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Discriminating between GARCH models for option pricing by their ability to compute accurate VIX measures

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Abstract

In this paper, we discuss the pricing performances of a large collection of GARCH models by questioning the global synergy between the choice of the affine/non-affine GARCH specification, the use of competing alternatives to the Gaussian distribution, the selection of an appropriate pricing kernel and the choice of different estimation strategies based on several sets of financial information. Furthermore, the study answers an important question in relation to the correlation between the performance of a pricing scheme and its ability to forecast VIX dynamics. VIX analysis clearly appears as a parsimonious first-stage filter to discard the worst GARCH option pricing models.

JEL classification: C52, G13.

Keywords: GARCH option pricing models, GARCH implied VIX, estimation strategies, non-monotonic stochastic discount factors.

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Introduction

Over the past three decades, ARCH/GARCH type models, initiated by Engle (1982) and Bollerslev (1986), and their various extensions (see for example Chorro *et al.* (2015) Chapter 2) have become an important toolkit in the financial literature. Concerning the pricing of derivatives, Duan (1995) was the first paper to propose a coherent theoretical framework, namely the locally risk-neutral valuation relationship (LRNVR), to obtain risk-neutral dynamics of Gaussian GARCH models. This methodology was popularized in Heston & Nandi (2000) where a discrete time affine GARCH-type model with Gaussian innovations was able to replicate one of the key features observed in continuous time literature (Heston (1993)): the fact that the no-arbitrage price of classical European options had semi-closed-form expression.[1] Since then, these two seminal works have been extended in various directions and, when using GARCH-type models to price options, the modeler is now facing several important empirical choices namely the volatility structure, the distribution of the conditional returns, the risk-neutral framework, and the estimation strategy. What is more, to test the empirical validity of these choices, cumbersome numerical analysis have to be performed using extensive historical options data.

The aim of the present study is two-fold. Firstly, based on the most recent advances in this topic, it attempts to shed light on the interlinkages between the four key factors of GARCH option pricing models by questioning in details the global synergy between the choice of the affine/non-affine GARCH specification, the use of competing alternatives to the Gaussian distribution, the selection of an appropriate pricing kernel and the choice of different estimation strategies based on several sets of financial information. Up to our knowledge, this global approach is unique in the literature where in general one or two factors are questioned ceteris paribus. Secondly, the paper questions the correlation between the performance of a pricing scheme and its ability to forecast VIX dynamics and we clearly establish that the performance of a model in fitting VIX time series gives a very good indication on related pricing performances at a very reasonable computational cost. VIX analysis appears in this way as a very interesting and parsimonious first-stage evaluation to discard the worst GARCH option pricing models without using extensive historical options data.

More precisely, in order to improve the numerical performances of the seminal Duan's option pricing model, four complementary areas have been explored in the literature:

1. Use more realistic GARCH processes coping with asymmetric volatility responses,
2. Use non-Gaussian distributions to deal with conditional skewness and kurtosis,
3. Use different risk-neutralization processes compatible with the preceding points,

4. Use, when it is possible, more information than just that of the log-returns to estimate the model.

The two first points are now a classic topic and many extensions have been proposed to cope with these well-documented stylized facts. The asymmetric effects of positive and negative shocks of equal magnitude on conditional volatility, the so-called leverage effect, may be captured using a large family of extended GARCH models the most popular being probably the exponential EGARCH of Nelson (1991), the NGARCH model of Engle & Ng (1993), the GJR-GARCH of Glosten *et al.* (1993), the threshold GARCH of Zakoian (1994), and the affine HN-GARCH by Heston & Nandi (2000). However, the leverage parameter of preceding specifications is not sufficient to capture all the skewness and kurtosis levels in standardized residuals. Therefore, Gaussian hypothesis for the conditional distribution of log-returns has to be relaxed and a myriad of possible choices may be used to take into account all the mass in the tails and the asymmetry (Chorro *et al.* (2015) Chapter 2). Among them, the Generalized Hyperbolic (Chorro *et al.* (2012), Badescu *et al.* (2011)) family and its Normal Inverse Gaussian (NIG) subclass (Stentoft (2008), Badescu *et al.* (2015)), the Inverse Gaussian (IG) distribution (Christoffersen *et al.* (2006a)), or the mixture of Gaussian (Badescu *et al.* (2008)) clearly improve forecasting performances of related GARCH models.

Once a competing model has been chosen, the choice of the so-called stochastic discount factor (SDF) to obtain risk-neutral dynamics is fundamental. For this third point, two constraining factors apply: this SDF has to be sufficiently flexible to provide explicit risk-neutral dynamics for a large variety of GARCH structures and innovation distributions and rich enough to produce good pricing performances. Since the seminal paper of Duan, several tools have been developed to select an equivalent martingale measure (see for example Chorro *et al.* (2015) Chapter 3).[2]

Finally, one of the main advantages of GARCH models, with respect for example to stochastic volatility ones[3], is that they may be efficiently estimated using a conditional version of the maximum likelihood estimation and a dataset of log-returns. In particular, since, in the case of exponential-affine or extended Girsanov principle SDF, the associated risk-neutral dynamics are explicit transforms of the historical ones, only log-returns information is needed to compute or approximate European option prices.[4] Even so, when an extra piece of financial information (price of plain vanilla options, the VIX index for the S&P500,...) is available it can be of interest to integrate it, in an efficient way, to the estimation process to reduce pricing errors. Therefore, following Christoffersen *et al.* (2012) it is now classically possible to build for some affine GARCH models (at the very least for the HN-GARCH Heston & Nandi (2000) and the IG-GARCH Christoffersen *et al.* (2006a) where semi-closed form expressions for option prices are obtained) a joint maximum likelihood based on log-returns and option prices. In this setting, the affine structure of the model is mandatory: if prices are evaluated using Monte-Carlo methods, computing the likelihood function may be cumbersome. In a recent study, Hao & Zhang (2013) have computed VIX index formulas implied by various non-affine asymmetric Gaussian GARCH models. They presented closed-form formulas for the VIX index associated with five classical non-affine Gaussian GARCH models when Duan (1995) LRNVR is used. Based on this result, Kanniainen *et al.* (2014) proposed a fair comparison between affine and non-affine Gaussian GARCH specifications using log-returns and VIX information in the estimation.[5] For two affine GARCH models Chorro & Fanirisoa (2019) and Papantonis (2016) proved that incorporating both the physical return dynamics of the index and risk-neutral dynamics of the VIX to estimate the parameters of GARCH option pricing models provides competitive pricing errors at a very low computational cost.[6]

This paper attempts to fill several gaps in the GARCH option pricing literature, in particular, from an empirical point of view.

Firstly, in the spirit of Christoffersen *et al.* (2004) the aim of our study is to provide an intensive comparison analysis of empirical performances, in VIX index or options valuation, between different GARCH-type models using Gaussian or non-Gaussian distributions under different classes of risk-neutral measures. Furthermore, particular attention is granted on the choice of the information set (VIX, options, returns) in the estimation process. To keep the empirical analysis manageable, we only focus our attention on four classical parsimonious GARCH(1,1) structures: HN-GARCH by Heston & Nandi (2000), GJR-GARCH by Glosten *et al.* (1993), NGARCH by Engle & Ng (1993), and IG-GARCH by Christoffersen *et al.* (2006a).[7] One advantage of this choice is to question the difference between affine and non-affine models. As a natural non-Gaussian alternative we favor the so-called NIG distribution not only because it is known to fit statistical properties of asset returns remarkably but also because, combined with Esscher and EGP SDF, pricing equations may be solved explicitly.[8] Furthermore, monotonic and non-monotonic pricing kernels (Monfort & Pégoraro (2012) and Chorro & Fanirisoa (2019)) are considered for Gaussian and IG distributions.

To our knowledge, in the existing literature, empirical studies questioned, in general, the impact of the distribution (Christoffersen *et al.* (2006a), Chorro *et al.* (2012)), the choice of the SDF (Badescu *et al.* (2011), Christoffersen *et al.* (2013), Chorro & Fanirisoa (2019)) or the estimation strategy (Hao & Zhang (2013), Kanniainen *et al.* (2014), Papantonis (2016), Lalancette & Simonato (2017)) on pricing performances, but none of them consider all these factors at the same time. For example, in Christoffersen *et al.* (2004) and Kanniainen *et al.* (2014) the authors study different GARCH structures with different estimation strategies, but restrict themselves to the Gaussian setting while in Chorro & Fanirisoa (2019) the authors focus on different SDF and estimation strategies only for the IG-GARCH model. Our study is a means of making a contribution to understand the combined impact of these complementary aspects (21 combinations of GARCH-distribution-SDF-estimation are tested), instead of providing restrictive pairwise comparisons, and to conclude that the combination of all them is fundamental to producing competitive valuation errors.

Secondly, we also explore in this paper if it is possible to partly classify a large family of GARCH option pricing models by their ability to simply reproduce the VIX index. In fact, the correlation between the option pricing performances of a model and its ability to compute accurate VIX measures is a natural question that appears in many talks and discussions among experts but, up to our knowledge, it is not clearly and rigorously addressed in the literature. Our methodology is inspired by the work of Hao & Zhang (2013) that intuitively explained the poor pricing performances of Gaussian GARCH models (risk-neutralized using the LRNVR) by their inefficiency to capture the variance risk premium. In this paper, we not only extend their conclusion exploring its robustness for non-Gaussian distributions and non-standard SDF[9] but also supporting our findings with a deep empirical study based on pricing errors associated with a large real-world dataset of option prices. Here, a challenging aspect is to make VIX analysis a first-stage filter to discard the worst GARCH option pricing models. From purely numerical aspects, such a conclusion would be very interesting to back-test these models in an efficient way, using only VIX information, when available, instead of complex option datasets.

This paper is structured along the following lines. In section 1 we first provide a partial presentation of all competing GARCH frameworks used in the empirical part. More precisely, we consider four GARCH structures for modeling volatility as a time-varying process: HN-GARCH, GJR, NGARCH, and IG-GARCH. Then, in section 2, we recap the main risk-neutralized frameworks adopted in this study. Next, in section 3, we derive the related VIX index formulas. Section 4 deals with the estimation challenge, presenting methodologies based on different information sets and the related numerical results in terms of VIX approximation and option pricing. We conclude in section 5.

1. Competing GARCH models

We consider a financial asset with a market price at time t given by S_t and we denote by $Y_t = \log\left(\frac{S_t}{S_{t-1}}\right)$ the associated log-returns defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ represents the historical probability measure. Information filtration $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ is generated by log-returns supposing that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. In what follows, we consider a general dynamics for the stock price process:

$$\begin{aligned} Y_t &= r + m_t + \sqrt{h_t} z_t \\ h_t &= F\left(z_{t-1}, h_{t-1}, \theta^V\right) \end{aligned} \tag{1}$$

where the z_t are i.i.d centered and reduced random variables depending on a vector of parameters θ^D, m_t is the predictable time-varying excess of returns, r is the risk-free rate and F is a mapping, compatible with realistic GARCH(1,1) volatility models that depends on a vector of parameters θ^V. From now on, the initial value h_0 of the conditional volatility is supposed to be constant and fixed at its unconditional level depending on the persistence of the model Ψ (i.e the coefficient in front of h_t in $\mathbb{E}_{\mathbb{P}}\left[h_{t+1} \mid \mathcal{F}_{t-1}\right]$).

For our empirical horse-race we favor four particular GARCH specifications often used in the literature to cope with volatility clustering and leverage effect. Moreover, these four GARCH-type models belong to two important families: affine and non-affine frameworks. While affine GARCH models are often used because they yield a semi-closed form solution for prices of European equity options, it is now well-documented (see for example Christoffersen *et al.* (2006b)) that non-affine ones provide a better fit to financial data. One important aspect of our empirical study will be to question once again this duality. Following Kanniainen *et al.* (2014) we choose the widely recognized NGARCH Engle & Ng (1993), GJR-GARCH Glosten *et al.* (1993), and affine HN-GARCH Heston & Nandi (2000) models and we add the IG-GARCH of Christoffersen *et al.* (2006a) (see also Chorro & Fanirisoa (2019)) that is a notable example of an affine model within a non-Gaussian setting. In the next sections we briefly recall the definitions and the main properties of these specifications.

1.1. *Affine competitors*

Since the seminal work of Heston (1993), affine models, that led to semi-closed form expressions for option prices, are the keystone of almost all numerical studies. In the discrete time literature, the HN-GARCH Heston & Nandi (2000) and the IG-GARCH of Christoffersen *et al.* (2006a) are two important contributions. More precisely, the historical dynamics are given by:

- ***The HN-GARCH model***

$$\begin{cases} Y_t &= r + \lambda_0 h_t + \sqrt{h_t} z_t \\ h_t &= a_0 + a_1 \left(z_{t-1} - \gamma \sqrt{h_{t-1}} \right)^2 + b_1 h_{t-1} \end{cases} \tag{2}$$

with $a_0 > 0$, $a_1 \geq 0$, $b_1 \geq 0$

- ***The IG-GARCH model***

$$\begin{cases} Y_t &= r + \nu h_t + \eta z_t \\ h_t &= w + b h_{t-1} + c z_{t-1} + a \frac{h_{t-1}^2}{z_t} \end{cases} \tag{3}$$

with $w > 0$, $b \geq 0$, $c \geq 0$, and $a \geq 0$.

In the HN-GARCH model the z_t are supposed to be Gaussian while in the IG-GARCH they follow an Inverse Gaussian distribution with degree of freedom $\delta_t = \frac{h_t}{\eta^2}$ whose probability density function is given by

$$d_{z_t}(z) = \mathbf{1}_{\{z>0\}} \frac{\delta_t}{\sqrt{2\pi z^3}} e^{-\left(\sqrt{z} - \delta_t / \sqrt{z}\right)^2 / 2}. \tag{4}$$

The persistence (that will be an important quantity to express associated VIX index formula) of the HN-GARCH (resp. IG-GARCH) is given by $\Psi = b_1 + a_1 \gamma^2$ (resp. $\Psi = b + \dfrac{c}{\eta^2} + a\eta^2$). Under these two hypotheses on the distributions of innovations, it is easy to prove for both models that the conditional moment generating function $\mathbb{G}^{\mathbb{P}}_{\log(S_T)|\mathcal{F}_t}(u) = E_{\mathbb{P}}[S_T^u \mid \mathcal{F}_t]$ of the log asset price under the physical measure can be written in the following log-linear form $\mathbb{G}^{\mathbb{P}}_{\log(S_T)|\mathcal{F}_t}(u) = S_t^u e^{A_t + B_t h_{t+1}}$ where the coefficients A_t and B_t can be obtained by working backward from the maturity date of the option and using terminal conditions $A_T = B_T = 0$. More precisely, for the HN-GARCH model,

$$\begin{cases} A_t &= ru + A_{t+1} + a_0 B_{t+1} - \frac{1}{2} log(1 - 2a_1 B_{t+1}) \\ B_t &= -\frac{1}{2} u + b_1 B_{t+1} + \left(\frac{u^2}{2} - 2a_1 \gamma B_{t+1} u + a_1 B_{t+1} \gamma^2 \right) (1 - 2a_1 B_{t+1})^{-1} \end{cases} \tag{5}$$

and for the IG-GARCH model

$$\begin{cases} A_t &= A_{t+1} + ur + w B_{t+1} - \frac{1}{2} log(1 - 2a\eta^4 B_{t+1}) \\ B_t &= b B_{t+1} + u\nu + \eta^{-2} - \eta^{-2} \sqrt{(1 - 2a\eta^4 B_{t+1})(1 - 2c B_{t+1} - 2u\eta)}. \end{cases} \tag{6}$$

Moreover, one important empirical consequence for the pricing of European call options is that the very particular form of the conditional moment generating function of $log(S_T)$ leads to the existence of semi-closed form expressions for prices which allow us to use Fast Fourier Transform (FFT) methodology and option information in the estimation procedure as explained in Chorro *et al.* (2015) Chap 4.

