^* \subset \underline{Z}^o = \{0\}$, $q \in \mathcal{NA} \cap {}^c\mathcal{NAC}$ and $\mathcal{NAC} = -\mathbb{R}_{++}$ hold.

Claim 1 *The following Assertions hold:*

(i) $\forall \lambda \in \mathbb{R}^{\mathbf{S}}_{++}$, $\mathcal{NA}(\lambda) \neq \varnothing$*;*

(ii) $\mathcal{NA} \not\subseteq \mathcal{NAC}$*, in general.*

Proof Assertion (i) Let $\lambda := (\lambda_s) \in \mathbb{R}^{\mathbf{S}}_{++}$ be given. Non restrictively from Cornet-De Boisdeffre (2009), we had assumed that the payoff and information structure, $[V, (S_i)]$, was arbitrage-free (i.e., $\mathcal{NA} \neq \varnothing$). Hence, we let $\overline{q} \in \mathcal{NA}$ and $(\lambda_{is}) \in \times_{i \in I} \mathbb{R}^{S_i}_{++}$ be given, such that $\overline{q} = \sum_{s \in S_i} \lambda_{is} V(s)$, for each $i \in I$. Since $\overline{q}$ is a no-arbitrage price, the relation $\overline{q}^1 = 0$ holds (for $\overline{q} \cdot z = 0$ holds for every $z \in Z^o$, from the definitions).

Assertion (i) From Assumption $A4$, $J^2 := \dim \underline{Z}^{o\perp} > 0$. We refer to the notations of sub-Section 2.3 and let $V(\underline{\mathbf{S}}) := (V(s))_{s\in\underline{\mathbf{S}}}$ be the $\underline{\mathbf{S}} \times J$ extracted sub-matrix of V, defined by its rows, $V(s)$, in states $s \in \underline{\mathbf{S}}$. We show that $rank\ V(\underline{\mathbf{S}}) = J^2$ or, equivalently, that the relation $V(\underline{\mathbf{S}}) \cdot z \neq 0$ holds, whenever $z \in \underline{Z}^{o\perp}\backslash\{0\}$. Indeed, the joint relations $V(\underline{\mathbf{S}}) \cdot z = 0$ and $z \in \underline{Z}^{o\perp}$ imply, from the definitions: $z \in \underline{Z}^o \cap \underline{Z}^{o\perp} = \{0\}$. For each $i \in I$, we define $q_i \in \mathbb{R}^J$ by $q_i = \sum_{s\in S_i\backslash\underline{\mathbf{S}}} \lambda_{is} V(s)^2$, if $S_i \neq \underline{\mathbf{S}}$, and $q_i = 0$ otherwise.

Since $q_i \in \underline{Z}^{o\perp}$ and $rank\ V(\underline{\mathbf{S}}) = J^2$, there exists a vector $(\mu_{is}) \in \mathbb{R}^{\underline{\mathbf{S}}}$, which we set as given, such that $q_i = \sum_{s\in\underline{\mathbf{S}}} \mu_{is}\ V(s)$, for every $i \in I$. For $N \in \mathbb{N}$ large enough, the relations $|\frac{\mu_{is}}{N}| < \lambda_s$ hold, for every pair $(i, s) \in I \times \underline{\mathbf{S}}$. Then, we let $\gamma_i := (\gamma_{is}) \in \mathbb{R}^{S_i}_{++}$ be defined, for each $i \in I$, by $\gamma_{is} = \frac{\lambda_{is}}{N}$, for every $s \in S_i\backslash\underline{\mathbf{S}}$, and $\gamma_{is} = \lambda_s - \frac{\mu_{is}}{N}$, for every $s \in \underline{\mathbf{S}}$. By construction, the individual state prices, $\gamma_i := (\gamma_{is}) \in \mathbb{R}^{S_i}_{++}$, defined for each $i \in I$, support a no-arbitrage price, $q \in \mathcal{NA}(\lambda)$. □

Assertion (ii) was shown on the heuristic example above. □

3 The existence theorem and proof

Along Theorem 1, any symmetric state price collection supports an equilibrium:

Theorem 1 *Let $\lambda \in \mathbb{R}^{\underline{\mathbf{S}}}_{++}$ be given. A standard economy, $\mathcal{E}$, admits an equilibrium,* $(p, q, [(x_i, z_i]) \in P \times \mathcal{NA}(\lambda) \times (\times_{i\in I} B_i(p, q))$.

