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# Why do we postpone annuity purchases?<sup>☆</sup>

Hippolyte d'Albis<sup>a,\*</sup>, Andrei Kalk<sup>b</sup>

<sup>a</sup>*Paris School of Economics, CNRS, France*

<sup>b</sup>*University of Vienna, Austria*

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## Abstract

This paper seeks to explain why annuity purchases are postponed to a later age. We consider an overlapping generations model with uncertain lifetime and two types of annuities. It is shown that, if the economy is dynamically inefficient, individuals demand annuities without delay. However, if it is efficient, annuity purchases are postponed. We also show that these results are robust to several extensions.

*Keywords:* Annuity purchases, Overlapping generations, Postponement, Annuity pricing  
*JEL classification:* D14, E13, G22

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## 1 Introduction

Empirical evidence for developed countries suggests that most annuity purchases are postponed to advanced ages. In the UK, for example, the modal age at purchase of annuities is between 60 and 65 (Einav et al., 2010). Moreover, less than 15% of annuitants in the UK market buy the product before the age of 60 (Taylor, 2004). In the US, nearly three-quarters of annuitants are aged 61 to 80 (Mitchell et al., 1999). Such observations stand in stark contrast to the prediction of the theoretical model proposed by Brugiavini (1993). Consumers indeed should purchase annuities only when they are young, provided that the information about longevity arrives late in life. This puzzling result has been replicated and extended in more recent studies by Sheshinski (2010) and Steinorth (2012). In particular, Steinorth (2012) has shown that early annuities can be driven out of the market if individuals have more information about their future survival type than annuity providers.

In this paper, we provide a new explanation for the phenomenon of delayed annuity purchases using a model similar to that of Brugiavini (1993) while considering general annuity pricing. We examine the timing of annuity purchases in an overlapping generations economy with production, where agents live for either two or three periods. Following Brugiavini (1993), we assume that agents can invest in annuities at an early age and receive the

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\*Corresponding author.

*Email addresses:* [hda1bis@psemail.eu](mailto:hda1bis@psemail.eu) (Hippolyte d'Albis), [andr.kalk@univie.ac.at](mailto:andr.kalk@univie.ac.at) (Andrei Kalk)

return in the third period of their life upon survival. In addition, they can purchase bonds in the beginning and annuities at an older age. A key feature of our model is that annuity premiums are determined in general equilibrium for perfectly competitive annuity markets. This idea can be found in the work of [d'Albis and Etner \(2018\)](#), who have studied illiquid annuities in an OLG economy. As in their model, annuity pricing in our framework may involve redistribution between cohorts.

We show that in the simple case where the probability of survival is known in advance to annuity providers and agents, annuities are generically purchased only once during the lifetime. In a *dynamically inefficient* economy, i.e., one characterized by over-accumulation of capital, agents purchase annuities early in life. On the contrary, if the economy is *dynamically efficient*, our basic model predicts that agents postpone the purchase of annuities to a later age. The intuition behind the results is based on a comparison of two saving strategies: investing a unit of consumption in annuities when young or investing it in bonds and then in annuities when old. Choosing the first strategy, agents receive the return which, in fact, represents a uniform transfer from the next generation of annuity holders. Thus, agents can benefit from population growth by purchasing annuities early in life. As long as the growth rate is higher (lower) than the long-run interest rate, this strategy is more (less) profitable than the other with delayed annuity purchases.

To check the robustness of our results, we also discuss some extensions and modifications of the basic model. The literature has argued that the irreversibility of annuity purchases ([Milevsky and Young, 2007](#)) and overly pessimistic beliefs about survival probabilities ([Sheshinski, 2008](#)) may reduce incentives to purchase annuities earlier in life. Correspondingly, we find that our results remain unchanged when annuity contracts are, in general, irreversible or agents have subjective survival expectations. This suggests that macroeconomic conditions play a major role in explaining why younger cohorts do not participate in the annuity market. The same applies to the additional assumption that the future survival probability is initially uncertain. Indeed, we show that in this case, contrary to [Brugiavini \(1993\)](#), all agents should postpone the purchase of annuities to an older age if the rate of interest is sufficiently higher than the rate of population growth.

