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Discriminating between GARCH models for option pricing by their ability to compute accurate VIX measures

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Abstract

In this paper, we discuss the pricing performances of a large collection of GARCH models by questioning the global synergy between the choice of the affine/non-affine GARCH specification, the use of competing alternatives to the Gaussian distribution, the selection of an appropriate pricing kernel and the choice of different estimation strategies based on several sets of financial information. Furthermore, the study answers an important question in relation to the correlation between the performance of a pricing scheme and its ability to forecast VIX dynamics. VIX analysis clearly appears as a parsimonious first-stage filter to discard the worst GARCH option pricing models.

JEL classification: C52, G13.

Keywords: GARCH option pricing models, GARCH implied VIX, estimation strategies, non-monotonic stochastic discount factors.

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Introduction

An increasing body of literature on time series analysis has been developed for the challenges of modeling volatility using the GARCH framework. The literature goes back to Engle (1982) with the class of autoregressive conditional heteroscedastic (ARCH) models. The latter represented the first theoretical attempt in order to model volatility as an endogenous time-varying process which incorporated conditional variance clustering to bring the model closer to reality. Nevertheless, except for when the maximum lag of the ARCH model is large, this approach fails to reproduce the decay rate of the unconditional autocorrelation function of squared log-returns observed in financial time series. Bollerslev (1986) proposed the parsimonious Generalized ARCH (GARCH) model¹ allowing for a much more flexible lag structure through a moving average component. Over time, original parametric specifications of the conditional variance have been generalized and extended in various directions to increase the practical flexibility of the model and incorporate in particular asymmetric effects and non-Gaussian innovations (see for example Chorro *et al.* (2015) Chapter 2).

Concerning the pricing of derivatives, Duan (1995) was the first paper to propose a coherent theoretical framework, namely the locally risk-neutral valuation relationship (LRNVR), to obtain risk-neutral dynamics of Gaussian GARCH models. This methodology was popularized by Heston and Nandi in Heston & Nandi (2000) where a discrete time affine GARCH-type model with Gaussian innovations was able to replicate one of the key features observed in continuous time literature (Heston (1993)): the fact that the no-arbitrage price of classical European options had semi-closed-form expression.²

In order to improve the numerical performances of Duan's option pricing model, three complementary areas may be explored:

1. Use more realistic GARCH processes coping with asymmetric volatility responses and non-Gaussian conditional distributions,
2. Use different risk-neutralization processes compatible with the preceding point,
3. Use, when it is possible, more information than just that of the log-returns to estimate the model.

The first point is now a classic topic and many extensions have been proposed to cope with these well-documented stylized facts. The asymmetric effects of positive and negative shocks of equal magnitude on conditional volatility, the so-called leverage effect, may be captured using a large family of extended GARCH models the most popular being probably the exponential EGARCH of Nelson (1991), the NGARCH model of Engle & Ng (1993), the GJR-GARCH of Glosten *et al.* (1993), the threshold GARCH

of Zakoian (1994), and the affine HN-GARCH by Heston & Nandi (2000). However, the leverage parameter of preceding specifications is not sufficient to capture all the skewness and kurtosis levels in standardized residuals. Therefore, Gaussian hypothesis for the conditional distribution of log-returns has to be relaxed and a myriad of possible choices may be used to take into account all the mass in the tails and the asymmetry (Chorro *et al.* (2015) Chapter 2). Among them, the Generalized Hyperbolic (Chorro *et al.* (2012), Badescu *et al.* (2011)) family and its Normal Inverse Gaussian (NIG) subclass (Stentoft (2008), Badescu *et al.* (2015)), the Inverse Gaussian (IG) distribution (Christoffersen *et al.* (2006)), or the mixture of Gaussian (Badescu *et al.* (2008)) clearly improve forecasting performances of related GARCH models.

Once a competing model has been chosen, the choice of the so-called stochastic discount factor (SDF) to obtain risk-neutral dynamics is fundamental. For this second point, two constraining factors apply: this SDF has to be sufficiently flexible to provide explicit risk-neutral dynamics for a large variety of GARCH structures and innovation distributions and rich enough to produce good pricing performances. Since the seminal paper of Duan, several tools have been developed to select an equivalent martingale measure (see for example Chorro *et al.* (2015) Chapter 3).³

One of the main advantages of GARCH models, with respect for example to stochastic volatility ones, is that they may be efficiently estimated using a conditional version of the maximum likelihood estimation and a dataset of log-returns. In particular, since, in the case of exponential-affine or extended Girsanov principle SDF, the associated risk-neutral dynamics are explicit transforms of the historical ones, only log-returns information is needed to compute or approximate European option prices.⁴ Even so, when an extra piece of financial information (price of plain vanilla options, the VIX index for the S&P500,...) is available it can be of interest to integrate it, in an efficient way, to the estimation process to reduce pricing errors. Therefore, following Christoffersen *et al.* (2012) it is now classically possible to build for some affine GARCH models (at the very least for the HN-GARCH Heston & Nandi (2000) and the IG-GARCH Christoffersen *et al.* (2006) where semi-closed form expressions for option prices are obtained) a joint maximum likelihood based on log-returns and option prices. In this setting, the affine structure of the model is mandatory: if prices are evaluated using Monte-Carlo methods, computing the likelihood function may be cumbersome. In a recent study, Hao & Zhang (2013) have computed VIX index formulas implied by various non-affine asymmetric Gaussian GARCH models. They presented closed-form formulas for the VIX index associated with five classical non-affine Gaussian GARCH models when Duan (1995) LRNVR is used. Based on this result, Kanninen *et al.* (2014) proposed a fair comparison between affine and non-affine Gaussian GARCH specifications using log-returns and VIX information

in the estimation.⁵ For two affine GARCH models Chorro & Fanirisoa (2016) and Papantonis (2016) proved that incorporating both the physical return dynamics of the index and risk-neutral dynamics of the VIX to estimate the parameters of GARCH option pricing models provides competitive pricing errors at a very low computational cost.⁶

This paper attempts to fill several gaps in the GARCH option pricing literature, in particular, from an empirical point of view.

Firstly, in the spirit of Christoffersen *et al.* (2004) the aim of our study is to provide an intensive comparison analysis of empirical performances, in VIX index or options valuation, between different GARCH-type models using Gaussian or non-Gaussian distributions under different classes of risk-neutral measures. Furthermore, particular attention is granted on the choice of the information set (VIX, options, returns) in the estimation process. To keep the empirical analysis manageable, we only focus our attention on four classical parsimonious GARCH(1,1) structures: HN-GARCH by Heston & Nandi (2000), GJR-GARCH by Glosten *et al.* (1993), NGARCH by Engle & Ng (1993), and IG-GARCH by Christoffersen *et al.* (2013).⁷ One advantage of this choice is to question the difference between affine and non-affine models. As a natural non-Gaussian alternative we favor the so-called NIG distribution not only because it is known to fit statistical properties of asset returns remarkably but also because, combined with Esscher and EGP SDF, pricing equations may be solved explicitly.⁸ Furthermore, monotonic and non-monotonic pricing kernels (Monfort & Pégoraro (2012) and Chorro & Fanirisoa (2016)) are considered for Gaussian and IG distributions. To our knowledge, in the existing literature, empirical studies questioned, in general, the impact of the distribution (Christoffersen *et al.* (2006), Chorro *et al.* (2012)), the choice of the SDF (Badescu *et al.* (2011), Christoffersen *et al.* (2013), Chorro & Fanirisoa (2016)) or the estimation strategy (Hao & Zhang (2013), Kannianen *et al.* (2014), Papantonis (2016), Lalancette & Simonato (2017)) on pricing performances, but few of them consider all these factors at the same time.⁹ Our study is a means of making a contribution to understand the combined impact of these complementary aspects (24 combinations of GARCH-distribution-SDF-estimation are tested).

Secondly, inspired by the work of Hao & Zhang (2013) that explained poor pricing performances of Gaussian GARCH models by their inefficiency to capture the variance risk premium, we also explore in this paper if it is possible to partly classify GARCH option pricing models by their ability to simply reproduce the VIX index. In fact, up to our knowledge, the correlation between the option pricing performances of a model and its ability to compute accurate VIX measures is not clearly established in the literature. A challenging aspect of the present study is to make VIX analysis a first-stage filter to

discard the worst GARCH option pricing models. From purely numerical aspects, such a conclusion would be very interesting to back-test these models in an efficient way, using only VIX information, when available, instead of complex option datasets.

This paper is structured along the following lines. In section 1 we first provide a partial presentation of all competing GARCH frameworks used in the empirical part. More precisely, we consider four GARCH structures for modeling volatility as a time-varying process: HN-GARCH, GJR, NGARCH, and IG-GARCH. Then, in section 2, we recap the main risk-neutralized frameworks adopted in this study. Next, in section 3, we derive the related VIX index formulas. Section 4 deals with the estimation challenge, presenting methodologies based on different information sets and the related numerical results in terms of VIX approximation and option pricing. We conclude in section 5.

1. Competing GARCH models

We consider a financial asset with a market price at time t given by S_t and we denote by $Y_t = \log \left(\frac{S_t}{S_{t-1}} \right)$ the associated log-returns defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where \mathbb{P} represents the historical probability measure. Information filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is generated by log-returns supposing that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. In what follows, we consider a general dynamics for the stock price process:

$$\begin{aligned} Y_t &= r + m_t + \sqrt{h_t} z_t \\ h_t &= F(z_{t-1}, h_{t-1}, \theta^V) \end{aligned} \tag{1}$$

where the z_t are i.i.d centered and reduced random variables depending on a vector of parameters θ^D , m_t is the predictable time-varying excess of returns, r is the risk-free rate and F is a mapping, compatible with realistic GARCH(1,1) volatility models that depends on a vector of parameters θ^V .¹⁰ For our empirical horse-race we favor four particular GARCH specifications often used in the literature to cope with volatility clustering and leverage effect. Moreover, these four GARCH-type models belong to two important families: affine and non-affine frameworks. While affine GARCH models are often used because they yield a semi-closed form solution for prices of European equity options, it is now well-documented that non-affine ones provide a better fit to financial data. One important aspect of our empirical study will be to question once again this duality. Following Kanninen *et al.* (2014) we choose the widely recognized NGARCH Engle & Ng (1993), GJR-GARCH Glosten *et al.* (1993), and affine HN-GARCH Heston & Nandi (2000) models and we add the IG-GARCH of Christoffersen *et al.* (2006) (see also Chorro & Fanirisoa (2016)) that is a notable example of an affine model within a non-Gaussian setting. In the next sections we briefly recall the definitions and the main

properties of these specifications.

1.1. Affine competitors

Since the seminal work of Heston (1993), affine models, that led to semi-closed form expressions for option prices, are the keystone of almost all numerical studies. In the discrete time literature, the HN-GARCH Heston & Nandi (2000) and the IG-GARCH of Christoffersen *et al.* (2006) are two important contributions. More precisely, for the HN-GARCH model the historical dynamics is given by

$$\begin{cases} Y_t &= r + \lambda_0 h_t + \sqrt{h_t} z_t \\ h_t &= a_0 + a_1 \left(z_{t-1} - \gamma \sqrt{h_{t-1}} \right)^2 + b_1 h_{t-1} \end{cases} \quad (2)$$

with $a_0 > 0$, $a_1 \geq 0$, $b_1 \geq 0$ and for the IG-GARCH specification by

$$\begin{cases} Y_t &= r + \nu h_t + \eta z_t \\ h_t &= w + b h_{t-1} + c z_{t-1} + a \frac{h_{t-1}^2}{z_t} \end{cases} \quad (3)$$

with $w > 0$, $b \geq 0$, $c \geq 0$, and $a \geq 0$.

In the HN-GARCH model the z_t are supposed to be Gaussian while in the IG-GARCH they follow an Inverse Gaussian distribution with degree of freedom $\delta_t = \frac{h_t}{\eta^2}$.¹¹ The persistence (that will be an important quantity to express associated VIX index formula) of the HN-GARCH (resp. IG-GARCH) is given by $\Psi = b_1 + a_1 \gamma^2$ (resp. $\Psi = b + \frac{c}{\eta^2} + a \eta^2$). Under these two hypotheses on the distributions of innovations, it is easy to prove for both models that the conditional moment generating function $\mathbb{G}_{\log(S_T)|\mathcal{F}_t}^{\mathbb{P}}(u) = E_{\mathbb{P}}[S_T^u | \mathcal{F}_t]$ of the log asset price under the physical measure can be written in the following log-linear form $\mathbb{G}_{\log(S_T)|\mathcal{F}_t}^{\mathbb{P}}(u) = S_t^u e^{A_t + B_t h_{t+1}}$ where the coefficients¹² A_t and B_t can be obtained by working backward from the maturity date of the option and using terminal conditions $A_T = B_T = 0$. Moreover, one important empirical consequence for the pricing of European call options is that the very particular form of the conditional moment generating function of $\log(S_T)$ leads to the existence of semi-closed form expressions for prices which allow us to use Fast Fourier Transform (FFT) methodology and option information in the estimation procedure as explained in Chorro *et al.* (2015) Chap 4.

1.2. Non-affine competitors

In order to propose asymmetric extensions of the original GARCH(1,1) model, one possibility is to modify the so-called news impact curve (NIC) introduced in Engle & Ng (1993). For this purpose, we may shift a symmetric NIC to the right or consider curves centered at 0 allowing for slopes of different magnitudes on either side of the origin.