1.2. Non-affine competitors

In order to propose asymmetric extensions of the original GARCH(1,1) model, one possibility is to modify the so-called news impact curve (NIC) introduced in Engle & Ng (1993). For this purpose, we may shift a symmetric NIC to the right or consider curves centered at 0 allowing for slopes of different magnitudes on either side of the origin. These two approaches were used by Engle & Ng (1993) and Glosten *et al.* (1993) in order to introduce respectively the popular NGARCH and GJR models. In both cases, a single leverage parameter constrains the response of the conditional variance to depend on the sign of a shock:

- ***The NGARCH model***

$$\begin{cases} Y_t &= r + \lambda_0\sqrt{h_t} - log(E_{\mathbb{P}}[e^{\sqrt{h_t}z_t}]) + \sqrt{h_t}z_t \\ h_t &= a_0 + b_1 h_{t-1} + a_1 h_{t-1}\left(z_{t-1} - \gamma\right)^2 \end{cases} \tag{7}$$

with $a_0 > 0$, $b_1 \geq 0$, $a_1 \geq 0$

- ***The GJR model***

$$\begin{cases} Y_t &= r + \lambda_0\sqrt{h_t} - \frac{h_t}{2} + \sqrt{h_t}z_t \\ h_t &= a_0 + h_{t-1}\left[b_1 + a_1\left(z_{t-1}\right)^2 + \gamma \max\left(0, -\left(z_{t-1}\right)\right)^2\right] \end{cases} \tag{8}$$

with $a_0 > 0$, $b_1 \geq 0$, $a_1 \geq 0$, and $\gamma \geq 0$.

The persistence of the NGARCH (resp. GJR) is given by $\Psi = b_1 + a_1\left(1 + \gamma^2\right)$ (resp. $\psi = b_1 + a_1 + \frac{\gamma}{2}$). Contrary to models presented in the preceding section, here, conditional moment generating function is not an exponential-affine function of the one step ahead volatility. To compute option prices we use in general Mont-Carlo approximations.

To conclude this subsection, let us discuss the main reasons for the choice of the conditional excess return in (7) and (8). For the GJR model, we take the classical Duan (1995) specification $m_t = \lambda_0\sqrt{h_t} - \frac{h_t}{2}$ while in the NGARCH model we follow Badescu *et al.* (2019) and take $m_t = \lambda_0\sqrt{h_t} - log(E_{\mathbb{P}}[e^{\sqrt{h_t}z_t}])$. These choices may appear arbitrary because in the non-affine setting they are not restricted to having an affine function of the conditional variance. Nevertheless, we can first remark that for Gaussian innovations both coincide. Then, as remarked in Hao & Zhang (2013), VIX implied formulas are available in this non-affine setting at the very least for Gaussian innovations. Finally, for the NGARCH model with NIG innovations, this very particular form may lead to a closed form expression for the model implied VIX as explained in Badescu *et al.* (2019). This property is remarkable because up to our knowledge this is the unique example in the literature of an explicit VIX index formula within a non-Gaussian and non-affine setting.

1.3. A flexible alternative to Gaussian distribution

It is now a well-known fact that forecasting performances of GARCH-type models are improved when using non-Gaussian innovations. Historically, several interesting distributions were proposed to better account for the deviation from normality. In the present section we have decided to mainly focus our attention on the Normal Inverse Gaussian (NIG) distribution. This four-parameter family of distributions has been extensively used during the last decade in discrete time literature, especially for pricing issues (Stentoft (2008), Badescu *et al.* (2011), Guégan *et al.* (2013), Badescu *et al.* (2019)): for $(\alpha, \beta, \delta, \mu)$ fulfilling $0 < |\beta| < \alpha$ and $\delta > 0$, the density of the NIG $(\alpha, \beta, \delta, \mu)$ is given by

$$d_{NIG}\left(z, \alpha, \beta, \delta, \mu\right) = \frac{\alpha}{\pi}\, e^{\delta\left(\sqrt{\alpha^2-\beta^2}+\beta\left(\frac{z-\mu}{\delta}\right)\right)}\, \frac{K_1\left(\alpha\delta\sqrt{1+\left(\frac{z-\mu}{\delta}\right)^2}\right)}{\sqrt{1+\left(\frac{z-\mu}{\delta}\right)^2}}$$

(where K_1 is the modified Bessel function of the third kind with index one) and the associated cumulant generating function by

$$\kappa_{NIG}(z) = \mu z + \delta\sqrt{\alpha^2-\beta^2} - \delta\sqrt{\alpha^2-(\beta+z)^2}.$$

The mean and the variance of this distribution are respectively given by

$$m = \mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}, \qquad \sigma^2 = \frac{\delta\alpha^2}{\left(\sqrt{\alpha^2-\beta^2}\right)^3}. \tag{9}$$

Therefore, from the stability of the NIG family under affine transforms, it is possible to obtain a centered version with unit variance considering

$$\text{NIG}\left(\tilde{\alpha},\tilde{\beta},\tilde{\delta},\tilde{\mu}\right)=\frac{\text{NIG}\left(\alpha,\beta,\delta,\mu\right)-m}{\sigma} \tag{10}$$

where $\tilde{\alpha}=\sigma\alpha$, $\tilde{\beta}=\sigma\beta$, $\tilde{\delta}=\frac{\delta}{\sigma}$ and $\tilde{\mu}=\frac{-m}{\sigma}+\frac{\mu}{\sigma}$.

2. Stochastic discount factors and risk-neutral dynamics

From the beginning of the 80's (see Chorro *et al.* (2015) Chap 3 and references therein), general methods providing arbitrage-free price processes via the notion of equivalent martingale measure (EMM) have been investigated both in discrete or continuous time frameworks. Furthermore, the choice of such an EMM is known to be equivalent to the specification of the so-called one-period stochastic discount factor (SDF). Since markets described by GARCH models are incomplete, there is a priori an infinite number of SDF available for pricing derivatives and a great challenge is to select tractable candidates for their strong economic foundations and/or empirical performances. In this section, we present the main paths to risk-neutralization that will be implemented in the numerical part to obtain arbitrage-free price approximations in Gaussian or non-Gaussian settings. More specifically, starting from the Duan (1995) approach particularly well-adapted to Gaussian residuals, we briefly recall the main lines of the recent advances in modeling SDF dynamics to cope with non-Gaussian innovations (Elliott & Madan (1998) extended Girsanov principle (EGP) and Siu *et al.* (2004) conditional Esscher transform) and/or have better representations of volatility risk (Monfort & Pégoraro (2012), Chorro & Fanirisoa (2019)). Here, the objective is not to provide a self-contained presentation of these classical tools but to remind about the main intuitions behind Gaussian (see Table A1 in Appendix A) and non-Gaussian (see Table A2 in Appendix A) risk-neutral dynamics that will be compared in the empirical part. We refer the interested reader to the technical Appendix A describing in details all the SDF implicitly used in the paper.

As in the preceding section, we consider a GARCH-type specification for the log-returns

$$\begin{aligned} Y_t &= r + m_t + \sqrt{h_t} z_t \\ h_t &= F\left(z_{t-1}, h_{t-1}, \theta^V\right) \end{aligned} \tag{11}$$

where the z_t are i.i.d centered random variables with unit variance.

- ***Duan's Local Risk-Neutral Valuation Relationship***

Supposing that the z_t are i.i.d $\mathcal{N}(0,1)$, Duan (1995) was the first to provide a coherent theoretical CCAPM framework to obtain risk-neutral dynamics in a GARCH environment independently of the underlying GARCH structure. More precisely, if $\mathbb{Q}$ is an EMM fulfilling LRNVR (a set of assumptions made on the utility function and the aggregated consumption growth that preserves both Gaussianity and volatility) then

$$\begin{aligned} Y_t &= r - \tfrac{h_t}{2} + \sqrt{h_t} z_t^* \\ h_t &= F\left(z_{t-1}^* - \tfrac{m_{t-1}}{\sqrt{h_{t-1}}} - \tfrac{\sqrt{h_{t-1}}}{2}, h_{t-1}, \theta^V \right) \end{aligned} \tag{12}$$

where the z_t^* are i.i.d $\mathcal{N}(0,1)$ under $\mathbb{Q}$. For Gaussian models presented in the preceding section, risk-neutral dynamics deduced from the Duan's argument are given in Table A1 in Appendix A. In the non-affine GJR and NGARCH setting, prices may be obtained from (12) using Monte-Carlo approximations while in the affine HN case semi-closed form formulas are available. Nevertheless, Duan's framework relies on Gaussian hypotheses and cannot be adapted with simplicity to more general distributions.

- ***The Extended Girsanov principle***

Duan's framework relies on Gaussian hypotheses and cannot be adapted with simplicity to more general distributions. Based on this observation, Elliott & Madan (1998) proposed a very simple way to select a SDF based on a Girsanov-type transformation that preserves returns distribution after the change of measure by only shifting the conditional mean to fulfill the martingale restriction. Such a pricing kernel has also been justified from its consistency with risk-adjusted cost minimizing hedging strategies, and under the EMM $\mathbb{Q}^{EGP}$ we have

$$\begin{aligned} Y_t &= r + m_t - \nu_t + \sqrt{h_t} z_t^* \\ h_t &= F(z_{t-1}^* - \tfrac{\nu_{t-1}}{\sqrt{h_{t-1}}}, h_{t-1}, \theta^V) \end{aligned} \tag{13}$$

where z_t^* follows the same law as z_t under $\mathbb{P}$ and where ν_t fulfills $e^{\nu_t} = e^{-r} E_{\mathbb{P}} \left[e^{Y_t} \mid \mathcal{F}_{t-1} \right]$. When the z_t are assumed to be Gaussian, we recover the same dynamics as in (12). Moreover, following Badescu *et al.* (2019), for NIG innovations this is a tractable framework, especially when combined with the NGARCH model to obtain a closed-form formula for the associated VIX index. In fact, the restriction imposed on the conditional mean in (7) provides explicit computations. Nevertheless, one of the major drawback of this approach, that may explain partly poor pricing performances of this method for long maturity options (see Badescu *et al.* (2008) and Badescu *et al.* (2011)), is the fact that from $\mathbb{P}$ to $\mathbb{Q}^{EGP}$ only the conditional mean is affected while the conditional variance, skewness and, kurtosis are the same.

- ***Exponential affine SDF: the conditional Esscher transform***

The conditional Esscher transform introduced in the GARCH setting by Siu *et al.* (2004) and Gouriéroux & Monfort (2007) is probably one of the best-known tool to select efficiently EMM. The associated SDF M^{Ess} is exponential-affine of log-returns and the predictable associated coefficients of affinity are uniquely determined by the pricing equations related to the bond and the risky asset. In contrast to Duan's approach a wide variety of return innovations may be chosen at the very least within the class of mixture or infinitely divisible distributions (see Chorro *et al.* (2015) Chap 3.4). Even if this tool coincides with the LRNVR in the Gaussian case, it allows for strongly non-linear relations between historical and risk-neutral volatility in the non-Gaussian setting. Furthermore, explicit risk-neutral dynamics (see Table A2 in Appendix A) may be obtained for the IG-GARCH model (3) and GARCH-type models with NIG innovations. In particular, if we suppose in (11) a NIG $\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\mu}\right)$ for the z_t, we obtain (Badescu *et al.* (2011)) the following dynamics under the Esscher EMM:

$$\begin{aligned} Y_t &= r + m_t + \sqrt{h_t} z_t^* \\ h_t &= F\left(z_{t-1}^*, h_{t-1}, \theta^V\right) \end{aligned} \tag{14}$$

where z_t^* follows, under $\mathbb{Q}^{Ess}$, a NIG$(\tilde{\alpha}, \tilde{\beta} + \sqrt{h_t}\theta_t^q, \tilde{\delta}, \tilde{\mu})$ with a predictable parameter θ_t^q having an explicit form (see Appendix A).

- ***Quadratic and U-shaped SDF***

As remarked in Monfort & Pégoraro (2012), the exponential-affine hypothesis concerning the SDF only allows for an equity risk premium and it may be interesting to partly solve empirical puzzles of option prices taking into account a second-order variance risk premium. To achieve this, the authors introduced an exponential-quadratic SDF M^{Qua} that extends M^{ess} adding a second moment-based source of risk information. Moreover, under Gaussian hypothesis, this new change of measure preserves the tractability of the model with a risk-neutral dynamics given by

$$\begin{aligned} Y_t &= r - \tfrac{h_t^*}{2} + \sqrt{h_t^*} z_t^* \\ h_t^* &= \pi F\left(\sqrt{\pi}(z_{t-1}^* - \tfrac{m_{t-1}}{\sqrt{h_{t-1}^*}} - \tfrac{\sqrt{h_{t-1}^*}}{2}), \tfrac{h_{t-1}^*}{\pi}, \theta^V\right) \end{aligned} \tag{15}$$

where the z_t^* are i.i.d $\mathcal{N}(0,1)$ under $\mathbb{Q}^{Qua}$ and π is the proportional wedge between risk-neutral and historical volatilities assumed to be constant across time (see Appendix A for more details). As a consequence, for the HN model (2), the dynamics under $\mathbb{Q}^{Qua}$ remains in the same family of affine GARCH models, preserving analytic properties of the HN specification in terms of option pricing. Inspired by this new methodology, Chorro

& Fanirisoa (2019) (see also Babaoglu *et al.* (2018)) proposed an exponential-hyperbolic SDF M^{Ushp} that is able to cope with the same remarkable features in the case of the IG-GARCH model (3) (see Appendix A).

To conclude this section, let us precisely describe all related GARCH option pricing models that will be tested in the empirical part: in the affine family, the classical Heston & Nandi (2000) and the IG-GARCH model (Christoffersen *et al.* (2006a) and Chorro & Fanirisoa (2019)) will be combined with exponential-affine and U-shaped SDF risk-neutralization processes. In these cases, Monte-Carlo methods won't be used to approximate the price of plain vanilla options. To relax the constraints on variance dynamics and conditional distributions related to affine specifications, we will also study two classical non-affine structures namely the GJR and NGARCH models with Gaussian or NIG innovations. In the Gaussian case the dynamics will be risk-neutralized using the LRNVR or the quadratic SDF while under NIG hypotheses, exponential-affine and EGP assumptions will be favored. This great variety of models and SDF will allow us to question several key aspects of GARCH option pricing modeling. Finally, for sake of concision and simplicity all the risk-neutral dynamics used in this study are gathered in Table A1 for Gaussian innovations and in Table A2 otherwise.