The findings presented in this paper are consistent with previous empirical studies for advanced economies, which usually experience low population growth. [Brown \(2001\)](#), for instance, using the sample of American households, has shown that the value of annuitization is increasing with age. Similarly, the fact that annuities are less attractive earlier in life seems to be supported by [Bütler et al. \(2013\)](#), who have studied male workers close to retirement in Switzerland. They have found that the probability of annuitizing pension wealth is lower for men retiring at a younger age. It is noteworthy that, in contrast to these results, obtained in a natural economic context, in a survey with hypothetical choices, one observes a significant negative effect of age on the probability of choosing an annuity ([Schreiber and Weber, 2016](#)). This gives a reason to think that macroeconomic conditions matter for the decision to delay annuity purchases.

More broadly, our paper contributes to the literature that demonstrates that macroeconomic frameworks are useful for analyzing lifecycle decisions in general and the demand for annuities in particular (see [Prettner and Canning \(2014\)](#), [Heijdra et al. \(2014, 2017\)](#)).

The rest of the paper is organized as follows. In Section 2, we present the basic model and characterize the demand for annuities in a steady state. Section 3 explores the robustness of our results to a bequest motive, subjective evaluation of longevity risk, **endogenous longevity**, and irreversibility of annuity purchases. Section 4 deals with the case of uncertain survival as in [Brugiavini \(1993\)](#). Finally, we discuss some additional issues in Section 5 and conclude in Section 6.

## 2 The basic model

Our basic framework builds on the three-period model of [Brugiavini \(1993\)](#). The key distinguishing feature of our approach is that we use general annuity pricing by analyzing an overlapping generations economy. Furthermore, contrary to Brugiavini's model, we assume that the information on life expectancy is complete. We will relax this assumption later.

### 2.1 Demographics

We consider an economy inhabited by overlapping generations of agents with uncertain lifetime. Time is discrete and indexed by  $t \in \{0, 1, \dots\}$ . Every agent is born at age 0 and lives for either two or three periods. The probability of survival between periods 2 and 3 is denoted by  $p \in (0, 1)$ . At date  $t$ , population consists of  $N_{i,t}$  agents, with index  $i \in \{0, 1, 2\}$  denoting their age. Thus, the number of newly born agents at date  $t$  is  $N_{0,t}$ , **the number of agents of age 1 at date  $t+1$**  is  $N_{1,t+1} = N_{0,t}$ , and **the number of agents of age 2 at date  $t+2$**  is  $N_{2,t+2} = pN_{1,t+1} = pN_{0,t}$ . Population grows at a constant rate  $n > -1$ :

$$N_{0,t} = (1 + n)N_{0,t-1}. \quad (1)$$

### 2.2 Annuity markets

There are two types of annuity contracts available for agents: short-term and long-term. The short-term contract can be purchased at age 1, whereas the long-term contract has to be purchased at age 0. Provided that agents are alive, both contracts recover the capital invested and pay a return at age 2. We denote the individual demand for short- and long-term annuities at date  $t$  by  $a_{1,t}$  and  $a_{0,t}$ , respectively.

The survival probability  $p$  is assumed to be known with certainty to annuity providers and agents. **We also assume that there is one market for each type of annuity contract and those markets are perfectly competitive. Then, with free entry, annuity providers make zero profit separately for short- and long-term annuities. We will now characterize the two respective zero-profit conditions and how they determine the market returns of annuities.**