These two approaches were used by Engle & Ng (1993) and Glosten *et al.* (1993) in order to introduce respectively the popular NGARCH and GJR models. In both cases, a single leverage parameter constrains the response of the conditional variance to depend on the sign of a shock. In the NGARCH model the dynamics¹³ of the risky asset under historical probability is given by

$$\begin{cases} Y_t &= r + \lambda_0 \sqrt{h_t} - \log(E_{\mathbb{P}}[e^{\sqrt{h_t} z_t}]) + \sqrt{h_t} z_t \\ h_t &= a_0 + b_1 h_{t-1} + a_1 h_{t-1} (z_{t-1} - \gamma)^2 \end{cases} \quad (4)$$

with $a_0 > 0$, $b_1 \geq 0$, $a_1 \geq 0$ and for the GJR model by

$$\begin{cases} Y_t &= r + \lambda_0 \sqrt{h_t} - \frac{h_t}{2} + \sqrt{h_t} z_t \\ h_t &= a_0 + h_{t-1} [b_1 + a_1 (z_{t-1})^2 + \gamma \max(0, -(z_{t-1}))^2] \end{cases} \quad (5)$$

with $a_0 > 0$, $b_1 \geq 0$, $a_1 \geq 0$, and $\gamma \geq 0$. The persistence of the NGARCH (resp. GJR) is given by $\Psi = b_1 + a_1 (1 + \gamma^2)$ (resp. $\psi = b_1 + a_1 + \frac{\gamma}{2}$). Contrary to models presented in the preceding section, here, conditional moment generating function is not an exponential-affine function of the one step ahead volatility. To compute option prices we use in general Monte Carlo approximations. Nevertheless, as remarked in Hao & Zhang (2013) VIX implied formulas are available in this non-affine setting at the very least for Gaussian innovations (and other very particular cases as Badescu *et al.* (2019)).

1.3. A flexible alternative to Gaussian distribution

It is now a well-known fact that forecasting performances of GARCH-type models are improved when using non-Gaussian innovations. Historically, several interesting distributions were proposed to better account for the deviation from normality. In the present section we have decided to mainly focus our attention on the Normal Inverse Gaussian (NIG) distribution. This four-parameter family of distributions has been extensively used during the last decade in discrete time literature, especially for pricing issues (Stentoft (2008), Badescu *et al.* (2011), Guégan *et al.* (2013), Badescu *et al.* (2019)): for $(\alpha, \beta, \delta, \mu)$ fulfilling $0 < |\beta| < \alpha$ and $\delta > 0$, the density of the NIG $(\alpha, \beta, \delta, \mu)$ is given by¹⁴

$$d_{NIG}(z, \alpha, \beta, \delta, \mu) = \frac{\alpha}{\pi} e^{\delta \left(\sqrt{\alpha^2 - \beta^2} + \beta \left(\frac{z - \mu}{\delta} \right) \right)} \frac{K_1 \left(\alpha \delta \sqrt{1 + \left(\frac{z - \mu}{\delta} \right)^2} \right)}{\sqrt{1 + \left(\frac{z - \mu}{\delta} \right)^2}}$$

where K_1 is the modified Bessel function of the third kind with index one. The mean and the variance of this distribution are respectively given by

$$m = \mu + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}, \quad \sigma^2 = \frac{\delta\alpha^2}{\left(\sqrt{\alpha^2 - \beta^2}\right)^3}. \quad (6)$$

Therefore, from the stability of the NIG family under affine transforms, it is possible to obtain a centered version with unit variance considering

$$\text{NIG}(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\mu}) = \frac{\text{NIG}(\alpha, \beta, \delta, \mu) - m}{\sigma} \quad (7)$$

where $\tilde{\alpha} = \sigma\alpha$, $\tilde{\beta} = \sigma\beta$, $\tilde{\delta} = \frac{\delta}{\sigma}$ and $\tilde{\mu} = \frac{-m}{\sigma} + \frac{\mu}{\sigma}$.

2. Stochastic discount factors and risk-neutral dynamics

From the beginning of the 80's (see Chorro *et al.* (2015) Chap 3 and references therein), general methods providing arbitrage-free price processes via the notion of equivalent martingale measure (EMM) have been investigated both in discrete or continuous time frameworks. Furthermore, the choice of such an EMM is known to be equivalent to the specification of the so-called one-period stochastic discount factor (SDF). Since markets described by GARCH models are incomplete, there is a priori an infinite number of SDF available for pricing derivatives and a great challenge is to select tractable candidates for their strong economic foundations and/or empirical performances. In this section, we present the main paths to risk-neutralization that will be implemented in the numerical part to obtain arbitrage-free price approximations in Gaussian or non-Gaussian settings. More specifically, starting from the Duan (1995) approach particularly well-adapted to Gaussian residuals, we briefly recall the main lines of the recent advances in modeling SDF dynamics to cope with non-Gaussian innovations (Elliott & Madan (1998) extended Girsanov principle (EGP) and Siu *et al.* (2004) conditional Esscher transform) and/or have better representations of volatility risk (Monfort & Pégoraro (2012), Chorro & Fanirisoa (2016)).¹⁵

As in the preceding section, we consider a GARCH-type specification for the log-returns

$$\begin{aligned} Y_t &= r + m_t + \sqrt{h_t} z_t \\ h_t &= F(z_{t-1}, h_{t-1}, \theta^V) \end{aligned} \quad (8)$$

where the z_t are i.i.d centered random variables with unit variance.

Supposing that the z_t are i.i.d $\mathcal{N}(0, 1)$, Duan (1995) was the first to provide a coherent

theoretical CCAPM framework to obtain risk-neutral dynamics in a GARCH environment independently of the underlying GARCH structure. More precisely, if \mathbb{Q} is an EMM fulfilling LRNVR¹⁶ then

$$\begin{aligned} Y_t &= r - \frac{h_t}{2} + \sqrt{h_t} z_t^* \\ h_t &= F \left(z_{t-1}^* - \frac{m_{t-1}}{\sqrt{h_{t-1}}} - \frac{\sqrt{h_{t-1}}}{2}, h_{t-1}, \theta^V \right) \end{aligned} \quad (9)$$

where the z_t^* are i.i.d $\mathcal{N}(0, 1)$ under \mathbb{Q} . For Gaussian models presented in the preceding section, risk-neutral dynamics deduced from the Duan's argument are given in Table 3. In the non-affine GJR and NGARCH setting, prices may be obtained from (9) using Monte Carlo approximations while in the affine HN case semi-closed form formulas are available. Nevertheless, Duan's framework relies on Gaussian hypotheses and cannot be adapted with simplicity to more general distributions. Based on this observation, Elliott & Madan (1998) proposed a very simple way to select a SDF based on a Girsanov-type transformation that preserves returns distribution after the change of measure by only shifting the conditional mean to fulfill the martingale restriction:¹⁷ under the EMM \mathbb{Q}^{EGP} we have

$$\begin{aligned} Y_t &= r + m_t - \nu_t + \sqrt{h_t} z_t^* \\ h_t &= F \left(z_{t-1}^* - \frac{\nu_{t-1}}{\sqrt{h_{t-1}}}, h_{t-1}, \theta^V \right) \end{aligned} \quad (10)$$

where z_t^* follows the same law as z_t under \mathbb{P} and where ν_t fulfills $e^{\nu_t} = e^{-r} E_{\mathbb{P}} [e^{Y_t} | \mathcal{F}_{t-1}]$. When the z_t are assumed to be Gaussian, we recover the same dynamics as in (9). Moreover, following Badescu *et al.* (2019), for NIG innovations this is a tractable framework, especially when combined with the NGARCH model to obtain a closed-form formula for the associated VIX index.¹⁸ Nevertheless, one of the major drawback of this approach, that may explain partly poor pricing performances of this method for long maturity options (see Badescu *et al.* (2008) and Badescu *et al.* (2011)), is the fact that from \mathbb{P} to \mathbb{Q}^{EGP} only the conditional mean is affected while the conditional variance, skewness and, kurtosis are the same. The conditional Esscher transform introduced in the GARCH setting by Siu *et al.* (2004) and Gouriéroux & Monfort (2007) is probably one of the best-known tool to select efficiently EMM. The associated SDF M^{ess} is exponential-affine of log-returns and the predictable associated coefficients of affinity are uniquely determined by the pricing equations related to the bond and the risky asset. In contrast to Duan's approach a wide variety of return innovations may be chosen at the very least within the class of mixture or infinitely divisible distributions (see Chorro *et al.* (2015) Chap 3.4). Even if this tool coincides with the LRNVR in the Gaussian case, it allows for strongly non-linear relations between historical and risk-neutral volatility in the non-Gaussian setting. Furthermore, explicit¹⁹ risk-neutral dynamics (see Table 4) may be obtained for the IG-GARCH model (3) and GARCH-type models with NIG innovations.

In particular, if we suppose in (8) a NIG $(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\mu})$ for the z_t , we obtain (Badescu *et al.* (2011)) the following dynamics under the Esscher EMM:

$$\begin{aligned} Y_t &= r + m_t + \sqrt{h_t} z_t^* \\ h_t &= F(z_{t-1}^*, h_{t-1}, \theta^V) \end{aligned} \quad (11)$$

where z_t^* follows, under \mathbb{Q}^{Ess} , a NIG $(\tilde{\alpha}, \tilde{\beta} + \sqrt{h_t} \theta_t^q, \tilde{\delta}, \tilde{\mu})$ with a predictable parameter θ_t^q having an explicit form. As remarked in Monfort & Pégoraro (2012), the exponential-affine hypothesis concerning the SDF only allows for an equity risk premium and it may be interesting to partly solve empirical puzzles of option prices taking into account a second-order variance risk premium. To achieve this, the authors introduced an exponential-quadratic SDF M^{Qua} that extends M^{ess} adding a second moment-based source of risk information. Moreover, under Gaussian hypothesis, this new change of measure preserves the tractability of the model with a risk-neutral dynamics given by

$$\begin{aligned} Y_t &= r - \frac{h_t^*}{2} + \sqrt{h_t^*} z_t^* \\ h_t^* &= \pi F\left(\sqrt{\pi}(z_{t-1}^* - \frac{m_{t-1}}{\sqrt{h_{t-1}^*}} - \frac{\sqrt{h_{t-1}^*}}{2}), \frac{h_{t-1}^*}{\pi}, \theta^V\right) \end{aligned} \quad (12)$$

where the z_t^* are i.i.d $\mathcal{N}(0, 1)$ under \mathbb{Q}^{Qua} and π is the proportional wedge between risk-neutral and historical volatilities assumed to be constant across time.²⁰ As a consequence, for the HN model (2), the dynamics under \mathbb{Q}^{Qua} remains in the same family of affine GARCH models, preserving analytic properties of the HN specification in terms of option pricing. Inspired by this new methodology, Chorro & Fanirisoa (2016) proposed an exponential-hyperbolic SDF M^{Ushp} that is able to cope with the same remarkable features in the case of the IG-GARCH model (3).

To conclude this section, let us precisely describe all related GARCH option pricing models that will be tested in the empirical part: in the affine family, the classical Heston & Nandi (2000) and the IG-GARCH model (Christoffersen *et al.* (2006) and Chorro & Fanirisoa (2016)) will be combined with exponential-affine and U-shaped SDF risk-neutralization processes. In these cases, Monte-Carlo methods won't be used to approximate the price of plain vanilla options. To relax the constraints on variance dynamics and conditional distributions related to affine specifications, we will also study two classical non-affine structures namely the GJR and NGARCH models with Gaussian or NIG innovations. In the Gaussian case the dynamics will be risk-neutralized using the LRNVR or the quadratic SDF while under NIG hypotheses, exponential-affine and EGP assumptions will be favored. This great variety of models and SDF will allow us to question several key aspects of GARCH option pricing modeling. Finally, for sake of concision and simplicity all the risk-neutral dynamics used in this study are gathered in Table 3 for Gaussian innovations and in Table 4 otherwise.

3. Model implied CBOE VIX

Considered as the investor's expectation of volatility (see Carr & Wu (2006)), the CBOE VIX index can be characterized as a forecast of the 30-day risk-neutral volatility (or 22 working days) of the S&P500 index. In this section, we denote by Vix_t a daily-based proxy for VIX_t which is the daily-adjusted expression of the expected arithmetic average of variance (see Hao & Zhang (2013)):

$$Vix_t = \frac{1}{\tau} \left(\frac{VIX_t}{100} \right)^2 = \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{T_c} \int_t^{t+T_c} h_u du \mid \mathcal{F}_t \right] \approx \frac{1}{T_c} \sum_{j=1}^{T_c} \mathbb{E}_{\mathbb{Q}} [h_{t+j} \mid \mathcal{F}_t] \quad (13)$$

where $\tau = 250$, $T_c = 22$ represents the maturity in days and \mathbb{Q} is an EMM. Depending on the choice of the risk-neutral dynamics and using iterative properties of conditional expectation, the term $\mathbb{E}_{\mathbb{Q}} [h_{t+j} \mid \mathcal{F}_t]$ can be explicitly computed for a large class of Gaussian (Hao & Zhang (2013)) and non-Gaussian (Chorro & Fanirisoa (2016), Badescu *et al.* (2019)) GARCH models. In general, $\mathbb{E}_{\mathbb{Q}} [h_{t+j} \mid \mathcal{F}_t]$ can be expressed as a linear function of historical volatility at time $t + 1$, unconditional variance, and variance persistence²¹ under the selected EMM. If we can obtain analytic expressions, we have the following general form for $\mathbb{E}_{\mathbb{Q}} [h_{t+j} \mid \mathcal{F}_t]$ and Vix_t :

$$\begin{cases} \mathbb{E}_{\mathbb{Q}} [h_{t+j} \mid \mathcal{F}_t] = h_{t+1} [\Psi^*]^{j-1} + \tilde{h}_0 \left[1 - (\Psi^*)^{j-1} \right] \\ Vix_t = h_{t+1} \frac{1 - (\Psi^*)^{T_c}}{(1 - \Psi^*) T_c} + \tilde{h}_0 \left(1 - \frac{1 - (\Psi^*)^{T_c}}{(1 - \Psi^*) T_c} \right) \end{cases} \quad (14)$$

where expressions of \tilde{h}_0 and Ψ^* for particular models and SDF are reported in Table 5. In fact, for Gaussian models under the LRNVR and for affine models with exponential-affine or U-shaped SDF we have closed form expressions. For example, in the case of the HN model, we obtain $\tilde{h}_0 = \frac{a_0 + a_1}{1 - \Psi^*}$ and $\Psi^* = b_1 + a_1(\gamma + \lambda_0 + \frac{1}{2})^2$ when an exponential-affine SDF is used while we obtain $\tilde{h}_0 = \frac{a_0 + \pi a_1}{1 - \Psi^*}$ and $\Psi^* = b_1 + \pi^2 a_1 \left(\frac{\gamma}{\pi} + \frac{\lambda_0}{\pi} + \frac{1}{2} \right)^2$ under the quadratic SDF.