3. Model implied CBOE VIX

Considered as the investor's expectation of volatility (see Carr & Wu (2006)), the CBOE VIX index can be characterized as a forecast of the 30-day risk-neutral volatility (or 22 working days) of the S&P500 index. In this section, we denote by Vix_t a daily-based proxy for VIX_t which is the daily-adjusted expression of the expected arithmetic average of variance (see Hao & Zhang (2013)):

$$\mathrm{Vix}_t = \frac{1}{\tau}\left(\frac{\mathrm{VIX}_t}{100}\right)^2 = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{T_c}\int_t^{t+T_c} h_u du \mid \mathcal{F}_t\right] \approx \frac{1}{T_c}\sum_{j=1}^{T_c}\mathbb{E}_{\mathbb{Q}}\left[h_{t+j} \mid \mathcal{F}_t\right] \tag{16}$$

where $\tau = 250$, $T_c = 22$ represents the maturity in days and $\mathbb{Q}$ is an EMM. Depending on the choice of the risk-neutral dynamics and using iterative properties of conditional expectation, the term $\mathbb{E}_{\mathbb{Q}}\left[h_{t+j} \mid \mathcal{F}_t\right]$ can be explicitly computed for a large class of Gaussian (Hao & Zhang (2013)) and non-Gaussian (Chorro & Fanirisoa (2019), Badescu *et al.* (2019)) GARCH models. In general, $\mathbb{E}_{\mathbb{Q}}\left[h_{t+j} \mid \mathcal{F}_t\right]$ can be expressed as a linear function of historical volatility at time $t+1$, risk-neutral unconditional variance $\tilde{h}_0$, and risk-neutral variance persistence Ψ^* under the selected EMM. If we can obtain analytic expressions, we have the following general form for $\mathbb{E}_{\mathbb{Q}}\left[h_{t+j} \mid \mathcal{F}_t\right]$ and Vix_t:

$$\begin{cases} \mathbb{E}_{\mathbb{Q}}\left[h_{t+j} \mid \mathcal{F}_t\right] = h_{t+1}\left[\Psi^*\right]^{j-1} + \tilde{h}_0\left[1-\left(\Psi^*\right)^{j-1}\right] \\ \mathrm{Vix}_t = h_{t+1}\dfrac{1-\left(\Psi^*\right)^{T_c}}{\left(1-\Psi^*\right)T_c} + \tilde{h}_0\left(1-\dfrac{1-\left(\Psi^*\right)^{T_c}}{\left(1-\Psi^*\right)T_c}\right) \end{cases} \tag{17}$$

where expressions of $\tilde{h}_0$ and Ψ^* for particular models and SDF are reported in Table A3 in Appendix A. In fact, for Gaussian models under the LRNVR and for affine models with exponential-affine or U-shaped SDF we have closed form expressions. For example, in the case of the HN model, we obtain $\tilde{h}_0 = \dfrac{a_0+a_1}{1-\Psi^*}$ and $\Psi^* = b_1 + a_1(\gamma+\lambda_0+\frac{1}{2})^2$ when an exponential-affine SDF is used while we obtain $\tilde{h}_0 = \dfrac{a_0+\pi a_1}{1-\Psi^*}$ and $\Psi^* = b_1 + \pi^2 a_1\left(\dfrac{\gamma}{\pi}+\dfrac{\lambda_0}{\pi}+\dfrac{1}{2}\right)^2$ under the quadratic SDF.

Unfortunately, in the case of NIG innovations (a notable exception is the NIG NGARCH model associated with the EGP of Badescu *et al.* (2019)) or when an exponential-quadratic SDF is used with the Gaussian NGARCH and GJR structures we do not have closed-form formulas for the implied Vix_t. However, as explained in Lalancette & Simonato (2017) we can still use Monte-Carlo simulations to approximate conditional expectation $\mathbb{E}_{\mathbb{Q}}\left[h_{t+j} \mid \mathcal{F}_t\right]$ and Vix_t.

4. Methodology and empirical results

In this section, we present the main points emerging from this analysis. First, we carry out numerical experiments to analyze pricing performances of all competing GARCH models, focusing on affine/non-affine structures, the risk-neutralization process and the estimation methodology. A pool of 21 possible combinations (Model/SDF/Estimation) will thereby be tested to try to understand the impact of underlying factors. Furthermore, a second experiment aims to question the possibility of partly ranking GARCH option pricing models by their ability to simply reproduce VIX dynamics, instead of using a heavy set of option data. More specifically, after a brief description of the data, we present the main lines of classical joint likelihood estimation methodologies based on Option-Returns or VIX-Returns data (see for example Kanniainen *et al.* (2014) and reference therein) and that of the two-step estimation strategy recently introduced in Chorro & Fanirisoa (2018) for NIG-GARCH processes. Then, when closed-form expressions for option prices are not available, we recall how Monte-Carlo approximations may be implemented efficiently in the GARCH framework using the powerful and simple adjustment proposed by Duan & Simonato (1998). Finally, this section ends with a presentation of the results based on our empirical findings.

4.1. Data description

The present study used S&P500 daily returns and VIX data from January 07, 1999 to December 22, 2010, which are composed of 2718 observations covering about 12 years. We plotted in Figure 1 the S&P500 and CBOE VIX indexes with their log-returns series while Table 1 displayed associated summary statistics. This information set was used to implement both classical conditional maximum likelihood strategies and joint estimation strategies based on returns and VIX information.

We also used a dataset of options written on the S&P500 obtained from Bloomberg. Due to the number of option pricing models to test in this section, we restricted ourselves to Wednesday's contracts and we classically apply to our dataset the same filters as described in Bakshi *et al.* (1997). Therefore, it concerned 4563 options contracts whose prices were quoted during the period spanning from January 2nd, 2009 to April 15, 2012. We divided the option data set into two subsets: one in which model parameters are estimated (to implement for the affine models the joint likelihood estimation based on returns and options) and another subset used to compare pricing performances of models. The first subset, used for the in-sample estimation and comparison, is called Dataset A from January 2nd, 2009 until December 22, 2010 and contains 2714 contracts. However, the second subset for the out-of-sample comparison is called Dataset B and contains 1849 contracts with 67-Wednesdays from January 03, 2011 until April 15, 2012. This will be used to test the out-of-sample ability to capture the behavior of the index option smile (VIX data from January 03, 2011 until April 15, 2012 are also used in the empirical part to test the ability of GARCH option pricing models to forecast VIX dynamics). Summary statistics for option data are reported in Table 2 for both Dataset A and B: this table shows the number of contracts, the average price, and the average implied volatility across moneynesses and times to maturity. The patterns in the Dataset B are clearly similar to those in the in-sample Dataset A.

Depending on the chosen estimation strategy, the in-sample dataset of returns is combined with in-sample VIX data or Dataset A to estimate the model as explained in the next section. Furthermore, usual in and out-of-sample option pricing performances are studied: we use in-sample estimated parameters to compute approximate prices (from FFT or Monte-Carlo approximations depending on the structure of the model) for the contracts in Dataset A and B to analyze associated errors. In the out-of sample exercise presented above, we assumed that model's parameters are constant over the whole sample period (Dataset B). Obviously, this may appear as unrealistic and unfair for the simulation and relaxing this assumption will highlight the robustness of our conclusions. Therefore, in a complementary numerical experiment, we allowed model parameters to change over time through a rolling window estimation strategy for the 67 Wednesdays in the Dataset B assuming a constant window of 12 years (resp. 2 years) for log-returns and VIX data (resp. for options). For each Wednesday in dataset B, we estimated each model and used corresponding parameters to price options next Wednesday.[10]

4.2. Estimation methodologies

In this section we denote by ϑ the set of risk-neutral parameters associated with historical dynamics (1). When conditional Esscher transform or extended Girsanov principle are used to obtain risk-neutral dynamics we simply have $\vartheta = (\theta^D, \theta^V)$ while $\vartheta = (\theta^D, \theta^V, \pi)$ in the case of U-shaped pricing kernels where θ^D is the vector of innovation parameters, θ^V represents the volatility parameters, and π is the proportional wedge between risk-neutral and historical volatilities supposed to be constant. Moreover, we denote by T (resp. N) the number of VIX and log-returns daily observations (resp. N the cardinal of the set of option market prices) involved in the estimation process. One of the main advantages of the GARCH machinery is that historical model parameters (θ^D, θ^V) may be easily obtained, from a simple log-returns dataset, using a conditional version of the classical maximum likelihood estimator maximizing

$$\log L_R(\theta^D, \theta^V) = \sum_{t=1}^{T} \log \left(\frac{1}{\sqrt{h_t}} f_{\theta^D} \left(\frac{Y_t - (r + m_t)}{\sqrt{h_t}} \right) \right)$$

where f_{θ^D} is the probability density function of the model innovations. However, the proportional wedge between historical and risk-neutral volatility π cannot be estimated only using returns data. Moreover, during the last decade, several empirical studies underlined the real interest to incorporate in the estimation process VIX or option information, when available, to improve related pricing performances. Therefore, we present below two joint likelihood estimation strategies used in the empirical part:

- ***Joint estimation strategy using Option-Returns information***

We consider a set of option market prices $(\hat{c}_1, ..., \hat{c}_N)$ and define associated weighted Vega errors $\epsilon_i = \dfrac{c_i - \hat{c}_i}{\hat{V}_i}$ where c_i and $\hat{V}_i$ are the model prices and the Black and Scholes Vega associated with $\hat{c}_i$. Following Trolle & Schwartz (2009), we suppose that the (ϵ_i) are i.i.d centered Gaussian variables with variance $\dfrac{1}{N}\sum_{i=1}^{N} \epsilon_i^2$. Therefore, the associated option log-likelihood is given by

$$\log L_{Op}(\vartheta) = -\frac{1}{2}\sum_{i=1}^{N}\left[\log\left(\frac{1}{N}\sum_{i=1}^{N}\epsilon_i^2\right) + \frac{\epsilon_i^2}{\frac{1}{N}\sum_{i=1}^{N}\epsilon_i^2}\right]$$

and we obtain the joint Option-Returns likelihood (see Christoffersen *et al.* (2013)):

$$\frac{T+N}{2}\frac{\log L_R((\theta^D,\theta^V))}{T} + \frac{T+N}{2}\frac{\log L_{Op}(\vartheta)}{N}. \tag{18}$$

One of the major drawbacks of this approach is the requirement to evaluate several times the objective function (18) in the maximization process. In the case of affine GARCH models presented above, independently of the choice of the exponential-affine or exponential U-shaped SDF, closed-form expressions for option prices are available and make this process computationally acceptable. As noticed in Section 3, for most of Gaussian GARCH specifications and for the NIG NGARCH model combined with the EGP it is possible to obtain closed-form expressions for the implied VIX. Therefore, as provided by Kanniainen *et al.* (2014), a similar strategy based on VIX information and not on options one may be implemented.

- ***Joint estimation strategy using VIX-Returns information***

To build the VIX log-likelihood we suppose with Kanniainen *et al.* (2014) (see also Chorro & Fanirisoa (2019) or Badescu *et al.* (2019)) that VIX pricing errors $u_t = \text{VIX}_t^{Market} - \text{VIX}_t^{Model}$ follow autoregressive disturbances $u_t = \varrho u_{t-1} + e_t$ where $(e_t)_t$ are i.i.d Gaussian random variables with mean zero and variance Σ^2 and where $\mid \varrho \mid < 1$ to ensure stationarity. Consequently the VIX log-likelihood is given by

$$\begin{aligned}\log L_{\text{VIX}}(\vartheta,\varrho) &= -\frac{T}{2}\left(\log(2\pi) + \log(\Sigma(1-\varrho^2))\right) + \frac{1}{2}\left(\log(1-\varrho^2)\right)\\ &\quad -\frac{1}{2\Sigma}\left(u_1^2 + \sum_{t=2}^{T}\frac{(u_t-\varrho u_{t-1})^2}{1-\varrho^2}\right)\end{aligned} \tag{19}$$

and we obtain the joint VIX-returns likelihood $\left(\log L_R(\theta^D,\theta^V) + \log L_{\text{VIX}}(\vartheta,\varrho)\right)$.

Finally, a last estimation strategy will be used in the empirical part for non-affine GARCH models with NIG innovations. This strategy, first introduced in Chorro & Fanirisoa (2018), derives from a very simple finding: under Gaussian hypotheses, non-affine GARCH models have outstanding properties (closed-form expressions for the model implied VIX) that fail when NIG innovations are involved. Therefore, inspired by the so-called quasi-maximum likelihood (QML) estimator, a two-step approach is possible to take benefit of these remarkable features in a Gaussian environment:

- ***Two-step estimation strategy using VIX-Returns***

  As in the QML approach, this two-step strategy estimates separately volatility and distribution parameters assuming Gaussian innovations in the first step. We start from a GARCH-type model with NIG innovations

Step 1: We assume that the $(z_t)_t$ are i.i.d $\mathcal{N}(0,1)$ under $\mathbb{P}$ and that, in this situation, we have a closed-form formula for the VIX index. Subsequently, we can estimate the vector of volatility parameters θ^V using the joint VIX-Returns likelihood.

Step 2: From the i.i.d residuals $\left(z_1\left(\hat{\theta}^V\right), \cdots, z_T\left(\hat{\theta}^V\right)\right)$ that may be extracted from the previous step, the distribution vector of parameters θ^D is obtained maximizing

$$\sum_{t=1}^{T} -\frac{\log(h_t)}{2} + \log\left[f_{\theta^D}\left(\frac{Y_t - (r + m_t)}{\sqrt{h_t}}\right)\right]$$

where f_{θ^D} is the density function of a centered NIG random variable with unit variance as introduced in Section 1.3.

This estimation strategy permits to introduce VIX information in the estimation process of NIG-NGARCH and NIG-GJR models without using Monte-Carlo approximation to compute the objective function of the optimizer. This approach not only reduces the computational time of estimation but also allows to split an optimization exercise with 10 variables into sub-problems of smaller dimensions[11].

- ***Estimation results***

To summarize, in our empirical study, the HN model with Gaussian innovations and the IG-GARCH model (risk-neutralized using Esscher or U-shaped SDF) will be estimated using the returns, the joint VIX-Returns and the joint Option-Returns likelihoods. The GJR and NGARCH models with Gaussian innovations (risk-neutralized using Esscher SDF) will be estimated using the returns and the joint VIX-Returns likelihood. The NGARCH with NIG innovations (risk-neutralized using EGP) will be estimated using the joint VIX-Returns likelihood. The GJR and NGARCH models with NIG innovations (risk-neutralized using Esscher SDF) will be estimated using the returns and the two-step estimation strategy. The GJR and NGARCH models with Gaussian innovations (risk-neutralized using the quadratic SDF) will be estimated using the joint VIX-Returns likelihood.[12] In Table 3 we review the numerical approximations used in the paper for each model, each SDF and each estimation strategy to compute the objective function in the estimation process.

The estimated parameter values and their respective standard errors, obtained from using the different sets of information, are reported in Table 4 (resp. Table 5) for

Gaussian GARCH models combined with the exponential-affine (resp. the quadratic) SDF. For NIG parameters, the results of the two-step estimation exercises are presented in Table 6, while Table 7 shows estimates for the IG-GARCH model under both M^{Ess} and M^{Ushp}. Finally, for the NIG-NGARCH model risk-neutralized using the EGP, the joint VIX-Returns likelihood estimates are illustrated in Table 8. In all cases, results are roughly in the same range as those obtained in many other previous empirical studies.

We notice for the IG-GARCH model that parameter estimates are remarkably stable across the different approaches. Concerning the other GARCH specifications, instead of focusing on the individual values of each parameter, we remark that global features of each model (persistence, leverage effect parameter) differ only a little from one strategy to another. For example, we can deduce from Tables 4 and 5 that in the case of the GJR GARCH specification we obtain historical (resp. risk-neutral) persistences around 0.986 (resp. around 0.996) and a leverage parameter γ between 0.022 and 0.023. We classically obtain high historical persistences and all models and estimation approaches clearly indicate the leverage effect. Moreover, in the case of the two U-shaped pricing kernels, the proportional wedge between the risk-neutral and the historical volatilities is significantly estimated to be greater than 1, with values ranging between 1.24 and 1.72 (see Tables 5 and 7) for the Gaussian HN and the IG-GARCH models, as observed in empirical studies. Last but not least, as remarked in Kanniainen *et al.* (2014), for the joint VIX-Returns estimation strategy, the autocorrelation coefficient ϱ is uniformly close to 1 with a minimum value of 0.81 for the Gaussian HN model combined with the quadratic SDF.