We henceforth set as given $\lambda := (\lambda_s) \in \mathbb{R}^{\underline{\mathbf{S}}}_{++}$ and let $\overline{q} = \sum_{s\in\underline{\mathbf{S}}} \lambda_s V(s)$.

The proof's main argument is the Gale-Mas-Colell (1975, 1979) fixed-point-like theorem. We apply the theorem to lower semi-continuous reaction correspondences, defined over a convex compact set, which formally represent agents' behaviours.

Thus, sub-Section 3.1 introduces an auxiliary compact economy, derived from the economy $\mathcal{E}$. Sub-Section 3.2 defines the reaction correspondences in that economy and applies the GMC theorem. A so-called (with slight abuse) "*fixed point*" obtains. Sub-Section 3.3 derives from this fixed point an equilibrium of the initial economy.

In particular, in the symmetric information case, the following proof, which defines budget sets symmetrically across agents, suggests an alternative to Cass'.

3.1 An auxiliary compact economy with modified budget sets

Using the notations of sub-Section 2.3, we let $Q := \{q \in Z^* : \|q\| \leqslant 1\}$

and define the following sets, for every $(i, p, q) \in I \times P \times Q$:

$$B_i^1(p,q) := \{\ (x,z) \in X_i{\times}Z_i : p_0 \cdot (x_0 - e_{i0}) + q \cdot z + \textstyle\sum_{s \in \underline{\mathbf{S}}} \lambda_s p_s{\cdot}(x_s - e_{is}) \leqslant 1,$$

$$p_s{\cdot}(x_s - e_{is}) \leqslant V(s){\cdot}z,\ \forall s \in \underline{\mathbf{S}},\ \ and\ \ p_s^i{\cdot}(x_s - e_{is}) \leqslant V(s){\cdot}z,\ \forall s \in S_i \backslash \underline{\mathbf{S}}\ \};$$

$$\mathcal{A}(p,q) := \{\ [(x_i, z_i)] \in \times_{i \in I}\ B_i(p, q + \overline{q}) :\ \textstyle\sum_{i \in I}(x_{is} - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}}',\ (\sum_{i \in I}\ z_i) \in Z^o\ \}.$$

The latter set meets the following boundary condition:

Lemma 1 $\exists r > 0 : \forall (p,q) \in P \times Q,\ \forall\ [(x_i, z_i)] \in \mathcal{A}(p,q),\ \sum_{i \in I}(\|x_i\| + \|z_i\|)\ < r$

Proof : See the Appendix. □

Along Lemma 1, for every $(i, p := (p_s), q) \in I \times P \times Q$, we let $X_i^* := \{x \in X_i : \|x\| \leqslant r\}$ and $Z_i^* := \{z \in Z_i : \|x\| \leqslant r\}$, and define the following convex compact sets:

$$B_i'(p,q) := \{\ (x,z) \in X_i^*{\times}Z_i^* : p_0 \cdot (x_0 - e_{i0}) + q \cdot z + \textstyle\sum_{s \in \underline{\mathbf{S}}} \lambda_s p_s{\cdot}(x_s - e_{is}) \leqslant \gamma_{(p,q)},$$

$$p_s{\cdot}(x_s - e_{is}) \leqslant V(s){\cdot}z,\ \forall s \in \underline{\mathbf{S}},\ \ and\ \ p_s^i{\cdot}(x_s - e_{is}) \leqslant V(s){\cdot}z,\ \forall s \in S_i \backslash \underline{\mathbf{S}}\ \},$$

where $\gamma_{(p,q)} := 1 - \min(1, \|p\| + \|q\|)$, so that $B_i'(p,q) \subset B_i^1(p,q)$.