Unfortunately, in the case of NIG innovations (a notable exception is the NIG NGARCH model associated with the EGP of Badescu *et al.* (2019)) or when an exponential-quadratic SDF is used with the Gaussian NGARCH and GJR structures we do not have closed-form formulas for the implied Vix_t . However, as explained in Lalancette & Simonato (2017) we can still use Monte Carlo simulations to approximate conditional expectation $\mathbb{E}_{\mathbb{Q}} [h_{t+j} \mid \mathcal{F}_t]$ and Vix_t .

4. Methodology and empirical results

In this section, we present the main points emerging from this analysis. First, we carry out numerical experiments to analyze pricing performances of all competing GARCH models, focusing on affine/non-affine structures, the risk-neutralization process and the estimation methodology. A pool of 24 possible combinations (Model/SDF/Estimation) will thereby be tested to try to understand the impact of underlying factors. Furthermore, a second experiment aims to question the possibility of partly ranking GARCH option pricing models by their ability to simply reproduce VIX dynamics, instead of using a heavy set of option data. More specifically, after a brief description of the data, we present the main lines of classical joint likelihood estimation methodologies based on Option>Returns or VIX>Returns data (see for example Kanniainen *et al.* (2014) and reference therein) and that of the two-step estimation strategy recently introduced in Chorro & Fanirisoa (2018) for NIG-GARCH processes. Then, when closed-form expressions for option prices are not available, we recall how Monte Carlo approximations may be implemented efficiently in the GARCH framework using the powerful and simple adjustment proposed by Duan & Simonato (1998). Finally, this section ends with a presentation of the results based on our empirical findings.

4.1. Data description

The present study used S&P500 daily returns and VIX data from January 07, 1999 to December 22, 2010, which are composed of 2718 observations covering about 12 years. We plotted in Figure 1 the S&P500 and CBOE VIX indexes with their log-returns series while Table 1 displayed associated summary statistics.²² This information set was used to implement both classical conditional maximum likelihood strategies and joint estimation strategies based on returns and VIX information.

We also used a dataset of options written on the S&P500 obtained from Bloomberg. Due to the number of option pricing models to test in this section, we restricted ourselves to Wednesday's contracts.²³ Therefore, it concerned 4563 options contracts whose prices were quoted during the period spanning from January 2nd, 2009 to April 15, 2012. We divided the option data set into two subsets: one in which model parameters are estimated (to implement for the affine models the joint likelihood estimation based on returns and options) and another subset used to compare pricing performances of models. The first subset, used for the in-sample estimation and comparison, is called Dataset A from January 2nd, 2009 until December 22, 2010 and contains 2714 contracts. However, the second subset for the out-of-sample comparison is called Dataset B and contains 1849 contracts with 67-Wednesdays from January 03, 2011 until April 15, 2012. This will be used to test the out-of-sample ability to capture the

behavior of the index option smile. Summary statistics for option data are reported in Table 2 for both Dataset A and B: this table shows the number of contracts, the average price, and the average implied volatility across moneynesses and times to maturity. The patterns in the Dataset B are clearly similar to those in the in-sample Dataset A.

Depending on the chosen estimation strategy, the in-sample dataset of returns is combined with in-sample VIX data or Dataset A to estimate the model as explained in the next section. Furthermore, usual in and out-of-sample option pricing performances are studied: we use in-sample estimated parameters to compute approximate prices (from FFT or Monte-Carlo approximations depending on the structure of the model) for the contracts in Dataset A and B to analyze associated errors. In the out-of-sample exercise presented above, we assumed that model's parameters are constant over the whole sample period (Dataset B). Obviously, this may appear as unrealistic and unfair for the simulation and relaxing this assumption will highlight the robustness of our conclusions. Therefore, in a complementary numerical experiment, we allowed model parameters to change over time through a rolling window estimation strategy for the 67 Wednesdays in the Dataset B.²⁴ For each Wednesday in dataset B, we estimated each model and use corresponding parameters to price options next Wednesday.²⁵

4.2. Estimation methodologies

In this section we denote by ϑ the set of risk-neutral parameters associated with historical dynamics (1). When conditional Esscher transform or extended Girsanov principle are used to obtain risk-neutral dynamics we simply have $\vartheta = (\theta^D, \theta^V)$ while $\vartheta = (\theta^D, \theta^V, \pi)$ in the case of U-shaped pricing kernels.²⁶ Moreover, we denote by T (resp. N) the number of VIX and log-returns daily observations (resp. N the cardinal of the set of option market prices) involved in the estimation process. One of the main advantages of the GARCH machinery is that historical model parameters (θ^D, θ^V) may be easily obtained, from a simple log-returns dataset, using a conditional version of the classical maximum likelihood estimator maximizing

$$\log L_R(\theta^D, \theta^V) = \sum_{t=1}^T \log \left(\frac{1}{\sqrt{h_t}} f_{\theta^D} \left(\frac{Y_t - (r + m_t)}{\sqrt{h_t}} \right) \right)$$

where f_{θ^D} is the probability density function of the model innovations. However, the proportional wedge between historical and risk-neutral volatility π cannot be estimated only using returns data. Moreover, during the last decade, several empirical studies underlined the real interest to incorporate in the estimation process VIX or option information, when available, to improve related pricing performances. There-

fore, we present below two joint likelihood estimation strategies used in the empirical part:

Joint estimation strategy using Option>Returns information: We consider a set of option market prices $(\hat{c}_1, \dots, \hat{c}_N)$ and define associated weighted Vega errors $\epsilon_i = \frac{c_i - \hat{c}_i}{\hat{V}_i}$ where c_i and \hat{V}_i are the model prices and the Black and Scholes Vega associated with \hat{c}_i . Following Trolle & Schwartz (2009), we suppose that the (ϵ_i) are i.i.d centered Gaussian variables with variance $\frac{1}{N} \sum_{i=1}^N \epsilon_i^2$. Therefore, the associated option log-likelihood is given by

$$\log L_{Op}(\vartheta) = -\frac{1}{2} \sum_{i=1}^N \left[\log \left(\frac{1}{N} \sum_{i=1}^N \epsilon_i^2 \right) + \frac{\epsilon_i^2}{\frac{1}{N} \sum_{i=1}^N \epsilon_i^2} \right]$$

and we obtain the joint Option>Returns likelihood (see Christoffersen *et al.* (2013)):

$$\frac{T + N \log L_R((\theta^D, \theta^V))}{2} + \frac{T + N \log L_{Op}(\vartheta)}{2N}. \quad (15)$$

One of the major drawbacks of this approach is the requirement to evaluate several times the objective function (15) in the maximization process. In the case of affine GARCH models presented above, independently of the choice of the exponential-affine or exponential U-shaped SDF, closed-form expressions for option prices are available and make this process computationally acceptable. As noticed in Section 3, for most of Gaussian GARCH specifications and for the NIG NGARCH model combined with the EGP it is possible to obtain closed-form expressions for the implied VIX. Therefore, as provided by Kannianen *et al.* (2014), a similar strategy based on VIX information and not on options one may be implemented.

Joint estimation strategy using VIX>Returns information: To build the VIX log-likelihood we suppose with Kannianen *et al.* (2014) (see also Chorro & Fanirisoa (2016) or Badescu *et al.* (2019)) that VIX pricing errors $u_t = \text{VIX}_t^{\text{Market}} - \text{VIX}_t^{\text{Model}}$ follow autoregressive disturbances $u_t = \varrho u_{t-1} + e_t$ where $(e_t)_t$ are i.i.d Gaussian random variables with mean zero and variance Σ^2 and where $|\varrho| < 1$ to ensure stationarity. Consequently the VIX log-likelihood is given by

$$\begin{aligned} \log L_{VIX}(\vartheta, \varrho) &= -\frac{T}{2} (\log(2\pi) + \log(\Sigma(1 - \varrho^2))) + \frac{1}{2} (\log(1 - \varrho^2)) \\ &\quad - \frac{1}{2\Sigma} \left(u_1^2 + \sum_{t=2}^T \frac{(u_t - \varrho u_{t-1})^2}{1 - \varrho^2} \right) \end{aligned} \quad (16)$$

and we obtain the joint VIX-returns likelihood $(\log L_R(\theta^D, \theta^V) + \log L_{VIX}(\vartheta, \varrho))$.

Finally, a last estimation strategy will be used in the empirical part for particular

GARCH models with NIG innovations. This strategy, first introduced in Chorro & Fanirisoa (2018), derives from a very simple finding: under Gaussian hypotheses, some GARCH-type models have outstanding properties (closed-form expressions for VIX and/or option prices) that fail when NIG innovations are involved. Therefore, inspired by the so-called quasi-maximum likelihood (QML) estimator, a two-step approach is possible to take benefit of these remarkable features in a Gaussian environment:

Two-step estimation strategies using VIX>Returns or Option>Returns information: As in the QML approach, this two-step strategy estimates separately volatility and distribution parameters assuming Gaussian innovations in the first step. We start from a GARCH-type model with NIG innovations

- Step 1: We assume that the $(z_t)_t$ are i.i.d $\mathcal{N}(0, 1)$ under \mathbb{P} . Subsequently, we can estimate the vector of volatility parameters θ^V as follows :
 - In the cases where we have a closed-form formula for option prices, $\hat{\theta}^V$ may be obtained by maximizing the joint Option>Returns likelihood (15).
 - In the cases where we do not have a closed-form formula for option prices but we have a closed-form formula for the VIX index then $\hat{\theta}^V$ may be obtained maximizing the joint VIX>Returns likelihood.
- Step 2: From the i.i.d residuals $\left(z_1 \left(\hat{\theta}^V\right), \dots, z_T \left(\hat{\theta}^V\right)\right)$ that may be extracted from the previous step, the distribution vector of parameters θ^D is obtained maximizing

$$\sum_{t=1}^T -\frac{\log(h_t)}{2} + \log \left[f_{\theta^D} \left(\frac{Y_t - (r + m_t)}{\sqrt{h_t}} \right) \right]$$

where f_{θ^D} is the density function of a centered NIG random variable with unit variance as introduced in Section 1.3.

To summarize, in our empirical study, the HN model with Gaussian innovations and the IG-GARCH model (risk-neutralized using Esscher or U-shaped SDF) will be estimated using the returns, the joint VIX>Returns and the joint Option>Returns likelihoods. The GJR and NGARCH models with Gaussian innovations (risk-neutralized using Esscher SDF) will be estimated using the returns and the joint VIX>Returns likelihood. The HN, GJR, NGARCH with NIG innovations (risk-neutralized using Esscher SDF) will be estimated using the returns and the two-step estimation strategy. The GJR and NGARCH models with Gaussian innovations (risk-neutralized using the quadratic SDF) will be estimated using the joint VIX>Returns likelihood.²⁷

The estimated parameter values and their respective standard errors, obtained from using the different sets of information, are reported in Table 6 (resp. Table 14) for

Gaussian GARCH models combined with the exponential-affine (resp. the quadratic) SDF. For NIG parameters, the results of the two-step estimation exercises are presented in Table 10, while Table 20 shows estimates for the IG-GARCH model under both M^{Ess} and M^{Ushp} . Finally, for the NIG-NGARCH model risk-neutralized using the EGP, the joint VIX>Returns likelihood estimates are illustrated in Table 18. In all cases, results are roughly in the same range as those obtained in many other previous empirical studies.

We notice for the IG-GARCH model that parameter estimates are remarkably stable across the different approaches. Concerning the other GARCH specifications, instead of focusing on the individual values of each parameter, we remark that global features of each model (implied persistence, leverage effect parameter) differ only a little from one strategy to another.²⁸ We classically obtain high historical persistences and all models and estimation approaches clearly indicate the leverage effect. Moreover, in the case of the two U-shaped pricing kernels, the proportional wedge between the risk-neutral and the historical volatilities is significantly estimated to be greater than 1, with values ranging between 1.24 and 1.72 (see Tables 14 and 20) for the Gaussian HN and the IG-GARCH models, as observed in empirical studies. Last but not least, as remarked in Kanninen *et al.* (2014), for the joint VIX>Returns estimation strategy, the autocorrelation coefficient ρ is uniformly close to 1 with a minimum value of 0.81 for the Gaussian HN model combined with the quadratic SDF.

Concerning parameters of the NIG distribution, we can see from Tables 10 and 18 that the observed (negative) values of skewness vary from -0.01 to -0.34 and that observed kurtosis vary from 1.42 to 2.62. These values provide evidence by their departure from normality and they are in the same range as those obtained in previous studies (see for example Badescu *et al.* (2011)).