Concerning parameters of the NIG distribution, we can see from Tables 6 and 8 that the observed (negative) values of skewness vary from -0.01 to -0.34 and that observed excess kurtosis vary from 1.42 to 2.62. These values provide evidence by their departure from normality and they are in the same range as those obtained in previous studies (see for example Badescu *et al.* (2011)).

4.3. Criteria for the option and VIX pricing analysis

Once a particular GARCH model has been properly estimated using a well-chosen set of historical financial information, we obtain explicitly from Tables A1 and A2 in Appendix A the related risk-neutral dynamics depending on the choice of the underlying SDF. For the HN-GARCH model with Gaussian innovations (Heston & Nandi (2000) and Monfort & Pégoraro (2012)) and the IG-GARCH model (Christoffersen *et al.* (2006a) and Chorro & Fanirisoa (2019)), under both exponential-affine and U-shaped SDF, we have quasi-closed-form solutions for pricing vanilla European options efficiently from FFT methodology (see for example Chorro *et al.* (2015) Chap 4.2) that massively decrease the required time to price a full option book. For other non-affine specifications, prices are approximated using Monte-Carlo simulation using 15000 trajectories. To test the quality of these price approximations we will use, in the empirical part, the in (Dataset A), out (Dataset B) and Wednesday (rolling window strategy) Implied Volatility Root Mean Squared Error (IVRMSE[13]) that measure the discrepancy between model and option prices:

$$IVRMSE = \sqrt{\frac{1}{N}\sum_i \left(\frac{c_i - \hat{c}_i}{\hat{V}_i}\right)^2}$$

where c_i is the option price given by the model, $\hat{c}_i$ the corresponding market price and $\hat{V}_i$ the Black and Scholes Vega associated with $\hat{c}_i$. Here, following for example Christoffersen *et al.* (2012), the volatility updating rule is simply deduced from returns to get option prices given by a model. Moreover, another interesting economic criteria will be the magnitude of the average annualized volatility risk premium (VRP) as defined in Papantonis (2016) in order to understand why an equity risk premium is in general not sufficient to produce realistic price levels. Finally, in order to discuss the correlation between option pricing performances and the capacity of implied VIX to fit the market VIX, we will use the measures of adequacy introduced in Qiang *et al.* (2015), namely, the mean percentage error (MPE_{VIX}), the mean percentage absolute error (MAE_{VIX}) and the root mean squared error ($RMSE_{\text{VIX}}$) defined below:

$$\begin{aligned}&MPE_{\text{VIX}} = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{\text{VIX}_j^{Model}}{\text{VIX}_j^{Market}} - 1\right),\ MAE_{\text{VIX}} = \frac{1}{N}\sum_{j=1}^{N}\left(\left|\frac{\text{VIX}_j^{Model}}{\text{VIX}_j^{Market}} - 1\right|\right)\\ &\text{and } RMSE_{\text{VIX}} = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(\text{VIX}_j^{Model} - \text{VIX}_j^{Market}\right)^2}.\end{aligned} \tag{20}$$

In Table 3 we review in details all the numerical approximations used in the paper for each model, each SDF and each estimation strategy to compute the out-of-sample performance measures.

4.4. Empirical findings

Our study relies on 21 combinations of GARCH-distribution-SDF-estimation. To make the presentation much more readable, we group them into five different categories: the Gaussian-GARCH models combined with M^{Ess}, the NIG-GARCH models combined with M^{Ess}, the Gaussian-GARCH models combined with M^{Qua}, the IG-GARCH model and the NIG-NGARCH model risk-neutralized using the EGP. For each group, we present in a specific table (see Tables 9, 10, 11, 12 and 13) option and VIX fitting performances based on the criteria introduced in the preceding section. Furthermore, we report for each model the related estimation time (the CPU time was obtained with a $2,4$ GHz Intel Core i9 processor and 32 GB RAM 2400 MHz DDR4) and the variance risk premium as defined in Papantonis (2016). These tables also provide, for a selected subclass containing more than one element, internal pairwise comparisons in terms of of out-of-sample and weekly out-of-sample option valuation errors. We complete these results in the numerical Appendix B giving internal pairwise comparisons in terms of computational time of estimation and in-sample pricing performances (see Tables B1, B2, B3 and B4). Finally, general results are provided to allow for broader conclusions: in Table 14, out-of-sample performances of the best models in each category are compared while we can find in Table 15 a summary of VIX and option performance measures of the 21 competitors and their corresponding rankings. Regarding results presented in Table 15, we can easily notice that ranks related to option (resp. to VIX) valuation are mostly independent of the choice of the underlying criteria selected from in sample, out-of-sample or weekly out-of sample IVRMSE (resp. from RMSE, MPE or MAE) with Spearman rank correlation coefficient greater than 0.9. Thus, in the following, numerical comparisons will rest on out-of-sample IVRMSE and VIX RMSE. We start our analysis at a group level.

• ***Pricing performances of Gaussian GARCH models with an exponential-affine SDF***

We deduce from Table 9 that, when they are estimated only using returns, pricing performances of Gaussian GARCH models seem to be independent of the choice of the GARCH structure with IVRMSE ranging from 0.07648 to 0.07770 under Duan's LRNVR. When an extra piece of financial information is introduced into the estimation process, we obtain the smallest IVRMSE of 0.065 for the non-affine specifications especially the GJR model. This is in line with the existing literature that favors non-affine Gaussian stochastic volatility models (see Christoffersen *et al.* (2006b), Kanniainen *et al.* (2014) and references therein). Table 10 leads to similar conclusions in the NIG environment while Table 11 confirms the slight superiority of non-affine Gaussian specifications when using an exponential-quadratic SDF. Nevertheless, option valuation errors under Gaussian distribution and exponential-affine SDF are the worst of all competitors. A plausible explanation comes from the fact that these models generate very small variance risk premia (see Table 9) which are not in line with empirical observations. In fact, as reported in Tables 10, 11, 12 and 13, when we use non-Gaussian alternatives and/or U-shaped pricing kernels we recover VRP between -2.867% (for the NIG-GJR model estimated using returns only) and -3.75411% (for the IG-GARCH model estimated using Option-Returns information) that are in line with a bulk of empirical studies (Papantonis (2016)). For Gaussian distribution and exponential-affine SDF, the variance risk is neglected and an equity risk premium is not sufficient to produce realistic price levels.

• ***Pricing performances of GARCH models with NIG distribution***

In Table 10, the overall IVRMSE is between 0.0592 and 0.0690 for NIG-GARCH models risk-neutralized with the Esscher SDF with values that are all smaller than corresponding values for Gaussian innovations. The minimal IVRMSE of 0.0592 is obtained in the case of the NIG-NGARCH model by estimating with the two-step estimation strategy using VIX-Returns information as introduced in Chorro & Fanirisoa (2018). Not surprisingly, a finer modeling approach of conditional skewness improves considerably the quality of price approximations. The two-step estimation strategy using VIX-Returns information helps to substantially improve performances at a parsimonious computational cost. The improvement (of around 14% for non-affine specifications) from using VIX information is also fundamental in this framework because returns based estimation strategy only leads to IVRMSE ranging from 0.0689 to 0.0690. This is confirmed in Table 13 for the NIG-NGARCH model associated with the EGP with an IVRMSE of 0.05935.

• ***Pricing performances of Gaussian GARCH models with a quadratic SDF***

Working with non-Gaussian residuals is not the only way to generate more realistic VRP than Gaussian-GARCH ones. We present in table 11 the IVRMSE of different Gaussian-GARCH models when an exponential-quadratic SDF is used to price options. It is worth noting that in this approach, it is not possible to directly estimate models from returns market quotes because the extra parameter π is involved in the risk-neutral dynamics. We obtain good IVRMSE between 0.06006 and 0.06331 that consistently outperform the Gaussian counterpart with exponential-affine SDF. Even if they are slightly worse than corresponding values for NIG-GARCH models for out-of-sample IVRMSE, the hierarchy is reversed when considering the next week pricing errors build on the rolling window estimation strategy. Both a modeling approach based on realistic conditional skewness and a modeling approach incorporating a variance premium in the pricing kernel seem to capture valuable empirical features. Therefore, a natural question is how is it possible that these two aspects are more complementary rather than competitive? The IG-GARCH model appears as an interesting candidate to tackle this issue.

• ***Pricing performances of the IG-GARCH model***

For the IG-GARCH model, we obtain (see Table 12) out-of-sample IVRMSE between 0.067427 (in the case of the Esscher SDF estimated using returns only) and 0.056641 (for the U-shaped SDF and Returns-Options estimation strategy). Once again, a dataset of returns is not sufficient to produce competitive results. Furthermore, when the joint VIX-Returns estimation process is performed we obtain an IVRMSE of 0.057568 that is much closer to the best value at a considerably shorter computation time. The U-shaped pricing kernel of Chorro & Fanirisoa (2019) outperforms by around 7% the Esscher SDF in a conditionally Inverse-Gaussian environment and produces the best performances observed in this section: conditional skewness is a key factor of GARCH option pricing models that becomes outstanding when associated with a non-standard SDF.

- ***Pricing performances: a global analysis***

When using GARCH option pricing models, the modeler is faced with four degrees of freedom: the GARCH structure, the distribution of the innovations, the pricing kernel, and the estimation strategy. Now we conclude the analysis of option pricing errors brought together in Table 15 with more general considerations on the impact of each factor caeteris paribus. Let us start with marginal effects: the impact of the choice of a non-affine GARCH structure accounting for the leverage effect is small with a 2.2% improvement in favor of the GJR model. In the same way, in the case of the NIG-NGARCH model estimated using returns and VIX information, the Esscher and the extended Girsanov principle SDF give rise to almost identical results with a difference of 1.4% for the benefit of the exponential-affine parameterization (see also Badescu *et al.* (2011) and Badescu *et al.* (2015) that deliver the same conclusion). Finally, using an estimation strategy based on options and returns information only improves by around 1% the IVRMSE with respect to its VIX-Returns counterpart (however, this improvement is around 10.5% when using returns only) as already observed in Chorro & Fanirisoa (2019). Nevertheless, for this latter point we have to keep in mind that this slight 1% upgrade comes at a very high computational cost as reported in Tables B1, B2, B3 and B4 of the numerical Appendix B. More decisively, the NIG distribution reduces the valuation error of around $11,6\%$ in comparison to Gaussian innovations while, in the affine family, the IG-GARCH model outperforms by 10.6% the Gaussian HN model. Concerning, the choice of the SDF, we clearly observe, both in the Gaussian and in the Inverse-Gaussian case that U-shaped parameterizations yield respectively to 13% and 7% lower IVRMSE (see Christoffersen *et al.* (2013), Chorro & Fanirisoa (2019), and Badescu *et al.* (2017) for similar findings). In the light of these observations, it is not surprising to see from Table 14 that, when we compare out-of-sample pricing errors between the best models of each sub-group, the most interesting performances are delivered by a model with non-Gaussian innovations, risk-neutralized using a U-shaped SDF and estimated maximizing the joint VIX-Returns log-likelihood, namely, the IG-GARCH model. What is more, we can observe (see Table B5 of the numerical Appendix B) that this conclusion still holds if we compare, for the best competitors, out-of-sample IVRMSE desegregated by moneynesses and time to maturities.

We conclude that, when it is possible, the combination of all these factors is fundamental to producing competitive valuation errors.[14] The best model is not the most richly parameterized but a parsimonious one able to cope with classical stylized facts in terms of historical dynamics and risk representation.

• ***Deduce pricing performances from VIX analysis***

Even if the ultimate criterion to compare GARCH option pricing models is the value of the pricing errors associated with a large real-world dataset of option prices, its computation may lead to large numerical issues in particular when Monte-Carlo approximations are needed. This is true, not only during the estimation stage, but also to compute the objective function. To conclude this section, we question the possibility of deducing option pricing performances of a GARCH model from its capacity to forecast VIX dynamics. In Table 15, we have reported the ranks of the 21 models considered in this article regarding VIX and options adequacy measures introduced in Section 4.3. For example,[15] when we measure the relationship between rankings obtained from out-of-sample pricing errors and VIX RMSE we obtain a significant Spearman's rank correlation coefficient of 0.90. Moreover, top ten models obtained using VIX RMSE criterion are mainly as highly ranked as using options based criterion. The most important conclusion is that the ranking of models is well-preserved independently of the chosen option or VIX adequacy measure: examining the performance of a model in fitting VIX time series gives a very good indication on related pricing performances at a very reasonable computational cost. VIX analysis appears in this way as a very interesting and parsimonious first-stage evaluation to discard the worst GARCH option pricing models.

5. Conclusion

In this paper, we have examined pricing performances of a large collection of GARCH models by questioning the global synergy between the choice of the affine/non-affine GARCH specification, the use of competing alternatives to the Gaussian distribution, the selection of an appropriate SDF and the choice of different estimation strategies based on several sets of financial information and on standard minimization algorithms. Therefore, 21 combinations of GARCH/distribution/SDF/estimation are tested using a large option dataset written on the S&P500. To do this, an intensive empirical comparison is performed not only based on in and out-of-sample pricing performances, but also using a weekly rolling window strategy where the model is estimated each Wednesday to price options one week later. Uniformly for these three criteria, the IG-GARCH model risk-neutralized using a U-shaped pricing kernel provides the best results. This gives evidence for the importance of using a non-Gaussian distribution combined with a non-standard stochastic discount factor that takes account for the variance risk premium. Of course, to estimate the variance risk aversion parameter, historical returns are not sufficient and an extra financial information is required. At this point, we have found that the joint VIX-Returns likelihood estimation provides competitive pricing errors at a very interesting computational cost with respect to option based estimation processes. This latter finding holds for all models considered in this paper. For non-affine GARCH specifications, we found that, under NIG innovations, very interesting pricing errors are obtained when, and only when, VIX information is incorporated into the estimation strategy. This is efficiently possible for the NGARCH model using the EGP risk-neutralization process or using the two-step estimation strategy developed in Chorro & Fanirisoa (2018).

Finally, we have questioned in this study the possibility to deduce option pricing performances of a GARCH model from its capacity to forecast VIX dynamics. When we ranked models using options or VIX criteria we obtained a highly significant Spearman's rank correlation coefficient of 0.90. Therefore, examining the performance of a model in fitting VIX time series gives a very good indication on related pricing performances at a very reasonable computational cost. VIX analysis appears in this way as a very interesting and parsimonious first-stage evaluation to discard the worst GARCH option pricing models.

Appendix A. Technical Appendix: Stochastic discount factors, risk-neutrals dynamics and model implied VIX

We propose in this technical appendix, the primer concerning the different stochastic discount factors (SDF) used in this paper to obtain the risk-neutral dynamics provided in Tables A1 and A2. This knowledge may be of interest to compute prices under the historical probability even if in the present paper all the prices have been computed or simulated under different risk-neutral probabilities. We refer the interested reader to Chorro *et al.* (2015) (Chap. 3) and Chorro & Fanirisoa (2019) for complete derivations.

Starting from a financial time series of log-returns $Y_t = \log\left(\frac{S_t}{S_{t-1}}\right)$ defined on a filtered historical probability space $\left(\Omega, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, \mathbb{P}\right)$ it is well-documented that the existence of an equivalent martingale measure $\mathbb{Q}$ is equivalent to the existence of the so-called one period SDF $(M_t)_{t\in\{1,\dots,T\}}$ that fulfills the pricing equations related to the bond and the risky asset:

$$\begin{cases} E_{\mathbb{P}}[e^r M_{t+1} \mid \mathcal{F}_t] = 1 \\ E_{\mathbb{P}}[e^{Y_{t+1}} M_{t+1} \mid \mathcal{F}_t] = 1. \end{cases} \tag{A1}$$

For GARCH models, combining discrete time modeling and continuous conditional distributions, the choice of such a SDF is not unique and several classical candidates are available in the financial literature.