Let  $R_t$  be the gross interest rate in the economy at date  $t$ . At date  $t-1$ , annuity providers collect agents' savings for each contract type and **lend them to firms** at the rate  $R_t$ . This yields  $N_{1,t-1}a_{1,t-1}R_t$  for the short-term contract and  $N_{0,t-1}a_{0,t-1}R_t$  for the long-term contract at date  $t$ . Since there is no profit, the two amounts are fully redistributed among agents who are currently aged 2 and purchased the annuity contracts before. Let us denote by  $R_t/\pi_t^S$  the yield on short-term annuities demanded by agents of age 2 at date  $t-1$ . Similarly, let

$R_t/\pi_t^L$  be the yield on long-term annuities demanded by agents of age 2 at date  $t - 2$ . Then, the zero-profit conditions for the two types of annuity contracts are expressed as

$$N_{1,t-1}a_{1,t-1}R_t = N_{2,t}a_{1,t-1}\frac{R_t}{\pi_t^S}, \quad (2)$$

$$N_{0,t-1}a_{0,t-1}R_t = N_{2,t}a_{0,t-2}\frac{R_t}{\pi_t^L}, \quad (3)$$

Applying our demographic equations, we obtain that annuity premiums  $1/\pi_t^S$  and  $1/\pi_t^L$  (defined for non-zero annuity demands) satisfy<sup>1</sup>

$$\pi_t^S = p, \quad \pi_t^L = \frac{pa_{0,t-2}}{(1+n)a_{0,t-1}}. \quad (4)$$

Thus, the short-term annuity is the standard annuity, with the inverse of the premium equal to the survival probability, as in [Yaari \(1965\)](#). As for the long-term annuity, the inverse of its premium depends not only on the survival probability but also on the population growth rate and the amounts of annuities purchased by previous generations. **For perfectly competitive annuity markets, which can be seen as an equilibrium feature of our model, it is clear that long-term annuities induce an intergenerational transfer.<sup>2</sup> This transfer goes in the same direction as that observed for a public pay-as-you-go pension system (i.e., from the young to the old), yet annuities themselves are products managed by private companies. It is also worth emphasizing that for  $a_{0,t-2} = a_{0,t-1}$ , the long-term annuity is more profitable than the short-term annuity if the rate of population growth is positive.**

### 2.3 Consumption and savings

For simplicity, we assume that agents do not consume at age 0. The expected lifetime utility of the agent born at date  $t$  is given by

$$u(c_{1,t+1}) + \theta pu(c_{2,t+2}), \quad (5)$$

where  $\theta \in (0, 1)$  is the discount factor and  $c_{i,t+i} \geq 0$  is the consumption at age  $i = 1, 2$ . The instantaneous utility function  $u$  is assumed to be continuously differentiable, increasing, concave and to satisfy  $\lim_{x \rightarrow 0} u'(x) = +\infty$ .

At age 0, the agent receives a wage income  $w_t$  that can be invested in long-term annuities  $a_{0,t}$  and riskless bonds, **which are direct loans from agents to firms**. We denote the agent's

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<sup>1</sup>Note that as long as annuity providers can observe the actual survival probability of an agent, agents can be heterogeneous in their survival. In this case, annuity premiums will be determined by the zero-profit condition for each survival type, and all our results will hold.

<sup>2</sup>In a real-world context, such redistribution through long-term annuities may, for instance, occur at the firm level from active to retired employees who committed to a group annuity contract.

demand for bonds by  $z_t$ .<sup>3</sup> Then, the budget constraint of the agent at age 0 is simply

$$a_{0,t} + z_t = w_t. \quad (6)$$

At age 1, the agent earns interest income from holding bonds  $z_t R_{t+1}$ , out of which she consumes  $c_{1,t+1}$  and purchases short-term annuities  $a_{1,t+1}$ :

$$c_{1,t+1} + a_{1,t+1} = z_t R_{t+1}. \quad (7)$$

Finally, at age 2, if the agent is alive, she consumes the return and capital invested in the two types of annuities:

$$c_{2,t+2} = a_{0,t} \frac{R_{t+2}}{\pi_{t+2}^L} + a_{1,t+1} \frac{R_{t+2}}{\pi_{t+2}^S}. \quad (8)$$

