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ABSTRACT

We study the problem of an upper approximation of a TU-game by a k-additive game under the constraint that both games yield the same Shapley value. The best approximation is obtained by minimizing the sum of excesses with respect to the original game, which yields an LP problem. We show that for any game with at most 4 players all vertices of the polyhedron of feasible solutions are optimal, and we give an explicit formula of the value of the LP problem for a particular class of games.

1. Introduction

From the mathematical point of view, TU-games are set functions on a finite set N of n elements, vanishing at the empty set. Apart from their use in game theory where they model the best outcome which can be obtained by a set (coalition) of players, they are largely used in Operations Research in general under the name of pseudo-Boolean functions [1], in combinatorial optimization, reliability theory, voting theory, and decision theory under the name of capacities (see [2] for a survey).

TU-games (referred to hereafter as games) require $2^n - 1$ coefficients to be defined, which makes their usage difficult in practice. This is why polynomial approximations by simpler functions are sought. Seminal papers on this topic are [3] and [1], where the following problem is studied:

$$\min_{w \in \mathbb{R}^n} \sum_{S \subset N} a_S (v(S) - w(S))^2 \text{ subject to } w(N) = v(N), \quad (1)$$

where v is a game on N, w is an additive game on N (i.e., which can be considered as a n-dimensional vector, letting $w(S) = \sum_{i \in S} w_i$, $w(\emptyset) = 0$), and $a_S \in \mathbb{R}$ for all S. [3] solves the problem when $a_S > 0$ for all $S \subseteq N$ and $a_S = a_T$ whenever $|S| = |T|$, while [1] considers the unconstrained version and $a_S = 1$ for all S. However, they also solve the approximation problem where w can take a more general form, called *2-additive* (approximation of degree 2 in their terminology), i.e., w is defined by n^2 coefficients. More general versions and variants have been studied in [4] (approximation by a k-additive game, or of degree k, yielding a model of complexity of order n^k), [5], [6], [7, 8], etc. (see a survey in [2]).

We propose here an alternative to least square approximation. In many situations, the notion of *dominance* is often important and meaningful. By this, we mean that the approximating game w should dominate v, i.e., $w(S) \geqslant v(S)$ for any S, while coinciding on N. In game theory, this amounts to saying that in the approximating game, players are guaranteed to receive at least as much as they could have received in the original game. In decision theory, a probability measure dominating a capacity is called *compatible* or *coherent*, and is a central notion when using imprecise probabilities [9]. When the approximating game w is additive, the set of dominating w is called the *core*, which is a central notion in game theory, decision theory and combinatorial optimization (see, e.g., [10, 2]). More generally, when the approximating game is k-additive, the set of dominating w forms the k-additive core; see, e.g., [11, 12] and [13].

As the k-additive core is a huge set, it remains to select the "best" approximation in some sense. Our proposition is to both 1) minimize the L_1-distance between v and w, and 2) impose that w and v have the same Shapley value (more generally, any positive sharing value ϕ^q). As $w \geqslant v$ pointwise, minimizing the L_1-distance amounts to minimizing the sum of excesses $w(S) - v(S)$, $S \subseteq N$. The constraint that v and w should satisfy $\phi^q(v) = \phi^q(w)$ can be interpreted as that v and w have the same first-order approximation, since $\phi^q(\cdot)$ can be seen as an additive game.

When $k = 1$, the problem has either a trivial solution, which is $\phi^q(v)$ itself, or no solution. Indeed, the (1-additive) core may be empty, and even if it is not, there is no guarantee in general that $\phi^q(v)$ is a core element. For example, it is well known that the Shapley value may not be in the core when v is not convex. However, when $k \geqslant 2$ it has been shown that 1) the k-additive core is never empty, and 2) for any positive sharing value ϕ^q there always exists an element w of the k-additive core such that $\phi^q(v) = \phi^q(w)$. This proves that our problem has always a feasible solution, and also an optimal solution as we will show. Since the k-additive core expands when k is increasing, it is then natural to consider $k = 2$ as a prevailing choice.

Section 2 introduces the basic concepts, in particular sharing values. In Section 3, we express the above approximation problem as an LP problem, and prove that it has an optimal solution. The case $k = 2$ is treated in more detail and also illustrated in Section 4, because 2-additive games can be represented by a weighted undirected graph. As TU-games can be seen as weighted hypergraphs, this gives in addition a way to approximate a weighted hypergraph by a weighted graph. In Section 5, we analyze the properties of the LP

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problem. It is found that for at most 4 players all vertices of the polyhedron of feasible solutions are optimal. In addition, for arbitrary n, an explicit formula of the optimal value of the objective function is given for a particular class of games, as well as a geometric characterization of the set of optimal solutions.