4.3. *Criteria for the option and VIX pricing analysis*

Once a particular GARCH model has been properly estimated using a well-chosen set of historical financial information, we obtain explicitly from Tables 3 and 4 the related risk-neutral dynamics depending on the choice of the underlying SDF. For the HN-GARCH model with Gaussian innovations (Heston & Nandi (2000) and Monfort & Pégoraro (2012)) and the IG-GARCH model (Christoffersen *et al.* (2006) and Chorro & Fanirisoa (2016)), under both exponential-affine and U-shaped SDF, we have quasi-closed-form solutions for pricing vanilla European options efficiently from FFT methodology (see for example Chorro *et al.* (2015) Chap 4.2) that massively decrease the required time to price a full option book. For other non-affine specifications, prices are approximated using Monte Carlo simulation using 15000 trajectories.²⁹ To test the quality of these

price approximations we will use, in the empirical part, the in (Dataset A), out (Dataset B) and Wednesday (rolling window strategy) Implied Volatility Root Mean Squared Error (IVRMSE³⁰) that measure the discrepancy between model and option prices:

$$IVRMSE = \sqrt{\frac{1}{N} \sum_i \left(\frac{c_i - \hat{c}_i}{\hat{V}_i} \right)^2}$$

where c_i is the option price given by the model, \hat{c}_i the corresponding market price and \hat{V}_i the Black and Scholes Vega associated with \hat{c}_i . Moreover, another interesting economic criteria will be the magnitude of the average annualized volatility risk premium (VRP) as defined in Papantonis (2016) in order to understand why an equity risk premium is in general not sufficient to produce realistic price levels. Finally, in order to discuss the correlation between option pricing performances and the capacity of implied VIX to fit the market VIX, we will use the measures of adequacy introduced in Qiang *et al.* (2015), namely, the mean percentage error (MPE_{VIX}), the mean percentage absolute error (MAE_{VIX}) and the root mean squared error ($RMSE_{VIX}$) defined below:

$$\begin{aligned} MPE_{VIX} &= \frac{1}{N} \sum_{j=1}^N \left(\frac{VIX_j^{Model}}{VIX_j^{Market}} - 1 \right), \quad MAE_{VIX} = \frac{1}{N} \sum_{j=1}^N \left(\left| \frac{VIX_j^{Model}}{VIX_j^{Market}} - 1 \right| \right) \\ \text{and } RMSE_{VIX} &= \sqrt{\frac{1}{N} \sum_{j=1}^N (VIX_j^{Model} - VIX_j^{Market})^2}. \end{aligned} \quad (17)$$

4.4. Empirical findings

Our study relies on 24 combinations of GARCH-distribution-SDF-estimation. To make the presentation much more readable, we group them into five different categories: the Gaussian-GARCH models combined with M^{Ess} , the NIG-GARCH models combined with M^{Ess} , the Gaussian-GARCH models combined with M^{Qua} , the NIG-NGARCH model risk-neutralized using the EGP, and the IG-GARCH model. For each group, we present in a specific table (see Tables 7, 11, 15, 19 and 21) option and VIX fitting performances based on the criteria introduced in the preceding section. Furthermore, we report for each model the related estimation time and the variance risk premium as defined in Papantonis (2016). For a selected subclass containing more than one element, we provide internal pairwise comparisons in terms of computational time of estimation and in-sample pricing performances (see Tables 8, 12, 16 and 22) and in terms of out-of-sample and weekly out-of-sample option valuation errors (see Tables 9, 13, 17 and 23). Finally, general results are provided to allow for broader conclusions: in Table 24, out-of-sample performances of the best models in each category are compared while we can find in Table 25 (resp. Table 26) a summary of VIX and option performance

measures of the 24 competitors (resp. the corresponding ranking). Regarding results presented in Table 26 we can easily notice that ranks related to option (resp. to VIX) valuation are mostly independent of the choice of the underlying criteria selected from in sample, out-of-sample or weekly out-of sample IVRMSE (resp. from RMSE, MPE or MAE). Thus, in the following, numerical comparisons will rest on out-of-sample IVRMSE and VIX RMSE. We start our analysis at a group level.

We deduce from Table 7 that, when they are estimated only using returns, pricing performances of Gaussian GARCH models seem to be independent of the choice of the GARCH structure with IVRMSE ranging from 0.07648 to 0.07770 under Duan's LRNVR. When an extra piece of financial information is introduced into the estimation process, we obtain the smallest IVRMSE of 0.065 for the non-affine specifications especially the GJR model. This is in line with the existing literature that favors non-affine Gaussian stochastic volatility models (see Kannianen *et al.* (2014) and references therein). Table 11 leads to similar conclusions in the NIG environment while Table 15 confirms the slight superiority of non-affine Gaussian specifications when using an exponential-quadratic SDF. Nevertheless, option valuation errors under Gaussian distribution and exponential-affine SDF are the worst of all competitors. A plausible explanation comes from the fact that these models generate very small variance risk premia (see Table 7) which are not in line with empirical observations. In fact, as reported in Tables 11, 15, 19 and 21, when we use non-Gaussian alternatives and/or U-shaped pricing kernels we recover VRP between -2.867% (for the NIG-GJR model estimated using returns only) and -3.75411% (for the IG-GARCH model estimated using Option>Returns information) that are in line with a bulk of empirical studies (Papantonis (2016)). For Gaussian distribution and exponential-affine SDF, the variance risk is neglected and an equity risk premium is not sufficient to produce realistic price levels.

In Table 11, the overall IVRMSE is between 0.05929 and 0.07004 for NIG-GARCH models risk-neutralized with the Esscher SDF with values that are all smaller than corresponding values for Gaussian innovations. The minimal IVRMSE of 0.05929 is obtained in the case of the NIG-NGARCH model by estimating with the two-step estimation strategy using VIX>Returns information as introduced in Chorro & Fanirisoa (2018). Not surprisingly, a finer modeling approach of conditional skewness improves considerably the quality of price approximations. The two-step estimation strategy using VIX>Returns information helps to substantially improve performances at a parsimonious computational cost. The improvement (of around 14% for non-affine specifications) from using VIX information is also fundamental in this framework because returns based estimation strategy only leads to IVRMSE ranging from 0.06894 to 0.07004. This is

confirmed in Table 19 for the NIG-NGARCH model associated with the EGP with an IVRMSE of 0.05935.

Working with non-Gaussian residuals is not the only way to generate more realistic VRP than Gaussian-GARCH ones. We present in table 15 the IVRMSE of different Gaussian-GARCH models when an exponential-quadratic SDF is used to price options. It is worth noting that in this approach, it is not possible to directly estimate models from returns market quotes because the extra parameter π is involved in the risk-neutral dynamics. We obtain good IVRMSE between 0.06006 and 0.06331 that consistently outperform the Gaussian counterpart with exponential-affine SDF. Even if they are slightly worse than corresponding values for NIG-GARCH models for out-of-sample IVRMSE, the hierarchy is reversed when considering the next week pricing errors build on the rolling window estimation strategy. Both a modeling approach based on realistic conditional skewness and a modeling approach incorporating a variance premium in the pricing kernel seem to capture valuable empirical features. Therefore, a natural question is how is it possible that these two aspects are more complementary rather than competitive? The IG-GARCH model appears as an interesting candidate to tackle this issue.

For the IG-GARCH model, we obtain (see Table 21) out-of-sample IVRMSE between 0.067427 (in the case of the Esscher SDF estimated using returns only) and 0.056641 (for the U-shaped SDF and Returns-Options estimation strategy). Once again, a dataset of returns is not sufficient to produce competitive results. Furthermore, when the joint VIX>Returns estimation process is performed we obtain an IVRMSE of 0.057568 that is much closer to the best value at a considerably shorter computation time. The U-shaped pricing kernel of Chorro & Fanirisoa (2016) outperforms by around 7% the Esscher SDF in a conditionally Inverse-Gaussian environment and produces the best performances observed in this section: conditional skewness is a key factor of GARCH option pricing models that becomes outstanding when associated with a non-standard SDF.

When using GARCH option pricing models, the modeler is faced with four degrees of freedom: the GARCH structure, the distribution of the innovations, the pricing kernel, and the estimation strategy. Now we conclude the analysis of option pricing errors brought together in Table 25 with more general considerations on the impact of each factor *caeteris paribus*. Let us start with marginal effects: the impact of the choice of a non-affine GARCH structure accounting for the leverage effect is small with a 2.2% improvement in favor of the *GJR* model. In the same way, in the case of the NIG-NGARCH model estimated using returns and VIX information, the Esscher and the extended Girsanov principle SDF give rise to almost identical results with a difference

of 1.4% for the benefit of the exponential-affine parameterization (see also Badescu *et al.* (2011) and Badescu *et al.* (2015) that deliver the same conclusion). Finally, using an estimation strategy based on options and returns information only improves by around 1% the IVRMSE with respect to its VIX>Returns counterpart (however, this improvement is around 10.5% when using returns only) as already observed in Chorro & Fanirisoa (2016). Nevertheless, for this latter point we have to keep in mind that this slight 1% upgrade comes at a very high computational cost as reported in Tables 8, 12, 16 and 22. More decisively, the NIG distribution reduces the valuation error of around 11,6% in comparison to Gaussian innovations while, in the affine family, the IG-GARCH model outperforms by 10.6% the Gaussian HN model. Concerning, the choice of the SDF, we clearly observe, both in the Gaussian and in the Inverse-Gaussian case that U-shaped parameterizations yield respectively to 13% and 7% lower IVRMSE (see Christoffersen *et al.* (2013), Chorro & Fanirisoa (2016), and Badescu *et al.* (2017) for similar findings). In the light of these observations, it is not surprising to see from Table 24 that, when we compare out-of-sample pricing errors between the best models of each sub-group, the most interesting performances are delivered by a model with non-Gaussian innovations, risk-neutralized using a U-shaped SDF and estimated maximizing the joint VIX>Returns log-likelihood, namely, the IG-GARCH model. We conclude that, when it is possible, the combination of all these factors is fundamental to producing competitive valuation errors.³¹ The best model is not the most richly parameterized but a parsimonious one able to cope with classical stylized facts in terms of historical dynamics and risk representation.

Even if the ultimate criterion to compare GARCH option pricing models is the value of the pricing errors associated with a large real-world dataset of option prices, its computation may lead to large numerical issues in particular when Monte Carlo approximations are needed. This is true, not only during the estimation stage, but also to compute the objective function. To conclude this section, we question the possibility of deducing option pricing performances of a GARCH model from its capacity to forecast VIX dynamics. In Table 26, we have reported the ranks of the 24 models considered in this article regarding VIX and options adequacy measures introduced in Section 4.3. For example,³² when we measure the relationship between rankings obtained from out-of-sample pricing errors and VIX RMSE we obtain a significant Spearman's rank correlation coefficient of 0.92. Moreover, top ten models obtained using VIX RMSE criterion are mainly as highly ranked as using options based criterion. The most important conclusion is that the ranking of models is well-preserved independently of the chosen option or VIX adequacy measure: examining the performance of a model in fitting VIX time series gives a very good indication on related pricing performances at a very reasonable computational cost. VIX analysis appears in this way as a very interesting and parsimonious first-stage evaluation to discard the worst GARCH option pricing models.

5. Conclusion

In this paper, we have examined pricing performances of a large collection of GARCH models by questioning the global synergy between the choice of the affine/non-affine GARCH specification, the use of competing alternatives to the Gaussian distribution, the selection of an appropriate SDF and the choice of different estimation strategies based on several sets of financial information and on standard minimization algorithms. Therefore, 24 combinations of GARCH/distribution/SDF/estimation are tested using a large option dataset written on the S&P500. To do this, an intensive empirical comparison is performed not only based on in and out-of-sample pricing performances, but also using a weekly rolling window strategy where the model is estimated each Wednesday to price options one week later. Uniformly for these three criteria, the IG-GARCH model risk-neutralized using a U-shaped pricing kernel provides the best results. This gives evidence for the importance of using a non-Gaussian distribution combined with a non-standard stochastic discount factor that takes account for the variance risk premium. Of course, to estimate the variance risk aversion parameter, historical returns are not sufficient and an extra financial information is required. At this point, we have found that the joint VIX>Returns likelihood estimation provides competitive pricing errors at a very interesting computational cost with respect to option based estimation processes. This latter finding holds for all models considered in this paper. For non-affine GARCH specifications, we found that, under NIG innovations, very interesting pricing errors are obtained when, and only when, VIX information is incorporated into the estimation strategy. This is efficiently possible for the NGARCH model using the EGP risk-neutralization process or using the two-step estimation strategy developed in Chorro & Fanirisoa (2018).

Finally, we have questioned in this study the possibility to deduce option pricing performances of a GARCH model from its capacity to forecast VIX dynamics. When we ranked models using options or VIX criteria we obtained a highly significant Spearman's rank correlation coefficient of 0.92. Therefore, examining the performance of a model in fitting VIX time series gives a very good indication on related pricing performances at a very reasonable computational cost. VIX analysis appears in this way as a very interesting and parsimonious first-stage evaluation to discard the worst GARCH option pricing models.

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Notes

¹Alternative approaches were proposed by Taylor (1986) and Heston (1993) using stochastic volatility models where information from the volatility structure is needed to estimate parameters of the model.

²In the Duan's framework, the coefficients of the GARCH risk-neutral dynamics are just functions of the historical ones, and so may be directly estimated from the log-returns. Nevertheless, the closed-form expression permits to efficiently use available option information to calibrate the model.

³The exponential-affine SDF, M^{ess} , developed by Bühlmann *et al.* (1996) and Siu *et al.* (2004), which is based on a conditional extension of the pioneering work of Esscher (1932), and the SDF given by the extended Girsanov principle of Elliott & Madan (1998) are probably the two best known. In particular, they coincide with Duan LRNVR in the Gaussian setting. Let us also remark that extended and non-monotonic versions of the exponential-affine SDF are available for particular choices of distributions as the exponential-quadratic SDF M^{Qua} of Monfort & Pégoraro (2012) (see also Christoffersen *et al.* (2013)) for Gaussian innovations and the exponential U-shaped stochastic discount factor M^{Ush} proposed by Chorro & Fanirisoa (2016) for the Inverse-Gaussian GARCH model.

⁴This is not true for M^{Qua} or M^{Ush} because, in this case, a risk-neutral parameter (the constant proportional wedge between historical and risk-neutral volatilities) has to be evaluated.

⁵Recently, a large number of studies have further investigated the ability of the VIX index as an input variable for volatility to forecast option prices. Considered as an expected volatility series, the VIX was proposed by Whaley (1993) and introduced by the CBOE in 1993 to serve as a market volatility indicator. The VIX captures how much the investor is willing to pay to deal with investment risks. In previous empirical papers on the importance of the VIX index, the attention focus has primarily been on the impact and the correlation of the VIX index with the stock market and returns volatility. Giot *et al.* (2005) and Sarwar (2012) have established empirical results that suggest an asymmetric relationship between stock market returns and VIX. Cochrane *et al.* (2012) observed the adequacy of the VIX index as an important factor in the determination of stock market returns and also of volatility.