In our framework, we focus on the demand for annuities. Therefore, we assume that the agent cannot borrow against them, which would otherwise be equivalent to the purchase of life insurance (Bernheim, 1991). Thus, the following constraints are imposed:

$$a_{0,t} \geq 0, \quad a_{1,t+1} \geq 0. \quad (9)$$

Note that although we allow for borrowing against bonds, non-negativity of consumption and demand for annuities imply  $z_t \geq 0$  because of (7). Hence, we eliminate the possibility for the agent to purchase an infinite amount of annuities at age 0 by borrowing an infinite amount of bonds.

The agent's problem is to choose  $(c_{1,t+1}, c_{2,t+2}, z_t, a_{0,t}, a_{1,t+1})$  that maximizes the lifetime utility (5) subject to the budget constraints (6)-(8), and the non-borrowing constraints (9). The agent has correct expectations about future interest rates and annuity premiums, but does not anticipate that her behavior can affect (4). Denoting by  $\lambda_t \geq 0$  and  $\mu_{t+1} \geq 0$  the Kuhn-Tucker multipliers associated with (9), the first-order conditions of this problem are given by

$$\begin{cases} u'(c_{1,t+1})R_{t+1} - \theta p u'(c_{2,t+2}) \frac{R_{t+2}}{\pi_{t+2}^L} = \lambda_t, \\ u'(c_{1,t+1}) - \theta p u'(c_{2,t+2}) \frac{R_{t+2}}{\pi_{t+2}^S} = \mu_{t+1}, \end{cases} \quad (10)$$

while the complementary slackness conditions are given by

$$\lambda_t a_{0,t} = 0, \quad \mu_{t+1} a_{1,t+1} = 0. \quad (11)$$

#### 2.4 Production

There is a representative firm producing a single final good every period. The technology is represented by a neoclassical production function  $F(K_t, L_t)$ , where  $K_t$  is the capital stock and  $L_t$  is the amount of labor used at date  $t$ . The function  $F$  is assumed to be increasing and

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<sup>3</sup>In the next section, when discussing the issue of a bequest motive, we also consider the case where the agent can purchase bonds at age 1 and 2. In the model with no bequest motive, purchasing bonds at these ages is never optimal.

strictly concave in each argument, continuously differentiable and homogeneous of degree one. Agents supply inelastically one unit of labor at age 0, so that  $L_t = N_{0,t}$  in equilibrium for each  $t$ . Capital depreciates at rate  $\delta \in [0, 1]$ . We also assume that factor markets are competitive, implying that the interest factor,  $R_t$ , and the wage,  $w_t$ , satisfy

$$R_t = 1 + f'(k_t) - \delta, \quad w_t = f(k_t) - f'(k_t)k_t, \quad (12)$$

where  $k_t := K_t/N_{0,t}$  is the capital stock per worker and  $f(k_t) := F(k_t, 1)$  is the intensive-form production function. In equilibrium, the capital stock at date  $t + 1$  is equal to aggregate savings at date  $t$ . This condition can be written in intensive terms as follows:

$$k_{t+1} = \frac{z_t + a_{0,t}}{1+n} + \frac{a_{1,t}}{(1+n)^2}. \quad (13)$$

## 2.5 Equilibrium

An **equilibrium path** of the economy described above is a sequence

$$\{c_{1,t}, c_{2,t}, z_t, a_{0,t}, a_{1,t}, \pi_{t+1}^S, \pi_{t+1}^L, R_{t+1}, w_{t+1}, k_{t+1}\}_{t=0}^{\infty} \quad (14)$$

that, for a given initial value of  $k_0$ , satisfies the annuity pricing conditions (4), the budget constraints (6)-(8), the optimality conditions (10) and (12), the capital market-clearing condition (13), and the complementary slackness conditions (11). In what follows, we focus on a steady-state equilibrium defined **as an equilibrium path** where all variables are constant. Therefore, we will omit the time index when referring to the steady-state value of the considered variable.