The conditional Esscher transform

We assume for the SDF a particular parametric form (exponential affine of the log-returns): $\forall t \in \{1, ..., T\}$,

$$M_t^{Ess} = e^{\theta_t^q Y_t + \xi_t^q} \tag{A2}$$

where θ_t^q and ξ_t^q are $\mathcal{F}_{t-1}$ measurable random variables.
For the dynamics considered in (1) it is in general possible to find explicitly θ_t^q and ξ_t^q that depend on the conditional distribution. In particular (see Chorro *et al.* (2015) (Chap. 3.4)), when the z_t are i.i.d $\mathcal{N}(0,1)$,

$$\begin{cases} \theta_t^q = -(\frac{1}{2} + \frac{m_t}{h_t}) \\ \xi_t^q = -\theta_t^q (r + m_t) - (\theta_t^q)^2 \frac{h_t}{2} - r \end{cases} \tag{A3}$$

and when the z_t are i.i.d $NIG\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\mu}\right)$,

$$\begin{cases} \theta_t^q = \frac{-1}{2} - \frac{\tilde{\alpha}\tilde{\beta}\sqrt{\tilde{\delta}}}{\sqrt{h_t}\tilde{\varrho}^{\frac{3}{2}}} - \frac{1}{2}\sqrt{\frac{\left(\tilde{\alpha}m_t + \sqrt{\tilde{\delta}h_t}\tilde{\beta}\tilde{\varrho}\right)^2}{h_t\tilde{\delta}\tilde{\varrho}^3}\left(\frac{4\tilde{\alpha}^4\tilde{\delta}^2}{h_t\tilde{\delta}\tilde{\varrho}^3 + \left(\tilde{\alpha}m_t + \sqrt{\tilde{\delta}h_t}\tilde{\beta}\tilde{\varrho}\right)^2} - 1\right)}. \\ \xi_t^q = -\tilde{\mu}\theta_t^q - \tilde{\delta}\tilde{\varrho} + \tilde{\delta}\sqrt{\tilde{\alpha}^2 - \left(\tilde{\beta} + \theta_t^q\right)^2} - r \end{cases} \tag{A4}$$

where $\tilde{\varrho} = \sqrt{\tilde{\alpha}^2 - \tilde{\beta}^2}$.
If we consider the IG-GARCH model (3), we obtain

$$\begin{cases} \theta_t^q = \frac{1}{2}\left[\eta^{-1} - \frac{1}{\nu^2\eta^3}\left[1 + \frac{\nu^2\eta^3}{2}\right]^2\right] \\ \xi_t^q = -r(\theta_t^q + 1) - \theta_t^q \nu h_t - \left[\delta_t\left(1 - \sqrt{(1 - 2\theta_t^q \eta)}\right)\right]. \end{cases} \tag{A5}$$

The Extended Girsanov principle

Following Badescu *et al.* (2019), we can see that for the particular NGARCH model (7) considered in the paper, when the z_t are i.i.d $NIG\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\mu}\right)$, the SDF coming from the Extended Girsanov principle is given by

$$M_t^{EGP} = e^{-r}\frac{d_{NIG}\left(z_t + \lambda_0, \tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\mu}\right)}{d_{NIG}\left(z_t, \tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\mu}\right)} \tag{A6}$$

where d_{NIG} is the density function of the NIG distribution given in section 1.3.

The Quadratic SDF for Gaussian GARCH models

In Monfort & Pégoraro (2012), the authors proposed, for Gaussian GARCH models of the form (1), to extend the exponential affine form of the SDF (A2) adding a quadratic term in the exponential to cope with more general empirical shapes. More precisely, they supposed that

$$M_t^{Qua} = e^{\xi_t^q + \theta_{1,t}^q Y_t + \theta_{2,t}^q Y_t^2} \tag{A7}$$

where ξ_t^q, $\theta_{1,t}^q$ and $\theta_{2,t}^q$ are $\mathcal{F}_{t-1}$ measurable random variables. Obviously, when $\theta_{2,t} = 0$ we recover M^{Ess}. This SDF depends on three variables and has to solve the two pricing equations (A1). In order to obtain a unique solution to this problem, we need to impose an extra constraint. Supposing a constant proportional wedge $\pi = \frac{h_t^*}{h_t}$ between the risk-neutral and the historical volatilities we obtain (see Chorro *et al.* (2015) (Section 3.5))

$$\begin{cases} \theta_{2,t}^q = \frac{\pi - 1}{2\pi h_t} \\ \\ \theta_{1,t}^q = -2r\theta_{2,t}^q - \frac{1}{2} - \frac{m_t}{h_t} \\ \\ \xi_t^q = -r - \frac{h_t(\theta_{1,t}^q)^2 + 2(r+m_t)\theta_{1,t}^q + 2\theta_{2,t}^q(r+m_t)^2}{2(1-2\theta_{2,t}^q h_t)} + \frac{1}{2}log(1 - 2\theta_{2,t}^q h_t). \end{cases} \tag{A8}$$

This new risk-neutral parameter π cannot be estimated using only information from the log-returns.

The U-shaped SDF for the IG-GARCH model

In Chorro & Fanirisoa (2019), the authors proposed an exponential-hyperbolic SDF for the IG-GARCH model (3) that may be seen as the analogous of (A7) for the IG distribution. They considered

$$M_t^{Ushp} = e^{\theta_t^q Y_t + \varepsilon_t^q + \frac{\rho_t^q}{y_t}} = e^{\theta_t^q Y_t + \varepsilon_t^q + \frac{\eta\rho_t^q}{Y_t - r - \nu h_t}} \tag{A9}$$

and supposing a constant proportional wedge π between the risk-neutral and the historical volatilities we obtain (see Chorro & Fanirisoa (2019) (proposition 2.3))

$$\begin{cases} \theta_t^q = \frac{1}{2\eta} - \dfrac{1}{2}\left[\sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1+\sqrt{1+\frac{8\nu}{27\pi}}\right)} + \sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1-\sqrt{1+\frac{8\nu}{27\pi}}\right)}\right] \\ \rho_t^q = \frac{\delta_t^2}{2}\left[1 - \dfrac{\nu^2\eta^4}{(1-2\theta_t^q\eta)\left[1-\left(\sqrt{1-2\eta^*}\right)\right]^2}\right] \\ \varepsilon_t^q = -r-(r+\nu h_t)\theta_t^q - \delta_t + \sqrt{(\delta_t^2-2\rho_t^q)(1-2\eta\theta_t^q)} - log(\delta_t) + \frac{1}{2}log(\delta_t^2 - 2\rho_t^q)). \end{cases} \tag{A10}$$

where $\eta^* = \sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1+\sqrt{1+\frac{8\nu}{27\pi}}\right)} + \sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1-\sqrt{1+\frac{8\nu}{27\pi}}\right)}$.

Tables A1, A2 and A3 provide in details the Gaussian and non-Gaussian risk-neutral dynamics considered in the paper and the related (when available) closed form expressions for the model implied VIX.

Table A1: Summary information on risk-neutral dynamics of each Gaussian GARCH model analyzed in this paper.

Model	Risk-neutral dynamics	Risk-neutral parameters and distribution	References
<u>HN-GARCH</u> :			
Gaus-Ess	$Y_t = r - \frac{h_t}{2} + \sqrt{h_t} z_t^*$	$z_t^* \hookrightarrow N(0,1)$	Heston & Nandi (2000)
	$h_t = a_0 + a_1 \left(z_{t-1}^* - \gamma^* \sqrt{h_{t-1}} \right)^2 + b_1 h_{t-1}$	$\gamma^* = \gamma + \lambda_0 + \frac{1}{2}$	Kanniainen *et al.* (2014)
Gaus-Qua	$Y_t = r - \frac{h_t^*}{2} + \sqrt{h_t^*} z_t^*$	$z_t^* \hookrightarrow N(0,1)$, $\pi = \frac{h_t^*}{h_t}$	Monfort & Pégoraro (2012)
	$h_t^* = \pi a_0 + \pi^2 a_1 \left(z_{t-1}^* - \gamma^* \sqrt{h_{t-1}^*} \right)^2 + b_1 h_{t-1}^*$	$\gamma^* = \left(\frac{\gamma}{\pi} + \frac{\lambda_0}{\pi} + \frac{1}{2} \right)$	Chorro *et al.* (2015) (Chap. 3.5)
<u>GJR-GARCH</u> :			
Gaus-Ess	$Y_t = r - \frac{h_t}{2} + \sqrt{h_t} z_t^*$		
	$h_t = a_0 + h_{t-1} \left[b_1 + a_1 \left(z_{t-1}^* - \lambda_0 \right)^2 \right]$	$z_t^* \hookrightarrow N(0,1)$	Kanniainen *et al.* (2014)
	$+\gamma h_{t-1} \max\left(0; -\left(z_{t-1}^* - \lambda_0\right)\right)^2$		
Gaus-Qua	$Y_t = r - \frac{h_t^*}{2} + \sqrt{h_t^*} z_t^*$	$z_t^* \hookrightarrow N(0,1)$, $\pi = \frac{h_t^*}{h_t}$	Monfort & Pégoraro (2012)
	$h_t^* = a_0^* + h_{t-1}^* b_1 + a_1^* h_{t-1}^* \left(z_{t-1}^* - \lambda_t^* \right)^2$	$a_0^* = a_0 \pi$, $a_1^* = a_1 \pi$, $\gamma^* = \gamma \pi$	Chorro *et al.* (2015) (Chap. 3.5)
	$+\gamma^* h_{t-1}^* \max\left(0; -\left(z_{t-1}^* - \lambda_t^*\right)\right)^2$	$\lambda_t^* = \frac{\lambda_0}{\sqrt{\pi}} - \frac{\sqrt{h_{t-1}^*}}{2} \left(\frac{1}{\pi} - 1 \right)$	
<u>NGARCH</u> :			
Gaus-Ess	$Y_t = r - \frac{h_t}{2} + \sqrt{h_t} z_t^*$	$z_t^* \hookrightarrow N(0,1)$	
	$h_t = a_0 + b_1 h_{t-1} + a_1 h_{t-1} \left(z_{t-1}^* - \gamma^* \right)^2$	$\gamma^* = \lambda_0 + \gamma$	Kanniainen *et al.* (2014)
Gaus-Qua	$Y_t = r - \frac{h_t^*}{2} + \sqrt{h_t^*} z_t^*$	$z_t^* \hookrightarrow N(0,1)$,	Monfort & Pégoraro (2012)
	$h_t^* = a_0^* + b_1 h_{t-1}^* + a_1^* h_{t-1}^* \left(z_{t-1}^* - \gamma_t^* \right)^2$	$\pi = \frac{h_t^*}{h_t}$, $a_0^* = a_0 \pi$, $a_1^* = a_1 \pi$	Chorro *et al.* (2015)
		$\gamma_t^* = \frac{\gamma}{\sqrt{\pi}} + \frac{\lambda_0}{\sqrt{\pi}} - \frac{\sqrt{h_{t-1}^*}}{2} \left(\frac{1}{\pi} - 1 \right)$	

Table A2: Summary information on risk-neutral dynamics of each non-Gaussian GARCH model analyzed in this paper.

Model	Risk-neutral dynamics	Risk-neutral parameters[1] and distribution	References
GJR-GARCH :			
NIG-Ess	$Y_t = r + m_t + \sqrt{h_t} z_t^*$	$z_t^* \hookrightarrow NIG\left(\tilde{\alpha}, \tilde{\beta} + \sqrt{h_t}\theta_t^q, \tilde{\delta}, \tilde{\mu}\right)$	Badescu *et al.* (2011)
	$h_t = a_0 + h_{t-1}\left[b_1 + a_1 \left(z_{t-1}^*\right)^2\right]$	$\theta_t^q = -\frac{1}{2} - \frac{\tilde{\alpha}\tilde{\beta}\sqrt{\tilde{\delta}}}{\sqrt{h_t}\tilde{\varrho}^{\frac{3}{2}}} - \frac{1}{2}\Theta_t$	Chorro *et al.* (2012)
	$+\gamma h_{t-1} \max\left(0; -\left(z_{t-1}^*\right)\right)^2$		
	$m_t = \lambda_0 \sqrt{h_t} - \frac{1}{2} h_t$	$\tilde{\varrho} = \sqrt{\tilde{\alpha}^2 - \tilde{\beta}^2}$	
NGARCH :			
NIG-Ess	$Y_t = r + m_t + \sqrt{h_t} z_t^*$	$z_t^* \hookrightarrow NIG\left(\tilde{\alpha}, \tilde{\beta} + \sqrt{h_t}\theta_t^q, \tilde{\delta}, \tilde{\mu}\right)$	Badescu *et al.* (2011)
	$h_t = a_0 + b_1 h_{t-1} + a_1 h_{t-1}\left(z_{t-1}^* - \gamma\right)^2$	$\theta_t^q = -\frac{1}{2} - \frac{\tilde{\alpha}\tilde{\beta}\sqrt{\tilde{\delta}}}{\sqrt{h_t}\tilde{\varrho}^{\frac{3}{2}}} - \frac{1}{2}\Theta_t$	Chorro *et al.* (2012)
	$m_t = \lambda_0 \sqrt{h_t} - log(E_{\mathbb{P}}[e^{\sqrt{h_t} z_t}])$	$\tilde{\varrho} = \sqrt{\tilde{\alpha}^2 - \tilde{\beta}^2}$	
NIG-EGP	$Y_t = r + m_t - \nu_t + \sqrt{h_t} z_t^*,$	$z_t^* \hookrightarrow NIG\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\mu}\right)$	Badescu *et al.* (2019)
	$h_t = a_0 + b_1 h_{t-1} + a_1 h_{t-1}\left(z_{t-1}^* - (\lambda_0 + \gamma)\right)^2$	$e^{\nu_t} = e^{-r}\mathbb{E}_{\mathbb{P}}\left[e^{Y_t} \mid \mathcal{F}_{t-1}\right]$	
	$m_t = \lambda_0 \sqrt{h_t} - log(E_{\mathbb{P}}[e^{\sqrt{h_t} z_t}])$	$\nu_t = m_t + \tilde{\mu}\sqrt{h_t} + \sqrt{\tilde{\alpha}^2 - \tilde{\beta}^2} - \sqrt{\tilde{\alpha}^2 - (\tilde{\beta} + \tilde{\delta}\sqrt{h_t})^2}$	
IG-GARCH :			
IG-Ess	$Y_t = r + \nu^* h_{t+1}^* + \eta^* z_t^*$	$z_t^* \hookrightarrow IG\left(\frac{h_t^*}{(\eta^*)^2}\right)$	**?**
	$h_t^* = w^* + b^* h_{t-1}^* + c^* z_{t-1}^* + a^* \frac{(h_{t-1}^*)^2}{z_{t-1}^*}$	$\eta^* = \frac{\eta}{1 - 2\theta^*\eta},\ \theta^* = \frac{1}{2}\left[\eta^{-1} - \frac{1}{\nu^2\eta^3}\left[1 + \frac{\nu^2\eta^3}{2}\right]^2\right]$	
		$c^* = c\left(\frac{\eta^*}{\eta}\right)^{\frac{5}{2}},\ a^* = a\left(\frac{\eta^*}{\eta}\right)^{-\frac{5}{2}},\ \nu^* = \nu\left(\frac{\eta^*}{\eta}\right)^{-\frac{3}{2}},\ w^* = w\left(\frac{\eta^*}{\eta}\right)^{\frac{3}{2}},$	
IG-Ushp	$Y_t = r + \nu^* h_{t+1}^* + \eta^* z_t^*$	$z_t^* \hookrightarrow IG\left(\frac{h_t^*}{(\eta^*)^2}\right),\ \pi = \frac{h_t^*}{h_t}$	
	$h_t^* = w^* + b h_t^* + c^* y_t^* + a^* \frac{(h_t^*)^2}{z_t^*}$	$\eta^* = \sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1 + \sqrt{1 + \frac{8\nu}{27\pi}}\right)} + \sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1 - \sqrt{1 + \frac{8\nu}{27\pi}}\right)}$	Chorro & Fanirisoa (2019)
		$\nu^* = \frac{\nu}{\pi},\ w^* = w\pi,\ c^* = \frac{c\pi\eta^*}{\eta},\ a^* = \frac{a\eta}{\pi\eta^*}$	

$$\Theta_t = \sqrt{\frac{\left(\tilde{\alpha} m_t + \sqrt{\tilde{\delta} h_t}\tilde{\beta}\tilde{\varrho}\right)^2}{h_t\tilde{\delta}\tilde{\varrho}^3}\left(\frac{4\tilde{\alpha}^4\tilde{\delta}^2}{h_t\tilde{\delta}\tilde{\varrho}^3 + \left(\tilde{\alpha} m_t + \sqrt{\tilde{\delta} h_t}\tilde{\beta}\tilde{\varrho}\right)^2} - 1\right)}$$