⁶When closed-form expressions are not available, two recent studies proposed interesting alternatives. In Lalancette & Simonato (2017) the authors proposed, for the NGARCH model with Johnson S_U distributed driving noise, numerical approximations to make possible the computation of the implied VIX index using Monte-Carlo simulations. In Chorro & Fanirisoa (2018) a new estimation strategy for some non-Gaussian GARCH models is presented to include options or VIX information in the joint estimation

at a low computational cost.

⁷An equivalent study could be performed in a companion paper for Markov-switching Elliott *et al.* (2006), multi-component Christoffersen *et al.* (2008) and multiple-shock Christoffersen *et al.* (2012) GARCH models.

⁸Such a property is not fulfilled if we use, for example, a mixture of Gaussian distributions.

⁹For example, in Kanniainen *et al.* (2014) the authors study different GARCH structures with different estimation strategies, but restrict themselves to the Gaussian setting.

¹⁰From now on h_0 is supposed to be constant and fixed at its unconditional level depending on the persistence of the model Ψ .

¹¹The density function of the IG distribution is given by the one parameter family: $\mathbf{1}_{\{y>0\}} \frac{\delta}{\sqrt{2\pi y^3}} e^{-(\sqrt{y}-\delta/\sqrt{y})^2/2}$ where $\delta \in \mathbb{R}_+^*$.

¹²For the HN-GARCH model, $A_t = ru + A_{t+1} + a_0 B_{t+1} - \frac{1}{2} \log(1 - 2a_1 B_{t+1})$ and $B_t = -\frac{1}{2}u + b_1 B_{t+1} + \left(\frac{u^2}{2} - 2a_1 \gamma B_{t+1} u + a_1 B_{t+1} \gamma^2\right) (1 - 2a_1 B_{t+1})^{-1}$. for the IG-GARCH model, $A(t) = A_{t+1} + ur + w B_{t+1} - \frac{1}{2} \log(1 - 2a(\eta)^4 B_{t+1})$ and $B(t) = b B_{t+1} + uv + (\eta)^{-2} - (\eta)^{-2} \sqrt{(1 - 2a(\eta)^4 B_{t+1})(1 - 2c B_{t+1} - 2u\eta)}$.

¹³For the NGARCH model, we take $m_t = \lambda_0 \sqrt{h_t} - \log(E_{\mathbb{P}}[e^{\sqrt{h_t} z_t}])$ as proposed in Badescu *et al.* (2019). When innovations are Gaussian the cumulant moment generating function at the point z is equal to $\frac{z^2}{2}$ and we recover the same excess returns as in the GJR model. Nevertheless we will see that this very particular choice leads to a closed form expression for the VIX index associated with the NGARCH model with NIG innovations when extended Girsanov risk-neutralization process is used. This property is remarkable because up to our knowledge this is the unique example in the literature of an explicit VIX index formula within a non-Gaussian setting.

¹⁴Equivalently this distribution may be characterized by its very simple log-moment generation function given by $\kappa_{NIG}(z) = \mu z + \delta \sqrt{\alpha^2 - \beta^2} - \delta \sqrt{\alpha^2 - (\beta + z)^2}$. This simple expression will be used in (4) and also to obtain in a very simple way risk-neutral dynamics in GARCH-type models with NIG innovations.

¹⁵The purpose of this section is not to provide a self-contained presentation of these classical tools (Chorro *et al.* (2015) Chap 3) but to recall the main intuitions behind Gaussian (see Table 3) and non-Gaussian (see Table 4) risk-neutral dynamics that will be compared in the empirical part.

¹⁶A set of assumptions made on the utility function and the aggregated consumption growth that preserves both Gaussianity and volatility.

¹⁷Such a pricing kernel has also been justified from its consistency with risk-adjusted cost minimizing hedging strategies.

¹⁸In fact, the restriction imposed on the conditional mean in (4) provides explicit

computations.

¹⁹In general, to obtain risk-neutral dynamics, pricing equations have to be solved numerically (Chorro *et al.* (2012)) at any time. However for some interesting choices (Gaussian, IG, NIG among others) solutions are analytic.

²⁰The exponential-quadratic stochastic discount factor can be expressed as $M_t^{Qua} = e^{\theta_{2,t}Y_t^2 + \theta_{1,t}Y_t + \varepsilon_t}$ where $(\varepsilon_t, \theta_{1,t}, \theta_{2,t})$ are predictable coefficients. Obviously, when $\theta_{2,t} = 0$ we recover M^{ess} . The pricing equations for the bond and the risky asset impose some restrictions on these predictable coefficients that are not uniquely defined. If we want to obtain a unique solution to the pricing system an extra condition is needed. A natural candidate is to impose a constant proportional wedge $\pi = \frac{h_t^*}{h_t}$ between risk-neutral and historical conditional variances (see Chorro *et al.* (2015) Chap 3.5). This new risk-neutral parameter (that cannot be estimated only using any information from the log-returns) can help producing richer dynamics.

²¹Where the variance persistence is the coefficient in front of h_t in $\mathbb{E}_{\mathbb{Q}}[h_{t+1} | \mathcal{F}_t]$.

²²Let us remark that VIX data from January 03, 2011 until April 15, 2012 are also used in the empirical part to test the ability of GARCH option pricing models to forecast VIX dynamics.

²³We apply to our dataset the same filters as described in Christoffersen *et al.* (2012).

²⁴We assume constant windows of 12 years (resp. 2 years) for log-returns and VIX data (resp. for options).

²⁵We particularly use estimated in-sample parameters as initial values for the optimization performed the first Wednesday while we initialize parameters of the following Wednesday estimation process by using parameters obtained the previous week.

²⁶As defined in the preceding sections, θ^D is the vector of innovation parameters, θ^V represents the volatility parameters, and π is the proportional wedge between risk-neutral and historical volatilities supposed to be constant.

²⁷In this case, and only in this case, the methodology of Lalancette & Simonato (2017) will be used to approximate the VIX using Monte Carlo methods.

²⁸For example, we can deduce from Tables 6 and 14 that in the case of the GJR GARCH specification we obtain historical (resp. risk-neutral) persistences around 0.986 (resp. around 0.996) and a leverage parameter γ between 0.022 and 0.023.

²⁹An important point to emphasize here is the use in our study of the so-called empirical martingale simulation methodology (EMS) proposed by Duan & Simonato (1998) to reduce drastically the variance of Monte Carlo estimators. As remarked for example in Badescu *et al.* (2015), EMS is an essential tool to improve numerical efficiency of Monte Carlo methods especially in the GARCH setting and to use a reasonable number of simulations to compute option prices. Nevertheless, in spite of their efficiency when combined with EMS, Monte Carlo approximations for non-affine models make impossible to use option information in the estimation process at a realistic computational cost.

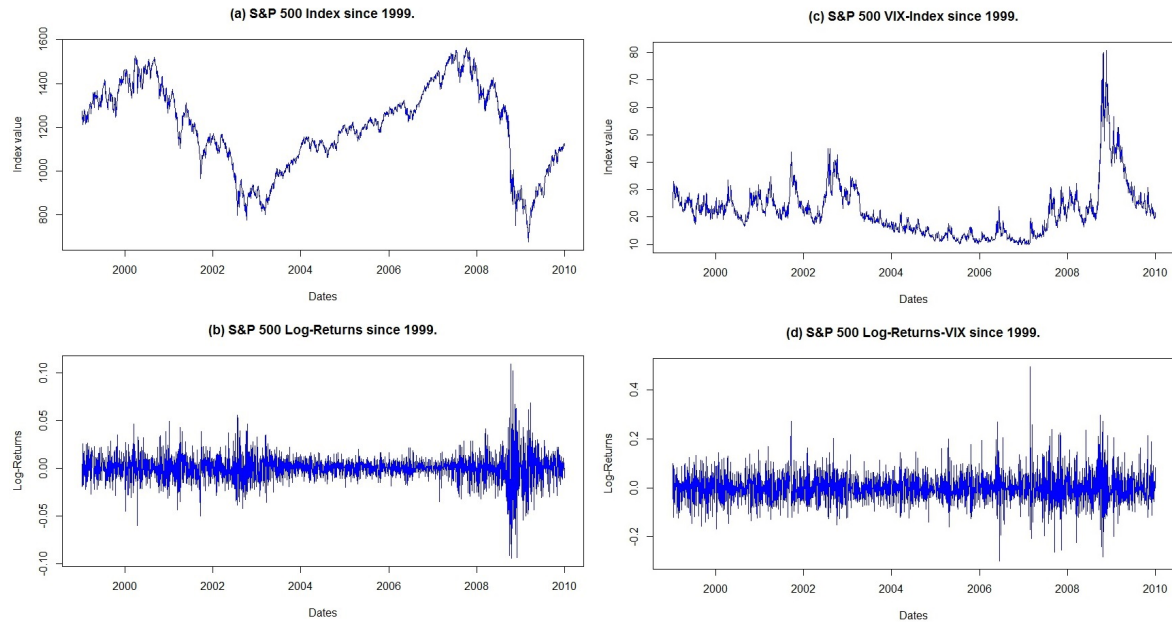
³⁰In the bulk of recent studies (Christoffersen *et al.* (2012), Kannianen *et al.* (2014), Chorro & Fanirisoa (2016), Badescu *et al.* (2017)) this indicator was used to measure pricing performances because Vega-weighted errors do not vary too much across maturities and moneyness contrary to price errors.

³¹It is also important to remark that noteworthy results are obtained with non-affine GARCH structures in NIG environment when VIX information is used in the estimation process. In this case, the residual error of around 3% comes from the necessity to use classical SDF to obtain risk-neutral dynamics.

³²In Table 26, we notice that the rankings related to options (or VIX) valuation are essentially independent of the choice of the adequacy measures. For example, Spearman's rank correlation coefficient between in and out-of sample pricing errors ranking methodologies is equal to 0.97 . Consequently, we focus our attention on out-of-sample pricing performances and VIX RMSE.

Figures

Fig. 1. S&P500 and VIX closing prices (top) and daily log-returns (bottom) from January 7, 1999 to December 22, 2010.



Tables

Table 1: Descriptive statistics of the S&P500 and VIX datasets covering the period January 7, 1999-December 22, 2010.

	Number of observations	Min	Max	Mean	Std Dev	Skewness	Kurtosis
Price index	2718	676.53	1565.15	1182.75	190.14	-0.0959	-0.6909
Log-returns	2718	-0.0947	0.1096	-0.0001	0.0139	-0.1214	7.3758
VIX index	2718	9.8900	80.8600	22.1859	9.6098	1.8853	5.6964
Log VIX	2718	-0.3506	0.4960	-0.0001	0.0613	0.5697	4.1682

Table 2: Properties of the in-sample (Dataset A) options data (2009-2010) and the out-of-sample (Dataset B) options data (2011-2012).

Option Dataset		Dataset A			Dataset B		
$\frac{T}{S/K}$		< 60	[60, 180]	> 180	< 60	[60, 180]	> 180
Number of call option contracts :		All			All		
0 < S/K < 0.975		178	607	286	1071	107	214
0.975 < S/K < 1.00		40	103	44	187	36	46
1.00 < S/K < 1.025		36	96	54	186	30	41
1.025 < S/K < 1.05		35	93	37	165	31	37
1.05 < S/K < 1.075		37	93	40	170	28	29
1.075 < S/K		122	546	267	935	79	138
All		448	1538	728	2714	311	505
Average call price :							
0 < S/K < 0.975		8.558	23.392	41.658	24.536	7.436	21.804
0.975 < S/K < 1.00		28.133	59.176	84.700	57.336	25.047	59.893
1.00 < S/K < 1.025		42.764	71.741	96.643	70.383	45.442	76.560
1.025 < S/K < 1.05		59.721	87.681	109.272	85.558	66.109	95.260
1.05 < S/K < 1.075		77.534	103.012	125.367	101.971	88.434	116.030
1.075 < S/K		133.310	170.220	187.118	163.549	147.551	178.710
All		58.337	85.870	107.460	83.889	63.336	91.376
Average implied volatility from call options :							
0 < S/K < 0.975		0.212	0.209	0.210	0.210	0.161	0.174
0.975 < S/K < 1.00		0.223	0.231	0.233	0.229	0.177	0.198
1.00 < S/K < 1.025		0.228	0.230	0.235	0.231	0.202	0.207
1.025 < S/K < 1.05		0.239	0.240	0.233	0.237	0.202	0.210
1.05 < S/K < 1.075		0.259	0.245	0.235	0.246	0.226	0.222
1.075 < S/K		0.308	0.267	0.255	0.277	0.260	0.235
All		0.245	0.237	0.234	0.238	0.204	0.207

Note: The table shows the number of contracts, the average price, and the average implied volatility across moneynesses and times to maturities.

Table 3: Summary information on risk-neutral dynamics of each Gaussian GARCH model analyzed in this paper.

