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► To cite this version:

Carmen Camacho, Alexandre Cornet. Soil pollution diffusion in a spatial agricultural economy. 2020. halshs-02652191v1

HAL Id: halshs-02652191

<https://shs.hal.science/halshs-02652191v1>

Preprint submitted on 29 May 2020 (v1), last revised 30 Mar 2021 (v2)

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WORKING PAPER N° 2020 – 31

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JEL Codes: C61, O44, Q15, Q56, R11

Keywords: Spatial dynamics, Ramsey model, Soil Pollution, Partial differential equations, Dynamic programming, Optimal Control

Soil pollution diffusion in a spatial agricultural economy.

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May 16, 2020

Abstract

There exists a pressing need to analyze the impact of agriculture on soil fertility. This paper develops a spatial growth model for an agricultural economy, in which pollution diffuses across space. In order to produce, the economy needs fertile soil, naturally bounded by the amount of available land. When regions have not yet reached their maximal soil fertility, they can locally invest in abatement in order to reduce soil pollution. Once a location reaches this maximum of fertile land, the economy is split in two: a fertile region and a polluted region. We analytically show how the polluted region can either stagnate at low levels of fertility, or catch up with the fertile region. Our results are numerically illustrated, including the resiliency of the economy to recover from pollution shocks.

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Journal of Economic Literature: C61, O44, Q15, Q56, R11.

1 Introduction

Every year, agriculture is responsible for a loss of 24 billion tonnes of fertile soil (UNCCD, 2017). This loss implies major risks in food security and crop sustainability worldwide. There is an urgent need to understand the impact of agriculture on soil fertility since food demand will keep increasing in the next decades, pushed by the global population growth. Although research on the economics of soil loss started in the late 1960's, this area has attracted relatively little attention until a very recent past. As Burt (1981) pointed out, "the most obvious reason for this apparent lack of interest in the subject is the view that advances in technology have made soil resources per se of less consequence for agricultural production". Indeed, in most economic growth papers, agricultural production is usually modeled as a linear function of land, implying that increases in technology or land lead to proportional increases in production. However, there are regions which stagnate at low levels of fertile soil. Hence, despite constant technology and the possibility to increase fertile soil, locations can remain below their production capacities, losing potential agricultural production. In the present paper, we develop a spatial growth model accounting for the diffusion of pollution into agricultural soils. We analyze and measure how pollution at a given location affects all other locations' fertility, and we obtain explicitly the optimal trajectories for fertile and polluted land not only across space, but also in time, taking into account the persistence of pollution.

Research on soil pollution dates back from the late 1960's and early 1970's. Among the first works, let us mention Ciriacy-Wantrup (1968), who models soil irreversible damage. Bunce (1974) analyzes investment in soil conservation, developing criteria to establish when, where and how much to invest. Adding the time dimension, Burt (1981) and McConnell (1983) apply control theory to the economics of soil conservation. Nevertheless, as pointed out by Burt, soil conservation did not receive the required attention since until the 1980's, technological progress in agriculture made soil resources less important in agricultural production. A second wave of theoretical research on the economics of soil developed in the 1980's and the 1990's, focusing on the linkages between various variables inherent to soil economics. For instance, Pope et al. (1983) solve a model for soil erosion where the depth of topsoil is considered, the farm's income and technological progress. Similarly Saliba (1985) studies the interactions between the choices in soil management, soil loss and farmland productivity. Noteworthy, among the papers of this second wave, there are region specific case-studies like Segarra et al. (1987) and Barbier (1990). More recently in the 2000s, the question of soil conservation shifted to the context of developing countries where it is thought that better soil management practices could lead to the highest potential gains.¹

The economics of soil conservation touches upon different trends in economics as agricul-

¹See for instance Antle et al. (2006), Hagos et al. (2006), de Graaf et al. (2008), Stephens et al. (2012), Barrett et al. (2015), Bevis et al. (2017), or Berazneva et al. (2018).

tural, environmental, development economics and dynamic programming. Here we study the economics of soil conservation using a spatial growth model where the diffusion of pollution is described using the Fick's law. According to the Fick's law the flux of pollution concentration at a location is proportional to the concentration gradient of pollution at this location (see Brock and Xepapadeas, 2008 and 2010, Smith et al., 2009, La Torre et al, 2015, or Camacho and Perez-Barahona, 2015). Our economy is made of a continuum of locations distributed along the unit circle. This is an agricultural economy where each location is endowed with a given amount of fertile soil. The unique agricultural good is locally managed, and it can be either directly consumed or reinvested to abate at least some of the pollution generated.

Our model is structurally close to Boucekkine et al. (2013) and Boucekkine et al. (2018), since production is linear in fertile soil. As in Camacho and Perez-Barahona (2015), pollution locally generated not only accumulates at the given site, but also diffuses across locations. Yet, there is a key difference with spatial models which study the diffusion of wealth or air pollution, and it is that fertile soil is naturally bounded. Thereby this paper presents an original problem to the economics literature and leads to the description of optimal policies considering that locations can succeed, or not, in making fertile all their local available land.

As in Boucekkine et al. (2018), the optimal trajectory of fertile soil grows at a constant rate when all locations are (at least partially) polluted. During this first phase, the policy maker takes into account spatial heterogeneity and local optimal consumption depends, among other factors, on its abatement efficiency. Despite being spatially heterogeneous, consumption grows at the same constant rate at all locations. A second phase towards convergence starts when all land is fertile in at least one location. In these fertile locations, the optimal choice of the policy maker is the autarky, in the sense that they consume what they produce. As a consequence, these locations only abate their own pollution. The other polluted region still experiences diffusion of pollution. Contrary to the well-known results that apply when the production factors are unbounded, the economy can reach a long-term situation with two distinct regions: a fertile region, where all locations have attained their maximum for fertile soil, next to which a polluted region where locations cannot reach their maximum level for fertile soil. During the convergence phase, its pollution is partially absorbed by the first region. However, once diffusion towards the fertile region stops, the polluted region may not be able to abate enough to eradicate pollution. If this happens, the polluted region can stagnate at low levels of fertile soil. What is striking is that the polluted region is not necessarily poor, nor technologically backward. Our numerical exercises show that in a spatially homogeneous economy, initial disparities may not vanish with time if the fertile region does not help absorbing foreign pollution. Additionally, let us mention that the policy maker has an active role in the transition period, indirectly redistributing consumption from fertile to polluted locations.

From the modeling perspective, this paper describes for the first time in this literature the

case of economies with frontiers that are not fixed, that is, they can change geographically with time, responding to the regions' optimal decisions. The frontier's dynamics are modeled using Stefan's equation. In Physics, Stefan's equation describes the evolution in time of the frontier between two phases of a matter. Typically, it has been applied to describe the frontier between the solid and liquid phases of water. Here, subsection 4 investigates on the evolution of the dynamic frontier between the two emerging regions as well as the possible existence of two distinct regions in the long-run.

In the numerical exercises, we assume that all locations have reached the maximum of fertile soil. All of a sudden, one region suffers a natural catastrophe, which makes increase polluted land. The exercises cover different cases depending on the magnitude of the shock, the size of the affected region and the operating technology. Overall, we show that if the economy uses an advanced technology, then it can absorb the shock with time. However, if the technology is not sophisticated enough, then the shock will diffuse, spatially affecting a large part of the economy in the medium and long-run.

The rest of the paper is organized as follows. The model is presented in section 2. Section 3 provides with the optimal solution for the policy maker problem, describing the transit between the two phases of the problem. In Section 4 we run different numerical exercises, which show some dynamic properties of the model. Section 5 concludes. All the proofs are gathered in the Appendix.

2 Soil pollution diffusion in a linear growth model

We consider a closed economy, where both land and households are distributed over the unit circle on the plane, $\mathcal{S} = \{(\sin \theta, \cos \theta) \in \mathbb{R}^2 : \theta \in [0, 2\pi]\}$. Each location $\theta \in \mathcal{S}$ is populated by $N(\theta)$ individuals and is endowed with an amount of land $L(t, \theta)$.

Assumption 1. *The spatial distribution of land, $L(\cdot)$ is given and independent of time. Moreover, $L(\cdot)$ is either a constant or a linear function of space.*

Assumption 1 implies that locations cannot increase their total land allocation. Moreover, we suppose that land is composed of both fertile and polluted soil, L_F and L_P respectively, such that $L(\theta) = L_P(t, \theta) + L_F(t, \theta)$. All locations produce a unique agriculture good from the labour of local available fertile soil, according to the following linear production function $Y(t, \theta) = A(\theta)L_F(t, \theta)$, where $A(\theta)$ is the local production technology at location θ .

We assume that pollution diffuses in the soil in a similar manner as air pollutants flow in the atmosphere in the well-known Gaussian Plume (see Sutton 1947a and 1947b). The dynamics of polluted land at one location is explained by three factors. First, pollution diffuses from more polluted locations to less polluted locations. The diffusion of soil pollution is therefore captured by $D \frac{\partial^2 L_P}{\partial \theta^2}(t, \theta)$, where D is the diffusion coefficient. For tractability

reasons, D is assumed both constant in time and homogeneous in space. Second, fertile soil deteriorates locally. Indeed, local production generates some pollutant, which transforms fertile soil into polluted soil. The local effect is measured as $\nu(\theta)A(\theta)L_F(t, \theta)$, where $\nu(\theta)$ is the local sensitivity of fertile soil with respect to pollution. For instance, $\nu(\theta)$ can be related to the utilisation of different production technologies more or less pollutant across space, to a higher or lower level of biodiversity at θ , etc. Third, polluted soil can be changed in fertile soil through investments in pollution abatement. Hence, if $C(t, \theta)$ denotes total consumption at location θ at time t , then each location θ invests an amount $A(\theta)L_F(t, \theta) - C(t, \theta) \geq 0$ in abatement. Let $\phi(\theta)$ be the local pollution abatement efficiency. Then the spatial dynamics of polluted soil writes:

$$\frac{\partial L_P}{\partial t}(t, \theta) = D \frac{\partial^2 L_P}{\partial \theta^2}(t, \theta) + \nu(\theta)A(\theta)L_F(t, \theta) - \phi(\theta)[A(\theta)L_F(t, \theta) - C(t, \theta)].$$

Therefore, under Assumption 1, one can write that $\frac{\partial L_F}{\partial t}(t, \theta) = -\frac{\partial L_P}{\partial t}(t, \theta)$, and the evolution of fertile soil is described for all $t > 0$ and $\theta \in \mathcal{S}$ by:

$$\begin{cases} \frac{\partial L_F}{\partial t}(t, \theta) = D \frac{\partial^2 L_F}{\partial \theta^2}(t, \theta) + A(\theta)[\phi(\theta) - \nu(\theta)]L_F(t, \theta) - c(t, \theta)N(\theta)\phi(\theta), \\ L_F(0, \theta) = L_F^0(\theta), \theta \in \mathcal{S}, \end{cases} \quad (1)$$

where $A(\theta)[\phi(\theta) - \nu(\theta)]$ is the net land productivity. Total consumption, $C(t, \theta)$, is the product of the per-capita consumption $c(t, \theta)$ and the location's time-independent exogenous population $N(\theta)$. Hereafter the paper utilizes and analyses only L_F . Nevertheless, it is straightforward to obtain the corresponding results for L_P .

Let us introduce the linear operator \mathcal{L} defined for all functions u in the Hilbert space of function from \mathcal{S} to \mathbb{R} by, $\mathcal{L}u(\theta) := Du''(\theta) + A(\theta)[\phi(\theta) - \nu(\theta)]u(\theta)$. Then (1) can be rewritten as:

$$\frac{\partial L_F}{\partial t}(t, \theta) = \mathcal{L}L_F(t, \theta) - c(t, \theta)N(\theta)\phi(\theta), \quad (2)$$

where the initial distribution of L_F is known, $L_F(0, \theta) = L_F^0(\theta)$ for all $\theta \in \mathcal{S}$.

Operator \mathcal{L} is well behaved when applied to twice differentiable functions, measurable in \mathcal{S} . We say a function ϕ defined on \mathcal{S} , regular and non-identically zero, is an eigenfunction of \mathcal{L} , with associated eigenvalue $\lambda \in \mathbb{R}$ if $\mathcal{L}\phi(x) = \lambda\phi(x)$. Coddington and Levinson (1955) proved that there exists a countable set of eigenvalues $\{\lambda_n\}_{n \geq 0}$, which can be ordered as a decreasing sequence. It can be proven that the first eigenvalue of \mathcal{L} , λ_0 , is an eigenvalue with multiplicity 1, all other eigenvalues have either multiplicity 1 or 2.² Besides, e_0 , the eigenfunction associated to λ_0 , is strictly positive on the unit circle and one can set $\int_0^{2\pi} e_0^2(\theta)d\theta = 1$. Moreover, the eigenfunctions of \mathcal{L} form an orthonormal basis in $L^2(\mathcal{S})$ and the sequence of eigenvalues tends to $-\infty$, while the eigenfunction e_n associated to λ_n has n zeros in $[0, 2\pi]$.³

²The multiplicity of an eigenvalue is the number of times it appears in this sequence.

³See Appendix A for a brief introduction to abstract dynamic systems in infinite dimensions. For further details see Coddington and Levinson (1955) or Brown et al. (2013).

Finally, we assume there exists a policy maker, whose aim is to maximize overall welfare. Welfare is measured as the time integral from time 0 of the present value of the spatial aggregate of individuals' utility. Here, utility depends solely on per capita consumption, c , and it is measured by a constant inter-temporal elasticity of substitution function of parameter $\sigma \in \mathbb{R}$. Knowing that the policy maker discounts time at a constant rate ρ , her problem writes as:

$$\max_c \int_0^\infty e^{-\rho t} \left(\int_0^{2\pi} \frac{c(t, \theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta \right) dt, \quad (3)$$

subject to (1) and:

$$0 \leq L_F(t, \theta) \leq L(\theta), \quad (4)$$

$$c(t, \theta) \geq 0, \text{ for all } \theta \in \mathcal{S}, \text{ and } t \geq 0. \quad (5)$$

Our model is structurally close to Boucekkine et al. (2013) and Boucekkine et al. (2018), since production is linear in fertile soil. As in previous works, all land is labored at all locations, independently of the distribution of population. Nevertheless, there is a key difference with all previous work from the literature: the production factor, here fertile soil, is naturally bounded. From this, we shall now give the following definition:

Definition 1. *The economy is said to be in Phase 1 (resp. Phase 2) when the constraint on fertile soil L_F expressed in (4) is not binding (resp. is binding).*

Thus, the present analysis faces new challenges as it leads to the description of optimal trajectories in two separate phases, that is, before and after all fertile soil is fertile in one location. Most remarkable, the policy maker will play an active role in Phase 1, during which no location has reached its boundary, by indirectly redistributing consumption from fertile to polluted locations. Finally, we face the theoretical challenge of describing the dynamic frontier dividing the region at the maximum of fertile soil, and the region still in transition. Here we make use of Stefan's equation in which the frontier evolves following the gradient of pollution. A detailed description of the Stefan's equation is provided in Section 3.2.

3 The optimal policy and the evolution of fertile soil

This section provides with the optimal solution to the policy maker problem introduced in the previous section. In particular, we show that the economy transits through two phases. In Phase 1, each location's land is divided between fertile and polluted soil. During such phase, the amount of fertile soil evolves steadily at a constant rate, which can either be positive or negative. Phase 2 starts whenever fertile soil at a single location attains the maximum or the minimum value, that is, $L(\theta)$ or 0. We characterize the entire path for optimal consumption and for fertile soil when the model's parameters are heterogeneous in space. We have relegated to the Appendix most technical material as well as all the article's proofs.

3.1 Dynamics within the bounds

Beforehand, we shall make the following assumption:

Assumption 2. $\rho > \lambda_0(1 - \sigma)$, where λ_0 the first eigenvalue of the problem $\mathcal{L}u = \lambda u$.

Be $\langle L_F \rangle(t) := \frac{1}{2\pi} \int_0^{2\pi} L_F(\eta, t) d\eta$ the instantaneous spatial mean value for a given fertile soil distribution at time t . We provide next with the optimal solution for fertile soil and consumption in Phase 1, that is, when no location have reached their local maximum of fertility level.⁴

Proposition 1. *Under Assumptions 1 and 2, the optimal spatial dynamics of fertile soil in Phase 1, $L_F^*(t, \theta)$ is the solution of:*

$$\frac{dL_F^*}{dt}(t, \theta) = \mathcal{L}L_F^*(t, \theta) - [\phi(\theta)N(\theta)]^{\frac{\sigma-1}{\sigma}} N(\theta)^{\frac{1}{\sigma}} (\alpha_0 e_0(\theta))^{-\frac{1}{\sigma}} \int_0^{2\pi} L_F^*(t, \eta) \alpha_0 e_0(\eta) d\eta,$$

where $L_F^*(t, \theta) < L(\theta)$ for all $t \geq 0$ and $\theta \in \mathcal{S}$, and the dynamics for its mean value writes as:

$$\langle L_F^* \rangle(t) = \langle L_F^0 \rangle e^{gt},$$

for all $t \geq 0$, where $L_F^0(\theta) = L_F(0, \theta)$ is the initial distribution of fertile soil.