2. Basic concepts

We consider $\mathcal{G}(N)$ the set of all TU-games on N, a finite set of n players. We recall that for any $\emptyset \neq S \subseteq N$, the *unanimity game* u_S is defined by $u_S(T) = 1$ if $T \supseteq S$, and 0 otherwise. It is well known that the set of unanimity games forms a basis of the $(2^n - 1)$-dim space $\mathcal{G}(N)$. The coordinates of a game $v \in \mathcal{G}(N)$ in this basis form its *Möbius transform*, denoted by m^v: $v = \sum_{S \subseteq N, S \neq \emptyset} m^v(S) u_S$, i.e., for every $T \subseteq N$

$$v(T) = \sum_{S \subseteq T} m^v(S).$$

A game v is *(at most) k-additive* for some fixed integer $k \in [1, n]$ if $m^v(S) = 0$ whenever $|S| > k$. Let $\mathcal{G}^k(N)$ denote the set of at most k-additive games on N. Such a game needs only a polynomial number of coefficients to be defined in the basis of unanimity games. Remarking that 1-additive games are additive games, which are also equivalent to payoff vectors, the k-additive core is defined by

$$\mathsf{C}^k(v) = \{w \in \mathcal{G}^k(N) : w(S) \geqslant v(S) \;\forall S \subset N, w(N) = v(N)\}.$$

It is a nonempty, convex and unbounded polyhedron for any game in $\mathcal{G}(N)$ and $k \geqslant 2$. It is the set of k-additive games dominating v under the constraint that the grand coalition receives a fixed amount.

A *sharing function* is a mapping $q : 2^N \setminus \{\emptyset\} \times N \to [0, 1]$ satisfying

$$q(S, i) = 0 \text{ if } i \notin S, \sum_{i \in S} q(S, i) = 1 \qquad (\emptyset \neq S \subseteq N).$$

q is *positive* if $q(S, i) > 0$ for all $i \in S$. We denote by $\mathcal{Q}(N)$ the set of all sharing functions and by $\mathcal{Q}_+(N)$ the set of all positive sharing functions. Let $v \in \mathcal{G}(N)$, and m^v be its Möbius transform. For any sharing function $q \in \mathcal{Q}(N)$ we define the payoff vector $\phi^q(v) \in \mathbb{R}^N$ by

$$\phi_i^q(v) = \sum_{S \ni i} m^v(S) q(S, i) \qquad (i \in N).$$

The *selectope* of v ([14]) is the set of all such payoff vectors:

$$\mathsf{S}(v) = \{\phi^q(v) \mid q \in \mathcal{Q}(N)\}.$$

Note that the selectope of v contains the Shapley value of v, corresponding to the *uniform sharing* $q(S, i) = 1/|S|$ for all $i \in S$. Also, all elements in the selectope are efficient, i.e., $\sum_{i \in N} \phi_i^q(v) = v(N)$. We denote by $\mathsf{Sh} : \mathcal{G}(N) \to \mathbb{R}^N$ the linear mapping assigning to each game v its Shapley value.

3. Upper approximation of TU-games

We are interested in the following problem:

> *(P1) Let v be a game in $\mathcal{G}(N)$. For a given integer $k \geqslant 2$, find a "best" approximation $w \in \mathsf{C}^k(v)$ s.t. $\mathsf{Sh}(v) = \mathsf{Sh}(w)$.*

A more general version is:

> *(P2) Let v be a game in $\mathcal{G}(N)$. For a given integer $k \geqslant 2$ and any positive sharing function q, find a "best" approximation $w \in \mathsf{C}^k(v)$ s.t. $\phi^q(v) = \phi^q(w)$.*

We know from [12] that for any $q \in \mathcal{Q}_+(N)$, any preimputation of v can be attained, i.e., for any $x \in X(v) := \{y \in \mathbb{R}^N \mid y(N) = v(N)\}$, there exists $w \in \mathsf{C}^k(v)$ such that $\phi^q(w) = x$. As $\mathsf{S}(v) \subseteq X(v)$, it follows that in (P2) it is always possible to find $w \in \mathsf{C}^k(v)$ such that $\phi^q(v) = \phi^q(w)$, i.e., (P2) has always a feasible solution.

We define the best approximation as the one which minimizes the sum of excesses of w over v, i.e., $\sum_{\emptyset \neq S \subset N} (w(S) - v(S))$. Observe that this amounts to minimize the distance between v and w in the sense of the L_1 norm.

First, we solve the general form (P2) and refine the result for (P1).

Solution of (P2) For this purpose, we use a result in [15] solving the so-called *inverse problem for the Shapley value*:

> *Given $v \in \mathcal{G}(N)$, find all $w \in \mathcal{G}(N)$ having the same Shapley value as v.*

We generalize this result to any ϕ^q (viewed as a mapping from $\mathcal{G}(N)$ to $\mathbb{R}^N$), $q \in \mathcal{Q}_+(v)$.