The auxiliary economy is alike that of Section 2, up to the change in budget correspondences, from B_i to B_i', for each $i \in I$, which meet the following Claim 2, and in agents' programs, replaced by the next sub-Section's reaction correspondences.

Claim 2 *For every $i \in I$, B'_i is upper semicontinuous.*

Proof Let $i \in I$ be given. The correspondences B'_i is, as standard, upper semicontinuous, for having a closed graph in a compact set. $\square$

3.2 The fixed-point-like argument

Budget sets were modified in sub-section 3.1, so that their interiors be non-empty. This was required to prove the lower semi-continuity of the reaction correspondences of Lemma 2, below. For every $(p, q) \in P \times Q$, these interior budget sets are as follows:

$$B''_i(p,q) := \{\ (x,z) \in X^*_i \times Z^*_i : p_0 \cdot (x_0 - e_{i0}) + q \cdot z + \textstyle\sum_{s \in \underline{\mathbf{S}}} \lambda_s p_s \cdot (x_s - e_{is}) < \gamma_{(p,q)},$$
$$p_s \cdot (x_s - e_{is}) < V(s) \cdot z,\ \forall s \in \underline{\mathbf{S}},\quad and\ \ p^i_s \cdot (x_s - e_{is}) < V(s) \cdot z,\ \forall s \in S_i \backslash \underline{\mathbf{S}}\ \},\ \text{for every } i \in I.$$

Claim 3 *The following Assertions hold, for each $i \in I$:*

(i) $\forall (p,q) \in P \times Q$, $B''_i(p,q) \neq \varnothing$;

(ii) *the correspondence B''_i is lower semicontinuous.*

Proof Let $(p,q) \in P \times Q$ and $i \in I$ be given. Assertion (i) From Assumptions *A2-A4* and the definition, we may always choose $(x,z) \in B''_i(p,q)$. $\square$

Assertion (ii) The convexity of $B''_i(p,q)$ yields, from Assertion (i), $B'_i(p,q) = \overline{B''_i(p,q)}$. Then, B''_i is lower semicontinuous for having an open graph in a compact set. $\square$

We now introduce an agent representing markets $(i = 0)$ and a reaction correspondence, for each agent, on the convex compact set, $\Theta := P \times Q \times (\times_{i \in I} X^*_i \times Z^*_i)$, so as to apply the GMC theorem. Thus, we let, for each $i \in I$ and all $\theta := (p, q, [(x_i, z_i)]) \in \Theta$:

$$\Psi_0(\theta) := \{\ (p', q') \in P \times Q : (q' - q) \cdot \textstyle\sum_{i \in I} z_i + \sum_{s \in \underline{\mathbf{S}}'} [(p'_s - p_s) \cdot \sum_{i \in I} (x_{is} - e_{is})] > 0\ \};$$

$$\Psi_i(\theta) := \begin{cases} B'_i(p,q) & if \quad (x_i, z_i) \notin B'_i(p,q) \\ B''_i(p,q) \cap P_i(x_i) \times Z^*_i & if \quad (x_i, z_i) \in B'_i(p,q) \end{cases}$$

Lemma 2 *For each $i \in I \cup \{0\}$, Ψ_i is lower semicontinuous.*

Proof See the Appendix. $\square$

Claim 4 *There exists $\theta^* := (p^*, q^*, [(x_i^*, z_i^*)]) \in \Theta$, such that:*

(i) $\forall (p, q) \in P \times Q,\ (q^* - q)\cdot \sum_{i \in I} z_i^* + \sum_{s \in \underline{\mathbf{S}}'} [(p_s^* - p_s)\cdot \sum_{i \in I} (x_{is}^* - e_{is})] \geqslant 0;$

(ii) $\forall i \in I,\ (x_i^*, z_i^*) \in B_i'(p^*, q^*)$ *and* $B_i''(p^*, q^*) \cap P_i(x_i^*) \times Z_i^* = \varnothing$.