Moreover, optimal consumption c^* can be expressed as:

$$c^*(t, \theta) = [\phi(\theta)\alpha_0 e_0(\theta)]^{-\frac{1}{\sigma}} e^{gt} \int_0^{2\pi} L_F^0(\eta) \alpha_0 e_0(\eta) d\eta, \quad (6)$$

where $\alpha_0 \in \mathbb{R}$ is defined as:

$$\alpha_0 = \left[\frac{\sigma}{\rho - \lambda_0(1 - \sigma)} \int_0^{2\pi} [\phi(\eta)e_0(\eta)]^{-\frac{1-\sigma}{\sigma}} N(\eta) d\eta \right]^{\frac{\sigma}{1-\sigma}}.$$

e_0 is the first eigenvector of the problem $\mathcal{L}u = \lambda u$ associated to the first eigenvalues λ_0 , and $g = \frac{\lambda_0 - \rho}{\sigma}$ is the growth rate of the economy.

Proposition 1 shows that both fertile soil mean and consumption grow at a constant and homogeneous rate g from $t = 0$. Hence, although consumption varies across locations, consumption grows at the same rate across space, and if g is positive, then consumption increases continuously with time at all locations. In contrast, as land is naturally bounded, fertile soil cannot increase forever. Some locations will attain their maximum in a finite time $\tau > 0$. At such time τ , the economy enters Phase 2, that is, when the fertile soil constraint (4) is binding at least at one location. We describe Phase 2 in the following section. Note

⁴Note that although we have separated the presentation of the optimal solution in the two phases for the sake of exposition, the optimization problem is solved globally from $t = 0$ taking into account the eventuality of reaching the upper bound for L_F at certain locations (see Appendix B). Using the optimal conditions found in Appendix B, Appendix C finds the explicit optimal solution in Phase 1, while Appendix F provides us with the optimal solution in Phase 2 that we provide in Section 3.2.

that, if $g < 0$, then fertile soil tends to zero at all locations, and the economy vanishes. This situation could also be considered as reaching Phase 2, although we opt here to focus on the case of a second phase with $g > 0$ and available fertile soil.

The proposition also reveals the existence of a sustainable level for the time discount rate $\tilde{\rho} = \lambda_0$. If the policy maker weights enough the future generations, $\rho < \tilde{\rho}$, she sacrifices current consumption to increase abatement and guarantee future generations' consumption. In this case $g > 0$ and at least some locations attain the maximum of fertile soil. If $\rho = \tilde{\rho}$, then $g = 0$ and the economy stagnates and remains at the initial situation. When the policy maker is not altruistic enough, $\rho > \tilde{\rho}$, then $g < 0$ and she prefers current consumption to abatement, endangering the future of fertile soil, production and consumption.

As in Boucekkine et al. (2018), the policy maker cannot disregard the heterogeneity of the economy in her optimal decision. Despite weighting the welfare of each individual equally, consumption per capita varies across locations. Note that c^* in (6) depends on the location characteristics via ϕ and e_0 . However, since the analytical form of e_0 is typically not computable, we cannot provide a detailed analytical description of the effect of the different factors on c^* . Similarly, when A , ϕ , ν and N are spatially distributed and when $D \neq 0$, the growth rate g in our general set-up depends on the first eigenvalue λ_0 of the \mathcal{L} operator, with $g = \frac{\lambda_0 - \rho}{\sigma}$, and does not have a closed-form formula for heterogeneous spatial characteristics.

Yet, under specific hypothesis on the model's parameters we can provide closed form formulas for the eigenvectors and eigenvalues of the operator. As a result, closed forms are also obtained for growth rate g , the distribution in consumption c^* and in fertile soil L_F^* . In particular, we develop next two applications of the general framework for which solutions are fully analytically tractable. First, we consider a diffusive homogeneous economy, where A , ϕ , ν and N are homogeneous, yet with potentially an heterogeneous initial fertile soil distribution. Secondly, we analyse the role of diffusion by comparing these results to the case of an unconnected economy made of spatially independent locations, that is, when pollution does not diffuse across locations, hence $D = 0$.

Proposition 2. *In a diffusive homogeneous economy, where the distributions for A , ϕ , ν and N are constant in space, optimal fertile soil in Phase 1 can be expressed as:*

$$L_F^*(t, \theta) = \langle L_F^0 \rangle e^{gt} + \sum_{n \geq 1} \langle L_F^0, e_n \rangle e_n(\theta) e^{\lambda_n t}, \quad \forall \theta \in \mathcal{S} \text{ and } \forall t \in [0, \tau[,$$

where L_F^0 is the initial fertile soil distribution, e_n the n^{th} eigenvector of the problem $\mathcal{L}u = \lambda u$ associated to the n^{th} eigenvalue λ_n , with $\lambda_n = A(\phi - \nu) - Dn^2$ for $n \geq 0$ and τ the time at which the economy reaches Phase 2.

Moreover, the optimal growth rate of consumption is independent of the pollution diffusion coefficient D and of population N and writes as:

$$g = \frac{\lambda_0 - \rho}{\sigma} = \frac{A(\phi - \nu) - \rho}{\sigma}.$$

For the homogeneous economy, the expression for the optimal fertile soil distribution is very convenient. Indeed, L_F^* can be written as the sum of its mean value, $\langle L_F^0 \rangle e^{gt}$, and a second term, $\sum_{n \geq 1} \langle L_F^0, e_n \rangle e_n e^{\lambda_n t}$, with zero mean. Since neither g nor L_F^0 depend on the diffusion coefficient D , then the first term does not depend on D either. Hence, the effects of pollution are gathered in the second term, that is, in the eigenvalues λ_n . Note that if $D > A(\phi - \nu)$, then $\lambda_n < 0$ for $n \geq 1$. Hence, if diffusion is strong enough, the second term vanishes to zero with time and fertile soil is only driven by its initial mean value in the long-run. This is all the more true as diffusion D increases.

Next let us consider the case of an homogeneous unconnected economy in which $D = 0$. Here, soil pollution no longer diffuses so that the quality of the soil is independent from one location to another. As a matter of fact, it is as if the economy was made of independent and unconnected locations where the activity of one location does not affects its neighbors. Hence, the spatial dynamic optimal control problem presented in Section 2 reduces to a simpler problem, which is to be solved for each location independently. In what follows we analyze and quantify the gains and losses when locations belong to a diffusive economy with positive growth. We define the consumption transfer distribution as:

$$\Gamma(t, \theta) := c_A(t, \theta) - c_D(t, \theta),$$

where c_A is the consumption distribution in the unconnected economy and c_D the consumption distribution in the diffusive economy. In other words, Γ measures the losses in consumption from entering a diffusive economy. Whenever $\Gamma(t, \theta)$ is positive at a given (t, θ) , location θ would be better off alone at time t , consumption-wise. Obviously, since the contrary is also true, the economy will be made of losers and winners at all times.

Proposition 3. *When A , ϕ , ν and N are homogeneous, the consumption profiles are:*

$$\begin{aligned} c_A(t, \theta) &= \frac{\rho + A(\phi - \nu)(\sigma - 1)}{N\sigma\phi} L_F^0(\theta) e^{gt}, \\ c_D(t, \theta) &= \frac{\rho + A(\phi - \nu)(\sigma - 1)}{N\sigma\phi} \langle L_F^0 \rangle e^{gt}, \end{aligned}$$

where, L_F^0 is the given initial fertile soil distribution and $\langle L_F^0 \rangle$ its mean value. The two consumption profiles grow at the same rate $g = \frac{A(\phi - \nu) - \rho}{\sigma}$ and the consumption transfer distribution depends on the initial distribution in fertile soil $L_F^0(\theta)$ and writes as:

$$\Gamma(t, \theta) = \frac{\rho + A(\phi - \nu)(\sigma - 1)}{N\sigma\phi} [L_F^0(\theta) - \langle L_F^0 \rangle] e^{gt}.$$

Proposition 3 proves that whether pollution diffuses or not, homogeneous economies grow at the same rate. In both cases, if the net productivity $A(\phi - \nu)$ is not large enough, that is if $A(\phi - \nu) < \rho$, then $g < 0$ and consumption decreases. Therefore, there is an abatement efficiency threshold $\frac{\rho}{A} + \nu$ above which the economy prospers. Hence, even in a diffusive economy, each location needs to abate efficiently so as to compensate its own pollution, or

else the economy collapses. In other words, no location can rely on positive spillovers from its neighbors, and this despite diffusion.

However, although consumption growth rates are identical, the distributions in consumption differ. In the unconnected economy, locations' consumption depends solely on their own initial endowment of fertile soil and spatial inequalities consumption-wise across locations persist in time. Note how consumption grows at a constant rate g and does not adjust to current values of fertile soil, building on the initial land endowment as reference. In contrast, consumption distribution in the diffusive economy is homogeneous in space, even starting from an heterogeneous initial distribution L_F^0 . Finally, Assumption 2 implies here that $\rho + A(\phi - \nu)(\sigma - 1) > 0$. Therefore, if at location θ , $L_F^0(\theta)$ is above the economy's initial mean value, then $\Gamma(t, \theta) > 0$, and thus location θ always transfers consumption (indirectly). Therefore, the policy maker induces larger abatement efforts to the initially more fertile and productive regions so they can absorb pollution from other locations, which thanks to that, can increase their consumption.

3.2 The dynamic frontier and the final adjustment

The second phase begins the moment one location's fertile soil attains its maximum. Locations at the maximum, divide production between consumption and abatement of instantaneously generated pollution before it settles. This way, fertile soil remains at its maximum value thereafter. Let us consider $\mathcal{S} = \underline{\mathcal{S}}(t) \cup \bar{\mathcal{S}}(t)$ where $\bar{\mathcal{S}}(t)$ is made of the locations that have reached the upper bound for fertility at time t and locations in $\underline{\mathcal{S}}(t)$ have not reached the boundary yet. Our results hold if $\bar{\mathcal{S}}(t)$ is a convex set or a finite union of convex sets. For simplicity reasons, we proceed as if $\bar{\mathcal{S}}(t)$ is made of a single convex set as Figure 1 shows. In the sequel, we shall omit the time index in $\underline{\mathcal{S}}(t)$ and $\bar{\mathcal{S}}(t)$.

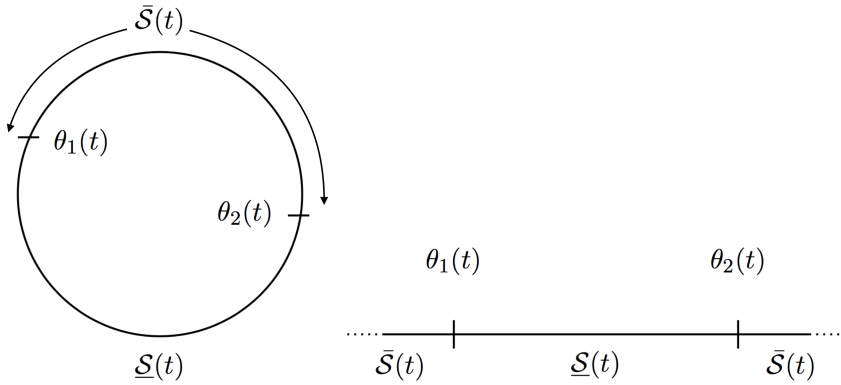


Figure 1: Description of the pollution frontier.

In Physics, the Stefan equation describes the evolution in time of the frontier between two phases of a matter. Typically, it has been applied to describe the frontier between the solid and liquid phases of water.⁵ In our problem, we assume that the pollution frontier delimiting

⁵The Stefan condition is expressed as a law of energy conservation, and in the case of water, the moving

the fertile region $\bar{\mathcal{S}}$ from the polluted region $\underline{\mathcal{S}}$ moves according to the Stefan equation, following the spatial gradient of polluted land. If at time $t \geq 0$ $\underline{\mathcal{S}}(t) = [\theta_1(t), \theta_2(t)] \subset \mathcal{S}$, then:

$$\dot{\theta}_1(t) = -D \frac{\partial L_P(t, \theta_1)}{\partial \theta} = D \frac{\partial L_F(t, \theta_1)}{\partial \theta} \quad \text{and} \quad \dot{\theta}_2(t) = -D \frac{\partial L_P(t, \theta_2)}{\partial \theta} = D \frac{\partial L_F(t, \theta_2)}{\partial \theta}.$$

Figure 2 provides with a graphical description of the dynamics of the pollution frontier.

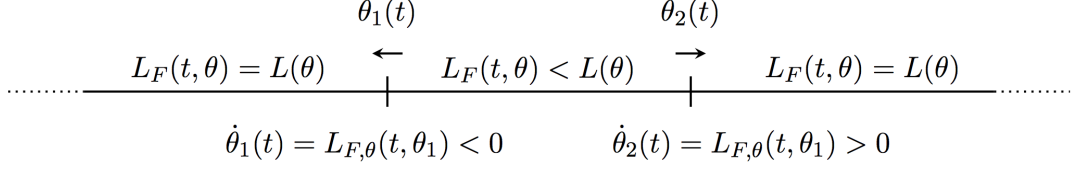


Figure 2: Dynamic pollution frontier

We now provide with the optimal consumption policy in Phase 2, that is, when at least one location fertility is binding, as well as the optimal fertile soil dynamics in the polluted region $\underline{\mathcal{S}}(t)$.

Proposition 4. *Suppose that at time $\tau \geq 0$, at least one location reaches the upper bound for fertile soil. Then, under Assumptions 1 and 2, the optimal spatial dynamics of fertile soil in Phase 2 depend on whether locations have reached their boundary. If at time $t \geq \tau$, $\theta \in \bar{\mathcal{S}}(t)$, then $L_F^*(t, \theta) = L(\theta)$ and optimal consumption is given by:*

$$c^*(t, \theta) = A(\theta) \frac{\phi(\theta) - \nu(\theta)}{\phi(\theta)} \frac{L(\theta)}{N(\theta)}, \quad \text{for all } t \geq \tau. \quad (7)$$

If on the contrary, $\theta \in \underline{\mathcal{S}}(t)$, then optimal consumption policy can be expressed as:

$$c^*(t, \theta) = (\phi \beta_0 e_0)^{-\frac{1}{\sigma}} \int_{\underline{\mathcal{S}}(t)} L_F^*(t, z) \beta_0 e_0(z) dz, \quad (8)$$

where e_0 is the first eigenvector of the problem $\mathcal{L}u = \lambda u$ on \mathcal{S} and β_0 a function of time such that:

$$\beta_0^{\frac{1-\sigma}{\sigma}} = \frac{\sigma}{\rho - \lambda_0(1 - \sigma)} \int_{\underline{\mathcal{S}}(t)} (\phi(z) e_0(z))^{-\frac{1-\sigma}{\sigma}} N(z) dz,$$

and the optimal fertile soil distribution in the polluted region $\underline{\mathcal{S}}$ is solution to the integro-differential equation:

$$\begin{aligned} \int_{\underline{\mathcal{S}}(t)} L_F^*(t, z) e_0(z) dz &= \int_{\underline{\mathcal{S}}(t)} L_F^*(\tau, z) e_0(z) dz e^{\frac{\lambda_0 - \rho}{\sigma}(t - \tau)} \\ &+ \int_{\tau}^t e^{\frac{\lambda_0 - \rho}{\sigma}(t - s)} \left(L(\theta_2(s)) e_0(2\pi) \dot{\theta}_2(s) - L(\theta_1(s)) e_0(0) \dot{\theta}_1(s) \right) ds, \end{aligned}$$

frontier depends on the change of temperature. Hence, if $u(x, t)$ denotes the temperature of water at location x in a space X at a given time $t \geq 0$, and $\xi(t)$ the frontier between solid and liquid water, then:

$$\eta \rho \dot{\xi}(t) = c_1 \frac{\partial u(x^-, t)}{\partial x} - c_2 \frac{\partial u(x^+, t)}{\partial x},$$

where $\xi(0) = 0$, so that initially all space is frozen, and water melts when temperature attains an upper bound, that is $u(x, t) \geq U$ for some $U \in \mathbb{R}$. c_1 and c_2 are the coefficients of heat conductivity, ρ is the density of the solid phase and η is the latent heat of melting per unit of mass.

where the evolution of the pollution frontier $[\theta_1(t), \theta_2(t)]$ in time is given by the initial condition $(\theta_1(\tau), \theta_2(\tau))$ and the dynamics $\dot{\theta}_1(t) = D \frac{\partial L_F^*(t, \theta_1)}{\partial \theta}$, $\dot{\theta}_2(t) = D \frac{\partial L_F^*(t, \theta_2)}{\partial \theta}$.