The solution of the inverse problem is simply $w = v + u$, where $u \in \mathsf{Ker}(\phi^q)$, the kernel of the mapping ϕ^q. A basis of the kernel is for example $B = (u_S^{\phi^q})_{S \subseteq N, |S| \geqslant 2}$ ($(2^n - n - 1)$-dim), with

$$u_S^{\phi^q} = u_S - \sum_{i \in S} q(S, i) u_{\{i\}}.$$

We go back to Problem (P2). We write $w = v + u$ with $u \in \mathsf{Ker}(\phi^q)$. Suppose $u \in \mathsf{Ker}(\phi^q)$ has coordinates $(\alpha_S)_{|S| \geqslant 2}$ in basis B. Then, in the basis of unanimity games,

$$u = \Bigg(\underbrace{\cdots \quad \alpha_S \quad \cdots}_{|S| \geqslant 2} \quad \underbrace{\cdots \quad -\sum_{\substack{T \ni i \\ |T| \geqslant 2}} q(T, i) \alpha_T \quad \cdots}_{i \in N} \Bigg). \tag{2}$$

Since w is k-additive, we must have $m^w(S) = 0$ for every $S \subseteq N$, $|S| > k$, which is equivalent to $m^u(S) = -m^v(S)$ for every $S \subseteq N$, $|S| > k$. From (2) we deduce

$$\alpha_S = -m^v(S) \qquad (S \subseteq N, |S| > k). \tag{3}$$

Since $w \in \mathsf{C}^k(v)$, we must ensure $w(S) \geqslant v(S)$ for all $S \subseteq N$, $S \neq \emptyset$, and $w(N) = v(N)$. This is equivalent to the condition $u(S) \geqslant 0$, for all $\emptyset \neq S \subsetneq N$, and $u(N) = 0$, the latter condition being ensured by the fact that $u(N) = \sum_{i\in N} \phi_i^q(u) = 0$. We have, using (2) and (3):

$$\begin{aligned}
u(S) &= \sum_{i\in S}\Big(-\sum_{\substack{T\ni i\\|T|\geqslant 2}} q(T,i)\alpha_T\Big) + \sum_{\substack{T\subseteq S\\|T|\geqslant 2}}\alpha_T\\
&= \sum_{\substack{T\subseteq S\\|T|\geqslant 2}}(-\alpha_T) + \sum_{i\in S}\Big(-\sum_{\substack{T\ni i\\T\not\subseteq S\\|T|\geqslant 2}} q(T,i)\alpha_T\Big) + \sum_{\substack{T\subseteq S\\|T|\geqslant 2}}\alpha_T\\
&= \sum_{i\in S}\Big(-\sum_{\substack{T\ni i\\T\not\subseteq S\\|T|\geqslant 2}} q(T,i)\alpha_T\Big)\\
&= \sum_{i\in S}\Big(\sum_{\substack{T\ni i\\T\not\subseteq S\\|T|>k}} q(T,i)m^v(T) - \sum_{\substack{T\ni i\\T\not\subseteq S\\2\leqslant|T|\leqslant k}} q(T,i)\alpha_T\Big).
\end{aligned}$$

Hence, the conditions on coefficients $\alpha_T, 2 \leqslant |T| \leqslant k$, are:

$$\sum_{i\in S}\sum_{\substack{T\ni i\\T\not\subseteq S\\2\leqslant|T|\leqslant k}} q(T,i)\alpha_T \leqslant \sum_{i\in S}\sum_{\substack{T\ni i\\T\not\subseteq S\\|T|>k}} q(T,i)m^v(T) \quad (\emptyset \neq S \subset N). \tag{4}$$

As $q(T,i) > 0$, this system of inequalities has always a solution (it suffices to take α_T sufficiently small) and defines an unbounded convex polyhedron.

Our objective function (to be minimized) is

$$z = \sum_{\emptyset\neq S\subset N}(w(S) - v(S)) = \sum_{\emptyset\neq S\subset N} u(S).$$

The minimization of z amounts to the maximization of z' (denoting cardinality by corresponding small letters):

$$\begin{aligned}
z' &= \sum_{\emptyset\neq S\subset N}\sum_{i\in S}\sum_{\substack{T\ni i\\T\not\subseteq S\\2\leqslant|T|\leqslant k}} q(T,i)\alpha_T\\
&= \sum_{\substack{T\subseteq N\\2\leqslant|T|\leqslant k}}\alpha_T\sum_{i\in T} q(T,i)\Big(\sum_{l=0}^{t-2}\binom{t-1}{l}\Big)2^{n-t}\\
&= \sum_{\substack{T\subseteq N\\2\leqslant|T|\leqslant k}}(2^{n-1} - 2^{n-t})\alpha_T.
\end{aligned}$$

Finally the "best" approximation is solution of the LP:

$$\begin{aligned}
\text{Maximize} \quad & \sum_{\substack{T\subseteq N\\2\leqslant|T|\leqslant k}}(2^{n-1} - 2^{n-t})\alpha_T\\
\text{s.t.} \quad & \sum_{i\in S}\sum_{\substack{T\ni i\\T\not\subseteq S\\2\leqslant|T|\leqslant k}} q(T,i)\alpha_T \leqslant \sum_{i\in S}\sum_{\substack{T\ni i\\T\not\subseteq S\\|T|>k}} q(T,i)m^v(T)\\
& (\emptyset \neq S \subset N).
\end{aligned}$$

Solution of (P1) When q is the uniform sharing function, some simplification appears in (4).