Proof Quoting Gale-Mas-Colell (1975, 1979): "*Given $X = \times_{i=1}^m Xi$, where X_i is a non-empty compact convex subset of $\mathbb{R}^n$, let $\varphi_i : X \to X_i$ be m convex (possibly empty) valued correspondences, which are lower semicontinuous. Then, there exists x in X such that for each i either $x_i \in \varphi_i(x)$ or $\varphi_i(x) = \varnothing$*". The correspondences Ψ_i, for each $i \in I \cup \{0\}$, meet all conditions of the above theorem and yield Claim 4. $\square$

3.3 An equilibrium of the economy $\mathcal{E}$

The above fixed point, θ^*, meets the following properties, proving Theorem 1:

Claim 5 *Given $\theta^* := (p^*, q^*,\ [(x_i^*, z_i^*)]\ ) \in \Theta$, along Claim 4, the following holds:*

(i) $\sum_{i \in I} z_i^{**} = 0$ *and* $\sum_{i \in I} (x_{is}^* - e_{is}) = 0$, $\forall s \in \underline{\mathbf{S}}'$;

(ii) for every $i \in I$, $(x_i^*, z_i^*) \in B_i'(p^*, q^*)$ *and* $B_i'(p^*, q^*) \cap P_i(x_i^*) \times Z_i^* = \varnothing$;

(iii) there exist $(z_i) \in \mathbb{R}^{J \times I}$ *and* $q \in \mathcal{NA}(\lambda)$, *such that* $(p^*, q,\ [(x_i^*, z_i)]\ )$ *is an equilibrium of the economy* $\mathcal{E}$ *and* $p^* \in \mathbb{R}_{++}^{L \times \underline{\mathbf{S}}'}$.

Proof Assertion (i) We show, first, that $\sum_{i \in I} z_i^{**} = 0$ (where, for each $i \in I$, z_i^{**} is the orthogonal projection of z_i^* on Z^*). Assume, by contraposition, that $\sum_{i \in I} z_i^{**} \neq 0$. Then, from Claim 4-$(i)$, the relations $q^* \cdot \sum_{i \in I} z_i^{**} = q^* \cdot \sum_{i \in I} z_i^* > 0$ and $\gamma_{(p^*, q^*)} = 0$ hold. Moreover, from Claim 4-(i), the relations $0 \leqslant \sum_{i \in I} p_s^* \cdot (x_{is}^* - e_{is})$ hold, for every $s \in \underline{\mathbf{S}}'$. From Claim 4-$(ii)$, the relations $p_0^* \cdot (x_{i0}^* - e_{i0}) + q^* \cdot z_i^* + \sum_{s \in \underline{\mathbf{S}}} \lambda_s\ p_s^* \cdot (x_{is}^* - e_{is}) \leqslant 0$ hold, for every $i \in I$. Summing them up (for $i \in I$) yields, from above:

$0 < p_0^* \cdot \sum_{i \in I} (x_{i0}^* - e_{i0}) + q^* \cdot \sum_{i \in I} z_i^* + \sum_{s \in \underline{\mathbf{S}}} \lambda_s \sum_{i \in I} p_s^* \cdot (x_{is}^* - e_{is}) \leqslant 0.$

This contradiction proves that $\sum_{i \in I} z_i^{**} = 0$.

Similarly, from Claim 4-(i), $p_s^* \cdot \sum_{i \in I} (x_{is}^* - e_{is}) \geqslant 0$ holds, for every $s \in \underline{\mathbf{S}}'$, and $p_0^* \cdot \sum_{i \in I} (x_{i0}^* - e_{i0}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_s \sum_{i \in I} p_s^* \cdot (x_{is}^* - e_{is}) > 0$ holds whenever $(\sum_{i \in I} (x_{is}^* - e_{is}))_{s \in \underline{\mathbf{S}}'} \neq 0$. Assume, by contraposition, that $\sum_{i \in I} (x_{is}^* - e_{is}) \neq 0$, for some $s \in \underline{\mathbf{S}}'$. Then, from Claim 4, the relation $\gamma_{(p^*, q^*)} = 0$ and the following budget constraints hold for each $i \in I$:

$p_0^* \cdot (x_{i0}^* - e_{i0}) + q^* \cdot z_i^* + \sum_{s \in \underline{\mathbf{S}}} \lambda_s p_s \cdot (x_{is} - e_{is}) \leqslant 0.$