Note that contrary to Phase 1, consumption in the polluted region $\underline{\mathcal{S}}$ does not grow at a constant rate. Among other things, its growth rate is linked to the evolution of the pollution frontier $[\theta_1(t), \theta_2(t)]$, that is the borders of $\underline{\mathcal{S}}$, and to abatement efficiency ϕ . In contrast, consumption in the fertile region $\bar{\mathcal{S}}$ is constant in time, yet heterogeneous in space, as it depends exclusively on locations' local characteristics. More productive locations, those less populated and those with a higher total land endowment consume more. Moreover, optimal consumption policy in $\bar{\mathcal{S}}$, can be rewritten as $\phi(AL - c^*N) = \nu AL$, ensuring that each location in the fertile region abates its own pollution, therefore avoiding pollution diffusion from the surrounding locations in $\underline{\mathcal{S}}$ and no longer contributing to the abatement of their pollution.

Finally, in the long-term in Phase 2, either all locations converge to their own boundary for fertile soil, either there exists a region in which pollution diffuses forever and in which fertile soil never reaches its maximum.

Proposition 5. *In Phase 2, the economy can reach a steady state in which $\mathcal{S} = \bar{\mathcal{S}} \cup \underline{\mathcal{S}}$ at all t , where $\theta_1(t) = \bar{\theta}_1$ and $\theta_2(t) = \bar{\theta}_2$, with $\bar{\theta}_1, \bar{\theta}_2 \in \mathbb{R}$. fertile soil equals total land in $\bar{\mathcal{S}}$, and consumption is given by (7). In the polluted region $\underline{\mathcal{S}}$, the steady state for fertile soil \bar{L}_F is solution to $D\bar{L}_F'' + A(\phi - \nu)\bar{L}_F = 0$.*

Proposition 5 presents one of the main findings of this paper: in the case of a diffusive economy, that is $D \neq 0$, the polluted region $\underline{\mathcal{S}}$ in Phase 2 may fall in a sort of environmental poverty trap and never catch up with the fertile region. This result is not necessary straightforward as it provides with the existence of an heterogeneous steady state in fertile soil, that is the emergence of fertility patterns, even in the case of an homogeneous economy. Note that the emergence of such soil patterns in the polluted region $\underline{\mathcal{S}}$ is a direct effect of pollution diffusivity as, if $D = 0$, then the steady state for fertile soil in $\underline{\mathcal{S}}$ is $\bar{L}_F = 0$.

4 Numerical experiments

The previous section provides with analytical solutions in fertile soil and consumption in the two phases of the economy. Here, we shall present numerical exercises to shed light on some of the remaining open questions that were not analytically accessible.

More precisely, subsection 4.1 studies the dependency of the optimal solution on the spatial heterogeneity of the model components in Phase 1. We explore the case of a technological central pole and analyse the effect on fertile soil spatial dynamics of variations in abatement efficiency, while considering both homogeneous and heterogeneous soil sensitivities. Subsection 4.2 proposes five exercises that describe the dynamics of the optimal solution in Phase 2, as well as the evolution of the pollution frontier. We first study the convergence towards the

fertile economy, that is, when all locations reach their maximum level of fertile soil. Then, we study the resiliency of the fertile economy to pollution shocks. We consider both a uniform technology and a central pole, and discuss on the capacity of the economy to recover from pollution shocks, depending on their size and on the available technology.⁶

Regarding our calibration, the discount rate ρ is 3% as in Boucekine et al. (2018) and Lopez (2008). Using micro-household data, Attanasio and Weber (1993) and Attanasio and Browning (1995) show that σ lies in an interval between 1.25 and 3.33. Here we assume that $\sigma = 2$ following Barro et al. (1995). The amount of available land is 10 at all locations. This means that $L_F(\theta) < 10$ at all locations in Phase 1, and that the economy reaches Phase 2 when at least one location reaches maximum fertile soil, that is here $L_F(\theta) = 10$. Population is also uniformly distributed, with $N(t, \theta) = 1$ for all t and θ . Finally, we set the diffusion parameter D to 1.

4.1 Optimal solution in Phase 1

We study the detrended consumption distribution $c_g^*(\theta) = e^{-gt}c^*(t, \theta)$ and the dynamics of fertile soil for two values of ϕ , the abatement efficiency, and different functional forms for ν , the soil sensitivity to pollution.

Effect of abatement efficiency ϕ . Let us consider here the case of a technological pole around π , and analyze the effect of the abatement efficiency when $\phi = 2$ and $\phi = 3$. Figure 4 shows the given distributions of productivity A , abatement efficiency ϕ , soil sensitivity ν and the resulting optimal detrended consumption c_g .

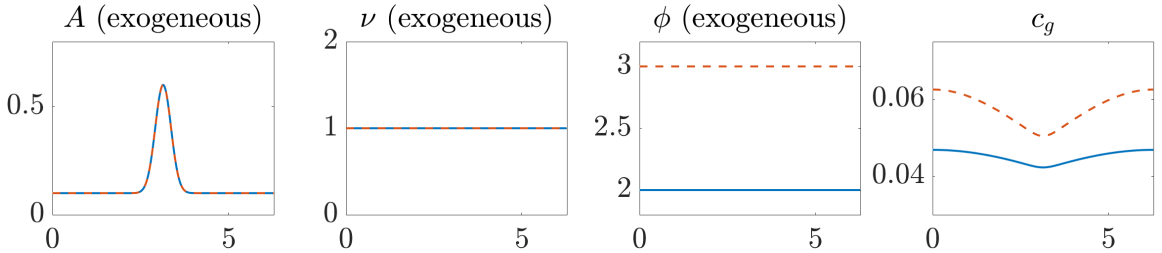


Figure 3: Impact of ϕ on c_g in Phase 1. $\phi = 3$ for dashed lines; $\phi = 2$ for solid lines.

Since net productivity $A(\phi - \nu)$ increases with ϕ , optimal consumption also increases with ϕ at each location. Figure 3 also shows that consumption is heterogeneous and that inequality in consumption becomes larger as abatement efficiency increases. Surprisingly, consumption is lower in the technologically advanced locations. There are two reasons for this phenomenon. First and independently of ϕ , pollution damage νAL_F is stronger in the technological pole. As a result, the social planner sacrifices consumption there in order to abate the local excess

⁶Subsection 4.1 uses the package Chebfun in Matlab. An algorithm has been developed in Matlab to simulate the optimal solution in Phase 2.

in pollution. Second, note that the central area can also abate more efficiently since $A\phi$ is larger there. For this reason the policy maker induces these locations to abate even further to act as a pollution sink along the optimal trajectory. In doing so, consumption diminishes locally while heterogeneity increases.

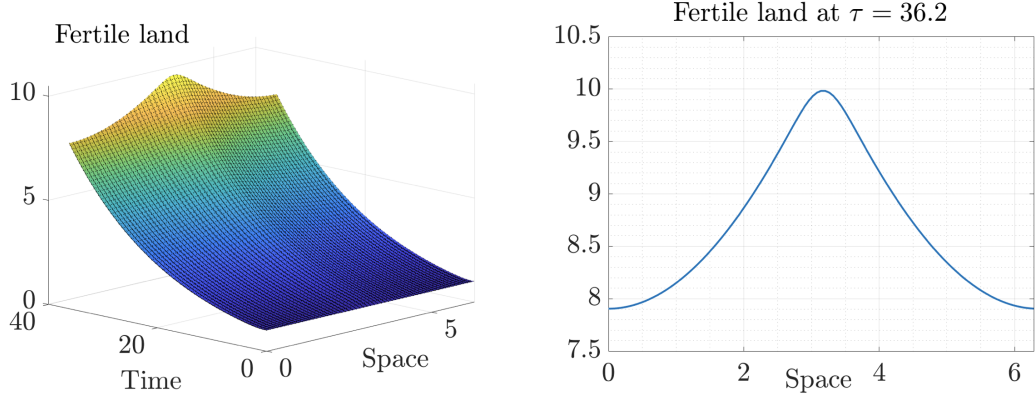


Figure 4: Dynamics of fertile soil in Phase 1 for $\phi = 2$ starting with $L_F^0 = 1$ for all θ .

As abatement increases, the technological pole accumulates more fertile soil and they are first to reach their fertility upper bound (see Figure 4). Moreover, as ϕ increases, this advantage of the core region over the periphery gets even more pronounced. Figure 5 shows that the more efficient the abatement technology is, the more unequal the economy (in terms of L_F). Yet, a higher ϕ reduces the length of Phase 1, compensating somehow the greater inequality it implies.

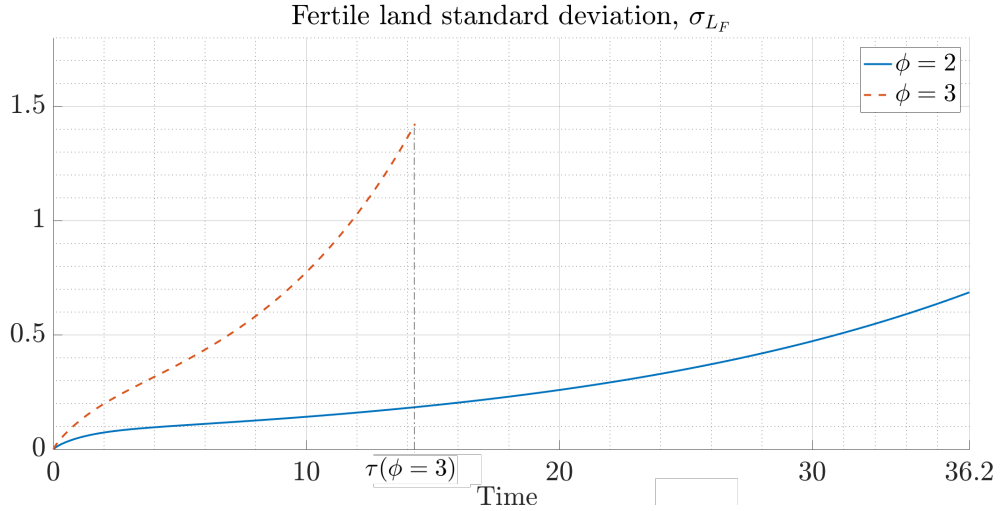


Figure 5: σ_{L_F} as a function of time for $\phi = 2$ and $\phi = 3$.

Proposition 1 proves that g is constant in time and spatially homogeneous in this Phase 1. Our numerical exercises show that g increases with ϕ . Indeed g grows from 6% to 14.1% when ϕ passes from 2 to 3. That is, an increase in 50% in the abatement efficiency implies an increase in 135% in the growth rate, pushing the economy towards Phase 2 at a faster pace.

Indeed, when $\phi = 2$ the economy reaches Phase 2 in 36.2 units of time, i.e. $\tau = 36.2$, while $\tau = 14.4$ when $\phi = 3$.

Soil sensitive to production. In this example locations with high productivity are also those where soil is more sensitive to pollution. This could reflect situations in which a very efficient production technology causes greater damages to soil, locally eroding the area's resilience to pollution. This dependence of soil sensitivity to productivity is modeled here using a power function: $\nu = \nu_0 A^\gamma$. Figure 6 shows the simulation results for two values of abatement efficiency $\phi = 10$ and $\phi = 11$.

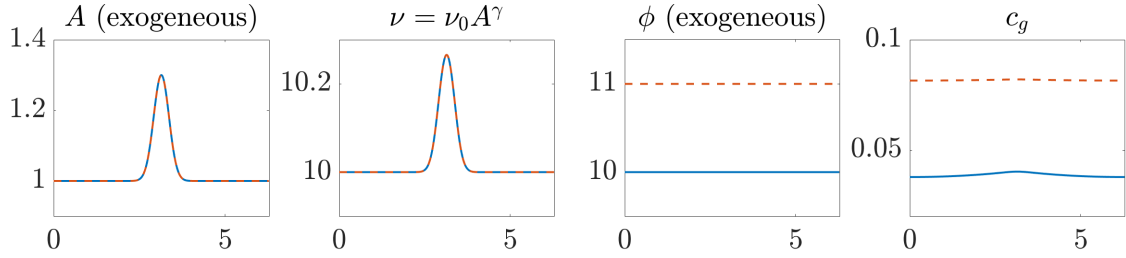


Figure 6: Impact of ϕ on c_g with spatially heterogeneous ν . Dashed lines correspond to $\phi = 11$; solid lines to $\phi = 10$.

As abatement efficiency increases, both detrended consumption and the growth rate increase as in the previous example. Yet, Figure 7 shows that, in this case, the fertile soil distribution when the economy reaches Phase 2 is U-shaped, which is in contrast with the inverted U-shaped distribution of the previous example. Here, abatement is not efficient enough in the sensitive and productive core region to absorb all the pollution that is locally generated. As a result, the first locations to reach the fertile economy are in the periphery, which are endowed with the least productive, yet least damaging, technology.

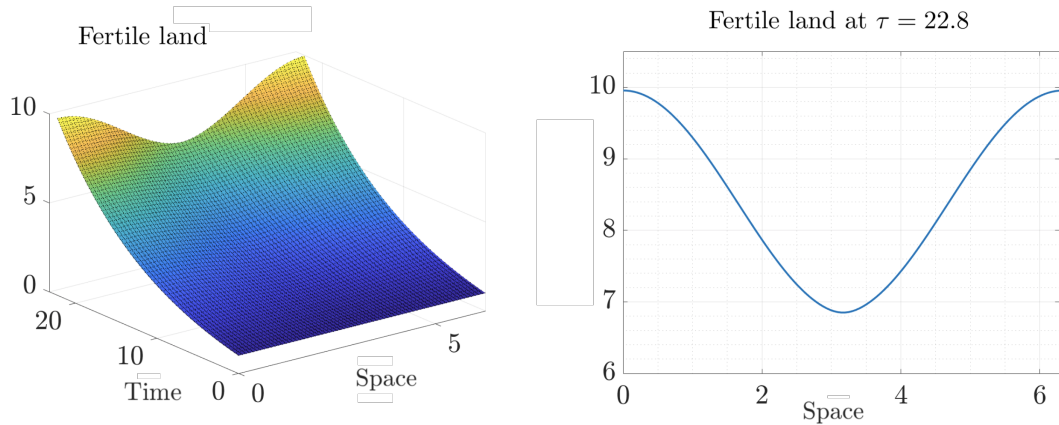


Figure 7: Dynamics of fertile soil in Phase 1 for $\phi = 11$ starting with $L_F^0 = 1$ for all θ .

Note that this behavior holds only for values of ϕ below a specific threshold. Above this threshold, abatement becomes efficient enough to absorb pollution from the core productive

region. In such case, the dynamics of fertile soil switches back to the inverted U-shape of the previous example in which the productive region reaches first the fertile economy. In either case, as long as the growth rate is positive, that is for $\lambda_0 > \rho$, one location reaches its fertility upper bound and the economy switches to Phase 2.

4.2 Fertile soil dynamics in Phase 2

Proposition 5 shows that despite spatial diffusion, fertile soil in the polluted region $\underline{\mathcal{S}}$ could remain below the maximum forever. We explore numerically the second phase of the dynamics, in which space is divided in two regions one of which has no pollution at all. The first exercise completes the first example of section 4.1, describing the dynamics of fertile soil from the moment Phase 2 is reached, until the fertile economy, that is, until fertile soil attains the maximum at all locations.

The second set of exercises investigates on the resiliency of the fertile economy towards pollution shocks and on the existence of a spatially heterogeneous fertile soil steady state. It will be assumed that starting from the zero pollution economy, a region receives a pollution shock. The shock could be thought as a natural or industrial catastrophe which locally pollutes soil on a limited geographic area such as the leak of pollutants from an industrial site or the explosion of a chemical plant.

Usually, when the technological coefficient is constant it is assumed that $A = 1$ for illustration purposes as in Boucekkine et al. (2018) or Camacho and Perez-Barahona (2015). For the same illustrative reason, when technology is constant we choose $A = 0.2$ to underline that the economy is agricultural and it uses a relative backward technology. In the cases where A is spatially heterogeneous, A will be an exponential function as we explain below. ϕ is equal to 2, ν is 0.2 and population is homogeneous and equal to 10.

4.2.1 Convergence towards the fertile economy

This exercise presents the continuation of the optimal solution studied in 4.1 and depicted in Figure 4, in which the central region attains Phase 2 at $\tau = 36.2$. From that moment onwards, locations at the maximum of fertile soil only abate to maintain their maximum level and diffusion only takes place across the borders of the fertile area. That is, locations with the largest amount of fertile soil stop absorbing pollution from neighboring locations. For locations in transition towards their maximum, this change comes as a shock and pollution rises. Let us underline that the passage from Phase 1 to Phase 2 was foreseen from $t = 0$ and it is by no means a shock, it is an optimal adaptation pattern. Figure 8 shows how at locations around $\theta = 0$ (or $\theta = 2\pi$), fertile soil drops from 7.9 to 7.4. Then abatement rates increase in $\underline{\mathcal{S}}$ and fertile soil starts growing again so that the polluted region catches up with the fertile one. The right panel in Figure 8 shows the growth rate of fertile soil. Note

that in contrast to Phase 1, the growth rate is not in the least constant in time nor spatially homogeneous. Locations where fertile soil had dropped in the most severe manner grow the fastest.