Model	Risk-neutral dynamics	Risk-neutral parameters and distribution	References
<u>HN-GARCH</u> :			
Gaus-Ess	$Y_t = r - \frac{h_t}{2} + \sqrt{h_t} z_t^*$ $h_t = a_0 + a_1 \left(z_{t-1}^* - \gamma^* \sqrt{h_{t-1}} \right)^2 + b_1 h_{t-1}$	$z_t^* \hookrightarrow N(0, 1)$ $\gamma^* = \gamma + \lambda_0 + \frac{1}{2}$	Heston & Nandi (2000)
Gaus-Qua	$Y_t = r - \frac{h_t^*}{2} + \sqrt{h_t^*} z_t^*$ $h_t^* = \pi a_0 + \pi^2 a_1 \left(z_{t-1}^* - \gamma^* \sqrt{h_{t-1}^*} \right)^2 + b_1 h_{t-1}^*$	$z_t^* \hookrightarrow N(0, 1), \pi = \frac{h_t^*}{h_t}$ $\gamma^* = \left(\frac{\gamma}{\pi} + \frac{\lambda_0}{\pi} + \frac{1}{2} \right)$	Kanninen <i>et al.</i> (2014) Monfort & Pégoraro (2012) Chorro <i>et al.</i> (2015) (Chap. 3.5)
<u>GJR-GARCH</u> :			
Gaus-Ess	$Y_t = r - \frac{h_t}{2} + \sqrt{h_t} z_t^*$ $h_t = a_0 + h_{t-1} \left[b_1 + a_1 \left(z_{t-1}^* - \lambda_0 \right)^2 \right] + \gamma h_{t-1} \max \left(0; - \left(z_{t-1}^* - \lambda_0 \right) \right)^2$	$z_t^* \hookrightarrow N(0, 1)$	Kanninen <i>et al.</i> (2014)
Gaus-Qua	$Y_t = r - \frac{h_t^*}{2} + \sqrt{h_t^*} z_t^*$ $h_t^* = a_0^* + h_{t-1}^* b_1 + a_1^* h_{t-1}^* \left(z_{t-1}^* - \lambda^* \right)^2 + \gamma^* h_{t-1}^* \max \left(0; - \left(z_{t-1}^* - \lambda^* \right) \right)^2$	$z_t^* \hookrightarrow N(0, 1), \pi = \frac{h_t^*}{h_t}$ $a_0^* = a_0 \pi, a_1^* = a_1 \pi, \gamma^* = \gamma \pi$ $\lambda^* = \frac{\lambda_0}{\sqrt{\pi}} - \frac{\sqrt{h_{t-1}^*}}{2} \left(\frac{1}{\pi} - 1 \right)$	Monfort & Pégoraro (2012) Chorro <i>et al.</i> (2015) (Chap. 3.5)
<u>NGARCH</u> :			
Gaus-Ess	$Y_t = r - \frac{h_t}{2} + \sqrt{h_t} z_t^*$ $h_t = a_0 + b_1 h_{t-1} + a_1 h_{t-1} \left(z_{t-1}^* - \gamma^* \right)^2$	$z_t^* \hookrightarrow N(0, 1)$ $\gamma^* = \lambda_0 + \gamma$	Kanninen <i>et al.</i> (2014)
Gaus-Qua	$Y_t = r - \frac{h_t^*}{2} + \sqrt{h_t^*} z_t^*$ $h_t^* = a_0^* + b_1 h_{t-1}^* + a_1^* h_{t-1}^* \left(z_{t-1}^* - \gamma_t^* \right)^2$	$z_t^* \hookrightarrow N(0, 1),$ $\pi = \frac{h_t^*}{h_t}, a_0^* = a_0 \pi, a_1^* = a_1 \pi$ $\gamma_t^* = \frac{\gamma}{\sqrt{\pi}} + \frac{\lambda_0}{\sqrt{\pi}} - \frac{\sqrt{h_{t-1}^*}}{2} \left(\frac{1}{\pi} - 1 \right)$	Monfort & Pégoraro (2012) Chorro <i>et al.</i> (2015)

Table 4: Summary information on risk-neutral dynamics of each non-Gaussian GARCH model analyzed in this paper.

Model	Risk-neutral dynamics	Risk-neutral parameters ¹ and distribution	References
<u>HN-GARCH</u> :			
NIG-Ess	$Y_t = r + m_t + \sqrt{h_t} z_t^*$ $h_t = a_0 + a_1 \left(z_{t-1}^* - \gamma \sqrt{h_{t-1}} \right)^2 + b_1 h_{t-1}$ $m_t = \lambda_0 \sqrt{h_t} - \frac{1}{2} h_t$	$z_t^* \hookrightarrow NIG \left(\tilde{\alpha}, \tilde{\beta} + \sqrt{h_t} \theta_t^q, \tilde{\delta}, \tilde{\mu} \right)$ $\theta_t^q = -\frac{1}{2} - \frac{\tilde{\alpha} \tilde{\beta} \sqrt{\tilde{\delta}}}{\sqrt{h_t} \varrho_t^{\frac{3}{2}}} - \frac{1}{2} \Theta_t,$ $\varrho = \sqrt{\tilde{\alpha}^2 - \tilde{\beta}^2}$	<p>Badescu <i>et al.</i> (2011)</p> <p>Chorro <i>et al.</i> (2012)</p>
<u>GJR-GARCH</u> :			
NIG-Ess	$Y_t = r + m_t + \sqrt{h_t} z_t^*$ $h_t = a_0 + h_{t-1} \left[b_1 + a_1 \left(z_{t-1}^* - \lambda_0 \right)^2 \right]$ $+ \gamma h_{t-1} \max \left(0; - \left(z_{t-1}^* - \lambda_0 \right) \right)^2$ $m_t = \lambda_0 \sqrt{h_t} - \frac{1}{2} h_t$	$z_t^* \hookrightarrow NIG \left(\tilde{\alpha}, \tilde{\beta} \sqrt{h_t} + \theta_t^q, \tilde{\delta}, \tilde{\mu} \right)$ $\theta_t^q = -\frac{1}{2} - \frac{\tilde{\alpha} \tilde{\beta} \sqrt{\tilde{\delta}}}{\sqrt{h_t} \varrho_t^{\frac{3}{2}}} - \frac{1}{2} \Theta_t$ $\varrho = \sqrt{\tilde{\alpha}^2 - \tilde{\beta}^2}$	<p>Badescu <i>et al.</i> (2011)</p> <p>Chorro <i>et al.</i> (2012)</p>
<u>NGARCH</u> :			
NIG-Ess	$Y_t = r + m_t + \sqrt{h_t} z_t^*$ $h_t = a_0 + b_1 h_{t-1} + a_1 h_{t-1} \left(z_{t-1}^* - \gamma \right)^2$ $m_t = \lambda_0 \sqrt{h_t} - \frac{1}{2} h_t$ $Y_t = r + m_t - \nu_t + \sqrt{h_t} z_t^*,$ $h_t = a_0 + b_1 h_{t-1} + a_1 h_{t-1} \left(z_{t-1}^* - \frac{\nu_{t-1}}{\sqrt{h_{t-1}}} - \gamma \right)^2$	$z_t^* \hookrightarrow NIG \left(\tilde{\alpha}, \tilde{\beta} \sqrt{h_t} + \theta_t^q, \tilde{\delta}, \tilde{\mu} \right)$ $\theta_t^q = -\frac{1}{2} - \frac{\tilde{\alpha} \tilde{\beta} \sqrt{\tilde{\delta}}}{\sqrt{h_t} \varrho_t^{\frac{3}{2}}} - \frac{1}{2} \Theta_t$ $\varrho = \sqrt{\tilde{\alpha}^2 - \tilde{\beta}^2}$ $z_t^* \hookrightarrow NIG \left(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\mu} \right)$ $e^{\nu_t} = e^{-r} \mathbb{E}_{\mathbb{P}} \left[e^{Y_t} \mid \mathcal{F}_{t-1} \right]$ $\nu_t = \tilde{\mu} \sqrt{h_t} + \sqrt{\tilde{\alpha}^2 - \tilde{\beta}^2} - \sqrt{\tilde{\alpha}^2 - (\tilde{\beta} + \tilde{\delta} \sqrt{h_t})^2}$	<p>Badescu <i>et al.</i> (2011)</p> <p>Chorro <i>et al.</i> (2012)</p> <p>Badescu <i>et al.</i> (2015)</p>
<u>IG-GARCH</u> :			
IG-Ess	$Y_t = r + \nu^* h_{t+1}^* + \eta^* z_t^*$ $h_t^* = w^* + b^* h_{t-1}^* + c^* z_{t-1}^* + a^* \frac{(h_{t-1}^*)^2}{z_{t-1}^*}$	$z_t^* \hookrightarrow IG \left(\frac{h_t}{\eta^*} \right)$ $\eta^* = \frac{\eta}{1 - 2\theta^* \eta}, \theta^* = \frac{1}{2} \left[\eta^{-1} - \frac{1}{\nu^2 \eta^3} \left[1 + \frac{\nu^2 \eta^3}{2} \right]^2 \right]$ $c^* = c \left(\frac{\eta^*}{\eta} \right)^{\frac{5}{2}}, a^* = a \left(\frac{\eta^*}{\eta} \right)^{-\frac{5}{2}}, \nu^* = \nu \left(\frac{\eta^*}{\eta} \right)^{-\frac{3}{2}}, w^* = w \left(\frac{\eta^*}{\eta} \right)^{\frac{3}{2}},$ $z_t^* \hookrightarrow IG \left(\frac{h_t}{\eta^*} \right), \pi = \frac{h_t}{h_t^*}$	Christoffersen <i>et al.</i> (2006)
IG-Ushp	$Y_t = r + \nu^* h_{t+1}^* + \eta^* z_t^*$ $h_t^* = w^* + b h_t^* + c^* y_t^* + a^* \frac{(h_t^*)^2}{z_t^*}$	$\eta^* = \sqrt[3]{\frac{\pi^2}{\nu^2} \left(-1 + \sqrt{1 + \frac{8\nu}{21\pi}} \right) + \sqrt[3]{\frac{\pi^2}{\nu^2} \left(-1 - \sqrt{1 + \frac{8\nu}{21\pi}} \right)}}$ $\nu^* = \frac{\nu}{\pi}, w^* = w\pi, c^* = \frac{c}{\eta}, a^* = \frac{a\eta}{\pi\eta^*}$	Chorro & Fanirisoa (2016)

$$\Theta_t = \sqrt{\frac{\left(\tilde{\alpha} m_t + \sqrt{\tilde{\delta} h_t \tilde{\beta} \varrho_t} \right)^2}{h_t \tilde{\delta} \varrho_t^3} - \frac{4 \tilde{\alpha}^4 \tilde{\delta}^2}{\left(h_t \tilde{\delta} \varrho_t^3 + \left(\tilde{\alpha} m_t + \sqrt{\tilde{\delta} h_t \tilde{\beta} \varrho_t} \right)^2 - 1 \right)}}$$

Table 5: GARCH implied VIX

GARCH models	\tilde{h}_0	Ψ^*
<u>HN-GARCH</u> :		
Gaussian-Ess	$\frac{a_0 + a_1}{1 - \Psi^*}$	$b_1 + a_1(\gamma + \lambda_0 + \frac{1}{2})^2$
Gaussian-Qua	$\frac{a_0 + \pi a_1}{1 - \Psi^*}$	$b_1 + \pi^2 a_1 \left(\frac{\gamma}{\pi} + \frac{\lambda_0}{\pi} + \frac{1}{2} \right)^2$
<u>GJR-GARCH</u> :		
Gaussian-Ess	$\frac{a_0}{1 - \Psi^*}$	$b_1 + [a_1 + \gamma N(\lambda_0)] (1 + \lambda_0^2) + \gamma \lambda_0 n(\lambda_0)$
<u>NGARCH</u> :		
Gaussian-Ess	$\frac{a_0}{1 - \Psi^*}$	$b_1 + a_1(1 + (\lambda_0 + \gamma)^2)$
NIG-EGP	$\frac{a_0}{1 - \Psi^*}$	$b_1 + a_1(1 + (\lambda_0 + \gamma)^2)$
<u>IG-GARCH</u> :		
Ess	$\frac{w + a(\eta)^4}{1 - \Psi^*}$	$b + \frac{c^*}{(\eta^*)^2} + a^* (\eta^*)^2$
Ushp	$\frac{w + \frac{a\eta}{\pi^2} (\eta^*)^3}{(1 - \psi^*)}$	$b + \frac{c^*}{(\eta^*)^2} + a^* (\eta^*)^2$

Note: We present expressions of the parameters \tilde{h}_0 and Ψ^* associated with the closed-form expression of Vix_t in equation 14 for different GARCH structures, stochastic discount factors and conditional distributions.

Table 6: Parameter estimates and standard errors of Gaussian GARCH models combined with the Esscher stochastic discount factor.

GARCH-type Information	HN-GARCH Returns	GJR-GARCH Returns	NGARCH Returns	HN-GARCH Opt-Ret	HN-GARCH Ret-VIX	GJR-GARCH Ret-VIX	NGARCH Ret-VIX
a_0	$3.854E - 08$	$3.049E - 06$	$1.677E - 06$	$1.859E - 07$	$3.757E - 12$	$4.966E - 06$	$3.557E - 06$
Stand.Dev	(0.0044)	(0.0011)	(0.0000)	(0.0009)	(0.0007)	(0.0004)	(0.0009)
a_1	$2.254E - 05$	$1.243E - 01$	$6.174E - 02$	$1.542E - 06$	$2.252E - 05$	$1.240E - 01$	$6.172E - 02$
Stand.Dev	(0.0001)	(0.0003)	(0.0014)	(0.0003)	(0.0002)	(0.0000)	(0.0007)
b_1	$8.272E - 01$	$8.509E - 01$	$8.446E - 01$	$6.500E - 01$	$9.117E - 01$	$8.504E - 01$	$7.956E - 01$
Stand.Dev	(0.0035)	(0.0006)	(0.0011)	(0.0030)	(0.0086)	(0.0015)	(0.0009)
γ	$5.379E + 01$	$2.208E - 02$	$1.174E + 00$	$4.586E + 02$	$1.423E + 01$	$2.314E - 02$	$4.701E - 08$
Stand.Dev	(0.0011)	(0.0025)	(0.0001)	(0.0095)	(0.0088)	(0.0005)	(0.0206)
λ_0	$1.020E + 00$	$2.288E - 01$	$8.911E - 07$	$8.596E + 00$	$1.513E + 00$	$1.989E - 01$	$8.452E - 01$
Stand.Dev	(0.0000)	(0.0055)	(0.0010)	(0.0008)	(0.0501)	(0.0033)	(0.0050)
ϱ	—	—	—	—	0.9992	0.8924	0.9542
Stand.Dev	—	—	—	—	(0.0106)	(0.0012)	(0.0110)

Note: Returns means MLE estimation procedure using only returns information, Opt-Ret means Joint MLE estimation using returns and options information, Ret-VIX means Joint MLE estimation using returns and VIX information. The estimation is based on log-returns and VIX datasets from January 7 1999 to December 22 2010 and on the in-sample dataset of options (2009-2010).