Table A3: GARCH implied VIX

GARCH models	$\tilde{h}_0$	Ψ^*
HN-GARCH :		
Gaussian-Ess	$\dfrac{a_0 + a_1}{1 - \Psi^*}$	$b_1 + a_1(\gamma + \lambda_0 + \frac{1}{2})^2$
Gaussian-Qua	$\dfrac{a_0 + \pi a_1}{1 - \Psi^*}$	$b_1 + \pi^2 a_1 \left(\dfrac{\gamma}{\pi} + \dfrac{\lambda_0}{\pi} + \dfrac{1}{2}\right)^2$
GJR-GARCH :		
Gaussian-Ess	$\dfrac{a_0}{1 - \Psi^*}$	$b_1 + [a_1 + \gamma N(\lambda_0)]\,(1 + \lambda_0^2) + \gamma\lambda_0 n(\lambda_0)$
NGARCH :		
Gaussian-Ess	$\dfrac{a_0}{1 - \Psi^*}$	$b_1 + a_1(1 + (\lambda_0 + \gamma)^2)$
NIG-EGP	$\dfrac{a_0}{1 - \Psi^*}$	$b_1 + a_1(1 + (\lambda_0 + \gamma)^2)$
IG-GARCH :		
Ess	$\dfrac{w + a\,(\eta)^4}{1 - \Psi^*}$	$b + \dfrac{c^*}{(\eta^*)^2} + a^*\,(\eta^*)^2$
Ushp	$\dfrac{w + \dfrac{a\eta}{\pi^2}\,(\eta^*)^3}{(1 - \psi^*)}$	$b + \dfrac{c^*}{(\eta^*)^2} + a^*\,(\eta^*)^2$

Note: We present expressions of the parameters $\tilde{h}_0$ and Ψ^* associated with the closed-form expression of Vix_t in equation 17 for different GARCH structures, stochastic discount factors and conditional distributions. In this table we denote by N (resp. n) the distribution (resp. the density) function of the standard Gaussian distribution.

Appendix B. Supplementary numerical results

Table B1: Model comparisons based on computational time of estimation and on in-sample IVRMSE given in Table 9.

GARCH-type	HN Ret	GJR Ret	NGARCH Ret	HN Opt-Ret	HN VIX-Ret	GJR VIX-Ret	NGARCH VIX-Ret
HN-Ret	–	-80.00	-150.00	$-9.004E+04$	20.000	-110.00	-40.00
GJR-Ret	4.072	–	-38.888	$-4.997E+04$	55.555	-16.666	22.222
NGARCH-Ret	4.561	0.509	–	$-3.595E+04$	68.000	16.000	44.000
HN-Opt-Ret	6.960	3.010	2.513	–	99.911	99.767	99.844
HN-VIX-Ret	3.179	-0.930	-1.448	-4.063	–	-162.50	-75.00
GJR-VIX-Ret	8.480	4.595	4.106	1.634	5.475	–	33.333
NGARCH-VIX-Ret	7.035	3.088	2.592	0.080	3.982	-1.579	–

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the computational time of estimation between the i-th and the j-th models, as example: $-80\% = 100 * (0.010 - 0.018)/0.010$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the in-sample IVRMSE between the j-th and the i-th models, as example:
$4.072\% = 100 * (0.059916 - 0.057476)/0.059916$.

Table B2: Model comparisons based on computational time of estimation and on in-sample IVRMSE given in Table 10.

GARCH-type	GJR Ret	NGARCH Ret	GJR VIX-Ret	NGARCH VIX-Ret
GJR-Ret	–	-29.16	-50.00	20.833
NGARCH-Ret	-3.038	–	-16.12	38.709
GJR-VIX-Ret (two-steps)	6.871	9.617	–	47.222
NGARCH-VIX-Ret (two-steps)	15.79	18.27	9.579	–

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the computational time of estimation between the i-th and the j-th models, as example: $-29.16\% = 100 * (0.024 - 0.031)/0.024$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the in-sample IVRMSE between the j-th and the i-th models, as example:
$-3.038\% = 100 * (0.0550 - 0.0567)/0.0550$.

Table B3: Model comparisons based on computational time of estimation and on in-sample IVRMSE given in Table 11.

GARCH-type	HN-Opt-Ret	HN-VIX-Ret	GJR-VIX-Ret	NGARCH-VIX-Ret
HN-Opt-Ret	–	99.816	89.802	90.693
HN-VIX-Ret	−0.528	–	−5442	−4957
GJR-VIX-Ret	0.626	1.149	–	8.736
NGARCH-VIX-Ret	3.640	4.146	3.033	–

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the computational time of estimation between the i-th and the j-th models, as example: $99.816\% = 100 * (10.326 - 0.019)/10.326$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the in-sample IVRMSE between the j-th and the i-th models, as example:
$-0.528\% = 100 * (0.05110 - 0.05137)/0.05110$.

Table B4: Model comparisons based on computational time of estimation and on in-sample IVRMSE given in Table 12.

IG-GARCH-type	Ess-Ret	Ess-Opt-Ret	Ushp-Opt-Ret	Ess-VIX-Ret	Ushp-VIX-Ret
Ess-Ret	–	$-2.5E+05$	$-2.8E+05$	57.777	3.333
Ess-Opt-Ret	15.08	–	−12.191	99.836	99.624
Ushp-Opt-Ret	19.89	5.663	–	99.853	99.665
Ess-VIX-Ret	14.49	−0.699	−6.745	–	−128.94
Ushp-VIX-Ret	19.29	4.957	−0.748	5.617	–

Note: The upper triangular part of the matrix illustrates relative difference (in percentage) of the computational time of estimation between the i-th and the j-th models, as example: $-2.564\ ^{+05}\% = 100 * (0.036 - 9.268)/0.036$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the in-sample IVRMSE between the j-th and the i-th models, as example:
$15.08\% = 100 * (0.054358 - 0.046160)/0.054358$.

Table B5: Out-of-sample IVRMSE, desegregated by moneynesses and time to maturities for the best competitors in each category.

IG-Opt-Ret-Ushp

	$0 < T < 20$	$20 < T < 80$	$80 < T < 180$	Total
$0 < S/K < 0.975$	0.105411	0.076586	**0.027593**	0.071749
$0.975 < S/K < 1.00$	**0.016052**	**0.013545**	**0.013763**	0.014200
$1.00 < S/K < 1.025$	**0.014311**	**0.013371**	**0.013306**	0.013551
$1.025 < S/K < 1.05$	**0.020052**	**0.014883**	**0.014644**	0.016088
$1.05 < S/K < 1.075$	0.043086	**0.017401**	0.015650	0.025040
$1.075 < S/K$	0.069161	0.055718	0.040288	0.054487
Total	0.072765	0.058147	0.028768	0.056638

IG-Ret-VIX-Ushp

	$0 < T < 20$	$20 < T < 80$	$80 < T < 180$	Total
$0 < S/K < 0.975$	0.643152	0.257923	0.049141	0.201291
$0.975 < S/K < 1.00$	0.049759	0.024902	0.018611	0.024267
$1.00 < S/K < 1.025$	0.036521	0.021830	0.016880	0.021019
$1.025 < S/K < 1.05$	0.034141	0.019331	0.015943	0.020250
$1.05 < S/K < 1.075$	**0.039549**	0.018797	0.015118	0.021402
$1.075 < S/K$	**0.050304**	0.043078	**0.022629**	0.038080
Total	0.079824	0.057921	0.025041	0.057587

NIG-NGARCH-Ret-VIX-Ess

	$0 < T < 20$	$20 < T < 80$	$80 < T < 180$	Total
$0 < S/K < 0.975$	**0.067470**	**0.047991**	0.054578	0.058037
$0.975 < S/K < 1.00$	0.039262	0.046477	0.029999	0.044425
$1.00 < S/K < 1.025$	0.045486	0.041796	0.042392	0.042746
$1.025 < S/K < 1.05$	0.045903	0.048465	0.035795	0.045551
$1.05 < S/K < 1.075$	0.067000	0.063004	0.045335	0.062825
$1.075 < S/K$	0.088499	0.060939	0.044677	0.062294
Total	0.067438	0.057175	0.046898	0.059387

G-NGARCH-Ret-VIX-Qua

	$0 < T < 20$	$20 < T < 80$	$80 < T < 180$	Total
$0 < S/K < 0.975$	1.240033	0.378643	0.077197	0.290360
$0.975 < S/K < 1.00$	0.071929	0.028683	0.021505	0.028633
$1.00 < S/K < 1.025$	0.038277	0.022620	0.017256	0.021892
$1.025 < S/K < 1.05$	0.037203	0.020372	0.016340	0.021372
$1.05 < S/K < 1.075$	0.059993	0.019695	**0.015078**	0.027155
$1.075 < S/K$	0.058712	**0.040631**	0.026832	0.039263
Total	0.112139	0.070373	0.030825	0.062414

Note: In this table, we provide the out-of-sample IVRMSE of the best competitors desegregated by moneynesses and time to maturities. Here we focus on the IG-GARCH combined with the U-shaped pricing kernel, on the best model built on the NIG distribution and on the best Gaussian GARCH model combined with the quadratic pricing kernel. The best values for each category are depicted in bold. We remark that the IG-GARCH model outperforms the NIG-NGARCH-Ret-VIX-Ess (resp. the G-NGARCH-Ret-VIX-Qua) one in 72 (resp. 82) percent of cases.

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Notes

[1]In the Duan's framework, the coefficients of the GARCH risk-neutral dynamics are just functions of the historical ones, and so may be directly estimated from the log-returns. Nevertheless, the closed-form expression permits to efficiently use available option information to calibrate the model.

[2]The exponential-affine SDF, M^{ess}, developed by Bühlmann *et al.* (1996) and Siu *et al.* (2004), which is based on a conditional extension of the pioneering work of Esscher (1932), and the SDF given by the extended Girsanov principle of Elliott & Madan (1998) are probably the two best known. In particular, they coincide with Duan LRNVR in the Gaussian setting. Let us also remark that extended and non-monotonic versions of the exponential-affine SDF are available for particular choices of distributions as the exponential-quadratic SDF M^{Qua} of Monfort & Pégoraro (2012) (see also Christoffersen *et al.* (2013)) for Gaussian innovations and the exponential U-shaped stochastic discount factor M^{Ush} proposed by Chorro & Fanirisoa (2019) for the Inverse-Gaussian GARCH model.

[3]See for example Taylor (1986) and Heston (1993) where information from the volatility structure is needed to estimate parameters of the model.

[4]This is not true for M^{Qua} or M^{Ush} because, in this case, a risk-neutral parameter (the constant proportional wedge between historical and risk-neutral volatilities) has to be evaluated.

[5]Recently, a large number of studies have further investigated the ability of the VIX index as an input variable for volatility to forecast option prices. Considered as an expected volatility series, the VIX was proposed by Whaley (1993) and introduced by the CBOE in 1993 to serve as a market volatility indicator. The VIX captures how much the investor is willing to pay to deal with investment risks. In previous empirical papers on the importance of the VIX index, the attention focus has primarily been on the impact and the correlation of the VIX index with the stock market and returns volatility. Giot *et al.* (2005) and Sarwar (2012) have established empirical results that suggest an asymmetric relationship between stock market returns and VIX. Cochrane *et al.* (2012) observed the adequacy of the VIX index as an important factor in the determination of stock market returns and also of volatility.

[6]When closed-form expressions are not available, two recent studies proposed interesting alternatives. In Lalancette & Simonato (2017) the authors proposed, for the NGARCH model with Johnson S_U distributed driving noise, numerical approximations to make possible the computation of the implied VIX index using Monte-Carlo simulations. In Chorro & Fanirisoa (2018) a new estimation strategy for some non-Gaussian GARCH models is presented to include options or VIX information in the joint estimation at a low computational cost.

[7]An equivalent study could be performed in a companion paper for Markov-switching Elliott *et al.* (2006), multi-component Christoffersen *et al.* (2008) and multiple-shock Christoffersen *et al.* (2012) GARCH models.

[8]Such a property is not fulfilled if we use, for example, a mixture of Gaussian distributions.

[9]Recent papers (see for example Qiang *et al.* (2015), Wang *et al.* (2017) and Zhang & Zhang (2020)) provide new estimation methodologies to improve the VIX forecasting performance of Gaussian GARCH models observed in Hao & Zhang (2013) while in Yang & Wang (2018) the authors favor the IG-GARCH model risk-neutralized using the conditional Esscher transform. In our study we focus not only on estimation strategies but also on non-standard distributions and SDF showing that the IG-GARCH model combined with a U-shaped pricing kernel delivers the best performances in forecasting the VIX index.

[10]We particularly use estimated in-sample parameters as initial values for the optimization performed the first Wednesday while we initialize parameters of the following Wednesday estimation process by using parameters obtained the previous week.

[11]We can see, in this situation, that the two-step estimation strategy provides Options IVRMSE and VIX RMSE with the same order of magnitude as those obtained from a direct joint estimation strategy using VIX-Returns information where the model implied VIX is computed from Monte-Carlo simulations under NIG residuals. Results in this direction are available upon request and will be the objective of a companion paper.

[12]In this case, and only in this case, the methodology of Lalancette & Simonato (2017) will be used to approximate VIX performance measures using Monte-Carlo methods.