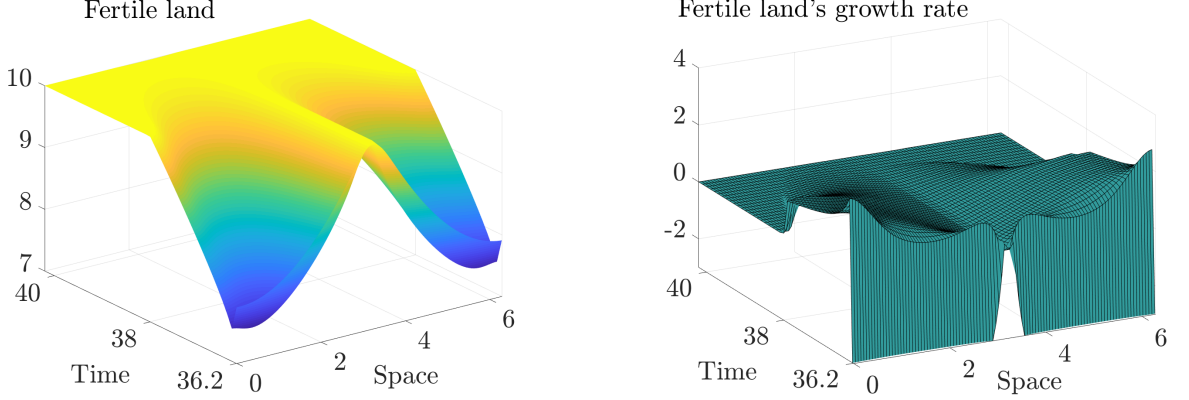


Figure 8: Dynamics in Phase 2 towards the fertile economy for $\phi = 2$.

4.2.2 Resiliency of the fertile economy to pollution shocks

In the following exercises, the economy is subject to different shocks which vary in size and coverage. First, let us compare the effect of a relatively small shock. The central region $[\frac{4.5}{10}2\pi, \frac{5.5}{10}2\pi]$ of size 1 receives a pollution shock, which induces a loss of 1% of its fertile soil, that is, $L_F(0, \theta) = 0.99L(\theta)$ for $\theta \in [\frac{4.5}{10}2\pi, \frac{5.5}{10}2\pi]$. The left column of Figure 9 describes the evolution in time of the pollution frontier between the fertile and the polluted regions after the shock, as well as the spatial dynamics of polluted land when technology A is spatially homogeneous, $A = 0.2$. On the right column, the figure shows the results when A follows an exponential spatial distribution, making the spatial center a technological pole. In this case, $A(\theta) = 0.20 + 2e^{-(\theta-\pi)^2}$. Advancing our results, we will conclude that a shock can be overcome only if the region under ecological stress is endowed with a sufficiently advanced technology.

Let us analyze the different stages of the evolution of a pollution shock. Right after the shock, polluted land increases, and this is independent on the technology the region uses as Figure 9 shows. Indeed, the remaining fertile soil has to face a large stock of pollution, trying to preserve consumption in as much as possible, sacrificing abatement temporarily. Additionally, we know that once pollution is present in the soil, it accumulates in time. As a result, after the initial shock, polluted land rockets. We begin by the case of the homogeneous technology, depicted on the left column of Figure 9.

In a second moment after the shock, pollution diffuses over space pushing the frontiers of the polluted region, so that the region with polluted land expands. As the top graphs show in Figure 9, the polluted region becomes stable in size for some time, reducing the level of pollution. Nevertheless, this situation is not stable. Indeed, the fall in polluted land is

followed by a very slow and subtle accumulation of pollution in all locations of the polluted region. Once a threshold of total pollution is reached, the region cannot abate enough. As a result, pollution starts diffusing spatially, pushing the frontiers of the polluted region outwards. At that moment there is a massive uprise in pollution everywhere. Indeed, not only local pollution accumulates in time by its own production, but it also receives pollution from neighboring locations. Again, locations tend to preserve consumption, diminishing abatement, which makes pollution increase locally. Noteworthy, in this example, locations at the maximum of fertile soil at the beginning, and which do not abate more than needed locally, end up being highly polluted by the diffusion mechanism.

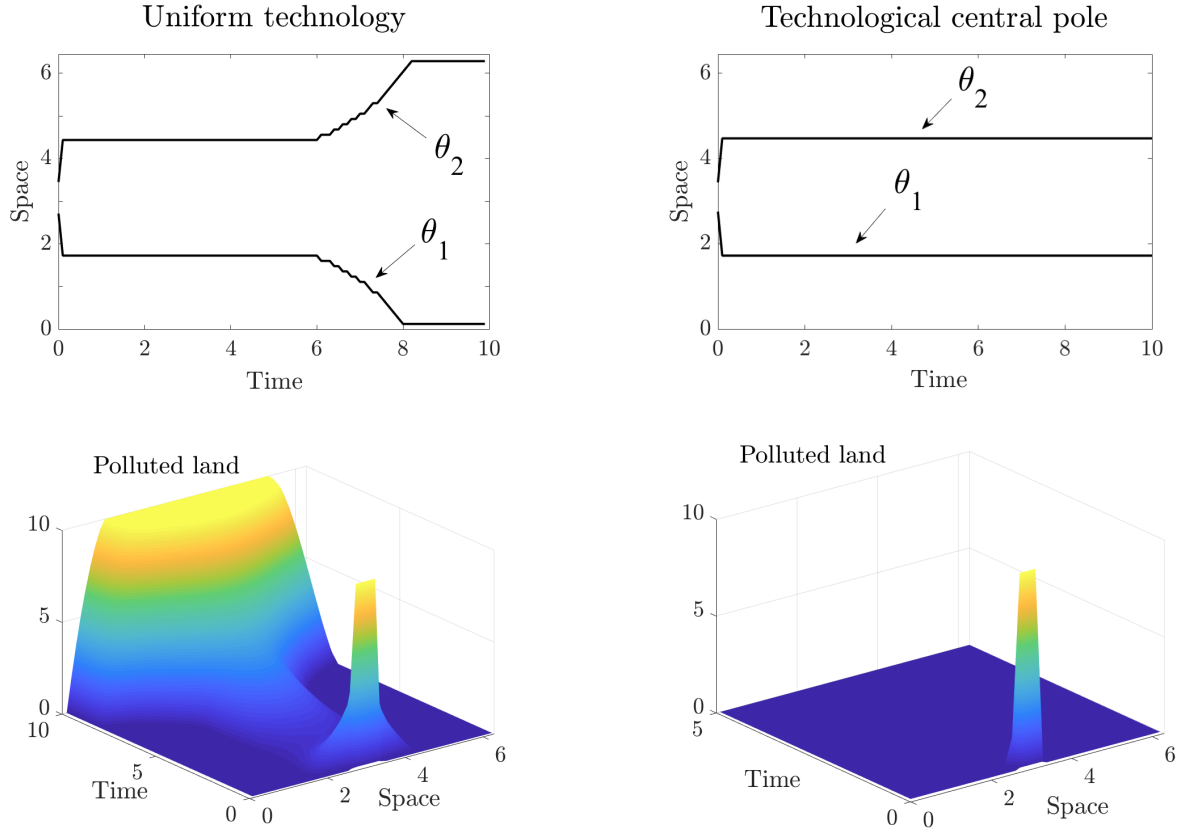


Figure 9: Spatial dynamics for initial shock $L_F(0, \theta) = 0.99L(\theta)$ for $\theta \in [\frac{4.5}{10}2\pi, \frac{5.5}{10}2\pi]$.

In sharp contrast, when the shocked region has a technological advantage, locations can overcome the shock by adapting their consumption and abatement during the initial moments (see graphs on the right of Figure 9). Although polluted land increases during the first moments, the central locations are more productive and can produce more using less land and hence polluting less. This mechanism counterbalances the shock, explaining why the central region absorbs all pollution in an extremely short period of time.

In a second example, we subject the economy to an enormous shock both in terms of the area covered and the amount of fertile soil lost. Indeed, in this example, $L_F(0, \theta) = 0.1L(\theta)$ for $\theta \in [\frac{2}{10}2\pi, \frac{8}{10}2\pi]$. That is, the catastrophic region represents 60% of the entire land, and

it loses 90% of its fertile soil. Figure 10 shows the results for both a homogeneous and a heterogeneous technology.

When technology is spatially homogeneous, the catastrophic region experiences a rise in pollution from the beginning. Augmenting further emissions of pollution, which add to the pre-existing pollution locally, and which diffuse spatially to the adjacent locations. A self-reinforcing polluting mechanism is triggered. Note that the central region remains at the highest levels of pollution (where almost all land is polluted) forever. Since technology is not advanced enough, the catastrophic region cannot produce enough for its population without worsening the environmental situation. Note how, as in the previous example, with time, pollution diffuses spatially and almost all the economy experiences high levels of pollution after 10 periods of time.

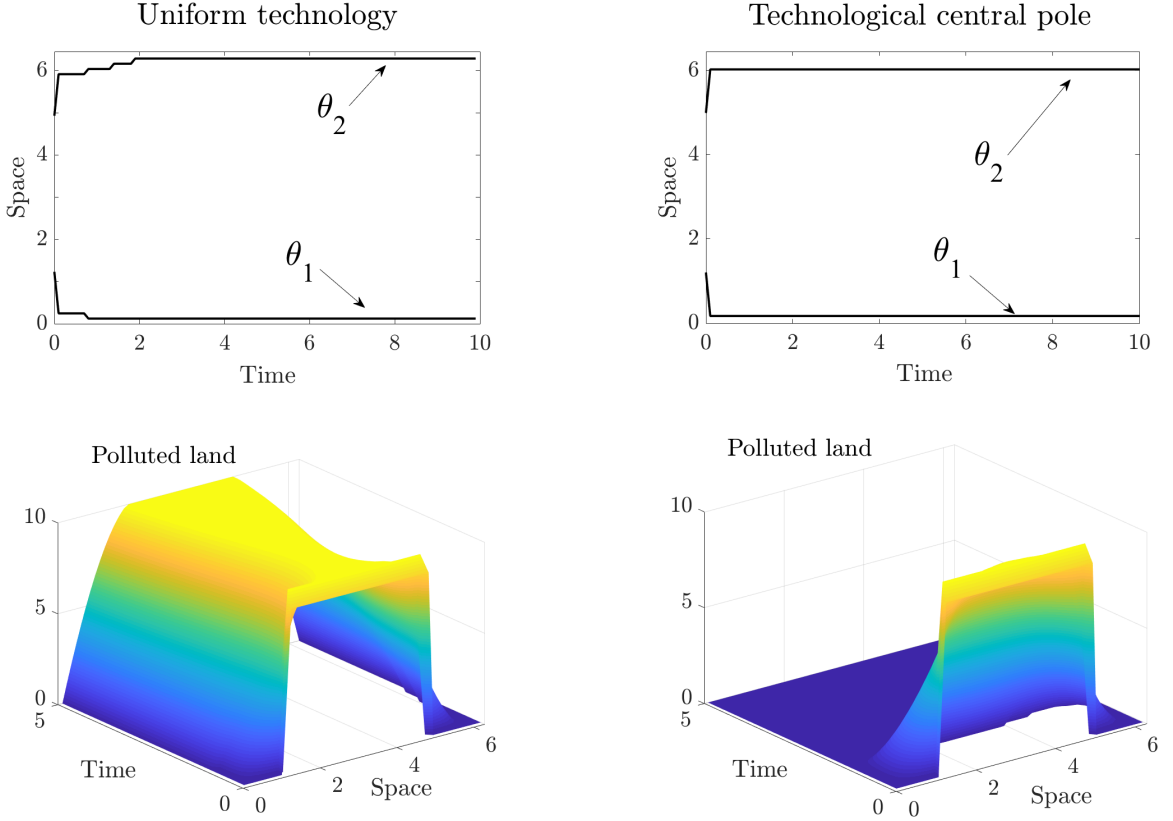


Figure 10: Spatial dynamics for initial shock $L_F(0, \theta) = 0.1L(\theta)$ with $\theta \in [\frac{2}{10}2\pi, \frac{8}{10}2\pi]$.

Finally, let us underline that when technology is sufficiently advanced, the catastrophic region can recover after an adjustment period even after an enormous shock (see right column in Figure 10). After the initial raise in pollution in the shocked region, all locations react, diminishing consumption, abating pollution until they reach the equilibrium. Here, all locations recover the maximum level of fertile soil. Nevertheless, given the size of the shock, recovery takes longer time than in the previous example.

Hence, our exercises have shown the existence of non homogeneous steady states for fertile soil, even when all the model's parameters are spatially homogeneous. As advanced, their

existence hinges on the size of the region and its technology. Indeed, a relatively small and backward region cannot absorb nor diffuse enough pollution, accumulating it locally instead.

5 Conclusion

This paper develops a spatial growth model accounting for the diffusion of pollution into soils. We provide with the explicit optimal trajectories for fertile and polluted soil in space and time, while considering the persistence of pollution into agricultural soils. We faced two major challenges. First, the production factor, fertile soil, is here, locally and naturally bounded by the amount of available land at each location. Second, space can be divided in regions, one of which could have attained the maximum of fertile soil. Borrowing the Stefan's equation from Physics, we have modeled the dynamics of the pollution frontier between the region in which fertile soil is at its maximum, and the region in which there is a share of polluted soil.

The boundedness of the production factor defines two phases in the dynamics of fertile soil. We described a first phase in which all locations have some polluted land, during which fertile soil and consumption grow at the same constant rate at all locations. Eventually, some locations eradicate all pollution and reach their maximum for fertile soil. Here the dynamics change radically. Locations at the maximum stop absorbing pollution coming from the yet struggling and polluted area. As a result, it may be the case that the polluted regions will never reach their potential maximum for fertile soil, remaining forever in a sort of environmental poverty trap, that is at a steady state with some share of polluted soil. This possibility has been analytically proven and numerically explored. We have indeed shown that economies could reach a spatially heterogeneous steady state after a catastrophic pollution shock when their technology is not advanced enough.

Therefore, and surprisingly enough, spatial pollution patterns can emerge even from an economy that is endowed with homogeneous technological and soil characteristics. The emergence of such heterogeneous soil patterns being the direct effect of pollution diffusivity in a spatially distributed agricultural economy.

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Appendices

A. Mathematical Preliminaries.

We represent (2) as an abstract dynamical system in infinite dimension.⁷ Its solutions belong to $L^2(\mathcal{S})$, defined as:

$$L^2(\mathcal{S}) = \{f : \mathcal{S} \longrightarrow \mathbb{R} \mid \int_0^{2\pi} |f(\theta)|^2 d\theta < \infty\}.$$

For any two functions f, g in $L^2(\mathcal{S})$, their inner product is defined as $\langle f, g \rangle = \int_0^{2\pi} f(\theta)g(\theta)d\theta$, and f ’s norm is then $\|f\|^2 = \langle f, f \rangle = \int_0^{2\pi} f^2(\theta)d\theta$. $L^2(\mathcal{S})$ endowed with the inner product $\langle \cdot, \cdot \rangle$ is a Hilbert space. Since we cannot take derivatives on all elements of $L^2(\mathcal{S})$, let us define three spaces that will help us defining the domain of \mathcal{L} :

$$\begin{aligned} \mathcal{H}_+^0(\mathcal{S}) &:= \{f \in L^2(\mathcal{S}) : f(\cdot) \geq 0, \text{ and } f(\cdot) \neq 0\}, \\ \mathcal{H}^1(\mathcal{S}) &:= \{f \in L^2(\mathcal{S}) : \exists f' \text{ in weak sense and belongs to } L^2(\mathcal{S})\}, \\ \mathcal{H}^2(\mathcal{S}) &:= \{f \in L^2(\mathcal{S}) : \exists f' \text{ in weak sense and belongs to } \mathcal{H}^1(\mathcal{S})\}. \end{aligned}$$

Hence, the domain of \mathcal{L} is $\mathcal{H}^2(\mathcal{S})$ and \mathcal{L} is well defined on this domain. Operator \mathcal{L} is self-adjoint since for any two $f, g \in \mathcal{H}^2(\mathcal{S})$, $\langle \mathcal{L}f, g \rangle = \langle f, \mathcal{L}^*g \rangle = \langle f, \mathcal{L}g \rangle$. Note first that the Laplacian operator is closed in its domain, $\mathcal{H}^2(\mathcal{S})$, and it generates a semigroup on $L^2(\mathcal{S})$. Since the second factor in \mathcal{L} is bounded, we can conclude that our operator \mathcal{L} is closed in its domain, $D(\mathcal{L}) = \mathcal{H}^2(\mathcal{S})$ and it generates a C_0 semigroup.

The Laplacian operator generates a C_0 semigroup in the following sense. Let \mathcal{A} be a closed linear operator defined on $L^2(\mathcal{S})$ (for simplicity), and consider the following initial value problem:

$$\begin{aligned} y'(t) &= \mathcal{A}y(t), \\ y(0) &= y_0 \in L^2(\mathcal{S}), \text{ given.} \end{aligned}$$

⁷A similar account can be found in Boucekkine et al. (2013). For a deeper insight see Bensoussan et al. (2007).