Table 7: Option pricing performances and VIX predictability (see section 4.3) of Gaussian-GARCH models combined with the Esscher stochastic discount factor.

GARCH-type Information	HN Returns	GJR Returns	NGARCH Returns	HN Opt-Ret	HN Ret-VIX	GJR Ret-VIX	NGARCH Ret-VIX
Model Properties							
Times (h)	0.010	0.018	0.025	9.014	0.008	0.021	0.014
$-VRP$ (in %)	$3.27E - 10$	$2.86E - 16$	$2.26E - 12$	$8.88E - 11$	$9.67E - 09$	$7.122E - 13$	$1.326E - 16$
Predictability of VIX							
MPE_{VIX}	0.0113	0.0106	-0.0088	-0.0092	-0.0023	-0.0016	-0.0012
MAE_{VIX}	0.0128	0.0125	0.0106	0.0104	0.0058	0.0059	0.0057
$RMSE_{VIX}$	0.2849	0.2771	0.2475	0.1869	0.1844	0.1740	0.1748
Pricing performances							
in-IVRMSE	0.0599	0.0574	0.0571	0.0557	0.0580	0.0548	0.0557
out-IVRMSE	0.0777	0.0764	0.0766	0.0733	0.0735	0.0650	0.0729
We-IVRMSE	0.0662	0.0651	0.0652	0.0610	0.0614	0.0592	0.0592

Note: The results are based on estimates provided in Table 6. The column **Model Properties** presents computational time of estimation in hours and Variance Risk Premium.

Table 8: Model comparisons based on computational time of estimation and on in-sample IVRMSE given in Table 7.

GARCH-type	HN		NGARCH		HN		GJR		NGARCH	
	Ret		Ret	Opt-Ret	VIX-Ret		VIX-Ret		VIX-Ret	
HN-Ret	—		−80.00	−9.004E + 04	20.000		−110.00		−40.00	
GJR-Ret	4.072		—	−4.997E + 04	55.555		−16.666		22.222	
NGARCH-Ret	4.561		0.509	−3.595E + 04	68.000		16.000		44.000	
HN-Opt-Ret	6.960		3.010	—	99.911		99.767		99.844	
HN-VIX-Ret	3.179		−0.930	−4.063	—		−162.50		−75.00	
GJR-VIX-Ret	8.480		4.595	1.634	5.475		—		33.333	
NGARCH-VIX-Ret	7.035		3.088	0.080	3.982		−1.579		—	

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the computational time of estimation between the i-th and the j-th models, as example: $-80\% = 100 * (0.010 - 0.018) / 0.010$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the in-sample IVRMSE between the j-th and the i-th models, as example: $4.072\% = 100 * (0.059916 - 0.057476) / 0.059916$.

Table 9: Model comparisons based on out-of-sample IVRMSE and Wednesday-IVRMSE given in Table 7.

GARCH-type	HN		GJR		NGARCH		HN		HN		GJR		NGARCH	
	Ret		Ret		Ret		Opt-Ret	VIX-Ret	VIX-Ret		VIX-Ret		VIX-Ret	
HN-Ret	—		1.565		1.393		5.540	5.394	5.394		16.340		6.054	
GJR-Ret	1.736		—		−0.174		4.045	3.890	3.890		15.010		4.560	
NGARCH-Ret	1.509		−0.230		—		4.212	4.057	4.057		15.160		4.727	
HN-Opt-Ret	7.923		6.297		6.512		—	−0.162	−0.162		11.430		0.536	
HN-VIX-Ret	7.259		5.621		5.838		−0.7212	—	—		11.570		0.697	
GJR-VIX-Ret	10.640		9.062		9.271		2.950	3.645	3.645		—		−12.30	
NGARCH-VIX-Ret	10.530		8.954		9.163		2.836	3.531	3.531		−0.118		—	

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the out-of-sample IVRMSE between the i -th and the j -th models, as example: $1.5650\% = 100 * (0.077701 - 0.076485)/0.077701$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the Wednesday-IVRMSE between the j -th and the i -th models, as example: $1.736\% = 100 * (0.06626 - 0.06511)/0.06626$.

Table 10: Parameter estimates and standard errors of the NIG distribution for GARCH models combined with the Esscher SDF.

GARCH-type Information	HN Returns	GJR Returns	NGARCH Returns	HN Opt-Ret	HN Ret-VIX	GJR Ret-VIX	NGARCH Ret-VIX
α	1.2501 (0.0004)	1.1550 (0.0108)	1.2702 (0.0036)	1.4630 (0.0005)	1.4365 (0.0008)	1.3589 (0.0001)	1.4536 (0.0009)
Stand.Dev	-0.0106 (0.0008)	-0.1432 (0.0057)	-0.0025 (0.0015)	-0.0061 (0.0008)	-0.0538 (0.0003)	-0.0058 (0.0023)	-0.0061 (0.0001)
β	1.4728 (0.0095)	1.0623 (0.0000)	1.6204 (0.0005)	1.4454 (0.0001)	1.3920 (0.0008)	1.5336 (0.0000)	1.4538 (0.0000)
Stand.Dev	2.7086 (0.0051)	0.1327 (0.0076)	1.9734 (0.0055)	2.1602 (0.0000)	11.6243 (0.0013)	7.9908 (0.0000)	2.0178 (0.0003)
μ							
Stand.Dev							

Note: These parameters have been obtained using the standard maximum-likelihood algorithm for the residuals extracted from Table 6.

Table 11: Option pricing performances and VIX predictability (see section 4.3) of NIG-GARCH models combined with the Esscher SDF.

GARCH-type Information	HN Returns	GJR Returns	NGARCH Returns	HN Opt-Ret	HN Ret-VIX	GJR Ret-VIX	NGARCH Ret-VIX
Model Properties							
Times (<i>h</i>)	0.016	0.024	0.031	9.071	0.017	0.036	0.019
<i>-VRP (in %)</i>	2.906	2.867	2.956	3.011	3.213	3.006	3.366
Predictability of VIX							
$MP_{E_{VIX}}$	-0.0054	-0.0043	-0.0029	-0.0010	-0.0006	0.0011	0.0010
MAE_{VIX}	0.0074	0.0062	0.0053	0.0054	0.0051	0.0047	0.0049
$RMS_{E_{VIX}}$	0.1884	0.1758	0.1703	0.1509	0.1314	0.1130	0.1308
Pricing performances							
in-IVRMSE	0.0573	0.0550	0.0567	0.0519	0.0521	0.0512	0.0463
out-IVRMSE	0.0700	0.0689	0.0690	0.0639	0.0648	0.0595	0.0592
We-IVRMSE	0.0595	0.0592	0.0586	0.0510	0.0514	0.0504	0.0504

Note: The results are based on estimates provided in Tables 6 and 10. The column **Model Properties** presents computational time of estimation in hours and Variance Risk Premium.

Table 12: Model comparisons based on computational time of estimation and on in-sample IVRMSE given in Table 11.

GARCH-type	HN		GJR		NGARCH		HN		HN-VIX-Ret		GJR		NGARCH	
	Ret		Ret		Ret		Opt-Ret		VIX-Ret		VIX-Ret		VIX-Ret	
HN-Ret	—		−50.00		−93.75		$-5.659E+04$		−6.250		−125.00		−18.75	
GJR-Ret	4.116		—		−29.16		$-3.769E+04$		29.166		−50.00		20.833	
NGARCH-Ret	1.203		−3.038		—		$-2.916E+04$		45.161		−16.12		38.709	
HN-Opt-Ret	9.404		5.515		8.301		—		99.812		99.603		99.790	
HN-VIX-Ret	9.094		5.192		7.987		−0.342		—		−111.76		−11.76	
GJR-VIX-Ret	10.70		6.871		9.617		1.435		1.771		—		47.222	
NGARCH-VIX-Ret	19.26		15.79		18.27		10.088		11.180		9.579		—	

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the computational time of estimation between the i-th and the j-th models, as example: $-50\% = 100 * (0.016 - 0.024) / 0.016$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the in-sample IVRMSE between the j-th and the i-th models, as example: $4.116\% = 100 * (0.057391 - 0.055029) / 0.057391$.

Table 13: Model comparisons based on out-of-sample IVRMSE and Wednesday-IVRMSE given in Table 11.

GARCH-type	HN		GJR		NGARCH		HN		GJR		NGARCH	
	Ret		Ret		Ret		Ret		VIX-Ret		VIX-Ret	
HN-Ret	—		1.565		1.483		8.669		7.360		14.960	
GJR-Ret	0.420		—		−0.088		7.217		5.887		13.610	
NGARCH-Ret	1.361		0.945		—		7.291		5.972		13.680	
HN-Opt-Ret	14.240		13.870		13.050		—		−1.433		6.888	
HN-VIX-Ret	13.560		13.200		12.370		−0.783		—		8.203	
GJR-VIX-Ret	15.260		14.900		14.090		1.195		1.964		—	
NGARCH-VIX-Ret	15.230		14.870		14.060		1.156		1.925		−0.039	

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the out-of-sample IVRMSE between the i -th and the j -th models, as example: $1.565\% = 100 * (0.070042 - 0.068946) / 0.070042$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the Wednesday-IVRMSE between the j -th and the i -th models, as example: $0.420\% = 100 * (0.05950 - 0.05925) / 0.05950$.

Table 14: Parameter estimates and standard errors of Gaussian GARCH models combined with the exponential-quadratic SDF.

GARCH-type Information	HN-GARCH Opt-Ret	HN-GARCH Ret-VIX	GJR-GARCH Ret-VIX	NGARCH Ret-VIX
a_0	5.7547E - 14	1.0014E - 12	4.966E - 06	1.780E - 06
Stand.Dev	(0.0009)	(0.0003)	(0.0008)	(0.0000)
a_1	1.5139E - 06	1.5048E - 06	1.241E - 01	3.877E - 02
Stand.Dev	(0.0368)	(0.0002)	(0.0003)	(0.0065)
b_1	6.500E - 01	6.5121E - 01	8.504E - 01	9.329E - 01
Stand.Dev	(0.0032)	(0.0066)	(0.0004)	(0.0000)
γ	4.5869E + 02	4.586E + 02	2.3142E - 02	1.277E - 07
Stand.Dev	(0.0036)	(0.0095)	(0.0002)	(0.0078)
λ_0	8.596E + 00	8.672E + 00	1.989E - 01	4.583E - 01
Stand.Dev	(0.0006)	(0.0036)	(0.0015)	(0.0004)
π	1.6723E + 00	1.722E + 00	1.2785E + 00	1.2413E + 00
Stand.Dev	(0.0048)	(0.0022)	(0.0012)	(0.0092)
ϱ	—	0.8099	0.9546	0.9170
Stand.Dev	—	(0.0003)	(0.06235)	(0.0023)

Note: Returns means MLE estimation procedure using only returns information, Ret-VIX means Joint MLE estimation using returns and options information, Ret-VIX means Joint MLE estimation using returns and VIX information. The estimation is based on log-returns and VIX datasets from January 7 1999 to December 22 2010 and on the in-sample dataset of options (2009-2010).

Table 15: Option pricing performances and VIX predictability (see section 4.3) of Gaussian-GARCH models combined with the exponential-quadratic SDF.

GARCH-type Information	HN-GARCH Opt-Ret	HN-GARCH Ret-VIX	GJR-GARCH Ret-VIX	NGARCH Ret-VIX
Model Properties				
Times (h)	10.326	0.019	1.053	0.961
$-VRP$ (in %)	3.2301	3.2475	3.3562	3.5628
Predictability of VIX				
MPE_{VIX}	-0.0012	-0.0004	-0.0003	-0.0004
MAE_{VIX}	0.0049	0.0047	0.0041	0.0041
$RMSE_{VIX}$	0.1445	0.1248	0.1119	0.1103
Pricing performances				
in- $IVRMSE$	0.0511	0.0513	0.0507	0.0492
out- $IVRMSE$	0.0627	0.0633	0.0628	0.0600
We- $IVRMSE$	0.0514	0.0515	0.0508	0.0493

Note: The results are based on estimates provided in Table 14. The column **Model Properties** presents computational time of estimation in hours and Variance Risk Premium.

Table 16: Model comparisons based on computational time of estimation and on in-sample IVRMSE given in Table 15. The upper triangular part of the matrix illustrates relative difference (in percentage) of the computational time of estimation between the i -th and the j -th models, as example: $99.816\% = 100 * (10.326 - 0.019) / 10.326$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the in-sample IVRMSE between the j -th and the i -th models, as example: $-0.528\% = 100 * (0.05110 - 0.05137) / 0.05110$.

GARCH-type	HN-Opt-Ret	HN-VIX-Ret	GJR-VIX-Ret	NGARCH-VIX-Ret
HN-Opt-Ret	—	99.816	89.802	90.693
HN-VIX-Ret	-0.528	—	-5442	-4957
GJR-VIX-Ret	0.626	1.149	—	8.736
NGARCH-VIX-Ret	3.640	4.146	3.033	—

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the computational time of estimation between the i -th and the j -th models, as example: $99.816\% = 100 * (10.326 - 0.019) / 10.326$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the in-sample IVRMSE between the j -th and the i -th models, as example: $-0.528\% = 100 * (0.05110 - 0.05137) / 0.05110$.

Table 17: Model comparisons based on out-of-sample IVRMSE and Wednesday-IVRMSE given in Table 15.

GARCH-type	HN-Opt-Ret	HN-VIX-Ret	GJR-VIX-Ret	NGARCH-VIX-Ret
HN-Opt-Ret	—	−0.892	−0.223	4.287
HN-VIX-Ret	−0.097	—	0.663	5.133
GJR-VIX-Ret	1.243	1.339	—	4.500
NGARCH-VIX-Ret	4.216	4.309	3.010	—

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the out-of-sample IVRMSE between the i -th and the j -th models, as example: $-0.892\% = 100 * (0.06275 - 0.06331) / 0.06275$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the Wednesday-IVRMSE between the j -th and the i -th models, as example: $-0.097\% = 100 * (0.05147 - 0.05152) / 0.05147$.