The following proposition characterizes the steady-state demand for short- and long-term annuities.

**Proposition 1.** *In the steady-state equilibrium, the demand for annuities satisfies*

$$\begin{cases} a_0 > 0 & \text{and } a_1 = 0 & \text{if } f'(k) < \delta + n, \\ a_0 = 0 & \text{and } a_1 > 0 & \text{if } f'(k) > \delta + n. \end{cases} \quad (15)$$

*A demand for annuities satisfying  $a_0 > 0$  and  $a_1 > 0$  can occur only if  $f'(k) = \delta + n$ .*

*Proof.* See Appendix A.

We see that in the steady-state equilibrium, annuities are generically demanded only once during the lifetime and that the timing of the purchase depends on macroeconomic conditions. Using the standard characterization of the dynamics in overlapping generations models (see, e.g., De la Croix and Michel (2002)), we know that a converging trajectory to the steady state is efficient when  $f'(k) \geq \delta + n$  and inefficient otherwise. Thus, Proposition 1 shows that annuity purchases occur without delay (i.e., at age 0) if the economy is dynamically inefficient. Conversely, annuity purchases are postponed (until age 1) if the economy is dynamically efficient. Since the latter situation is known to be Pareto optimal, we can

also conclude that delaying the purchase of annuities is optimal from a social welfare point of view.

The intuition of Proposition 1's result is as follows. Solving the utility maximization problem, the agent compares the expected returns of the possible saving strategies. The first one is to invest a unit of consumption in bonds at age 0 and then in short-term annuities at age 1. This strategy does not involve any redistribution between cohorts. The second one is to invest a unit of consumption in long-term annuities at age 0 and wait until age 2. As long-term annuities induce a transfer from the next generation, their return increases with the population growth rate. If the latter becomes larger than the steady-state interest rate, the second saving strategy becomes more profitable. This sort of reasoning is analogous to the one applied to the optimality of pay-as-you-go pension system. Proposition 1 shows that it applies to the optimal timing of annuity purchases. Note finally that in the special case where  $R = 1 + n$ , both strategies yield the same return for agents, and any configuration of annuity purchases is possible (except for the configuration in which both contracts are not demanded).

## 2.6 An example

To illustrate Proposition 1, we consider a simple example with a logarithmic instantaneous utility function and a Cobb-Douglas production function:

$$u(c_i) = \ln c_i, \quad f(k) = k^\alpha, \quad (16)$$

where  $\alpha \in (0, 1)$  is the share of capital income in total income. We show in Appendix B that with these functional forms, the steady-state level of capital per worker is given by a function  $k(n)$  such that

$$k(n) = \begin{cases} \left( \frac{1 + \sigma \frac{1-\delta}{1+n} + \sqrt{(1 + \sigma \frac{1-\delta}{1+n})^2 + \sigma \frac{4\alpha}{1-\alpha}}}{2(1+n)(1-\alpha)^{-1}} \right)^{\frac{1}{1-\alpha}} & \text{if } n < \underline{n}, \\ \left( \frac{\alpha}{\delta+n} \right)^{\frac{1}{1-\alpha}} & \text{if } \underline{n} \leq n \leq \bar{n}, \\ \left( \frac{1-\alpha}{1+n} \right)^{\frac{1}{1-\alpha}} & \text{if } n > \bar{n}, \end{cases} \quad (17)$$

where

$$\sigma := \frac{\theta p}{1 + \theta p}, \quad \underline{n} := \frac{\alpha - \delta(1 + \sigma)(1 - \alpha)}{(1 + \sigma)(1 - \alpha) - \alpha}, \quad \bar{n} := \frac{\alpha - \delta(1 - \alpha)}{1 - 2\alpha}. \quad (18)$$

Figure 1 depicts the relationship between the marginal product of capital  $f'(k(n)) = \alpha k(n)^{\alpha-1}$  and the population growth rate  $n$ .<sup>4</sup> Depending on the parameters' values, three cases are possible:

1. For  $n < \underline{n}$ , we have  $f'(k(n)) > \delta + n$ , implying, by Proposition 1, that annuities are purchased only at age 1 (i.e.,  $a_0 = 0$  and  $a_1 > 0$ ).