If the operator \mathcal{A} is bounded, then the solution to this problem is given by:

$$y(t) = e^{t\mathcal{A}}y_0 := \sum_{j=0}^{\infty} \frac{t^j \mathcal{A}^j y_0}{j!}.$$

If we define $Z_{\mathcal{A}}(t) = e^{t\mathcal{A}}$, then $Z_{\mathcal{A}}(t+s) = Z_{\mathcal{A}}(t)Z_{\mathcal{A}}(s)$, and $Z_{\mathcal{A}}(0) = 1$. As mentioned in p. 89 in Bensoussan (2007), there is a one-to-one correspondence between the group $(\mathbb{R}, +)$ and the subset $Z_{\mathcal{A}} = \{Z_{\mathcal{A}}(t) : t \in \mathbb{R}\}$ in $L(\mathcal{S})$, or in particular in $L^2(\mathcal{S})$ under composition “ \circ ”. So we say that $(Z_{\mathcal{A}}, \circ)$ is a group. When \mathcal{A} is unbounded like in the case of the Laplacian operator, then one can define $Z_{\mathcal{A}}$ only for $t \geq 0$ and $s \geq 0$ in general. We say then that $Z_{\mathcal{A}}$ is a semigroup.

The most useful property of semigroups hinges on the following relationship. The infinitesimal generator \mathcal{A} of Z is the linear operator in \mathcal{S} defined by:

$$D(\mathcal{A}) = \{x \in \mathcal{S} : \text{such that } \lim_{h \rightarrow 0^+} \frac{Z(h)x - x}{h} \text{ exists}\},$$

and $\mathcal{A}x = \lim_{h \rightarrow 0^+} \frac{Z(h)x - x}{h}$ for all $x \in D(\mathcal{A})$. Then, we have that:

$$\frac{d}{dt}Z(t)x = \mathcal{A}Z(t)x = Z(t)\mathcal{A}x.$$

Given the set of admissible trajectories for L_F , we can define the set of admissible strategies for c, \mathcal{C} as:

$$\mathcal{C}(L_F^0) = \{c \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}^t, L^2(\mathcal{S})^+) : L_F \in \mathcal{H}_0, \text{ for all } t \geq 0\}.$$

Finally, the objective functional is:

$$J(L_F^0; c) := \int_0^{\infty} e^{-\rho t} U(c) dt,$$

where:

$$U(c) = \int_{\mathcal{S}} \frac{c(t, \theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta.$$

The value function associated to our problem is then:

$$V(L_F^0) := \sup_c J(L_F^0; c).$$

B. Necessary optimal conditions.

Although our problem is close to Boucekkine et al. (2013) and Boucekkine et al. (2018), it differs from then in that our problem is bounded. We start by obtaining the set of necessary optimal conditions using the Ekeland variational principle. This set of conditions has the main shortcoming of including a reverse time parabolic partial differential equation, which renders our problem difficult to analyse. At this point, we resort to dynamic programming to obtain the optimal solution in the unbounded case. Fortunately, and since the optimal solution is unique (see Camacho and Perez-Barahona, 2015), we can prove that the solution previously obtained satisfies the set of optimal conditions when fertile land lies within its boundaries.

We shall prove that fertile land grows at a constant and spatially homogeneous rate, steadily increasing or decreasing with time. As a result, one location will eventually reach a boundary for fertile land. At that moment, the solution enters the second phase. Here, some of the locations

have attained the maximum level of fertile land, while others are still transiting. We characterize the optimal solution under the assumption that once locations reach the bound, they stop investing in pollution abatement.

Let us write the value function $V(\cdot)$ associated to problem (3), (1), (2), (4) and (5):

$$\begin{aligned} V(c, L_F, \psi, \mu) = & \int_0^\infty \int_0^{2\pi} e^{-\rho t} \frac{c(t, \theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta dt \\ & - \int_0^\infty \int_0^{2\pi} \psi(t) \left(\frac{\partial L_F}{\partial t}(t, \theta) - D \frac{\partial^2 L_F}{\partial \theta^2}(t, \theta) - A(\theta)[\phi(\theta) - \nu(\theta)] L_F(t, \theta) + c(t, \theta) N(\theta) \phi(\theta) \right) d\theta dt \\ & - \int_0^\infty \int_0^{2\pi} \mu(t, \theta) (L(\theta) - L_F(t, \theta)) d\theta dt - \int_0^\infty \int_0^{2\pi} \xi(t, \theta) L_F(t, \theta) d\theta dt. \end{aligned}$$

Then, assuming there exists an optimal solution and that any solution to our problem can be written as a deviation from the optimal solution we obtain:

$$c(t, \theta) = c^*(t, \theta) + \epsilon \mathcal{C}(t, \theta),$$

$$L_F(t, \theta) = L_F^*(t, \theta) + \epsilon \mathcal{L}_F(t, \theta).$$

V becomes then a function of ϵ . To optimize $V(\cdot)$ boils down to minimizing the deviation. At the optimal ϵ^* , $\frac{\partial V(\epsilon^*)}{\partial \epsilon} = 0$. We use that the auxiliary variables ψ, μ and ξ are \mathcal{C}^1 functions, so that $\psi(t, 0) = \psi(t, \pi)$, $\mu(t, 0) = \mu(t, \pi)$, $\xi(t, 0) = \xi(t, \pi)$; and $\psi_x(t, 0) = \psi_x(t, \pi)$, $\mu_x(t, 0) = \mu_x(t, \pi)$, $\xi_x(t, 0) = \xi_x(t, \pi)$. Since $L_F \in \mathcal{C}^1$, it is also true that $L_F(t, 0) = L_F(t, 2\pi)$ and $L_{F,x}(t, 0) = L_{F,x}(t, 2\pi)$. Then, after applying integration by parts:

$$\begin{aligned} \frac{\partial V(\epsilon)}{\partial \epsilon} = & \int_0^\infty \int_0^{2\pi} e^{-\rho t} c(t, \theta)^{-\sigma} \mathcal{C}(t, \theta) N(\theta) d\theta dt \\ & - \int_0^\infty \int_0^{2\pi} [-\psi_t(t, \theta) \mathcal{L}_F(t, \theta) - D \psi_{xx}(t, \theta) \mathcal{L}_F(t, \theta)] d\theta dt \\ & - \int_0^\infty \int_0^{2\pi} [-A(\theta)[\phi(\theta) - \nu(\theta)] \psi(t, \theta) \mathcal{L}_F(t, \theta) + \mathcal{C}(t, \theta) N(\theta) \phi(\theta)] d\theta dt \\ & + \int_0^\infty \int_0^{2\pi} [\mu(t, \theta) - \xi(t, \theta)] \mathcal{L}_F(t, \theta) d\theta dt, \end{aligned}$$

with $\lim_{t \rightarrow \infty} \psi(t, \theta) L_F(t, \theta) = 0$, for all $\theta \in \mathcal{S}$. Along with the transversality condition, we obtained the following condition on L_F and ψ :

$$\psi(t, \theta) \nabla_\theta L_F(t, \theta) \Big|_0^{2\pi} = \nabla_\theta \psi(t, \theta) L_F(t, \theta) \Big|_0^{2\pi}, \quad (9)$$

where ∇_θ is the space integral of \mathcal{L} . That is:

$$\nabla_\theta L_F(t, \theta) = \frac{\partial L_F(t, \theta)}{\partial \theta} + \int_{\theta_1(t)}^{\theta(t)} A(z) [\phi(z) - \nu(z)] L_F(t, z) dz,$$

for any $\theta \in \mathcal{S}$. In Phase 1, this condition does not provide us with further insight into the problem since L_F and ψ are continuous and continuously differentiable in the unit circle. Nevertheless, (9) adds crucial information about the border in Phase 2. Collecting the terms multiplying \mathcal{C} and \mathcal{L}_F ,

and detrending ψ , we obtain the following set of necessary optimal conditions for all $\theta \in \mathcal{S}$ and for all $t > 0$:

$$\begin{cases} \psi(t, \theta) = \frac{1}{\phi(\theta)} c(t, \theta)^{-\sigma}, \\ \psi_t(t, \theta) + D\psi_{xx}(t, \theta) + A(\theta)[\phi(\theta) - \nu(\theta)]\psi(t, \theta) - \rho\psi(t, \theta) + \mu(t, \theta) - \xi(t, \theta) = 0, \\ \mu(t, \theta)(L(\theta) - L_F(t, \theta)) = 0, \\ \xi(t, \theta)L_F(t, \theta) = 0. \end{cases}$$

Recalling that $\mathcal{L}\psi(t, \theta) = D\psi_{xx}(t, \theta) + A(\theta)[\phi(\theta) - \nu(\theta)]\psi(t, \theta)$, we can rewrite the second equation as:

$$\psi_t(t, \theta) + \mathcal{L}\psi(t, \theta) - \rho\psi(t, \theta) + \mu(t, \theta) - \xi(t, \theta) = 0.$$

Note that if L_F lies within its bounds, then both $\mu(t, \theta)$ and $\xi(t, \theta)$ are identically zero. Hence, we can provide a set of necessary conditions when fertile land is between the lower and the upper bound, at all locations:

$$(I) \begin{cases} \frac{\partial L_F}{\partial t}(t, \theta) = \mathcal{L}L_F(t, \theta) - c(t, \theta)N(\theta)\phi(\theta), \\ \psi(t, \theta) = \frac{1}{\phi(\theta)} c(t, \theta)^{-\sigma}, \\ \psi_t(t, \theta) + \mathcal{L}\psi(t, \theta) - \rho\psi(t, \theta) = 0, \\ \lim_{t \rightarrow \infty} \psi(t, \theta)L_F(t, \theta) = 0. \end{cases}$$

The second phase is triggered whenever a location reaches the upper bound for fertile land. We shall discuss the case when fertile land reaches the lower bound later. In this case, we assume that $\mathcal{S} = \underline{\mathcal{S}} \cup \bar{\mathcal{S}}$, where $\bar{\mathcal{S}}$ is the set of locations that have reached the upper bound. Hence, for all $\theta \in \bar{\mathcal{S}}$:

$$L_F(t, \theta) = L(\theta), \quad L_{F,t}(t, \theta) = 0 \quad \text{and} \quad L_{F,xx}(t, \theta) = 0,$$

which implies that $c(t, \theta) = A(\theta) \frac{\phi(\theta) - \nu(\theta)}{\phi(\theta)} \frac{L(\theta)}{N(\theta)}, \forall t \geq 0$.

However, when analyzing the set of optimal conditions in $\underline{\mathcal{S}}$, we need to pay special attention to the border. The border condition (9) becomes:

$$\psi(t, \theta) \nabla_{\theta} L_F(t, \theta) \big|_{\theta_1(t)}^{\theta_2(t)} = \nabla_{\theta} \psi(t, \theta) L_F(t, \theta) \big|_{\theta_1(t)}^{\theta_2(t)}. \quad (10)$$

Hence, in the second phase at all t :

$$(II) \begin{cases} \frac{\partial L_F}{\partial t}(t, \theta) = \mathcal{L}L_F(t, \theta) - c(t, \theta)N(\theta)\phi(\theta), \\ \psi(t, \theta) = \frac{1}{\phi(\theta)} c(t, \theta)^{-\sigma}, \\ \psi_t(t, \theta) + \mathcal{L}\psi(t, \theta) - \rho\psi(t, \theta) = 0, \forall \theta \in \underline{\mathcal{S}}, \\ \psi(t, \theta) \nabla_{\theta} L_F(t, \theta) \big|_{\theta_1(t)}^{\theta_2(t)} = \nabla_{\theta} \psi(t, \theta) L_F(t, \theta) \big|_{\theta_1(t)}^{\theta_2(t)}, \\ \lim_{t \rightarrow \infty} \psi(t, \theta)L_F(t, \theta) = 0, \\ \text{and} \\ L_F(t, \theta) = L(\theta), \quad L_{F,t}(t, \theta) = 0 \quad \text{and} \quad L_{F,xx}(t, \theta) = 0, \\ c(t, \theta) = A(\theta) \frac{\phi(\theta) - \nu(\theta)}{\phi(\theta)} \frac{L(\theta)}{N(\theta)}, \forall \theta \in \bar{\mathcal{S}}. \end{cases}$$

C. Proof of Proposition 1.

In Phase 1, the Hamilton-Jacobi-Bellman (HJB) equation associated to the previous optimal control problem can be written as:

$$\rho v(L_F) = \langle L_F, \mathcal{L}\nabla v(L_F) \rangle + \sup_c \left\{ \left\langle \frac{c^{1-\sigma}}{1-\sigma} N, \mathbb{1} \right\rangle - \langle \phi c N, \nabla v(L_F) \rangle \right\}.$$

Let $\hat{c} := \operatorname{argmax} \left\{ \left\langle \frac{c^{1-\sigma}}{1-\sigma} N, \mathbb{1} \right\rangle - \langle \phi c N, \nabla v(L_F) \rangle \right\}$ then \hat{c} verifies that:

$$\hat{c}^{-\sigma} N = \phi N \nabla v(L_F) \iff \hat{c} = [\phi \nabla v(L_F)]^{-\frac{1}{\sigma}}.$$

Therefore:

$$\begin{aligned} \sup_c \left\{ \left\langle \frac{c^{1-\sigma}}{1-\sigma} N, \mathbb{1} \right\rangle - \langle \phi c N, \nabla v(L_F) \rangle \right\} &= \left\langle \frac{\hat{c}^{1-\sigma}}{1-\sigma} N, \mathbb{1} \right\rangle - \langle \phi \hat{c} N, \nabla v(L_F) \rangle \\ &= \left\langle \frac{1}{1-\sigma} [\phi \nabla v(L_F)]^{-\frac{1}{\sigma}} N, \mathbb{1} \right\rangle - \langle \phi N [\phi \nabla v(L_F)]^{-\frac{1}{\sigma}}, \nabla v(L_F) \rangle \\ &= \frac{1}{1-\sigma} \langle [\phi \nabla v(L_F)]^{-\frac{1-\sigma}{\sigma}} N, \mathbb{1} \rangle - \langle [\phi \nabla v(L_F)]^{-\frac{1-\sigma}{\sigma}} N, \mathbb{1} \rangle = \frac{\sigma}{1-\sigma} \langle [\phi \nabla v(L_F)]^{-\frac{1-\sigma}{\sigma}} N, \mathbb{1} \rangle. \end{aligned}$$

We look for a solution that can be written as:

$$v = \frac{\langle L_F, \alpha_0 e_0 \rangle^{1-\sigma}}{1-\sigma},$$

with $\alpha_0 \in \mathbb{R}$. Therefore, $\nabla v(L_F) = \langle L_F, \alpha_0 e_0 \rangle^{-\sigma} \alpha_0 e_0$. Plugging this solution in the HJB equation leads to the following equation in α_0 :

$$\begin{aligned} \frac{\rho}{1-\sigma} \langle L_F, \alpha_0 e_0 \rangle^{1-\sigma} &= \langle L_F, \langle L_F, \alpha_0 e_0 \rangle^{-\sigma} \mathcal{L} \alpha_0 e_0 \rangle + \frac{\sigma}{1-\sigma} \langle L_F, \alpha_0 e_0 \rangle^{1-\sigma} \langle (\phi \alpha_0 e_0)^{-\frac{1-\sigma}{\sigma}} N, \mathbb{1} \rangle \\ &= \lambda_0 \langle L_F, \alpha_0 e_0 \rangle^{1-\sigma} + \frac{\sigma}{1-\sigma} \langle L_F, \alpha_0 e_0 \rangle^{1-\sigma} \int_0^{2\pi} (\phi(\eta) \alpha_0 e_0(\eta))^{-\frac{1-\sigma}{\sigma}} N(\eta) d\eta. \end{aligned}$$

Hence, dividing by $\langle L_F, \alpha_0 e_0 \rangle^{1-\sigma}$:

$$\begin{aligned} \frac{\rho}{1-\sigma} &= \lambda_0 + \frac{\sigma}{1-\sigma} \int_0^{2\pi} (\phi(\eta) \alpha_0 e_0(\eta))^{-\frac{1-\sigma}{\sigma}} N(\eta) d\eta \\ \iff \frac{\rho - \lambda_0(1-\sigma)}{\sigma} &= \alpha_0^{-\frac{1-\sigma}{\sigma}} \int_0^{2\pi} (\phi(\eta) e_0(\eta))^{-\frac{1-\sigma}{\sigma}} N(\eta) d\eta. \end{aligned}$$

Since by Assumption 2:

$$\rho - \lambda_0(1-\sigma) > 0,$$

then:

$$\alpha_0 = \left[\frac{\sigma}{\rho - \lambda_0(1-\sigma)} \int_0^{2\pi} (\phi(\eta) e_0(\eta))^{-\frac{1-\sigma}{\sigma}} N(\eta) d\eta \right]^{\frac{\sigma}{1-\sigma}}.$$

Moreover, we have that:

$$L'_F = \mathcal{L}L_F - (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}} \langle L_F, \alpha_0 e_0 \rangle.$$