$$u(S) = \sum_{\substack{T\not\subseteq S, T\cap S\neq\emptyset\\|T|>k}} m^v(T)\frac{|T\cap S|}{|T|} - \sum_{\substack{T\not\subseteq S, T\cap S\neq\emptyset\\2\leqslant|T|\leqslant k}}\alpha_T\frac{|T\cap S|}{|T|}.$$

Hence, the conditions on coefficients $\alpha_T, 2 \leqslant |T| \leqslant k$, are (denoting cardinality by corresponding small letters)

$$\sum_{\substack{\emptyset\neq T\subseteq S\\\emptyset\neq L\subseteq N\setminus S\\2\leqslant t+l\leqslant k}}\frac{t}{t+l}\alpha_{T\cup L} \leqslant \sum_{\substack{\emptyset\neq T\subseteq S\\\emptyset\neq L\subseteq N\setminus S\\t+l>k}}\frac{t}{t+l}m^v(T\cup L) \quad (\emptyset \neq S \subset N). \tag{5}$$

There is no simplification for the objective function. Hence, the best approximation for (P1) is given by solving the following LP:

$$\begin{aligned}
\text{Maximize} \quad & \sum_{\substack{T\subseteq N\\2\leqslant|T|\leqslant k}}(2^{n-1} - 2^{n-t})\alpha_T\\
\text{s.t.} \quad & \sum_{\substack{\emptyset\neq T\subseteq S\\\emptyset\neq L\subseteq N\setminus S\\2\leqslant t+l\leqslant k}}\frac{t}{t+l}\alpha_{T\cup L} \leqslant \sum_{\substack{\emptyset\neq T\subseteq S\\\emptyset\neq L\subseteq N\setminus S\\t+l>k}}\frac{t}{t+l}m^v(T\cup L)\\
& (\emptyset \neq S \subset N).
\end{aligned}$$

The case $k = 2$ Further simplifications can be obtained in the case of the 2-additive core (the most interesting case in practice). Indeed, the variables are α_{ij}, with $\{i,j\} \subseteq N$. The objective function becomes:

$$z' = \sum_{\{i,j\}\subseteq N} 2^{n-2}\alpha_{ij},$$

so that the solution of (P2) is given by solving the following LP:

$$\begin{aligned}
\text{Maximize} \quad & \sum_{\{i,j\}\subseteq N}\alpha_{ij}\\
\text{s.t.} \quad & \sum_{i\in S, j\notin S} q(\{i,j\},i)\alpha_{ij} \leqslant \sum_{i\in S}\sum_{\substack{T\ni i\\T\not\subseteq S\\|T|>2}} q(T,i)m^v(T)\\
& (\emptyset \neq S \subset N),
\end{aligned}$$

while the solution of (P1) is given by solving:

$$\begin{aligned}
\text{Maximize} \quad & \sum_{\{i,j\}\subseteq N}\alpha_{ij}\\
\text{s.t.} \quad & \sum_{\substack{i\in S\\j\notin S}}\frac{1}{2}\alpha_{ij} \leqslant \sum_{\substack{\emptyset\neq T\subseteq S\\\emptyset\neq L\subseteq N\setminus S\\t+l>2}}\frac{t}{t+l}m^v(T\cup L) \quad (\emptyset \neq S \subset N).
\end{aligned} \tag{6}$$

Graph representation of 2-additive games A 2-additive TU-game v can be represented by an undirected weighted graph, where the set of nodes is N, and there is a link $\{i,j\}$

between two distinct nodes i, j whenever $m^v(\{i,j\}) \neq 0$, the link being weighted by $m^v(\{i,j\})$. In addition, node i receives the weight $m^v(\{i\})$. Then, 2-additivity implies that

$$v(S) = \sum_{i \in S} m^v(\{i\}) + \sum_{\{i,j\} \subseteq S} m^v(\{i,j\}),$$

i.e., the worth $v(S)$ is the sum of the weights of the nodes in S plus the weights of the links inside the subgraph limited to S. We remark also that, as the Shapley value is the sharing value with uniform sharing,

$$\mathsf{Sh}_i(v) = m^v(i) + \frac{1}{2} \sum_{j \neq i} m^v(ij). \tag{7}$$

Interestingly, Deng and Papadimitriou [16] consider in their study of complexity of TU-games almost the same class of games, the only difference being that nodes have no self-loops (i.e., $m^v(i) = 0$ for all $i \in N$). They show that for this class of games, the Shapley value coincides with the nucleolus.

Going back to our approximation problem (P2), taking $k = 2$, we obtain that any TU-game v can be approximated by an undirected weighted graph (not unique in general) representing a 2-additive game w that dominates v under the constraint $v(N) = w(N)$, minimizing the sum of excesses and having the same Shapley value.

4. Example

Let us consider $n = 4$ and the game v, together with its Möbius transform as given in Table 1.