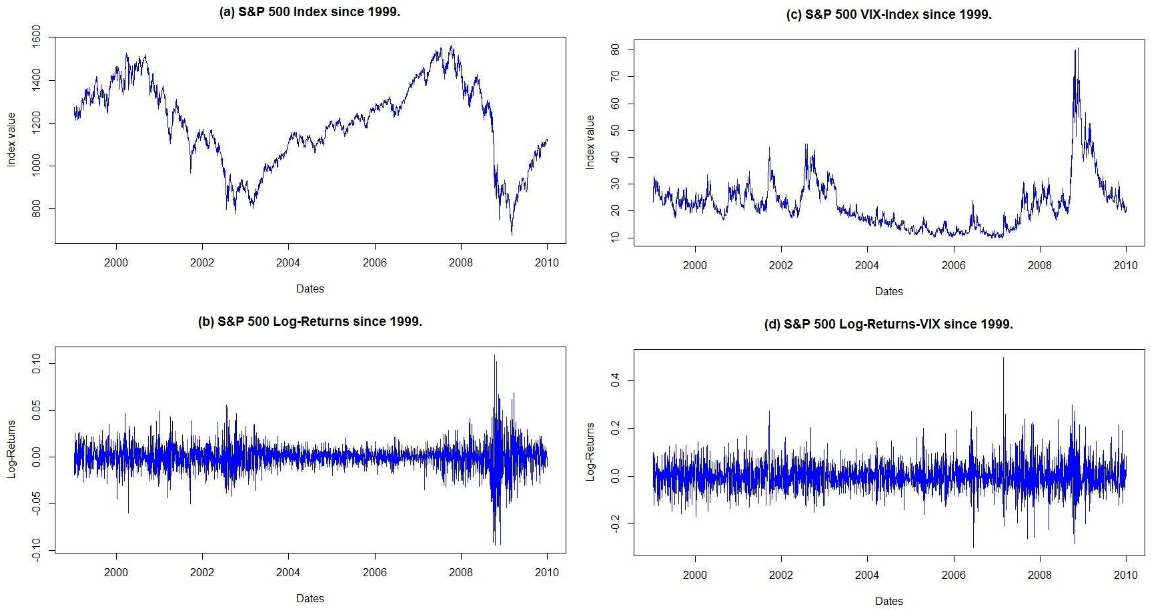
[13]In the bulk of recent studies (Christoffersen *et al.* (2012), Kanniainen *et al.* (2014), Chorro & Fanirisoa (2019), Badescu *et al.* (2017)) this indicator was used to measure pricing performances because Vega-weighted errors do not vary too much across maturities and moneyness contrary to price errors.

[14]It is also important to remark that noteworthy results are obtained with non-affine GARCH structures in NIG environment when VIX information is used in the estimation process. In this case, the residual error of around 3% comes from the necessity to use classical SDF to obtain risk-neutral dynamics.

[15]In Table 15, we notice that the rankings related to options (or VIX) valuation are essentially independent of the choice of the adequacy measures. For example, Spearman's rank correlation coefficient between in and out-of sample pricing errors ranking methodologies is equal to 0.96 . Consequently, we focus our attention on out-of-sample pricing performances and VIX RMSE.

Figures

Fig. 1. S&P500 and VIX closing prices (top) and daily log-returns (bottom) from January 7, 1999 to December 22, 2010.



Tables

Table 1: Descriptive statistics of the S&P500 and VIX datasets covering the period January 7, 1999-December 22, 2010.

	Number of observations	Min	Max	Mean	Std Dev	Skewness	Kurtosis
Price index	2718	676.53	1565.15	1182.75	190.14	−0.0959	−0.6909
Log-returns	2718	−0.0947	0.1096	−0.0001	0.0139	−0.1214	7.3758
VIX index	2718	9.8900	80.8600	22.1859	9.6098	1.8853	5.6964
Log VIX	2718	−0.3506	0.4960	−0.0001	0.0613	0.5697	4.1682

Table 2: Properties of the in-sample (Dataset A) options data (2009-2010) and the out-of-sample (Dataset B) options data (2011-2012).

Option Dataset	Dataset A				Dataset B			
T / S/K	< 60	$[60, 180]$	> 180	**All**	< 60	$[60, 180]$	> 180	**All**
Number of call option contracts :								
$0 < S/K < 0.975$	178	607	286	1071	107	419	214	740
$0.975 < S/K < 1.00$	40	103	44	187	36	80	46	162
$1.00 < S/K < 1.025$	36	96	54	186	30	75	41	146
$1.025 < S/K < 1.05$	35	93	37	165	31	75	37	143
$1.05 < S/K < 1.075$	37	93	40	170	28	72	29	129
$1.075 < S/K$	122	546	267	935	79	312	138	529
All	448	1538	728	2714	311	1033	505	1849
Average call price :								
$0 < S/K < 0.975$	8.558	23.392	41.658	24.536	7.436	21.804	42.351	23.863
$0.975 < S/K < 1.00$	28.133	59.176	84.700	57.336	25.047	59.893	84.423	56.454
$1.00 < S/K < 1.025$	42.764	71.741	96.643	70.383	45.442	76.560	103.004	75.002
$1.025 < S/K < 1.05$	59.721	87.681	109.272	85.558	66.109	95.260	119.433	93.600
$1.05 < S/K < 1.075$	77.534	103.012	125.367	101.971	88.434	116.030	139.506	114.656
$1.075 < S/K$	133.310	170.220	187.118	163.549	147.551	178.710	197.623	174.628
All	58.337	85.870	107.460	83.889	63.336	91.376	114.390	89.701
Average implied volatility from call options :								
$0 < S/K < 0.975$	0.212	0.209	0.210	0.210	0.161	0.174	0.182	0.172
$0.975 < S/K < 1.00$	0.223	0.231	0.233	0.229	0.177	0.198	0.205	0.194
$1.00 < S/K < 1.025$	0.228	0.230	0.235	0.231	0.202	0.207	0.211	0.207
$1.025 < S/K < 1.05$	0.239	0.240	0.233	0.237	0.202	0.210	0.213	0.208
$1.05 < S/K < 1.075$	0.259	0.245	0.235	0.246	0.226	0.222	0.211	0.220
$1.075 < S/K$	0.308	0.267	0.255	0.277	0.260	0.235	0.228	0.241
All	0.245	0.237	0.234	0.238	0.204	0.207	0.208	0.207

Note: The table shows the number of contracts, the average price, and the average implied volatility across moneynesses and times to maturities.

Table 3: When to use what: numerical approximations used in the paper for each model and each estimation strategy to compute the objective function in the estimation process and the out-of-sample performance measures.

GARCH	Estimation strategy		Performance measures	
Model	VIX	OPTIONS	VIX	OPTIONS
G.HN.Ret.Ess	–	–	CF	FFT
G.GJR.Ret.Ess	–	–	CF	FFT
G.NGARCH.Ret.Ess	–	–	CF	MC
G.HN.Op.Ret.Ess	–	FFT	CF	FFT
G.HN.Ret.VIX.Ess	CF	–	CF	FFT
G.GJR.Ret.VIX.Ess	CF	–	CF	MC
G.NGARCH.Ret.VIX.Ess	CF	–	CF	MC
NIG.GJR.Ret.Ess	–	–	MC	MC
NIG.NGARCH.Ret.Ess	–	–	MC	MC
NIG.GJR.Ret.VIX.Ess (two-step)	CF	–	MC	MC
NIG.NGARCH.Ret.VIX.Ess (two-step)	CF	–	MC	MC
G.HN.Op.Ret.Qua	–	FFT	CF	FFT
G.HN.Ret.VIX.Qua	CF	–	CF	FFT
G.GJR.Ret.VIX.Qua	MC	–	MC	MC
G.NGARCH.Ret.VIX.Qua	MC	–	MC	MC
NIG.NGARCH.Ret.VIX.EGP	CF	–	CF	MC
IG.Ret.Ess	–	–	CF	FFT
IG.Opt.Ret.Ess	–	FFT	CF	FFT
IG.Opt.Ret.Ushp	–	FFT	CF	FFT
IG.Ret.VIX.Ess	CF	–	CF	FFT
IG.Ret.VIX.Ushp	CF	–	CF	FFT

Note: In this table, the acronym CF means that we have used the closed form expressions provided in Table A3 to compute the model implied VIX, the acronym FFT that we have used the Fast Fourier Transform machinery with a number of discretization points of 2^{11} to compute option prices and the acronym MC that we have used Monte-Carlo simulations with 15000 paths to approximate the related expectations. An important point to emphasize here is the use in our study of the so-called empirical martingale simulation methodology (EMS) proposed by Duan & Simonato (1998) to reduce drastically the variance of Monte-Carlo estimators. As remarked for example in Badescu *et al.* (2015), EMS is an essential tool to improve numerical efficiency of Monte-Carlo methods especially in the GARCH setting and to use a reasonable number of simulations to compute option prices. For the interested reader, results with 125000 Monte-Carlo simulations and 2^{16} discretization points are available upon request and do not change the conclusion of our study.

Table 4: Parameter estimates and standard errors of Gaussian GARCH models combined with the Esscher stochastic discount factor.

GARCH-type Information	HN-GARCH Returns	GJR-GARCH Returns	NGARCH Returns	HN-GARCH Opt-Ret	HN-GARCH Ret-VIX	GJR-GARCH Ret-VIX	NGARCH Ret-VIX
a_0	$3.854E-08$	$3.049E-06$	$1.677E-06$	$1.859E-07$	$3.757E-12$	$4.966E-06$	$3.557E-06$
Stand.Dev	(0.0044)	(0.0011)	(0.0000)	(0.0009)	(0.0007)	(0.0004)	(0.0009)
a_1	$2.254E-05$	$1.243E-01$	$6.174E-02$	$1.542E-06$	$2.252E-05$	$1.240E-01$	$6.172E-02$
Stand.Dev	(0.0001)	(0.0003)	(0.0014)	(0.0003)	(0.0002)	(0.0000)	(0.0007)
b_1	$8.272E-01$	$8.509E-01$	$8.446E-01$	$6.500E-01$	$9.117E-01$	$8.504E-01$	$7.956E-01$
Stand.Dev	(0.0035)	(0.0006)	(0.0011)	(0.0030)	(0.0086)	(0.0015)	(0.0009)
γ	$5.379E+01$	$2.208E-02$	$1.174E+00$	$4.586E+02$	$1.423E+01$	$2.314E-02$	$4.701E-08$
Stand.Dev	(0.0011)	(0.0025)	(0.0001)	(0.0095)	(0.0088)	(0.0005)	(0.0206)
λ_0	$1.020E+00$	$2.288E-01$	$8.911E-07$	$8.596E+00$	$1.513E+00$	$1.989E-01$	$8.452E-01$
Stand.Dev	(0.0000)	(0.0055)	(0.0010)	(0.0008)	(0.0501)	(0.0033)	(0.0050)
ϱ	–	–	–	–	0.9992	0.8924	0.9542
Stand.Dev	–	–	–	–	(0.0106)	(0.0012)	(0.0110)

Note: Returns means MLE estimation procedure using only returns information, Opt-Ret means Joint MLE estimation using returns and options information, Ret-VIX means Joint MLE estimation using returns and VIX information. The estimation is based on log-returns and VIX datasets from January 7 1999 to December 22 2010 and on the in-sample dataset of options (2009-2010).

Table 5: Parameter estimates and standard errors of Gaussian GARCH models combined with the exponential-quadratic SDF.

GARCH-type Information	HN-GARCH Opt-Ret	HN-GARCH Ret-VIX	GJR-GARCH Ret-VIX	NGARCH Ret-VIX
a_0	$5.7547E-14$	$1.0014E-12$	$4.966E-06$	$1.780E-06$
Stand.Dev	(0.0009)	(0.0003)	(0.0008)	(0.0000)
a_1	$1.5139E-06$	$1.5048E-06$	$1.241E-01$	$3.877E-02$
Stand.Dev	(0.0368)	(0.0002)	(0.0003)	(0.0065)
b_1	$6.500E-01$	$6.5121E-01$	$8.504E-01$	$9.329E-01$
Stand.Dev	(0.0032)	(0.0066)	(0.0004)	(0.0000)
γ	$4.5869E+02$	$4.586E+02$	$2.3142E-02$	$1.277E-07$
Stand.Dev	(0.0036)	(0.0095)	(0.0002)	(0.0078)
λ_0	$8.596E+00$	$8.672E+00$	$1.989E-01$	$4.583E-01$
Stand.Dev	(0.0006)	(0.0036)	(0.0015)	(0.0004)
π	$1.6723E+00$	$1.722E+00$	$1.2785E+00$	$1.2413E+00$
Stand.Dev	(0.0048)	(0.0022)	(0.0012)	(0.0092)
ϱ	–	0.8099	0.9546	0.9170
Stand.Dev	–	(0.0003)	(0.06235)	(0.0023)

Note: Returns means MLE estimation procedure using only returns information, Ret-VIX means Joint MLE estimation using returns and options information, Ret-VIX means Joint MLE estimation using returns and VIX information. The estimation is based on log-returns and VIX datasets from January 7 1999 to December 22 2010 and on the in-sample dataset of options (2009-2010).

Table 6: Parameter estimates and standard errors of the NIG distribution for GARCH models combined with the Esscher SDF.

GARCH-type Information	GJR Returns	NGARCH Returns	GJR Ret-VIX	NGARCH Ret-VIX
α	1.1550	1.2702	1.3589	1.4536
Stand.Dev	(0.0108)	(0.0036)	(0.0001)	(0.0009)
β	-0.1432	-0.0025	-0.0058	-0.0061
Stand.Dev	(0.0057)	(0.0015)	(0.0023)	(0.0001)
δ	1.0623	1.6204	1.5336	1.4538
Stand.Dev	(0.0000)	(0.0005)	(0.0000)	(0.0000)
μ	0.1327	1.9734	7.9908	2.0178
Stand.Dev	(0.0076)	(0.0055)	(0.0000)	(0.0003)

Note: These parameters have been obtained using the standard maximum-likelihood algorithm for the residuals extracted from Table 4.

Table 7: Parameter estimates and standard errors of the IG-GARCH model combined with Esscher and U-shaped SDF.

Joint-Estimation	Returns	Returns-Option		Returns-VIX	
Model		M_t^{ess}	M_t^{Ushp}	M_t^{ess}	M_t^{Ushp}
Parameters:					
w	$1.2061E-06$	$9.7699E-06$	$1.0185E-05$	$2.2341E-06$	$5.3156E-05$
Stand.Dev	(0.0000)	(0.0006)	(0.0002)	(0.0002)	(0.0004)
b	$2.3052E-03$	$1.0159E-03$	$1.7211E-03$	$2.3184E-03$	$1.8603E-03$
Stand.Dev	(0.0000)	(0.0000)	(0.0001)	(0.0003)	(0.0003)
c	$4.9024E-05$	$4.5379E-05$	$4.5118E-05$	$4.8949E-05$	$4.8233E-05$
Stand.Dev	(0.0000)	(0.0000)	(0.0001)	(0.0009)	(0.0005)
a	$3.3174E+03$	$3.3317E+03$	$3.3174E+02$	$3.3174E+03$	$3.3174E+03$
Stand.Dev	(0.0000)	(0.0001)	(0.0007)	(0.0002)	(0.0005)
η	$-7.972E-03$	$-7.5314E-03$	$-7.4936E-03$	$-7.9552E-03$	$-7.928E-03$
Stand.Dev	(0.0000)	(0.0000)	(0.0000)	(0.0002)	(0.0010)
ν	$1.2584E+02$	$1.2594E+02$	$1.2573E+02$	$1.2583E+02$	$1.2584E+02$
Stand.Dev	(0.0009)	(0.0001)	(0.0003)	(0.0003)	(0.0001)
π	–	–	1.4005	–	1.6325
Stand.Dev	–	–	(0.0084)	–	(0.0035)
ϱ	–	–	–	$9.9552E-01$	$9.9386E-01$
Stand.Dev	–	–	–	(0.0070)	(0.0034)

Note: Returns means MLE estimation procedure using only returns information, Ret-VIX means Joint MLE estimation using returns and options information, Ret-VIX means Joint MLE estimation using returns and VIX information. The estimation is based on log-returns and VIX datasets from January 7 1999 to December 22 2010 and on the in-sample dataset of options (2009-2010).