Thus $\forall \Phi \in \mathcal{D}(\mathcal{L})$:

$$\begin{aligned} \frac{d}{dt} \langle L_F, \Phi \rangle &= \langle \mathcal{L}L_F, \Phi \rangle - \langle L, \alpha_0 e_0 \rangle \langle (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}}, \Phi \rangle \\ &= \langle L_F, \mathcal{L}\Phi \rangle - \langle L, \alpha_0 e_0 \rangle \langle (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}}, \Phi \rangle. \end{aligned}$$

Then, for $\Phi = \alpha_0 e_0$ we have that:

$$\begin{aligned} \frac{d}{dt} \langle L_F, \alpha_0 e_0 \rangle &= \langle L_F, \lambda_0 \alpha_0 e_0 \rangle - \langle L_F, \alpha_0 e_0 \rangle \langle (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}}, \alpha_0 e_0 \rangle \\ &= [\lambda_0 - \langle (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}}, \alpha_0 e_0 \rangle] \langle L_F, \alpha_0 e_0 \rangle. \end{aligned}$$

Therefore, be $g := \lambda_0 - \langle (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}}, \alpha_0 e_0 \rangle$. We have:

$$\frac{d}{dt} \langle L_F, \alpha_0 e_0 \rangle = g \langle L_F, \alpha_0 e_0 \rangle.$$

which can be solved as:

$$\langle L_F, \alpha_0 e_0 \rangle = \langle L_F^0, \alpha_0 e_0 \rangle e^{gt}, \quad (11)$$

where L_F^0 is the initial distribution for fertile soil. The optimal control on consumption can therefore be expressed as:

$$c^* = [\phi \nabla v(L_F)]^{-\frac{1}{\sigma}} = \phi^{-\frac{1}{\sigma}} [\langle L_F, \alpha_0 e_0 \rangle^{-\sigma} \alpha_0 e_0]^{-\frac{1}{\sigma}} = \phi^{-\frac{1}{\sigma}} \langle L_F, \alpha_0 e_0 \rangle (\alpha_0 e_0)^{-\frac{1}{\sigma}}.$$

Using (22) we obtain optimal consumption in terms of the initial distribution of fertile land, the model parameters and g :

$$c^* = \phi^{-\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}} \langle L_F^0, \alpha_0 e_0 \rangle e^{gt}.$$

Finally when obtaining the expression of α_0 we showed that:

$$\frac{\rho - \lambda_0(1 - \sigma)}{\sigma} = \int_0^{2\pi} (\phi(z)N(z)\alpha_0 e_0(z))^{-\frac{1-\sigma}{\sigma}} N(z)^{\frac{1}{\sigma}} dz.$$

Therefore, a simpler expression of g can be derived as:

$$g = \lambda_0 - \int_0^{2\pi} (\phi(z)N(z)\alpha_0 e_0(z))^{-\frac{1-\sigma}{\sigma}} N(z)^{\frac{1}{\sigma}} dz = \lambda_0 - \frac{\rho - \lambda_0(1 - \sigma)}{\sigma} = \frac{\lambda_0 - \rho}{\sigma}.$$

We claim that:

$$c^*(t, \theta) = (\phi(\theta)\alpha_0 e_0(\theta))^{-\frac{1}{\sigma}} e^{gt} \int_0^{2\pi} L_F^0(z)\alpha_0 e_0(z) dz,$$

is the optimal solution in Phase 1. Let us prove next that actually this solution verifies the optimal necessary conditions (I). Since $\psi(t, \theta) = \frac{1}{\phi(\theta)} c(t, \theta)^{-\sigma}$, then:

$$\psi(t, \theta) = \alpha_0 e_0(\theta) e^{-\sigma gt} \left(\int_0^{2\pi} L_F^0(z)\alpha_0 e_0(z) dz \right)^{-\sigma}, \quad (12)$$

and its partial derivatives are:

$$\begin{aligned} \psi_t(t, \theta) &= -\sigma g \psi(t, \theta), \\ \psi_x(t, \theta) &= \alpha_0 e'_0(\theta) e^{-\sigma gt} \left(\int_0^{2\pi} L_F^0(z)\alpha_0 e_0(z) dz \right)^{-\sigma}, \\ \psi_{xx}(t, \theta) &= \alpha_0 e''_0(\theta) e^{-\sigma gt} \left(\int_0^{2\pi} L_F^0(z)\alpha_0 e_0(z) dz \right)^{-\sigma}. \end{aligned}$$

Substituting into:

$$\psi_t(t, \theta) + \mathcal{L}\psi(t, \theta) - \rho\psi(t, \theta) + \mu(t, \theta) = 0,$$

checking that $\mathcal{L}\psi(t, \theta) = \lambda_0\psi(t, \theta)$, and letting $\mu(t, \theta) \equiv 0$, we do obtain that the solution in (24) satisfies the optimality condition whenever $g = \frac{\lambda_0 - \rho}{\sigma}$.

Finally, regarding the mean value of fertile soil, as $\langle L_F, \alpha_0 e_0 \rangle = \langle L_F^0, \alpha_0 e_0 \rangle e^{gt}$ then:

$$\int_0^{2\pi} L_F(t, \theta) e_0(\theta) d\theta = \int_0^{2\pi} L_F^0(\theta) e_0(\theta) e^{gt} d\theta \iff \int_0^{2\pi} [L_F(t, \theta) - L_F^0(\theta) e^{gt}] e_0(\theta) d\theta = 0.$$

Defining $\underline{e}_0 := \min_{[0, 2\pi]} e_0(\theta)$ and $\bar{e}_0 := \max_{[0, 2\pi]} e_0(\theta)$ leads to the following inequalities:

$$\underline{e}_0 \int_0^{2\pi} (L_F(t, \theta) - L_F^0(\theta)e^{gt}) d\theta \leq \int_0^{2\pi} (L_F(t, \theta) - L_F^0(\theta)e^{gt}) e_0(\theta) d\theta = 0,$$

and:

$$\bar{e}_0 \int_0^{2\pi} (L_F(t, \theta) - L_F^0(\theta)e^{gt}) d\theta \geq \int_0^{2\pi} (L_F(t, \theta) - L_F^0(\theta)e^{gt}) e_0(\theta) d\theta = 0.$$

Thereby, as $\forall \theta \in [0, 2\pi]$, $e_0(\theta) > 0$, then $\bar{e}_0 > 0$ and $\underline{e}_0 > 0$. We obtain the following inequalities:

$$\int_0^{2\pi} (L_F(t, \theta) - L_F^0(\theta)e^{gt}) d\theta \leq 0, \text{ and } \int_0^{2\pi} (L_F(t, \theta) - L_F^0(\theta)e^{gt}) d\theta \geq 0,$$

which implies that $\int_0^{2\pi} (L_F(t, \theta) - L_F^0(\theta)e^{gt}) d\theta = 0$ and $\langle L_F(t) \rangle = e^{gt} \langle L_F^0 \rangle$ after dividing by the length of the interval to obtain the result in terms of mean values.

D. Proof of Proposition 2.

We decompose fertile soil on the eigenvector basis as $L_F = \sum_{n \geq 0} \langle L_F, e_n \rangle e_n(\theta)$ and look for the expression of the coefficients $\langle L_F, e_n \rangle$ for $n \geq 0$.

Here, as the economy is supposed homogeneous, the distributions of A , ϕ and ν and N are homogeneous. Thus, the eigenvalue problem can be written as the following second order linear ordinary differential equation with constant coefficients $u'' + \frac{A(\phi - \nu) - \lambda}{D} u = 0$. Solutions of such problems are $u(\theta) = C_1 \cos\left(\theta \sqrt{\frac{A(\phi - \nu) - \lambda}{D}} + C_2\right)$, with $(C_1, C_2) \in \mathbb{R}^2$. Since our problem is 2π periodic, solutions need to verify $u(0) = u(2\pi)$. Hence:

$$\cos(C_2) = \cos\left(2\pi \sqrt{\frac{A(\phi - \nu) - \lambda}{D}} + C_2\right),$$

which holds if and only if:

$$2\pi \sqrt{\frac{A(\phi - \nu) - \lambda_n}{D}} + C_2 = C_2 + 2n\pi, \text{ with } n \in \mathbb{Z} \implies \sqrt{\frac{A(\phi - \nu) - \lambda_n}{D}} = n.$$

Therefore, the general expression of the n^{th} eigenvalue of the problem can be written as:

$$\lambda_n = A(\phi - \nu) - Dn^2,$$

and the expression of the n^{th} eigenvector is given by:

$$\begin{aligned} e_n(\theta) &= C_1 \cos\left(\theta \sqrt{\frac{A(\phi - \nu) - \lambda_n}{D}} + C_2\right) \\ &= C_1 \cos\left(\theta \sqrt{\frac{A(\phi - \nu) - A(\phi - \nu) + Dn^2}{D}} + C_2\right) \\ &= C_1 \cos(\theta n + C_2). \end{aligned}$$

- For $n = 0$, we have that $\lambda_0 = A(\phi - \nu)$. As a result:

$$g = \frac{\lambda_0 - \rho}{\sigma} = \frac{A(\phi - \nu) - \rho}{\sigma}.$$

Moreover, from the general expression of the eigenvectors we get that the first eigenvector e_0 is homogeneous in space in the case of an homogeneous economy. As a result, the first term $\langle L_F, e_0 \rangle e_0$ in the decomposition of fertile soil can be written as:

$$\langle L_F, e_0 \rangle e_0 = \langle L_F^0, e_0 \rangle e_0 e^{gt} = e_0^2 \int_0^{2\pi} L_F^0(\theta) d\theta e^{gt}.$$

As the eigenvectors form an orthonormal basis then $\int_0^{2\pi} e_0^2 d\theta = 1$ which leads in the homogeneous case to $e_0^2 = \frac{1}{2\pi}$. As a result:

$$\langle L_F, e_0 \rangle e_0 = \langle L_F^0, e_0 \rangle e_0 e^{gt} = \langle L_F^0 \rangle e^{gt}.$$

• For $n \geq 1$, as the dynamics of fertile soil write as:

$$L_F' = \mathcal{L}L_F - (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}} \langle L_F, \alpha_0 e_0 \rangle,$$

and as $\langle L_F, e_0 \rangle = \langle L_F^0, e_0 \rangle e^{gt}$ then:

$$\begin{aligned} \frac{d}{dt} \langle L_F, e_n \rangle &= \langle \mathcal{L}L_F, e_n \rangle - \langle L_F, \alpha_0 e_0 \rangle \langle (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}}, e_n \rangle \\ &= \langle L_F, \mathcal{L}e_n \rangle - \langle L_F, \alpha_0 e_0 \rangle \beta_n \\ &= \lambda_n \langle L_F, e_n \rangle - \langle L_F^0, \alpha_0 e_0 \rangle e^{gt} \beta_n, \end{aligned}$$

with $\beta_n = \langle (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}}, e_n \rangle$. As ϕ , N , and e_0 are homogeneous, we get for all $n \geq 1$ that:

$$\begin{aligned} \beta_n &= (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}} \int_0^{2\pi} e_n(\theta) d\theta \\ &= (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}} \int_0^{2\pi} C_1 \cos(\theta n + C_2) d\theta \\ &= (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\alpha_0 e_0)^{-\frac{1}{\sigma}} \left[\frac{C_1}{n} \sin(\theta n + C_2) \right]_0^{2\pi} \\ &= 0. \end{aligned}$$

Therefore, rewriting the dynamics of $\langle L_F, e_n \rangle$ for $n \geq 1$ with $\beta_n = 0$ leads to:

$$\frac{d}{dt} \langle L_F, e_n \rangle = \lambda_n \langle L_F, e_n \rangle,$$

which can be solved as: $\langle L_F, e_n \rangle = \langle L_F^0, e_n \rangle e^{\lambda_n t}$. Hence, fertile soil can be expressed as: $L_F(t, \theta) = \langle L_F^0 \rangle e^{gt} + \sum_{n \geq 1} \langle L_F^0, e_n \rangle e_n e^{\lambda_n t}$, which proves the claim.

E. Proof of Proposition 3.

We proceed to the proof in several steps:

1. We obtain the expression of consumption $c_A(t)$ in an unconnected economy. In such case where pollution does not diffuse into soil ($D = 0$), social planner solves separately at each location the following problem:

$$\max_{c_A(t)} \int_0^\infty \frac{c_A^{1-\sigma}}{1-\sigma} N e^{-\rho t} dt, \quad (13)$$

subject to:

$$\begin{cases} \dot{L}_F = A(\phi - \nu)L_F - \Phi N c_A, & t > 0, \theta \in S^1, \\ L_F(0) = L_F^0 \geq 0. \end{cases} \quad (14)$$

Under this specifications the Hamiltonian writes as $\mathcal{H} = \frac{c_A^{1-\sigma}}{1-\sigma} N e^{-\rho t} + \lambda[A(\phi - \nu)L_F - \Phi N c_A]$ and the F.O.C lead to:

$$\begin{cases} \frac{\partial \mathcal{H}}{\partial c_A} = 0 \implies c_A^{-\sigma} N e^{-\rho t} = \lambda \phi N, \\ \frac{\partial \mathcal{H}}{\partial L_F} = -\dot{\lambda} \implies -\dot{\lambda} = \lambda A(\phi - \nu). \end{cases}$$

Be $\tilde{\lambda} = \lambda e^{\rho t}$ then $\dot{\tilde{\lambda}} = \dot{\lambda} e^{\rho t} + \rho \tilde{\lambda}$ hence $-\dot{\tilde{\lambda}} + \rho \tilde{\lambda} = \tilde{\lambda} A(\phi - \nu)$ which leads to: $-\frac{\dot{\tilde{\lambda}}}{\tilde{\lambda}} = A(\phi - \nu) - \rho$. Moreover, as $\tilde{\lambda} = \frac{c_A^{-\sigma}}{\phi}$ and $\dot{\tilde{\lambda}} = -\frac{\sigma c_A^{-\sigma-1} \dot{c}_A}{\phi}$ then $\frac{\dot{\tilde{\lambda}}}{\tilde{\lambda}} = -\sigma \frac{\dot{c}_A}{c_A}$. Therefore, be $g := \frac{\dot{c}_A}{c_A}$, as a result we have that:

$$g = \frac{A(\phi - \nu) - \rho}{\sigma}$$

and $c_A(t, \theta) = c_A(0, \theta) e^{gt}$. Finally, as the dynamics of fertile soil writes for all $\theta \in \mathcal{S}$ as:

$$\dot{L}_F(t, \theta) = A(\phi - \nu)L_F(t, \theta) - \phi N c_A(t, \theta) \iff \frac{\dot{L}_F(t, \theta)}{L_F(t, \theta)} = A(\phi - \nu) - \phi N \frac{c_A(t, \theta)}{L_F(t, \theta)},$$

then, along the balanced growth path, $\frac{\dot{L}_F(t, \theta)}{L_F(t, \theta)} = g$ thus: $L_F(t, \theta) = L_F(0, \theta) e^{gt} = L_F^0(\theta) e^{gt}$ and:

$$c_A(t, \theta) = \frac{A}{N\sigma\phi} \left[\frac{\rho}{A} + (\phi - \nu)(\sigma - 1) \right] L_F(t, \theta) = \frac{A}{N\sigma\phi} \left[\frac{\rho}{A} + (\phi - \nu)(\sigma - 1) \right] L_F^0(\theta) e^{gt}.$$

2. Next, the expression for consumption in a diffusive economy c_D is given in Proposition 1 as:

$$c_D(t, \theta) = e^{gt} (\phi \alpha_0 e_0(\theta))^{-\frac{1}{\sigma}} \int_0^{2\pi} L_F^0(\eta) \alpha_0 e_0(\eta) d\eta,$$

with:

$$\alpha_0 = \left[\frac{\sigma}{\rho - A(\phi - \nu)(1 - \sigma)} \int_0^{2\pi} N(\phi e_0(\eta))^{-\frac{1-\sigma}{\sigma}} d\eta \right]^{\frac{\sigma}{1-\sigma}},$$

and $e_0(\cdot)$ the first eigenvector of the problem $\mathcal{L}u = \lambda u$ which is homogeneous in space in the homogeneous economy (see Appendix D). Therefore:

$$\begin{aligned} \alpha_0 e_0 &= \left[\frac{\sigma}{\rho - A(\phi - \nu)(1 - \sigma)} \int_0^{2\pi} N(\phi e_0)^{-\frac{1-\sigma}{\sigma}} d\eta \right]^{\frac{\sigma}{1-\sigma}} e_0 \\ &= \left[\frac{2\pi\sigma N(\phi e_0)^{-\frac{1-\sigma}{\sigma}}}{\rho - A(\phi - \nu)(1 - \sigma)} \right]^{\frac{\sigma}{1-\sigma}} e_0 = \frac{1}{\phi} \left[\frac{2\pi\sigma N}{\rho - A(\phi - \nu)(1 - \sigma)} \right]^{\frac{\sigma}{1-\sigma}}. \end{aligned}$$