Table 1
The game v with its Möbius transform

S	$i \in N$	12	13	14	23	24	34	123	124	134	234	N
$v(S)$	0	2	1	0	1	1	1	7	0	2	6	7
$m^v(S)$	0	2	1	0	1	1	1	3	-3	0	3	-2

v has no peculiar property (symmetry, k-additivity). Its Shapley value is

$$\mathsf{Sh}(v) = \begin{pmatrix} 1 & 5/2 & 3 & 1/2 \end{pmatrix}.$$

The system of inequalities (6) reads, after removing redundant ones:

$$\begin{array}{ll}
(S = 1, 234) & \alpha_{12} + \alpha_{13} + \alpha_{14} \leqslant -3 \\
(S = 2, 134) & \alpha_{12} + \alpha_{23} + \alpha_{24} \leqslant -1 \\
(S = 3, 124) & \alpha_{13} + \alpha_{23} + \alpha_{34} \leqslant 3 \\
(S = 4, 123) & \alpha_{14} + \alpha_{24} + \alpha_{34} \leqslant -3 \\
(S = 12) & \alpha_{13} + \alpha_{14} + \alpha_{23} + \alpha_{24} \leqslant 0 \\
(S = 14) & \alpha_{12} + \alpha_{13} + \alpha_{24} + \alpha_{34} \leqslant -2 \\
(S = 24) & \alpha_{12} + \alpha_{23} + \alpha_{14} + \alpha_{34} \leqslant 0
\end{array}$$

The corresponding polyhedron has 3 vertices which are:

$$\begin{aligned}
\alpha &= \begin{pmatrix} -2 & 0 & -1 & 3 & -2 & 0 \end{pmatrix} \\
\beta &= \begin{pmatrix} -2 & 1 & -2 & 2 & -1 & 0 \end{pmatrix} \\
\gamma &= \begin{pmatrix} -1 & 0 & -2 & 2 & -2 & 1 \end{pmatrix}.
\end{aligned}$$

They all maximize the objective function, hence the whole face defined by the convex hull of these 3 vertices forms the set of optimal solutions. Let us choose α and compute the corresponding w. We find by application of (2) the coordinates of u in the basis of unanimity games and then those of w as presented in Table 2. Similarly, if we choose β and γ, we get Tables 3 and 4, respectively.

Table 2
The game w with its Möbius transform: the vertex α

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$m^u(S)$	1	1	0	1	-2	0	-1	3	-2	0	-3	3	0	-3	2
$m^w(S)$	1	1	0	1	0	1	-1	4	-1	1	0	0	0	0	0
$w(S)$	1	1	0	1	2	2	1	5	1	2	7	1	3	6	7

Table 3
The game w with its Möbius transform: the vertex β

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$m^u(S)$	1	1	0	1	-2	1	-2	2	-1	0	-3	3	0	-3	2
$m^w(S)$	1	1	0	1	0	2	-2	3	0	1	0	0	0	0	0
$w(S)$	1	1	0	1	2	3	0	4	2	2	7	1	3	6	7

Table 4
The game w with its Möbius transform: the vertex γ

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$m^u(S)$	1	1	0	1	-1	0	-2	2	-2	1	-3	3	0	-3	2
$m^w(S)$	1	1	0	1	1	1	-2	3	-1	2	0	0	0	0	0
$w(S)$	1	1	0	1	3	2	0	4	1	3	7	1	3	6	7

For all three cases, w is indeed 2-additive, belongs to $\mathsf{C}^2(v)$, and has the same Shapley value as v, i.e., (7) holds, as it can be checked.

5. On the set of solutions of the LP problem for $k = 2$

We consider the case $k = 2$ and the Shapley value, i.e., the LP problem defined in (6). We remark that in the example of Section 4, all vertices of the polyhedron of feasible solutions are optimal. A natural question is whether this is a general property or a particular case. We show in the sequel that for $n \leqslant 4$ this is indeed always true, while a similar property holds for $n > 4$ for a particular class of games.

We write the system of inequalities defining the set of feasible solutions as follows:

$$\sum_{i \in S, j \notin S} \alpha_{ij} \leqslant b_S, \qquad (S \subseteq N, 1 \leqslant |S| \leqslant \lfloor \frac{n}{2} \rfloor) \tag{8}$$

where

$$b_S := \min\left(\sum_{\substack{\emptyset\neq T\subseteq S\\ \emptyset\neq L\subseteq N\setminus S\\ t+l>2}} \frac{t}{t+l} m^v(T\cup L), \sum_{\substack{\emptyset\neq T\subseteq N\setminus S\\ \emptyset\neq L\subseteq S\\ t+l>2}} \frac{t}{t+l} m^v(T\cup L)\right)$$

Note that by the definition of b_S and the fact that no inequality is a positive linear combination of the others and no equality is implied, the system is irredundant.

An important preliminary observation on the LP (6) is that it is not unbounded, i.e., an optimal vertex always exists. Indeed, recall that $w = v + u$ with v bounded and we look for w such that $\sum_S (w(S) - v(S)) = \sum_S u(S)$ is minimal. Therefore optimal u must be bounded and so is optimal w.