Table 8: Parameter estimates and standard errors of the NIG-NGARCH model combined with the extended Girsanov principle SDF.

Vol Parameters	a_0	a_1	b_1	γ	λ_0	ϱ
Values	$1.896E-06$	$3.877E-02$	$9.329E-01$	$7.110E-01$	$9.937E-02$	0.9163
Stand.Dev	(0.0035)	(0.0008)	(0.0000)	(0.0001)	(0.0000)	(0.0004)
NIG Parameters	α	β	δ	μ	–	–
Values	$2.961E+00$	$-9.441E-01$	1.5877	0.5341	–	–
Stand.Dev	(0.0002)	(0.0013)	(0.0000)	(0.0019)	–	–

Note: The estimation is based on Joint MLE estimation using returns and VIX information from January 7 1999 to December 22 2010.

Table 9: Option pricing performances and VIX predictability (see section 4.3) of Gaussian-GARCH models combined with the Esscher stochastic discount factor.

GARCH-type Information	HN Returns	GJR Returns	NGARCH Returns	HN Opt-Ret	HN Ret-VIX	GJR Ret-VIX	NGARCH Ret-VIX
Option pricing performances and VIX predictability							
Model Properties							
Times (h)	0.010	0.018	0.025	9.014	0.008	0.021	0.014
$-VRP$ (in %)	$3.27E-10$	$2.86E-16$	$2.26E-12$	$8.88E-11$	$9.67E-09$	$7.122E-13$	$1.326E-16$
Predictability of VIX							
MPE_{VIX}	0.0113	0.0106	-0.0088	-0.0092	-0.0023	-0.0016	-0.0012
MAE_{VIX}	0.0128	0.0125	0.0106	0.0104	0.0058	0.0059	0.0057
$RMSE_{VIX}$	0.2849	0.2771	0.2475	0.1869	0.1844	0.1740	0.1748
Pricing performances							
in-$IVRMSE$	0.0599	0.0574	0.0571	0.0557	0.0580	0.0548	0.0557
out-$IVRMSE$	0.0777	0.0764	0.0766	0.0733	0.0735	0.0650	0.0729
We-$IVRMSE$	0.0662	0.0651	0.0652	0.0610	0.0614	0.0592	0.0592
Model comparisons based on out-of-sample IVRMSE and Wednesday-IVRMSE							
GARCH-type	HN Returns	GJR Returns	NGARCH Returns	HN Opt-Ret	HN Ret-VIX	GJR Ret-VIX	NGARCH Ret-VIX
HN-Ret	–	1.565	1.393	5.540	5.394	16.340	6.054
GJR-Ret	1.736	–	-0.174	4.045	3.890	15.010	4.560
NGARCH-Ret	1.509	-0.230	–	4.212	4.057	15.160	4.727
HN-Opt-Ret	7.923	6.297	6.512	–	-0.162	11.430	0.536
HN-VIX-Ret	7.259	5.621	5.838	-0.7212	–	11.570	0.697
GJR-VIX-Ret	10.640	9.062	9.271	2.950	3.645	–	-12.30
NGARCH-VIX-Ret	10.530	8.954	9.163	2.836	3.531	-0.118	–

Note: The results are based on estimates provided in Table 4. The column *Model Properties* presents computational time of estimation in hours and Variance Risk Premium.

For the models comparisons, the upper triangular part of the matrix illustrates relative difference (in percentage) of the out-of-sample IVRMSE between the i-th and the j-th models, as example: $1.5650\% = 100 * (0.077701 - 0.076485)/0.077701$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the Wednesday-IVRMSE between the j-th and the i-th models, as example: $1.736\% = 100 * (0.06626 - 0.06511)/0.06626$.

Table 10: Option pricing performances and VIX predictability (see section 4.3) of NIG-GARCH models combined with the Esscher SDF.

GARCH-type Information	GJR Returns	NGARCH Returns	GJR Ret-VIX (two-step)	NGARCH Ret-VIX (two-step)
Option pricing performances and VIX predictability				
Model Properties				
Times (h)	0.024	0.031	0.036	0.019
$-VRP$ (*in* %)	2.867	2.956	3.006	3.366
Predictability of VIX				
MPE_{VIX}	−0.0043	−0.0029	0.0011	0.0010
MAE_{VIX}	0.0062	0.0053	0.0047	0.0049
$RMSE_{VIX}$	0.1758	0.1703	0.1130	0.1308
Pricing performances				
in-$IVRMSE$	0.0550	0.0567	0.0512	0.0463
out-$IVRMSE$	0.0689	0.0690	0.0595	0.0592
We-$IVRMSE$	0.0592	0.0586	0.0504	0.0504
Model comparisons based on out-of-sample IVRMSE and Wednesday-IVRMSE				
GARCH-type	GJR Returns	NGARCH Returns	GJR Ret-VIX (two-step)	NGARCH Ret-VIX (two-step)
GJR-Ret	–	−0.088	13.610	14.00
NGARCH-Ret	0.945	–	13.680	14.07
GJR-VIX-Ret (two-step)	14.900	14.090	–	0.45
NGARCH-VIX-Ret (two-step)	14.870	14.060	−0.039	–

Note: The results are based on estimates provided in Tables 4 and 6. The column *Model Properties* presents computational time of estimation in hours and Variance Risk Premium.

The upper triangular part of the matrix illustrates relative difference (in percentage) of the out-of-sample IVRMSE between the i-th and the j-th models, as example: $-0.088\% = 100 * (0.0689 - 0.0690)/0.0689$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the Wednesday-IVRMSE between the j-th and the i-th models, as example: $0.945\% = 100 * (0.0592 - 0.0586)/0.0592$.

Table 11: Option pricing performances and VIX predictability (see section 4.3) of Gaussian-GARCH models combined with the exponential-quadratic SDF.

GARCH-type Information	HN-GARCH Opt-Ret	HN-GARCH Ret-VIX	GJR-GARCH Ret-VIX	NGARCH Ret-VIX
Option pricing performances and VIX predictability				
Model Properties				
Times (h)	10.326	0.019	1.053	0.961
$-VRP$ (*in* %)	3.2301	3.2475	3.3562	3.5628
Predictibility of VIX				
MPE_{VIX}	-0.0012	-0.0004	-0.0003	-0.0004
MAE_{VIX}	0.0049	0.0047	0.0041	0.0041
$RMSE_{VIX}$	0.1445	0.1248	0.1119	0.1103
Pricing performances				
in-$IVRMSE$	0.0511	0.0513	0.0507	0.0492
out-$IVRMSE$	0.0627	0.0633	0.0628	0.0600
We-$IVRMSE$	0.0514	0.0515	0.0508	0.0493

Model comparisons based on out-of-sample IVRMSE and Wednesday-IVRMSE

GARCH-type	HN-GARCH Opt-Ret	HN-GARCH Ret-VIX	GJR-GARCH Ret-VIX	NGARCH Ret-VIX
HN-Opt-Ret	–	-0.892	-0.223	4.287
HN-VIX-Ret	-0.097	–	0.663	5.133
GJR-VIX-Ret	1.243	1.339	–	4.500
NGARCH-VIX-Ret	4.216	4.309	3.010	–

Note: The results are based on estimates provided in Table 5. The column *Model Properties* presents computational time of estimation in hours and Variance Risk Premium.
For the model comparisons, the upper triangular part of the matrix illustrates relative difference (in percentage) of the out-of-sample IVRMSE between the i-th and the j-th models, as example: $-0.892\% = 100 * (0.06275 - 0.06331)/0.06275$. The lower triangular part of the matrix illustrates relative difference (in percentage) of the Wednesday-IVRMSE between the j-th and the i-th models, as example: $-0.097\% = 100 * (0.05147 - 0.05152)/0.05147$.

Table 12: Option pricing performances and VIX predictability (see section 4.3) of the IG-GARCH model combined with Esscher and U-shaped SDF.

Joint-Estimation	Returns	Returns-Option		Returns-VIX	
SDF	M_t^{Ess}	M_t^{Ess}	M_t^{Ushp}	M_t^{Ess}	M_t^{Ushp}
Option pricing performances and VIX predictability					
Model Properties:					
Times (h)	0.036	9.2684	10.3984	0.0152	0.0348
VPR	3.1785	3.7541	3.5165	3.7042	3.4563
Predictability of VIX:					
MPE_{VIX}	-0.0011	-0.0004	-0.0003	-0.00008	-0.00019
MAE_{VIX}	0.0051	0.0043	0.0040	0.0039	0.0039
$RMSE_{VIX}$	0.1364	0.1315	0.1061	0.1010	0.0990
Pricing performances:					
in-IVRMSE	0.0543	0.0461	0.0435	0.0464	0.0438
out-IVRMSE	0.0674	0.0610	0.0566	0.0618	0.0575
we-IVRMSE	0.0514	0.0500	0.0480	0.0510	0.0480
Model comparisons based on out-of-sample IVRMSE and Wednesday-IVRMSE					
Joint-Estimation	Returns	Returns-Option		Returns-VIX	
SDF	M_t^{Ess}	M_t^{Ess}	M_t^{Ushp}	M_t^{Ess}	M_t^{Ushp}
Ess-Ret	–	9.446	16.000	8.295	14.620
Ess-Opt-Ret	2.759	–	7.234	-1.271	5.716
Ushp-Opt-Ret	6.722	4.076	–	-9.168	-1.637
Ess-VIX-Ret	0.891	-1.920	-6.251	–	6.899
Ushp-VIX-Ret	6.567	3.916	-0.166	5.726	–

Note: The results are based on estimates provided in Table 7. The column *Model Properties* presents the computational time of estimation in hours and the Variance Risk Premium.

For the model comparisons, the upper triangular part of the matrix illustrates relative difference (in percentage) of the out-of-sample IVRMSE between the i-th and the j-th models, as example: $9.446\% = 100 * (0.067427 - 0.061058)/0.067427$. The lower triangular part of the matrix illustrates the relative difference (in percentage) of Wednesday-IVRMSE between the j-th and the i-th models, as example: $2.759\% = 100 * (0.05147 - 0.05152)/0.05147$.

Table 13: Option pricing performances and VIX predictability (see section 4.3) of the NIG-NGARCH model combined with the extended Girsanov principle SDF.

VIX Performances	$-VRP$ $(in\ \%)$	MPE_{VIX}	MAE_{VIX}	$RMSE_{VIX}$
Values	3.3645	-0.0001	0.0039	0.1066
Pricing performances	Times (h)	in-IVRMSE	out-IVRMSE	we-IVRMSE
Values	0.0101	0.0480	0.0593	0.0488

Note: The results are based on estimates provided in Table 8.

Table 14: Model comparisons based on the out-of-sample IVRMSE and the Wednesday-IVRMSE for best competitors of each sub-group.

	GJR Gaus-Ess	NGARCH NIG-Ess	NGARCH Gaus-Qua	NGARCH NIG-EGP	IG Ushp
GJR-Gaus-Ess	–	8.780	7.603	8.695	11.440
NGARCH-NIG-Ess	14.81	–	−1.290	−0.092	2.913
NGARCH-Gaus-Qua	16.74	2.260	–	1.182	4.149
NGARCH-NIG-EGP	17.50	3.152	0.912	–	3.003
IG-Ushp	18.78	4.659	2.454	1.556	–

Note: Due to the weak difference between the results obtained using Opt-Ret or VIX-Ret information we favor the IVRMSE obtained from Joint MLE estimation using returns and VIX to reduce computational burden. The upper triangular part of the matrix illustrates relative difference (in percentage) of the out-of-sample IVRMSE between the i-th and the j-th models. The lower triangular part of the matrix illustrates relative difference (in percentage) of Wednesday-IVRMSE between the j-th and the i-th models.

Table 15: Option pricing performances and VIX predictability for the 21 competitors considered in this paper. The corresponding rankings are given in brackets.

GARCH	*IVRMSE*			*VIX*		
Model	in	out	We	RMSE	MAE	MPE
G.HN.Ret.Ess	0.05939 [21]	0.07770 [21]	0.06647 [21]	0.28494 [21]	0.01287 [21]	0.011305 [21]
G.GJR.Ret.Ess	0.05747 [19]	0.07733 [20]	0.06511 [19]	0.27710 [20]	0.01254 [20]	0.010604 [20]
G.NGARCH.Ret.Ess	0.05718 [18]	0.07661 [19]	0.06526 [20]	0.24753 [19]	0.01065 [19]	−0.00886 [18]
G.HN.Op.Ret.Ess	0.05574 [16]	0.07339 [17]	0.06101 [17]	0.18696 [18]	0.01044 [18]	−0.00929 [19]
G.HN.Ret.VIX.Ess	0.05801 [20]	0.07351 [18]	0.06145 [18]	0.18444 [17]	0.00588 [15]	−0.00237 [15]
G.GJR.Ret.VIX.Ess	0.05483 [13]	0.06500 [12]	0.05921 [14]	0.17404 [14]	0.00594 [16]	−0.00168 [14]
G.NGARCH.Ret.VIX.Ess	0.05570 [15]	0.07299 [16]	0.05928 [16]	0.17480 [15]	0.00574 [14]	−0.00127 [13]
NIG.GJR.Ret.Ess	0.05502 [14]	0.06894 [14]	0.05925 [15]	0.17585 [16]	0.00629 [17]	−0.00437 [17]
NIG.NGARCH.Ret.Ess	0.05670 [17]	0.06900 [15]	0.05869 [13]	0.17036 [13]	0.00533 [13]	−0.00296 [16]
NIG.GJR.Ret.VIX.Ess (two-step)	0.05124 [8]	0.05956 [4]	0.05042 [6]	0.11303 [7]	0.00471 [8]	0.001135 [10]
NIG.NGARCH.Ret.VIX.Ess (two-step)	0.04633 [4]	0.05929 [3]	0.05044 [7]	0.13082 [9]	0.00499 [11]	0.001045 [9]
G.HN.Op.Ret.Qua	0.05110 [7]	0.06275 [9]	0.05147 [10]	0.14451 [12]	0.00491 [10]	−0.00125 [12]
G.HN.Ret.VIX.Qua	0.05137 [10]	0.06331 [11]	0.05152 [12]	0.12488 [8]	0.00474 [9]	−0.00044 [7]
G.GJR.Ret.VIX.Qua	0.05124 [9]	0.06289 [10]	0.05108 [9]	0.11196 [6]	0.00410 [5]	−0.00036 [5]
G.NGARCH.Ret.VIX.Qua	0.05168 [11]	0.06240 [8]	0.05005 [4]	0.11032 [5]	0.00417 [6]	−0.00040 [6]
NIG.NGARCH.Ret.VIX.EGP	0.05016 [6]	0.06012 [5]	0.04967 [3]	0.10662 [4]	0.00395 [1]	−0.00013 [2]
IG.Ret.Ess	0.05435 [12]	0.06742 [13]	0.05147 [11]	0.13641 [11]	0.00511 [12]	−0.00118 [11]
IG.Opt.Ret.Ess	0.04616 [3]	0.06105 [6]	0.05005 [5]	0.13159 [10]	0.00432 [7]	−0.00049 [8]
IG.Opt.Ret.Ushp	0.04354 [1]	0.05664 [1]	0.04801 [1]	0.10616 [3]	0.00400 [4]	−0.00031 [4]
IG.Ret.VIX.Ess	0.04648 [5]	0.06183 [7]	0.05101 [8]	0.10108 [2]	0.00398 [3]	−0.00008 [1]
IG.Ret.VIX.Ushp	0.04387 [2]	0.05756 [2]	0.04809 [2]	0.09909 [1]	0.00396 [2]	−0.00019 [3]