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<sup>4</sup>In Figure 1, we denote  $\xi(n) := 2/[1 + \sigma \frac{1-\delta}{1+n} + \sqrt{(1 + \sigma \frac{1-\delta}{1+n})^2 + \sigma \frac{4\alpha}{1-\alpha}}]$  for brevity.



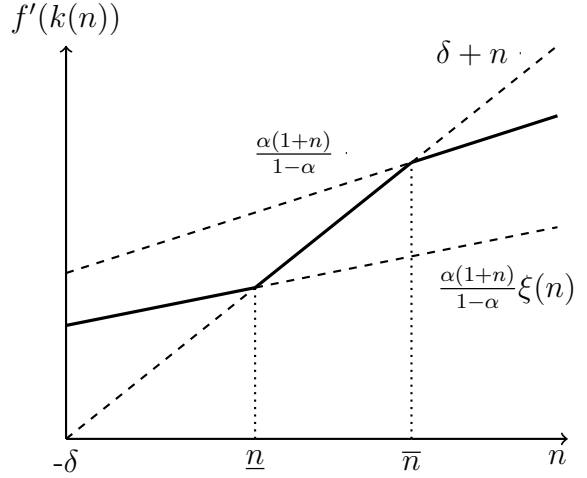


Figure 1: The relationship between  $f'(k(n))$  and  $n$ .

2. For  $\underline{n} \leq n \leq \bar{n}$ , we have  $f'(k(n)) = \delta + n$ . Then, all configurations of annuity purchases are possible. More specifically, one can check (see Appendix B) that annuities are purchased only at age 1, when  $n = \underline{n}$ , and only at age 0, when  $n = \bar{n}$ . When  $n$  lies in between the two extremes, annuities are purchased both at age 0 and age 1 (i.e.,  $a_0 > 0$  and  $a_1 > 0$ ).
3. For  $n > \underline{n}$ , we have  $f'(k(n)) < \delta + n$ , implying, by Proposition 1, that annuities are purchased only at age 0 (i.e.,  $a_0 > 0$  and  $a_1 = 0$ ).

With these three conditions, we can easily identify which annuity configuration is likely to be observed in an economy. We use standard parameter values. Annual discount factor is set to 0.99, depreciation to 2.5%, and population growth to 1%. Assuming that each period corresponds to 10 years, we then calculate  $\theta = 0.99^{10}$ ,  $\delta = 1 - 0.975^{10}$ , and  $n = 1.01^{10} - 1$ . In addition, we set  $\alpha = 1/3$  and  $p = 0.85$ .<sup>5</sup> Thus, we obtain  $n = 0.105 < \underline{n} = 0.192$ . That is, under plausible restrictions, the model predicts a postponement of annuity purchases.

### 3 Robustness

In this section, we argue that the result presented in Proposition 1 still holds when we consider alternative assumptions on individual preferences and on the characteristics of the annuities. In particular, we consider successively **four** cases: (i) bequest motive, (ii) subjective evaluation of longevity risk, (iii) **endogenous longevity**, and (iv) **irreversibility** of annuity purchases.

#### 3.1 Bequest motive

First, similarly to Yaari (1965), Davidoff et al. (2005), and Lockwood (2012), we allow agents to be altruistic and introduce a bequest motive into the model. It is worth mentioning

<sup>5</sup>Our results are not sensitive to the value of  $p$ .

that in the model with bequests proposed by [Lockwood \(2012\)](#), it has been found that there is little incentive to delay annuity purchases. The way we model a bequest motive is analogous to those considered in the literature.