We begin by considering the cases $n = 3$ and $n = 4$ (the case $n = 2$ being trivial). With $n = 3$, the above system has 3 variables and reduces to

$$\begin{aligned}
\alpha_{12} + \alpha_{13} &\leqslant b_1\\
\alpha_{12} + \alpha_{23} &\leqslant b_2\\
\alpha_{13} + \alpha_{23} &\leqslant b_3.
\end{aligned}$$

As a vertex is defined by 3 linearly independent equations, there is only one vertex, whose coordinates are obtained by solving the above system with equalities. By the above observation, this vertex must be optimal.

We proceed to the case $n = 4$. The irredundant system of inequalities reads:

$$\alpha_{12} + \alpha_{13} + \alpha_{14} \leqslant b_1 \tag{9}$$
$$\alpha_{12} + \alpha_{23} + \alpha_{24} \leqslant b_2 \tag{10}$$
$$\alpha_{13} + \alpha_{23} + \alpha_{34} \leqslant b_3 \tag{11}$$
$$\alpha_{14} + \alpha_{24} + \alpha_{34} \leqslant b_4 \tag{12}$$
$$\alpha_{13} + \alpha_{14} + \alpha_{23} + \alpha_{24} \leqslant b_{12} \tag{13}$$
$$\alpha_{12} + \alpha_{14} + \alpha_{23} + \alpha_{34} \leqslant b_{13} \tag{14}$$
$$\alpha_{12} + \alpha_{13} + \alpha_{24} + \alpha_{34} \leqslant b_{14} \tag{15}$$

A vertex satisfies at least 6 linearly independent equalities, and any subsystem of 6 equalities among the 7 is a linearly independent set.

Consider a vertex (exists by the previous observation) and suppose first that (13) to (15) are tight. This determines a 3-dim plane parallel to the objective function hyperplane since the sum of the three equalities yields:

$$\alpha_{12}+\alpha_{13}+\alpha_{14}+\alpha_{23}+\alpha_{24}+\alpha_{34} = 1/2(b_{12}+b_{13}+b_{23}).$$

Now, if this is not true, this vertex must satisfy (9) to (12) with equality. This determines a 2-dim plane, also parallel to the objective function hyperplane as the sum of the 4 equations yields:

$$\alpha_{12}+\alpha_{13}+\alpha_{14}+\alpha_{23}+\alpha_{24}+\alpha_{34} = 1/2(b_1+b_2+b_3+b_4).$$

Suppose $b_1 + b_2 + b_3 + b_4 < b_{12} + b_{13} + b_{23}$. Then, since vertices are feasible points, any vertex satisfies (9) to (12) with equality, and exactly one inequality in (13) to (15) is loose. As a consequence, there are at most 3 vertices and all of them are optimal.

Using the same argument, if $b_1 + b_2 + b_3 + b_4 > b_{12} + b_{13} + b_{23}$, there are at most 4 vertices, all satisfying (13) to (15) with equality and all optimal. Lastly, if $b_1+b_2+b_3+b_4 = b_{12} + b_{13} + b_{23}$, there is exactly one vertex, satisfying all 7 inequalities with equality.

We turn to the general case (n arbitrary). We observe that by summing all inequalities for a fixed $|S|$, we obtain an inequality whose frontier is parallel to the objective function, as for $n = 4$. Indeed, in the subsystem corresponding to $|S| = s$, each pair ij is seen for each S such that $j \notin S \ni i$ and for each S such that $i \notin S \ni j$, which yields $2\binom{n-2}{s-1}$ times in total. As this number does not depend on ij, it follows that summing all inequalities of the subsystem yields

$$\sum_{ij} \alpha_{ij} \leqslant \frac{1}{2\binom{n-2}{s-1}} \sum_{S:|S|=s} b_S. \tag{16}$$

The following property is fundamental.

Lemma 1. Consider the (P1) problem with $k = 2$, n arbitrary. For every cardinality $1 \leqslant s \leqslant \lfloor \frac{n}{2} \rfloor$,

$$\sum_{\substack{S\subseteq N\\ |S|=s}} b'_S = \binom{n-1}{s-1} \sum_{\substack{K\subseteq N\\ |K|\geqslant 3}} m^v(K)$$

with $b'_S = \sum_{\substack{\emptyset\neq T\subseteq S\\ \emptyset\neq L\subseteq N\setminus S\\ t+l>2}} \frac{t}{t+l} m^v(T\cup L).$

Proof. It can be checked that, for any fixed cardinality $1 \leqslant s \leqslant \lfloor \frac{n}{2} \rfloor$, regrouping terms leads to:

$$\sum_{\substack{S\subseteq N\\ |S|=s}} b'_S = \sum_{\substack{K\subseteq N\\ |K|\geqslant 3}} \left(\sum_{t=1\vee(s-n+k)}^{s\wedge k} \binom{n-k}{s-t}\binom{k-1}{t-1}\right) m^v(K).$$

The $\vee, \wedge$ in the summation only ensure that in binomial coefficients $\binom{i}{j}$, we have $0 \leqslant j \leqslant i$. In order to simplify the notation, we take by convention that the coefficient is zero if this does not hold. Then we have to show that for any fixed m, r such that $r \leqslant m$, the following general relation holds

$$\sum_{p=r-l}^{r} \binom{l}{r-p}\binom{m-l}{p} = \binom{m}{r}$$

for every $l = 0, \dots, m-2$, which will prove the desired result. We show this by induction on m and r, using the relation $\binom{i}{j} = \binom{i-1}{j} + \binom{i-1}{j-1}$.