Summing them up yields, from above:

$0 < p_0^* \cdot \sum_{i \in I} (x_{i0}^* - e_{i0}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_s \sum_{i \in I} p_s^* \cdot (x_{is}^* - e_{is}) \leqslant 0.$

This contradiction proves that $\sum_{i \in I} (x_{is}^* - e_{is}) = 0$, for every $s \in \underline{\mathbf{S}}'$. □

Assertion (ii). Let $i \in I$ be given. From Claim 4-(ii), we need only show: $B_i'(p^*, q^*) \cap P_i(x_i^*) \times Z_i^* = \varnothing$. Assume, by contraposition, there exists $(x_i, z_i) \in B_i'(p^*, q^*) \cap P_i(x_i^*) \times Z_i^*$. From Claim 3, there exists $(x_i', z_i') \in B_i''(p^*, q^*) \subset B_i'(p^*, q^*)$. By construction, the relations $(x_i^n, z_i^n) := [\frac{1}{n}(x_i', z_i') + (1 - \frac{1}{n})(x_i, z_i)] \in B_i''(p^*, q^*)$ hold, for every $n \in \mathbb{N}$. From Assumption *A3*, the relation $(x_i^N, z_i^N) \in P_i(x_i^*) \times Z_i^*$ also holds, for $N \in \mathbb{N}$ big enough, which implies: $(x_i^N, z_i^N) \in B_i''(p^*, q^*) \cap P_i(x_i^*) \times Z_i^*$. The latter contradicts Claim 4-(ii). □

Assertion (iii) The relation $p_s^* \in \mathbb{R}_{++}^{L \times \underline{\mathbf{S}}'}$ is standard from Assertions (i)-(ii) and Assumptions *A1-A2*. From Assertions (i)-(ii) and Assumption *A1*, agents' budget constraints hold with equality. Then, from Assertion (i), the relations $\sum_{i \in I} (x_{is}^* - e_{is}) = 0$, for $s \in \underline{\mathbf{S}}$, yield: $0 = \sum_{i \in I} p_s^* \cdot (x_{is}^* - e_{is}) = \sum_{i \in I} V(s) \cdot z_i^* = V(s) \cdot \sum_{i \in I} z_i^*$, for every $s \in \underline{\mathbf{S}}$.

Referring to the notations of sub-Section 2.3, the latter relations are written $(\sum_{i \in I} z_i^*) \in \underline{Z}^o$, whereas, from Assertion (i), $\sum_{i \in I} z_i^{**} = 0$. It follows that $\sum_{i \in I} z_i^* =$

$(\sum_{i\in I}\ z_i^{*1}) \in Z^o$. Let $q := q^* + \overline{q} := (q^* + \sum_{s\in \underline{\mathbf{S}}}\ \lambda_s V(s)) \in \mathcal{NA}(\lambda)$ be a given. From above, we set as given $(z_i^o) \in \times_{i\in I} Z_i^o$, such that $\sum_{i\in I}(z_i^* -\ z_i^o) = 0$, and denote $z_i = (z_i^* - z_i^o)$, for each $i \in I$. From the definitions, the following relations hold: $q \cdot z_i = q \cdot z_i^* = q^* \cdot z_i^{**} + \overline{q} \cdot z_i^*$ and $V(s) \cdot z_i = V(s) \cdot z_i^*$, for each $(i,s) \in I \times S_i$.

For each $i \in I$, the satiated budget constraints of $(x_i^*, z_i^*) \in B_i'(p^*, q^*)$ in states $s \in \underline{\mathbf{S}}$ imply: $\sum_{s\in \underline{\mathbf{S}}}\ \lambda_s\ p_s^* \cdot (x_{is}^* - e_{is}) = \overline{q} \cdot z_i^*$. At the first period, the same constraints are written, for each $i \in I$ and from above: $p_0^* \cdot (x_{i0}^* - e_{i0}) + q^* \cdot z_i^* + \sum_{s\in \underline{\mathbf{S}}}\ \lambda_s\ p_s^* \cdot (x_{is}^* - e_{is}) = p_0^* \cdot (x_{i0}^* - e_{i0}) + q \cdot z_i = \gamma_{(p^*,q^*)}$. Summing up the latter relations (for $i \in I$) yields, from Assertion (i): $0 = \sum_{i\in I}\ p_0^* \cdot (x_{i0}^* - e_{i0}) + q^* \cdot \sum_{i\in I} z_i^* + \sum_{s\in \underline{\mathbf{S}}}\ \lambda_s\ \sum_{i\in I}\ p_s^* \cdot (x_{is}^* - e_{is}) = \#I.\gamma_{(p^*,q^*)}$.