Thus, we can prove that $c_D(\cdot)$ is homogeneous in space and obtain the following expression:

$$\begin{aligned} c_D(t, \theta) &= e^{gt} (\phi \alpha_0 e_0)^{-\frac{1}{\sigma}} \int_0^{2\pi} L_F^0(\eta) \alpha_0 e_0 d\eta \\ &= e^{gt} \left[\frac{2\pi\sigma N}{\rho - A(\phi - \nu)(1 - \sigma)} \right]^{\frac{1}{\sigma-1}} \frac{1}{\phi} \left[\frac{2\pi\sigma N}{\rho - A(\phi - \nu)(1 - \sigma)} \right]^{\frac{\sigma}{1-\sigma}} \int_0^{2\pi} L_F^0(\eta) d\eta \\ &= e^{gt} \frac{\rho - A(\phi - \nu)(1 - \sigma)}{2\pi\sigma N\phi} \int_0^{2\pi} L_F^0(\eta) d\eta \\ &= \frac{\rho - A(\phi - \nu)(1 - \sigma)}{N\sigma\phi} \langle L_F^0 \rangle e^{gt}. \end{aligned}$$

3. Finally, we compute transfers. From the definition of $\tau(t, \theta)$ and using our previous results:

$$\begin{aligned} \tau(t, \theta) &= c_A(t, \theta) - c_D(t, \theta) = \frac{\rho + A(\phi - \nu)(\sigma - 1)}{N\sigma\phi} L_F^0(\theta) e^{gt} - \frac{\rho + A(\phi - \nu)(\sigma - 1)}{N\sigma\phi} \langle L_F^0 \rangle e^{gt} \\ &= \frac{\rho + A(\phi - \nu)(\sigma - 1)}{N\sigma\phi} [L_F^0(\theta) - \langle L_F^0 \rangle] e^{gt}. \end{aligned}$$

F. Proof of Proposition 4.

In order to obtain the optimal trajectories in $\underline{S}(t)$, we need to define the interior product in $\underline{S}(t)$. Note that the space changes with time and so will the interior product definition. For any two functions f, g in $\mathcal{H}^2(\underline{S}(t))$, we define their interior product as:

$$\langle f, g \rangle_{\underline{S}} = \int_{\theta_1(t)}^{\theta_2(t)} f(z)g(z)dz.$$

In Phase 2, the Hamilton-Jacobi-Bellman (HJB) equation associated to the previous optimal control problem can be written as:

$$\rho v(L_F) = \langle L_F, \mathcal{L}\nabla v(L_F) \rangle_{\underline{S}} + \sup_c \left\{ \langle \frac{c^{1-\sigma}}{1-\sigma} N, \mathbb{1} \rangle_{\underline{S}} - \langle \phi c N, \nabla v(L_F) \rangle_{\underline{S}} \right\}.$$

Let $\hat{c} := \operatorname{argmax} \left\{ \langle \frac{c^{1-\sigma}}{1-\sigma} N, \mathbb{1} \rangle_{\underline{S}} - \langle \phi c N, \nabla v(L_F) \rangle_{\underline{S}} \right\}$. As in Appendix 22, \hat{c} verifies that $\hat{c} = [\phi \nabla v(L_F)]^{-\frac{1}{\sigma}}$.

In Phase 2 as well, we look for a solution that can be written as $v = \frac{\langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{1-\sigma}}{1-\sigma}$, where β_0 is this time a continuous function of time. e_0 is the first eigenvector associated to eigenvalue l_0 of the eigenvalue problem: $\mathcal{L}u = lu$ on S .

Operator \mathcal{L} is self-adjoint only under certain assumptions. Indeed:

$$\begin{aligned} \langle \mathcal{L}L_F, \nabla v(L_F) \rangle_{\underline{S}} &= \int_{\theta_1(t)}^{\theta_2(t)} \mathcal{L}L_F(z) \nabla v(L_F)(z) dz \\ &= \nabla_{\theta} L_F(z) \nabla v(L_F) \Big|_{\theta_1(t)}^{\theta_2(t)} - \int_{\theta_1(t)}^{\theta_2(t)} \nabla_{\theta} L_F(z) \nabla_{\theta} (\nabla v(L_F)(z)) dz \\ &= \nabla_{\theta} L_F(z) \nabla v(L_F) \Big|_{\theta_1(t)}^{\theta_2(t)} - L_F(z) \nabla_{\theta} (\nabla v(L_F)) \Big|_{\theta_1(t)}^{\theta_2(t)} + \int_{\theta_1(t)}^{\theta_2(t)} L_F(z) \mathcal{L}\nabla v(L_F)(z) dz. \end{aligned}$$

Hence:

$$\langle \mathcal{L}L_F, \nabla v(L_F) \rangle_{\underline{S}} = \langle L_F, \mathcal{L}(\nabla v(L_F)) \rangle_{\underline{S}} + \nabla_{\theta} L_F(z) \nabla v(L_F) \Big|_{\theta_1(t)}^{\theta_2(t)} - L_F(z) \nabla_{\theta} (\nabla v(L_F)) \Big|_{\theta_1(t)}^{\theta_2(t)},$$

and our operator is self-adjoint if and only if:

$$\nabla_{\theta} L_F(z) \nabla v(L_F) \Big|_{\theta_1(t)}^{\theta_2(t)} = L_F(z) \nabla_{\theta} (\nabla v(L_F)) \Big|_{\theta_1(t)}^{\theta_2(t)}. \quad (15)$$

Note that the condition in (15) coincides with (10), implying that operator \mathcal{L} is self-adjoint in the subset of optimal solutions.

Proceeding like in the previous Appendix, we substitute \hat{c} into the HJB equation and impose that the solutions must be optimal, that is, they verify (15):

$$\begin{aligned} \frac{\rho}{1-\sigma} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{1-\sigma} &= \langle L_F, \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma} \mathcal{L}\beta_0 e_0 \rangle_{\underline{S}} + \frac{\sigma}{1-\sigma} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{1-\sigma} \langle (\phi \beta_0 e_0)^{-\frac{1-\sigma}{\sigma}} N, \mathbb{1} \rangle_{\underline{S}} \\ &= \lambda_0 \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{1-\sigma} + \frac{\sigma}{1-\sigma} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{1-\sigma} \int_{\underline{S}} (\phi(\eta) \beta_0 e_0(\eta))^{-\frac{1-\sigma}{\sigma}} N(\eta) d\eta. \end{aligned}$$

Hence, dividing by $\langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{1-\sigma}$:

$$\begin{aligned} \frac{\rho}{1-\sigma} &= \lambda_0 + \frac{\sigma}{1-\sigma} \int_{\underline{S}} (\phi(\eta) \beta_0 e_0(\eta))^{-\frac{1-\sigma}{\sigma}} N(\eta) d\eta \\ \iff \frac{\rho - \lambda_0(1-\sigma)}{\sigma} &= \beta_0^{-\frac{1-\sigma}{\sigma}} \int_{\underline{S}} (\phi(\eta) e_0(\eta))^{-\frac{1-\sigma}{\sigma}} N(\eta) d\eta. \end{aligned}$$

Again, by Assumption 2 $\rho - \lambda_0(1 - \sigma) > 0$, so that:

$$\beta_0^{\frac{1-\sigma}{\sigma}} = \frac{\sigma}{\rho - \lambda_0(1 - \sigma)} \int_{\underline{S}} (\phi(z)e_0(z))^{-\frac{1-\sigma}{\sigma}} N(z) dz = \frac{\sigma}{\rho - \lambda_0(1 - \sigma)} < (\phi(z)e_0(z))^{-\frac{1-\sigma}{\sigma}} N(z), \mathbf{1} \rangle_{\underline{S}}. \quad (16)$$

Hence:

$$\hat{c} = [\phi \nabla v(L_F)]^{-\frac{1}{\sigma}} = (\phi \beta_0 e_0)^{-\frac{1}{\sigma}} < L_F, \beta_0 e_0 \rangle_{\underline{S}}. \quad (17)$$

Taking logarithms and derivating, we obtain β_0 's growth rate:

$$\begin{aligned} \frac{1-\sigma}{\sigma} \frac{\beta_{0,t}(t)}{\beta_0(t)} &= \frac{d}{dt} \ln \int_{\underline{S}} (\phi(z)e_0(z))^{-\frac{1-\sigma}{\sigma}} N(z) dz \\ &= \frac{N(\theta_2(t))[\phi(\theta_2(t))e_0(\theta_2(t))]^{-\frac{1-\sigma}{\sigma}} \dot{\theta}_2(t) - N(\theta_1(t))[\phi(\theta_1(t))e_0(\theta_1(t))]^{-\frac{1-\sigma}{\sigma}} \dot{\theta}_1(t)}{\int_{\underline{S}} (\phi(z)e_0(z))^{-\frac{1-\sigma}{\sigma}} N(z) dz}. \end{aligned}$$

Replacing optimal consumption, \hat{c} , in the original state equation for L_F :

$$L'_F = \mathcal{L}L_F - \phi N (\phi \beta_0 e_0)^{-\frac{1}{\sigma}} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}. \quad (18)$$

Thus, following the same steps as in the previous appendix, we multiply the entire estate equation by $\beta_0 e_0$ and then compute the interior product. If we restrict our analysis to optimal solutions that verify (15), then:

$$\begin{aligned} \langle \frac{d}{dt} L_F, \beta_0 e_0 \rangle_{\underline{S}} &= \langle \mathcal{L}L_F, \beta_0 e_0 \rangle_{\underline{S}} - \langle (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\beta_0 e_0)^{-\frac{1}{\sigma}}, \beta_0 e_0 \rangle_{\underline{S}} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} \\ &= \langle L_F, \beta_0 \mathcal{L}e_0 \rangle_{\underline{S}} - \langle (\phi N \beta_0 e_0)^{\frac{-(1-\sigma)}{\sigma}} N^{\frac{1}{\sigma}}, \mathbf{1} \rangle_{\underline{S}} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} \\ &= \lambda_0 \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} - \langle (\phi N \beta_0 e_0)^{\frac{-(1-\sigma)}{\sigma}} N^{\frac{1}{\sigma}}, \mathbf{1} \rangle_{\underline{S}} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}. \end{aligned}$$

Note that we cannot solve the problem unless we transform $\langle \frac{d}{dt} L_F, \beta_0 e_0 \rangle_{\underline{S}}$. Indeed, when the interior product changes with time, it is not straightforward to proceed as in the previous case. Note that:

$$\begin{aligned} \frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} &= \frac{d}{dt} \int_{\theta_1(t)}^{\theta_2(t)} L_F \beta_0 e_0 dz \\ &= \int_{\theta_1(t)}^{\theta_2(t)} \frac{d}{dt} L_F \beta_0 e_0 dz + \int_{\theta_1(t)}^{\theta_2(t)} L_F \frac{d}{dt} \beta_0 e_0 dz + L_F(\theta_2(t)) \beta_0(t) e_0(\theta_2(t)) \dot{\theta}_2(t) \\ &\quad - L_F(\theta_1(t)) \beta_0(t) e_0(\theta_1(t)) \dot{\theta}_1(t) \\ &= \langle \frac{d}{dt} L_F, \beta_0 e_0 \rangle_{\underline{S}} + \langle L_F, \beta_{0,t} e_0 \rangle_{\underline{S}} + L_F(\theta_2(t)) \beta_0(t) e_0(\theta_2(t)) \dot{\theta}_2(t) \\ &\quad - L_F(\theta_1(t)) \beta_0(t) e_0(\theta_1(t)) \dot{\theta}_1(t). \end{aligned}$$

Then, we finally obtain that:

$$\begin{aligned} \langle \frac{d}{dt} L_F, \beta_0 e_0 \rangle_{\underline{S}} &= \frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} - \frac{\beta_{0,t}}{\beta_0} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} \\ &\quad - L_F(\theta_2(t)) \beta_0(t) e_0(\theta_2(t)) \dot{\theta}_2(t) + L_F(\theta_1(t)) \beta_0(t) e_0(\theta_1(t)) \dot{\theta}_1(t). \end{aligned} \quad (19)$$

If we replace (15) and (19) into (18):

$$\begin{aligned} \frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} &= \left[\lambda_0 - \langle (\phi N)^{\frac{\sigma-1}{\sigma}} N^{\frac{1}{\sigma}} (\beta_0 e_0)^{-\frac{1}{\sigma}}, \beta_0 e_0 \rangle_{\underline{S}} + \frac{\beta_{0,t}}{\beta_0} \right] \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} \\ &\quad + L_F(\theta_2(t)) \beta_0(t) e_0(\theta_2(t)) \dot{\theta}_2(t) - L_F(\theta_1(t)) \beta_0(t) e_0(\theta_1(t)) \dot{\theta}_1(t). \end{aligned} \quad (20)$$

Using (16), and removing the time variable for simplicity:

$$\frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} = \left[\frac{\lambda_0 - \rho}{\sigma} + \frac{\beta_{0,t}}{\beta_0} \right] \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} + \beta_0 \left(L_F(\theta_2) e_0(\theta_2) \dot{\theta}_2 - L_F(\theta_1) e_0(\theta_1) \dot{\theta}_1 \right). \quad (21)$$

We can solve (21) in two stages:

1. First, we solve the homogeneous equation as follows:

$$\begin{aligned} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} &= \langle L_F, \beta_0 e_0 \rangle_{\underline{S}(\tau)} e^{\int_{\tau}^t \left(\frac{\lambda_0 - \rho}{\sigma} + \frac{\beta_{0,t}}{\beta_0} \right) ds} \\ &= \langle L_F(\tau), \beta_0(\tau) e_0 \rangle e^{\frac{\lambda_0 - \rho}{\sigma} (t - \tau)} e^{\int_{\tau}^t \frac{\beta_{0,t}}{\beta_0} ds} = \langle L_F(\tau), \beta_0(\tau) e_0 \rangle e^{\frac{\lambda_0 - \rho}{\sigma} (t - \tau)} e^{\ln \beta_0|_{\tau}^t} \\ &= \langle L_F(\tau), \beta_0(\tau) e_0 \rangle e^{\frac{\lambda_0 - \rho}{\sigma} (t - \tau)} e^{\ln \beta_0|_{\tau}^t} = \langle L_F(\tau), \beta_0(\tau) e_0 \rangle e^{\frac{\lambda_0 - \rho}{\sigma} (t - \tau)} \frac{\beta_0(t)}{\beta_0(\tau)} \\ &= \langle L_F(\tau), \alpha_0 e_0 \rangle e^{\frac{\lambda_0 - \rho}{\sigma} (t - \tau)} \frac{\beta_0(t)}{\alpha_0}, \end{aligned} \quad (22)$$

where $L_F(\tau)$ is the distribution of fertile soil at the time the second phase is triggered and $\underline{S}(\tau) = \mathcal{S}$. By continuity, it is true that $\beta_0(\tau) = \alpha_0$.

2. Then we solve the non-homogeneous equation trying a solution of the type:

$$\langle L_F, \beta_0 e_0 \rangle_{\underline{S}} = m(t) e^{\frac{\lambda_0 - \rho}{\sigma} t} \beta_0(t). \quad (23)$$

If it is a solution, it has to verify (21), that is:

$$\begin{aligned} \frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} &= \dot{m}(t) e^{\frac{\lambda_0 - \rho}{\sigma} t} \beta_0 + m(t) \frac{\lambda_0 - \rho}{\sigma} e^{\frac{\lambda_0 - \rho}{\sigma} t} \beta_0 + m(t) e^{\frac{\lambda_0 - \rho}{\sigma} t} \beta_{0,t}(t) \\ &= e^{\frac{\lambda_0 - \rho}{\sigma} t} \beta_0 \left(\dot{m}(t) + m(t) \left(\frac{\lambda_0 - \rho}{\sigma} + \frac{\beta_{0,t}(t)}{\beta_0(t)} \right) \right), \end{aligned}$$

and on the other hand, it has to verify the original equation:

$$\frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} = \left[\frac{\lambda_0 - \rho}{\sigma} + \frac{\beta_{0,t}}{\beta_0} \right] \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} + \beta_0 \left(L_F(\theta_2) e_0(\theta_2) \dot{\theta}_2 - L_F(\theta_1) e_0(\theta_1) \dot{\theta}_1 \right).$$

Hence:

$$\begin{aligned} e^{\frac{\lambda_0 - \rho}{\sigma} t} \beta_0 \dot{m}(t) &= \beta_0 \left(L_F(\theta_2) e_0(\theta_2) \dot{\theta}_2 - L_F(\theta_1) e_0(\theta_1) \dot{\theta}_1 \right) \\ \dot{m}(t) &= e^{-\frac{\lambda_0 - \rho}{\sigma} t} \left(L_F(\theta_2) e_0(\theta_2) \dot{\theta}_2 - L_F(\theta_1) e_0(\theta_1) \dot{\theta}_1 \right), \end{aligned}$$

that is:

$$m(t) = \int_{\tau}^t e^{-\frac{\lambda_0 - \rho}{\sigma} s} \left(L_F(\theta_2(s)) e_0(\theta_2(s)) \dot{\theta}_2(s) - L_F(\theta_1(s)) e_0(\theta_1(s)) \dot{\theta}_1(s) \right) ds.$$

Plugging m back into (23), and taking into account that $\langle L_F, \beta_0 e_0 \rangle_{\underline{S}(\tau)} = \langle L_F(\tau), \alpha_0 e_0 \rangle$, we obtain that the solution for $\langle L_F, \beta_0 e_0 \rangle_{\underline{S}}$ is:

$$\begin{aligned} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} &= \frac{\beta_0(t)}{\alpha_0} \langle L_F(\tau), \alpha_0 e_0 \rangle e^{\frac{\lambda_0 - \rho}{\sigma} (t - \tau)} \\ &\quad + e^{\frac{\lambda_0 - \rho}{\sigma} t} \beta_0(t) \int_{\tau}^t e^{-\frac{\lambda_0 - \rho}{\sigma} s} \left(L_F(\theta_2(s)) e_0(\theta_2(s)) \dot{\theta}_2(s) - L_F(\theta_1(s)) e_0(\theta_1(s)) \dot{\theta}_1(s) \right) ds, \end{aligned}$$

or:

$$\begin{aligned} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} &= \beta_0(t) \langle L_F(\tau), e_0 \rangle e^{\frac{\lambda_0 - \rho}{\sigma} (t - \tau)} \\ &\quad + \beta_0(t) \int_{\tau}^t e^{\frac{\lambda_0 - \rho}{\sigma} (t - s)} \left(L_F(\theta_2(s)) e_0(\theta_2(s)) \dot{\theta}_2(s) - L_F(\theta_1(s)) e_0(\theta_1(s)) \dot{\theta}_1(s) \right) ds. \end{aligned}$$

With this solution for $\langle L_F, \beta_0 e_0 \rangle_{\underline{S}}$, we can obtain the optimal solution for consumption:

$$c^* = [\phi \nabla v(L_F)]^{-\frac{1}{\sigma}} = \phi^{-\frac{1}{\sigma}} [\langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma} \beta_0 e_0]^{-\frac{1}{\sigma}} = (\phi \beta_0 e_0)^{-\frac{1}{\sigma}} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}.$$

Let us prove next that c^* is the optimal solution in Phase 2, i.e. that c^* verifies the optimal necessary conditions (II). If c^* satisfies that $c^* = [\phi \nabla v(L_F)]^{-\frac{1}{\sigma}}$ and $c^* = [\phi \lambda]^{-\frac{1}{\sigma}}$, we need then that $\nabla v(L_F) = \lambda$. Hence, first we need to solve for λ .