We start with an induction on m, with fixed $r \leqslant m$. The result is easy to check with $n = 2$ and $r = 1, 2$. We have

$$\sum_{p=r-l}^{r} \binom{l}{r-p}\binom{m+1-l}{p} =$$

$$\sum_{p=r-l}^{r}\binom{l}{r-p}\left[\binom{m-l}{p-1}+\binom{m-l}{p}\right]=$$
$$\sum_{p'=r-1-l}^{r-1}\binom{l}{r-p'-1}\binom{m-l}{p'}+\binom{m}{r}=$$
$$\binom{m}{r-1}+\binom{m}{r}=\binom{m+1}{r},$$

where we have used twice the induction hypothesis and $p'=p-1$. We now proceed with the induction on r for a fixed m. For any $r<m$,

$$\sum_{p=r+1-l}^{r+1}\binom{l}{r+1-p}\binom{m-l}{p}=$$
$$\sum_{p'=r-l}^{r}\binom{l}{r-p'}\binom{m-l}{p'+1}=$$
$$\sum_{p'=r-l}^{r}\left[\binom{l}{r-p'}\binom{m-l-1}{p'+1}+\binom{l}{r-p'}\binom{m-l-1}{p'}\right]=$$
$$\sum_{p=r-l+1}^{r+1}\binom{l}{r-p+1}\binom{m-l-1}{p}+\binom{m-1}{r}=$$
$$\binom{m-1}{r+1}+\binom{m-1}{r}=\binom{m}{r+1},$$

with $p'=p-1$, and we have used the induction hypothesis on r with m and the fact that the result is true for $m-1$ and any $r\leqslant m-1$. □

Supposing that $b_S=b'_S$ for all S of a given cardinality s, the combination of (16) and Lemma 1 yields

$$\sum_{ij}\alpha_{ij}\leqslant\frac{n-1}{n-s}\sum_{|K|\geqslant 3}m^v(K). \tag{17}$$

We are now in position to establish our main result for this section.

Theorem 1. Let v be such that $m^v\geqslant 0$ and $b_S=b'_S$ for all $\varnothing\neq S\subset N$. Then the linear program has optimal value for the objective function

$$z^*=\sum_{|K|\geqslant 3}m^v(K)$$

and every vertex satisfying all inequalities for $|S|=1$ with equality is optimal. In other words, the set of optimal solutions is the intersection of the linear space defined by the equalities for $|S|=1$ and the positive orthant.

Proof. Observe that $\frac{n-1}{n-s}<\frac{n-1}{n-s'}$ iff $s<s'$. On the other hand, under the assumptions on v, as any feasible point satisfies (17) for all $s=1,\dots,n-1$, it follows that the optimal value of the objective function is bounded above by $\sum_{|K|\geqslant 3}m^v(K)$. Finding a feasible point attaining this upper bound suffices to prove the result. Consider a point satisfying all inequalities for $|S|=1$ with equality. Then $\sum_{ij}\alpha_{ij}=\sum_{|K|\geqslant 3}m^v(K)$. It remains to show that there are such points which are feasible. Consider any inequality pertaining to some S, $|S|>1$ and $|S|\leqslant\lfloor\frac{n}{2}\rfloor$:

$$\sum_{\substack{i\in S\\ j\notin S}}\alpha_{ij}\leqslant 2\sum_{\substack{\varnothing\neq T\subseteq S\\ \varnothing\neq L\subseteq N\setminus S\\ t+l>2}}\frac{t}{t+l}m^v(T\cup L). \tag{18}$$

Consider now any equality for $\{i\}$:

$$\sum_{j\neq i}\alpha_{ij}=2\sum_{\substack{\varnothing\neq L\subseteq N\setminus\{i\}\\ l>1}}\frac{1}{1+l}m^v(L\cup\{i\}).$$

Adding all those with $i\in S$ we obtain:

$$\sum_{i\in S}\sum_{j\neq i}\alpha_{ij}=2\sum_{i\in S}\sum_{\substack{\varnothing\neq L\subseteq N\setminus\{i\}\\ l>1}}\frac{1}{1+l}m^v(L\cup\{i\})$$

which can be rewritten as

$$\begin{aligned}\sum_{j\notin S}\sum_{i\in S}\alpha_{ij}+2\sum_{\{i,j\}\subseteq S}\alpha_{ij}\\ &= 2\sum_{\substack{\varnothing\neq L\subseteq N\setminus S\\ l>1}}\Bigg[\frac{1}{1+l}\sum_{i\in S}m^v(L\cup\{i\})+\\ &\quad\frac{2}{2+l}\sum_{\{i,j\}\subseteq S}m^v(L\cup\{i,j\})+\cdots+\frac{s}{s+l}m^v(L\cup S)\Bigg]\\ &= 2\sum_{\substack{\varnothing\neq T\subseteq S\\ \varnothing\neq L\subseteq N\setminus S\\ t+l>2}}\frac{t}{t+l}m^v(T\cup L).\end{aligned}$$