Then, all above relations, Assertion (ii) and the definitions of $q \in \mathcal{NA}(\lambda)$ and of buget sets (namely, $B_i'(p^*, q^*)$ and $B_i(p^*, q)$) imply: $(x_i^*, z_i) \in B_i(p^*, q)$, for each $i \in I$, and $[(x_i^*, z_i^*)] \in \mathcal{A}(p^*, q^*)$. The latter implies, from Lemma 1: $\sum_{i\in I}(\|x_i^*\| + \|z_i^*\|)\ < r$.

We now let $i \in I$ be given and show that (x_i^*, z_i) is optimal in $B_i(p^*, q)$. If not, there exists $(x_i, z_i') \in B_i(p^*, q) \cap P_i(x_i^*) \times \mathbb{R}^J$. From the definitions of q, Z_i^o, Z_i^*, from Assumption *A3* and above, we may assume that $(x_i, z_i') \in B_i'(p^*, q^*) \cap P_i(x_i^*) \times Z_i^*$, which contradicts Assertion (ii). Thus, we have shown that Conditions (a)-(b)-(c) of Definition 1 hold for the collection, $\mathcal{C} := (p^*, q, [(x_i^*, z_i)]) \in P \times \mathcal{NA}(\lambda) \times (\times_{i\in I} B_i(p^*, q))$. □

Appendix

Let $\mathcal{A}(p,q) := \{\ [(x_i, z_i)] \in \times_{i\in I}\ B_i(p, q + \overline{q}) :\ \sum_{i\in I}(x_{is} - e_{is}) = 0,\ \forall s \in \underline{\mathbf{S}}',\ (\sum_{i\in I}\ z_i) \in Z^o\ \}$, for every $(p,q) \in P \times Q$. These sets are bounded as follows:

Lemma 1 $\exists r > 0 : \forall (p,q) \in P \times Q,\ \forall\ [(x_i, z_i)] \in \mathcal{A}(p,q),\ \sum_{i\in I}(\|x_i\| + \|z_i\|)\ < r$

Proof Let $\delta = \sum_{i\in I} \|e_i\|$, $(p,q) \in P \times Q$ and $[(x_i, z_i)] \in \mathcal{A}(p,q)$ be given. The relations $x_{is} \in [0, \delta]^L$ hold, for every pair $(i,s) \in I \times \underline{\mathbf{S}}'$, from the market clearance conditions of $\mathcal{A}(p,q)$. From the fact that there exists $\alpha > 0$, such that $p_s^i \in [\alpha, +\infty[^L$, for every forecast, $(i,s) \in I \times S_i \backslash \underline{\mathbf{S}}$, it suffices to prove the following Assertion:

$\exists r' > 0,\ \forall (p,q) \in P \times Q,\ \forall\ [(x_i, z_i)] \in \mathcal{A}(p,q),\ \sum_{i \in I}\ \|z_i\|\ < r'$.