Since $\lambda(t, \theta) = \frac{1}{\phi(\theta)} c(t, \theta)^{-\sigma}$, then:

$$\lambda(t, \theta) = \beta_0 e_0 \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma}, \quad (24)$$

and its partial derivatives with respect to t is:

$$\lambda_t(t, \theta) = -\sigma \beta_0 e_0 \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma-1} \frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} + \beta_{0,t} e_0 \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma}.$$

Plus $\mathcal{L}\lambda(t, \theta) = \lambda_0 \lambda(t, \theta)$. Substituting into:

$$\lambda_t(t, \theta) + \mathcal{L}\lambda(t, \theta) - \rho \lambda(t, \theta) = 0,$$

we have:

$$\begin{aligned} & -\sigma \beta_0 e_0 \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma-1} \frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} + \beta_{0,t} e_0 \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma} + (\lambda_0 - \rho) \beta_0 e_0 \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma} = 0, \\ \iff & -\sigma \beta_0 e_0 \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma-1} \frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} + \left(\lambda_0 - \rho + \frac{\beta_{0,t}}{\beta_0} \right) \beta_0 e_0 \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma} = 0, \\ \iff & -\sigma \frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} + \left(\lambda_0 - \rho + \frac{\beta_{0,t}}{\beta_0} \right) \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} = 0, \\ \iff & \frac{d}{dt} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} = \frac{1}{\sigma} \left(\lambda_0 - \rho + \frac{\beta_{0,t}}{\beta_0} \right) \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}. \end{aligned}$$

Hence, the solution coming from the two approaches are identical if and only if:

$$\begin{aligned} & \left[\frac{\lambda_0 - \rho}{\sigma} + \frac{\beta_{0,t}}{\beta_0} \right] \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} + \beta_0 \left(L_F(\theta_2) e_0(\theta_2) \dot{\theta}_2 - L_F(\theta_1) e_0(\theta_1) \dot{\theta}_1 \right) \\ & = \frac{1}{\sigma} \left(\lambda_0 - \rho + \frac{\beta_{0,t}}{\beta_0} \right) \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}. \end{aligned}$$

The condition above can be simplified:

$$\frac{1 - \sigma}{\sigma} \frac{\beta_{0,t}}{\beta_0} \langle L_F, \beta_0 e_0 \rangle_{\underline{S}} = \beta_0 \left(L_F(\theta_2) e_0(\theta_2) \dot{\theta}_2 - L_F(\theta_1) e_0(\theta_1) \dot{\theta}_1 \right). \quad (25)$$

In order to better understand our results, we can substitute $\frac{\beta_{0,t}}{\beta_0}$ using:

$$\frac{1 - \sigma}{\sigma} \frac{\beta_{0,t}(t)}{\beta_0(t)} = \frac{N(\theta_2(t)) [\phi(\theta_2(t)) e_0(\theta_2(t))]^{-\frac{1-\sigma}{\sigma}} \dot{\theta}_2(t) - N(\theta_1(t)) [\phi(\theta_1(t)) e_0(\theta_1(t))]^{-\frac{1-\sigma}{\sigma}} \dot{\theta}_1(t)}{\int_{\underline{S}} (\phi(z) e_0(z))^{-\frac{1-\sigma}{\sigma}} N(z) dz}.$$

Equating both expressions:

$$\begin{aligned} & \dot{\theta}_2 \left(\beta_0 L_F(\theta_2) e_0(\theta_2) - \frac{\langle L_F, \beta_0 e_0 \rangle_{\underline{S}}}{\langle (\phi e_0)^{-\frac{1-\sigma}{\sigma}} N, \mathbb{1} \rangle_{\underline{S}}} N(\theta_2) (\phi(\theta_2) e_0(\theta_2))^{\frac{\sigma-1}{\sigma}} \right) \\ & = \dot{\theta}_1 \left(\beta_0 L_F(\theta_1) e_0(\theta_1) - \frac{\langle L_F, \beta_0 e_0 \rangle_{\underline{S}}}{\langle (\phi e_0)^{-\frac{1-\sigma}{\sigma}} N, \mathbb{1} \rangle_{\underline{S}}} N(\theta_1) (\phi(\theta_1) e_0(\theta_1))^{\frac{\sigma-1}{\sigma}} \right). \end{aligned} \quad (26)$$

We can write (26) writing $\dot{\theta}_i = DL_{F,\theta}(\theta_i)$, and using that $\psi(\theta) = \beta_0 e_0(\theta) \langle L_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma}$ and $\beta_0 = \left(\frac{\sigma}{\rho - \lambda_0(1-\sigma)} \right)^{\frac{\sigma}{1-\sigma}} \langle (\phi e_0)^{-\frac{1-\sigma}{\sigma}} N, \mathbb{1} \rangle_{\underline{S}}^{\frac{\sigma}{1-\sigma}}$:

$$\begin{aligned} & L_{F,\theta}(\theta_2) e_0(\theta_2) \left(L(\theta_2) - \frac{\sigma}{\rho - \lambda_0(1-\sigma)} N(\theta_2) \phi(\theta_2)^{\frac{\sigma-1}{\sigma}} \psi(\theta_2)^{\frac{-1}{\sigma}} \right) \\ &= L_{F,\theta}(\theta_1) e_0(\theta_1) \left(L(\theta_1) - \frac{\sigma}{\rho - \lambda_0(1-\sigma)} N(\theta_1) \phi(\theta_1)^{\frac{\sigma-1}{\sigma}} \psi(\theta_1)^{\frac{-1}{\sigma}} \right). \end{aligned} \quad (27)$$

(27) needs to be completed with the spatial boundary conditions that ensure that our operator is self-adjoint:

$$\nabla_{\theta} L_F(z) \nabla v(L_F) \big|_{\theta_1(t)}^{\theta_2(t)} = L_F(z) \nabla_{\theta} (\nabla v(L_F)) \big|_{\theta_1(t)}^{\theta_2(t)}, \quad (28)$$

that is:

$$L_{F,\theta}(\theta_2) \psi(\theta_2) - L_{F,\theta}(\theta_1) \psi(\theta_1) = L(\theta_2) \psi_{\theta}(\theta_2) - L(\theta_1) \psi_{\theta}(\theta_1). \quad (29)$$

Hence, the optimal solution is given by the following system made of a PDE for the shadow price of fertile land, two boundary conditions on λ and ψ_{θ} at θ_1 and θ_2 plus the transversality condition:

$$(III) \left\{ \begin{array}{l} \psi_t(t, \theta) + \mathcal{L}\psi(t, \theta) - \rho\psi(t, \theta) = 0, \quad \forall t \geq \tau, \text{ and } \forall \theta \in \underline{S}, \\ L_{F,\theta}(\theta_2) e_0(\theta_2) \left(L(\theta_2) - \frac{\sigma}{\rho - \lambda_0(1-\sigma)} N(\theta_2) \phi(\theta_2)^{\frac{\sigma-1}{\sigma}} \psi(\theta_2)^{\frac{-1}{\sigma}} \right) \\ \quad = L_{F,\theta}(\theta_1) e_0(\theta_1) \left(L(\theta_1) - \frac{\sigma}{\rho - \lambda_0(1-\sigma)} N(\theta_1) \phi(\theta_1)^{\frac{\sigma-1}{\sigma}} \psi(\theta_1)^{\frac{-1}{\sigma}} \right), \quad \forall t \geq \tau, \\ L_{F,\theta}(\theta_2) \psi(\theta_2) - L_{F,\theta}(\theta_1) \psi(\theta_1) = L(\theta_2) \psi_{\theta}(\theta_2) - L(\theta_1) \psi_{\theta}(\theta_1), \quad \forall t \geq \tau, \\ \lim_{t \rightarrow \infty} \psi L_F = 0, \quad \forall \theta \in \underline{S}. \end{array} \right.$$

G. Proof of Proposition 5.

If at time τ' , a steady state is attained, then $\dot{\theta}_1(t) = \dot{\theta}_2(t) = 0$ for all $t \geq \tau'$. Let us denote the steady state solution $(\bar{L}_F, \bar{\psi}, \bar{\theta}_1, \bar{\theta}_2)$. At the steady state, the left hand side of (29) is zero. Knowing that for $i = 1, 2$, $L_F(\bar{\theta}_i) = L(\bar{\theta}_i)$, the right hand side provides us with a condition on the border values:

$$L(\bar{\theta}_1) e_{0,\theta}(\bar{\theta}_1) = L(\bar{\theta}_2) e_{0,\theta}(\bar{\theta}_2).$$

\bar{L}_F is the solution of:

$$\mathcal{L}\bar{L}_F = \phi N (\phi \beta_0 e_0)^{-\frac{1}{\sigma}} \langle \bar{L}_F, \beta_0 e_0 \rangle_{\underline{S}},$$

with $\bar{L}_F(\bar{\theta}_1) = L(\bar{\theta}_1)$ and $\bar{L}_F(\bar{\theta}_2) = L(\bar{\theta}_2)$. Besides, we know that $\bar{L}_{F,\theta}(\bar{\theta}_2) = \bar{L}_{F,\theta}(\bar{\theta}_1) = 0$. And as before, we will obtain c using that $\bar{\psi} = \frac{1}{\phi(\theta)} \bar{c}^{-\sigma}$, then:

$$\mathcal{L}\bar{\psi} - \rho\bar{\psi} = 0. \quad (30)$$

The steady state value of consumption at the borders can be obtained from (26):

$$\bar{c}(\bar{\theta}_1) = \frac{\rho - \lambda_0(1 - \sigma)}{\sigma} \frac{L_F(\bar{\theta}_1)}{N(\bar{\theta}_1)\phi(\bar{\theta}_1)},$$

$$\bar{c}(\bar{\theta}_2) = \frac{\rho - \lambda_0(1 - \sigma)}{\sigma} \frac{L_F(\bar{\theta}_2)}{N(\bar{\theta}_2)\phi(\bar{\theta}_2)}.$$

Hence, the border conditions for $\bar{\lambda}$ in (30) are:

$$\bar{\psi}(\bar{\theta}_1) = \frac{1}{\phi(\bar{\theta}_1)^{1-\sigma}} \left(\frac{\rho - \lambda_0(1 - \sigma)}{\sigma} \frac{L_F(\bar{\theta}_1)}{N(\bar{\theta}_1)} \right)^{-\sigma},$$

$$\bar{\psi}(\bar{\theta}_2) = \frac{1}{\phi(\bar{\theta}_2)^{1-\sigma}} \left(\frac{\rho - \lambda_0(1 - \sigma)}{\sigma} \frac{L_F(\bar{\theta}_2)}{N(\bar{\theta}_2)} \right)^{-\sigma}.$$

Thereby, the steady state $(\bar{L}_F, \bar{\psi}, \bar{\theta}_1, \bar{\theta}_2)$ is the solution to:

$$(IV) \left\{ \begin{array}{l} \mathcal{L}\bar{L}_F = \phi N (\phi\beta_0 e_0)^{-\frac{1}{\sigma}} \langle \bar{L}_F, \beta_0 e_0 \rangle_{\underline{S}}, \\ \bar{L}_F(\bar{\theta}_1) = L(\bar{\theta}_1), \quad \bar{L}_F(\bar{\theta}_2) = L(\bar{\theta}_2), \\ L_{\theta}(\bar{\theta}_1) = 0, \quad L_{\theta}(\bar{\theta}_2) = 0, \\ \mathcal{L}\bar{\psi} - \rho\bar{\psi} = 0, \\ \bar{\psi}(\bar{\theta}_1) = \frac{1}{\phi(\bar{\theta}_1)^{1-\sigma}} \left(\frac{\rho - \lambda_0(1-\sigma)}{\sigma} \frac{L_F(\bar{\theta}_1)}{N(\bar{\theta}_1)} \right)^{-\sigma}, \\ \bar{\psi}(\bar{\theta}_2) = \frac{1}{\phi(\bar{\theta}_2)^{1-\sigma}} \left(\frac{\rho - \lambda_0(1-\sigma)}{\sigma} \frac{L_F(\bar{\theta}_2)}{N(\bar{\theta}_2)} \right)^{-\sigma}. \end{array} \right.$$

The steady state for consumption is obtained using that $\bar{\psi} = \frac{1}{\phi(\bar{\theta})} \bar{c}^{-\sigma}$.

Next, note that $\mathcal{L}\bar{\psi} - \rho\bar{\psi} = 0$. Moreover in Phase 2 we suppose that $\rho < \lambda_0 = \max_{\lambda \in \sigma(\mathcal{L})} \lambda$. Thus $\bar{\psi}$ is an eigenvector of \mathcal{L} and ρ its eigenvalue. As $\bar{\psi}$ is an eigenvector, we have that $\bar{\psi} \neq 0$, meaning that there exists a non-zero steady state that we characterize in the following. As the eigenvectors of \mathcal{L} are orthogonal then:

$$\begin{aligned} \langle \bar{\psi}, e_0 \rangle_{\underline{S}} = 0 &\iff \langle \frac{\bar{c}^{-\sigma}}{\phi}, e_0 \rangle_{\underline{S}} = 0 \text{ as } \bar{\psi} = \frac{\bar{c}^{-\sigma}}{\phi} \\ &\iff \langle \frac{\langle \bar{L}_F, \beta_0 e_0 \rangle_{\underline{S}} (\phi\beta_0 e_0)^{-\frac{1}{\sigma}}}{\phi}, e_0 \rangle_{\underline{S}} = 0 \text{ as } \bar{c} = \langle \bar{L}_F, \beta_0 e_0 \rangle_{\underline{S}} (\phi\beta_0 e_0)^{-\frac{1}{\sigma}} \\ &\iff \langle \bar{L}_F, \beta_0 e_0 \rangle_{\underline{S}}^{-\sigma} \beta_0 \langle e_0, e_0 \rangle_{\underline{S}} = 0 \\ &\iff \langle \bar{L}_F, \beta_0 e_0 \rangle_{\underline{S}} = 0 \text{ as } \langle e_0, e_0 \rangle_{\underline{S}} = 1 \end{aligned}$$

Finally, as \bar{L}_F is solution to:

$$\mathcal{L}\bar{L}_F = \phi N (\phi\beta_0 e_0)^{-\frac{1}{\sigma}} \langle \bar{L}_F, \beta_0 e_0 \rangle_{\underline{S}},$$

then from what precedes, as $\langle \bar{L}_F, \beta_0 e_0 \rangle_{\underline{S}} = 0$, we have that $\bar{L}_F \in \ker \mathcal{L}$ i.e. \bar{L}_F is solution to:

$$\mathcal{L}\bar{L}_F = 0 \iff D\bar{L}_F'' + A(\phi - \nu)\bar{L}_F = 0,$$

which proves the claim of Proposition 5.