Plugging this equality in (18), we find

$$2\sum_{\{i,j\}\subseteq S}\alpha_{ij}\geqslant 0.$$

As this must be true for any $|S|\geqslant 2$, it follows that the coordinates α_{ij} must be nonnegative for all i,j to ensure feasibility. This is possible since the right hand-side of the system of equalities is nonnegative. This proves the last assertion of the theorem. □

We end this section by some considerations on extremal rays of the polyhedron defined by (8). Recall that extremal rays are found by considering the system (8) where the right-hand side is replaced by 0 everywhere, and by finding a set of inequalities that yields a 1-dim solution set, which has to be feasible w.r.t. the remaining inequalities.

We claim that if r is an extremal ray, then the sum of its components r_{ij} cannot be zero. Indeed, suppose that r is an extremal ray with $\sum_{ij}r_{ij}=0$ and take α^* an optimal solution of the system (8). Then $\beta:=\alpha^*+cr$ is a solution of the system, for any $c\geqslant 0$. However, $\sum_{ij}\beta_{ij}=\sum_{ij}\alpha^*_{ij}+c\sum_{ij}r_{ij}=\sum_{ij}\alpha_{ij}$, from which it follows that β is optimal as well, for any $c \geqslant 0$. As there is no optimal unbounded solution, the claim is proved.

We use this fact to find all extremal rays, and illustrate it on the case $n = 4$. It follows that an extremal ray is the solution of a system where exactly 1 inequality in (9) to (12) is loose and exactly 1 inequality in (13) to (15) is loose. Then there are at most $3 \times 4 = 12$ extremal rays. It turns out that all the 12 are feasible so they are extremal rays. It can be checked that the extremal rays are of the form $\begin{pmatrix} 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$ where the 0's correspond to the ij's present 3 times in the selected equalities, the 1's to the ij's present 4 times, and the -1's to the ij's present 2 times. Our example corresponds to selecting (9) to (11), and (13), (14).

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References

[1] P. L. Hammer, R. Holzman, On approximations of pseudo-boolean functions, ZOR - Methods and Models of Operations Research 36 (1992) 3–21.

[2] M. Grabisch, Set Functions, Games and Capacities in Decision Making, Theory and Decision Library C – Game Theory, Social Choice, Decision Theory, and Optimization, Springer, 2016.

[3] A. Charnes, B. Golany, M. Keane, J. Rousseau, Extremal principle solutions of games in characteristic function form: core, Chebychev and Shapley value generalizations, in: J. Sengupta, G. Kadekodi (Eds.), Econometrics of Planning and Efficiency, Kluwer Academic Publisher, 1988, pp. 123–133.

[4] M. Grabisch, J.-L. Marichal, M. Roubens, Equivalent representations of set functions, Mathematics of Operations Research 25(2) (2000) 157–178.

[5] J.-L. Marichal, P. Mathonet, Weighted Banzhaf power and interaction indexes through weighted approximations of games, European Journal of Operational Research 211 (2011) 352–358.

[6] G. Ding, R. Lax, J. Chen, P. Chen, B. Marx, Transforms of pseudo-boolean random variables, Discrete Applied Mathematics 158 (2010) 13–24.

[7] L. M. Ruiz, F. Valenciano, J. M. Zarzuelo, The least square prenucleolus and the least square nucleolus, two values for TU games based on the excess vector, International Journal of Game Theory 25 (1996) 113–134.

[8] L. M. Ruiz, F. Valenciano, J. M. Zarzuelo, The family of least square values for transferable utility games, Games and Economic Behavior 24 (1998) 109–130.

[9] P. Walley, Statistical Reasoning with Imprecise Probabilities, Chapman and Hall, London, 1991.

[10] B. Peleg, P. Sudhölter, Introduction to the Theory of Cooperative Games, Kluwer Academic Publisher, 2003.

[11] P. Miranda, M. Grabisch, k-balanced games and capacities, European Journal of Operational Research 200(2) (2010) 1465–472.

[12] M. Grabisch, T. Li, On the set of imputations induced by the k-additive core, European Journal of Operational Research 214 (2011) 697–702.

[13] V. Vassil'ev, Polynomial cores of cooperative games, Optimizacia 21 (1978) 5–29.

[14] J. Derks, H. Haller, H. Peters, The selectope for cooperative games, International Journal of Game Theory 29 (2000) 23–38.

[15] U. Faigle, M. Grabisch, Bases and linear transforms of TU-games and cooperation systems, International Journal of Game Theory 45 (2016) 875–892.

[16] X. Deng, C. H. Papadimitriou, On the complexity of cooperative solution concepts, Mathematics of Operations Research 19(2) (1994) 257–266.

[17] D. Schmeidler, The nucleolus of a characteristic function game, SIAM Journal of Applied Mathematics 17 (1969) 1163–1170.