Assume, by contraposition, that, for every $k \in \mathbb{N}$, there exist $(p^k, q^k) \in P \times Q$ and $[(x_i^k, z_i^k)] \in \mathcal{A}(p^k, q^k)$, such that $\|z^k\| := \sum_{i \in I}\ \|z_i^k\|\ := \alpha_k > k$. For every $k \in \mathbb{N}$, let $z'^k := z^k/\alpha_k := (z_i^k/\alpha_k)$. The bounded sequence, $\{z'^k\}$, may be assumed to converge in a closed set, say to $z := (z_i) \in \times_{i \in I} Z_i$, such that $\|z\| = 1$. The relations $[(x_i^k, z_i^k)] \in \mathcal{A}(p^k, q^k)$ hold, for every $k \in \mathbb{N}$, and imply, for every $(i,s,k) \in I \times S_i \times \mathbb{N}$:

$V(s) \cdot z_i^k \geqslant -\delta$, hence, $V(s) \cdot z_i'^k \geqslant -\delta/k$ and, in the limit, $V(s) \cdot z_i \geqslant 0$, for each $s \in S_i$;

$\sum_{i \in I}\ z_i'^k \in Z^o$, hence, $\sum_{i \in I}\ z_i \in Z^o$.

Let $\sum_{i \in I}\ z_i = \sum_{i \in I}\ z_i^o$ for some $(z_i^o) \in \times_{i \in I} Z_i^o$, and $(z_i^*) := (z_i) - (z_i^o)$ be given. The following relations hold from above: $V(s) \cdot z_i^* \geqslant 0$, for all $(i,s) \in I \times S_i$ and $\sum_{i \in I} z_i^* = 0$.

The latter relations imply, from Cornet-De Boisdeffre (2002, p. 401): $(z_i^*) \in \times_{i \in I} Z_i^o$, that is, $z := (z_i) = 0$. This contradicts the relation $\|z\| = 1$ and proves Lemma 1. □

Lemma 2 *For each $i \in I \cup \{0\}$, Ψ_i is lower semicontinuous.*

Proof The correspondences Ψ_0 is lower semicontinuous for having an open graph.

We now set $i \in I$ and $\theta := (p, q, [(x_i, z_i)]) \in \Theta$ as given.

- Assume that $(x_i, z_i) \notin B_i'(p,q)$. Then, $\Psi_i(\theta) = B_i'(p,q)$.

Let V be an open set in $X_i^* \times Z_i^*$, such that $V \cap B_i'(p,q) \neq \varnothing$. It follows from the convexity of $B_i'(p,q)$ and the non-emptyness of the open set $B_i''(p,q)$ that $V \cap B_i''(p,q) \neq$

$\varnothing$. From Claim 3, there exists a neighborhood U of (p,q), such that $V \cap B_i'(p',q') \supset V \cap B_i''(p',q') \neq \varnothing$, for every $(p',q') \in U$.

Since $B_i'(p,q)$ is nonempty, closed, convex in the compact set $X_i^* \times Z_i^*$, there exist open sets V_1 and V_2 in $X_i^* \times Z_i^*$, such that $(x_i, z_i) \in V_1$, $B_i'(p,q) \subset V_2$ and $V_1 \cap V_2 = \varnothing$. From Claim 2, there exists a neighborhood $U_1 \subset U$ of (p,q), such that $B_i'(p',q') \subset V_2$, for every $(p',q') \in U_1$. Let $W = U_1 \times (\times_{j\in I} W_j)$, where $W_i := V_1$, and $W_j := X_j^* \times Z_j^*$, for each $j \in I \backslash \{i\}$. Then, W is a neighborhood of θ, such that $\Psi_i(\theta') = B_i'(p',q')$, and $V \cap \Psi_i(\theta') \neq \varnothing$, for every $\theta' \in W$. Thus, Ψ_i is lower semicontinuous at θ. $\square$

- Assume that $(x_i, z_i) \in B_i'(p,q)$, i.e., $\Psi_i(\theta) = B_i''(p,q) \cap P_i(x) \times Z_i^*$.

Lower semicontinuity results from the definition if $\Psi_i(\theta) = \varnothing$. Assume $\Psi_i(\theta) \neq \varnothing$. We recall that P_i (from Assumption *A3*) is lower semicontinuous with open values and that B_i'' has an open graph. As a corollary, the correspondence $(p', q'[(x_i', z_i')]) \in \Theta \mapsto B_i''(p',q') \cap P_i(x_i') \times Z_i^* \subset B_i'(p',q')$ is lower semicontinuous at θ. Then, from the latter inclusions, Ψ_i is lower semicontinuous at θ. $\square$

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