Table 18: Parameter estimates and standard errors of the NIG-NGARCH model combined with the extended Girsanov principle SDF.

Vol Parameters		a_0	a_1	b_1	γ	λ_0	ϱ
Values		$1.896E - 06$	$3.877E - 02$	$9.329E - 01$	$7.110E - 01$	$9.937E - 02$	0.9163
Stand.Dev		(0.0035)	(0.0008)	(0.0000)	(0.0001)	(0.0000)	(0.0004)
NIG Parameters		α	β	δ	μ	—	—
Values		$2.961E + 00$	$-9.441E - 01$	1.5877	0.5341	—	—
Stand.Dev		(0.0002)	(0.0013)	(0.0000)	(0.0019)	—	—

Note: The estimation is based on Joint MLE estimation using returns and VIX information from January 7 1999 to December 22 2010.

Table 19: Option pricing performances and VIX predictability (see section 4.3) of the NIG-NGARCH model combined with the extended Gorsanov principle SDF.

VIX Performances		$-VRP$ (in %)	MPE_{VIX}	MAE_{VIX}	$RMSE_{VIX}$
Values		3.3645	-0.0001	0.0039	0.1066
Pricing performances		Times (h)	in-IVRMSE	out-IVRMSE	we-IVRMSE
Values		0.0101	0.0480	0.0593	0.0488

Note: The results are based on estimates provided in Table 18.

Table 20: Parameter estimates and standard errors of the IG-GARCH model combined with Esscher and U-shaped SDF.

Joint-Estimation	Returns	Returns-Option		Returns-VIX	
Model		M_t^{ess}	M_t^{Ushp}	M_t^{ess}	M_t^{Ushp}
Parameters:					
w	1.2061E - 06 (0.0000)	9.7699E - 06 (0.0006)	1.0185E - 05 (0.0002)	2.2341E - 06 (0.0002)	5.3156E - 05 (0.0004)
Stand.Dev					
b	2.3052E - 03 (0.0000)	1.0159E - 03 (0.0000)	1.7211E - 03 (0.0001)	2.3184E - 03 (0.0003)	1.8603E - 03 (0.0003)
Stand.Dev					
c	4.9024E - 05 (0.0000)	4.5379E - 05 (0.0000)	4.5118E - 05 (0.0001)	4.8949E - 05 (0.0009)	4.8233E - 05 (0.0005)
Stand.Dev					
a	3.3174E + 03 (0.0000)	3.3317E + 03 (0.0001)	3.3174E + 02 (0.0007)	3.3174E + 03 (0.0002)	3.3174E + 03 (0.0005)
Stand.Dev					
η	-7.972E - 03 (0.0000)	-7.5314E - 03 (0.0000)	-7.4936E - 03 (0.0000)	-7.9552E - 03 (0.0002)	-7.928E - 03 (0.0010)
Stand.Dev					
ν	1.2584E + 02 (0.0009)	1.2594E + 02 (0.0001)	1.2573E + 02 (0.0003)	1.2583E + 02 (0.0003)	1.2584E + 02 (0.0001)
Stand.Dev					
π	-	-	1.4005 (0.0084)	-	1.6325 (0.0035)
Stand.Dev					
ϱ	-	-	-	9.9552E - 01 (0.0070)	9.9386E - 01 (0.0034)
Stand.Dev					

Note: Returns means MLE estimation procedure using only returns information, Ret-VIX means Joint MLE estimation using returns and options information, Ret-VIX means Joint MLE estimation using returns and VIX information. The estimation is based on log-returns and VIX datasets from January 7 1999 to December 22 2010 and on the in-sample dataset of options (2009-2010).

Table 21: Option pricing performances and VIX predictability (see section 4.3) of the IG-GARCH model combined with Esscher and U-shaped SDF.

Joint-Estimation		Returns	Returns-Option	Returns-VIX		
Model		M_t^{ess}	M_t^{Ushp}	M_t^{ess}	M_t^{Ushp}	
Model Properties:						
Times (h)		0.036	9.2684	10.3984	0.0152	0.0348
VPR		3.1785	3.7541	3.5165	3.7042	3.4563
Predictability of VIX:						
MPE_{VIX}		-0.0011	-0.0004	-0.0003	-0.00008	-0.00019
MAE_{VIX}		0.0051	0.0043	0.0040	0.0039	0.0039
$RMSE_{VIX}$		0.1364	0.1315	0.1061	0.1010	0.0990
Pricing performances:						
in-IVRMSE		0.0543	0.0461	0.0435	0.0464	0.0438
out-IVRMSE		0.0674	0.0610	0.0566	0.0618	0.0575
we-IVRMSE		0.0514	0.0500	0.0480	0.0510	0.0480

Note: The results are based on estimates provided in Table 20. The column **Model Properties** presents the computational time of estimation in hours and the Variance Risk Premium.

Table 22: Model comparisons based on computational time of estimation and on in-sample IVRMSE given in Table 21.

IG-GARCH-type	Ess-Ret	Ess-Opt-Ret	Ushp-Opt-Ret	Ess-VIX-Ret	Ushp-VIX-Ret
Ess-Ret	—	$-2.5E + 05$	$-2.8E + 05$	57.777	3.333
Ess-Opt-Ret	15.08	—	-12.191	99.836	99.624
Ushp-Opt-Ret	19.89	5.663	—	99.853	99.665
Ess-VIX-Ret	14.49	-0.699	-6.745	—	-128.94
Ushp-VIX-Ret	19.29	4.957	-0.748	5.617	—

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the computational time of estimation between the i -th and the j -th models, as example: $-2.564 + 05\% = 100 * (0.036 - 9.268)/0.036$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the in-sample IVRMSE between the j -th and the i -th models, as example: $15.08\% = 100 * (0.054358 - 0.046160)/0.054358$.

Table 23: Model comparisons based on the out-of-sample IVRMSE and the Wednesday-IVRMSE given in Table 21.

IG-GARCH-type	Ess-Ret	Ess-Opt-Ret	Ushp-Opt-Ret	Ess-VIX-Ret	Ushp-VIX-Ret
Ess-Ret	—	9.446	16.000	8.295	14.620
Ess-Opt-Ret	2.759	—	7.234	−1.271	5.716
Ushp-Opt-Ret	6.722	4.076	—	−9.168	−1.637
Ess-VIX-Ret	0.891	−1.920	−6.251	—	6.899
Ushp-VIX-Ret	6.567	3.916	−0.166	5.726	—

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the out-of-sample IVRMSE between the i-th and the j-th models, as example: $9.446\% = 100 * (0.067427 - 0.061058) / 0.067427$. The lower triangular part of the matrix illustrates the relative difference (in percentage) of Wednesday-IVRMSE between the j-th and the i-th models, as example: $2.759\% = 100 * (0.05147 - 0.05152) / 0.05147$.

Table 24: Model comparisons based on the out-of-sample IVRMSE and the Wednesday-IVRMSE for best competitors of each sub-group.

	GJR		NGARCH		NGARCH		NGARCH		IG	
	Gaus-Ess	NIG-Ess	Gaus-Ess	NIG-Ess	Gaus-Qua	NIG-EGP	Gaus-Qua	NIG-EGP	Ushp	Ushp
GJR-Gaus-Ess	—	8.780	7.603	8.695	11.440					
NGARCH-NIG-Ess	14.81	—	—1.290	—0.092	2.913					
NGARCH-Gaus-Qua	16.74	2.260	—	1.182	4.149					
NGARCH-NIG-EGP	17.50	3.152	0.912	—	3.003					
IG-Ushp	18.78	4.659	2.454	1.556	—					

Note: Due to the weak difference between the results obtained using Opt-Ret or VIX-Ret information we favor the IVRMSE obtained from Joint MLE estimation using returns and VIX to reduce computational burden. The upper triangular part of the matrix illustrates relative difference (in percentage) of the out-of-sample IVRMSE between the i-th and the j-th models. The lower triangular part of the matrix illustrates relative difference (in percentage) of Wednesday-IVRMSE between the j-th and the i-th models.

Table 25: Option pricing performances and VIX predictability for the 24 competitors considered in this paper.

GARCH	IVRMSE			VIX		
<i>Model</i>	<i>in</i>	<i>out</i>	<i>We</i>	<i>RMSE</i>	<i>MAE</i>	<i>MPE</i>
<i>G.HN.Ret.Ess</i>	0.05939	0.07770	0.06647	0.28494	0.01287	0.011305
<i>G.GJR.Ret.Ess</i>	0.05747	0.07733	0.06511	0.27710	0.01254	0.010604
<i>G.NGARCH.Ret.Ess</i>	0.05718	0.07661	0.06526	0.24753	0.01065	−0.00886
<i>G.HN.Op.Ret.Ess</i>	0.05574	0.07339	0.06101	0.18696	0.01044	−0.00929
<i>G.HN.Ret.VIX.Ess</i>	0.05801	0.07351	0.06145	0.18444	0.00588	−0.00237
<i>G.GJR.Ret.VIX.Ess</i>	0.05483	0.06500	0.05921	0.17404	0.00594	−0.00168
<i>G.NGARCH.Ret.VIX.Ess</i>	0.05570	0.07299	0.05928	0.17480	0.00574	−0.00127
<i>NIG.HN.Ret.Ess</i>	0.05739	0.07004	0.05950	0.18843	0.00748	−0.00546
<i>NIG.GJR.Ret.Ess</i>	0.05502	0.06894	0.05925	0.17585	0.00629	−0.00437
<i>NIG.NGARCH.Ret.Ess</i>	0.05670	0.06900	0.05869	0.17036	0.00533	−0.00296
<i>NIG.HN.Op.Ret.Ess</i>	0.05199	0.06397	0.05103	0.15090	0.00545	−0.00108
<i>NIG.HN.Ret.VIX.Ess</i>	0.05217	0.06488	0.05143	0.13141	0.00513	−0.00060
<i>NIG.GJR.Ret.VIX.Ess</i>	0.05124	0.05956	0.05042	0.11303	0.00471	0.001135
<i>NIG.NGARCH.Ret.VIX.Ess</i>	0.04633	0.05929	0.05044	0.13082	0.00499	0.001045
<i>G.HN.Op.Ret.Qua</i>	0.05110	0.06275	0.05147	0.14451	0.00491	−0.00125
<i>G.HN.Ret.VIX.Qua</i>	0.05137	0.06331	0.05152	0.12488	0.00474	−0.00044
<i>G.GJR.Ret.VIX.Qua</i>	0.05124	0.06289	0.05108	0.11196	0.00410	−0.00036
<i>G.NGARCH.Ret.VIX.Qua</i>	0.05168	0.06240	0.05005	0.11032	0.00417	−0.00040
<i>NIG.NGARCH.Ret.VIX.EGP</i>	0.05016	0.06012	0.04967	0.10662	0.00395	−0.00013
<i>IG.Ret.Ess</i>	0.05435	0.06742	0.05147	0.13641	0.00511	−0.00118
<i>IG.Op.Ret.Ess</i>	0.04616	0.06105	0.05005	0.13159	0.00432	−0.00049
<i>IG.Op.Ret.Ushp</i>	0.04354	0.05664	0.04801	0.10616	0.00400	−0.00031
<i>IG.Ret.VIX.Ess</i>	0.04648	0.06183	0.05101	0.10108	0.00398	−0.00008
<i>IG.Ret.VIX.Ushp</i>	0.04387	0.05756	0.04809	0.09909	0.00396	−0.00019

Table 26: Rankings of the 24 competitors considered in this paper in terms of option pricing performances and VIX predictability obtained in Table 25.

GARCH	IVRMSE			VIX		
<i>Model</i>	<i>in</i>	<i>out</i>	<i>We</i>	<i>RMSE</i>	<i>MAE</i>	<i>MPE</i>
<i>G.HN.Ret.Ess</i>	24	24	24	24	24	24
<i>G.GJR.Ret.Ess</i>	22	23	22	23	23	23
<i>G.NGARCH.Ret.Ess</i>	20	22	23	22	22	21
<i>G.HN.Op.Ret.Ess</i>	18	20	20	20	21	22
<i>G.HN.Ret.VIX.Ess</i>	23	21	21	19	17	17
<i>G.GJR.Ret.VIX.Ess</i>	15	14	16	16	18	16
<i>G.NGARCH.Ret.VIX.Ess</i>	17	19	18	17	16	15
<i>NIG.HN.Ret.Ess</i>	21	18	19	21	20	20
<i>NIG.GJR.Ret.Ess</i>	16	16	17	18	19	19
<i>NIG.NGARCH.Ret.Ess</i>	19	17	15	15	14	18
<i>NIG.HN.Op.Ret.Ess</i>	12	12	9	14	15	11
<i>NIG.HN.Ret.VIX.Ess</i>	13	13	11	10	13	9
<i>NIG.GJR.Ret.VIX.Ess</i>	8	4	6	7	8	12
<i>NIG.NGARCH.Ret.VIX.Ess</i>	4	3	7	9	11	10
<i>G.HN.Op.Ret.Qua</i>	7	9	12	13	10	14
<i>G.HN.Ret.VIX.Qua</i>	10	11	14	8	9	7
<i>G.GJR.Ret.VIX.Qua</i>	9	10	10	6	5	5
<i>G.NGARCH.Ret.VIX.Qua</i>	11	8	4	5	6	6
<i>NIG.NGARCH.Ret.VIX.EGP</i>	6	5	3	4	1	2
<i>IG.Ret.Ess</i>	14	15	13	12	12	13
<i>IG.Opt.Ret.Ess</i>	3	6	5	11	7	8
<i>IG.Opt.Ret.Ushp</i>	1	1	1	3	4	4
<i>IG.Ret.VIX.Ess</i>	5	7	8	2	3	1
<i>IG.Ret.VIX.Ushp</i>	2	2	2	1	2	3