We suppose that, apart from short-term annuities, the agent of age 1 who was born at date  $t$  can purchase risk-free bonds,  $z_{1,t+1} \geq 0$ , to leave a bequest in case of early death. Bequests are left to each of her  $1+n$  children and received at age 1. If the agent does not survive to age 2, the capitalized bequest  $z_{1,t+1}R_{t+2}$  is transferred to an offspring. If she survives, this amount can be used to finance own consumption or purchase further bonds,  $z_{2,t+2} \geq 0$ , which will be finally left as a bequest. As a result, the budget constraints at age 1 and 2 are modified as follows:

$$c_{1,t+1} + a_{1,t+1} + (1+n)z_{1,t+1} = z_t R_{t+1} + (1-p)z_{1,t}R_{t+1} + pz_{2,t+1}, \quad (19)$$

$$c_{2,t+2} + (1+n)z_{2,t+2} = a_{0,t} \frac{R_{t+2}}{\pi_{t+2}^L} + a_{1,t+1} \frac{R_{t+2}}{\pi_{t+2}^S} + (1+n)z_{1,t+1}R_{t+2}, \quad (20)$$

where  $z_{1,t}$  and  $z_{2,t+1}$  are the bequests received by the agent from her parent and taken by the former as given.

We adopt a “joy of giving” specification of the bequest motive, sometimes also referred to as impure or paternalistic altruism. The expected lifetime utility (5) is replaced by

$$u(c_{1,t+1}) + \theta pu(c_{2,t+2}) + \beta[(1-p)v(z_{1,t+1}R_{t+2}) + pv(z_{2,t+2})], \quad (21)$$

where  $v$  is utility from bequests and  $\beta \geq 0$  measures the degree of altruism. We assume that  $v$  is an increasing and strictly concave function with  $\lim_{x \rightarrow 0} v'(x) = +\infty$ . We obtain:

**Corollary 1.** *Proposition 1 still holds when agents have a bequest motive.*

*Proof.* See Appendix A.

Introducing a bequest motive leaves the timing of annuity purchases unchanged. The idea is that agents set aside the amount of bonds they want to bequeath and then finance consumption by annuitizing their wealth with the same strategy as described in the previous section.<sup>6</sup>

We also obtain that when annuities are purchased at age 1,  $z_1 R = z_2$  and thus annuitization exactly equals consumption at age 2. This is consistent with the findings of [Yaari \(1965\)](#) and [Davidoff et al. \(2005\)](#). However, when annuities are purchased at age 0, we have  $z_1 R < z_2$ , meaning that the capitalized value of the bequest increases with age. In this case, the level of annuitization is higher than the last period’s consumption.

### 3.2 Subjective evaluation of longevity risk

Another extension we consider is subjective beliefs about longevity. As discussed in [Sheshinski \(2008\)](#), overly pessimistic beliefs about survival probabilities may explain the

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<sup>6</sup>Note that the situation may be more complex in the model with dynastic altruism ([Barro, 1974](#)). This extension is beyond the scope of our paper (and barely studied in the literature on annuities).

observed low demand for annuities by young cohorts. Accordingly, suppose that agents in our model subjectively evaluate their survival probability as  $\tilde{p}$  that can be different from the actual  $p$ . The probability  $\tilde{p}$  is restricted to be identical across agents, although it is not necessary for the result below. **The agent's objective function** (5) becomes

$$u(c_{1,t+1}) + \theta\tilde{p}u(c_{2,t+2}). \quad (22)$$

On the other hand, we assume that annuity providers observe the actual survival probability, and the values of annuity premiums are based on the observed probability. Budget constraints are thus unchanged. We obtain the following result:

**Corollary 2.** *Proposition 1 still holds when agents subjectively evaluate their survival.*

*Proof.* See Appendix A.

The timing of annuity purchases is not affected by whether agents overestimate or underestimate their survival. Purchasing annuities early and/or late in life, the agent applies the same  $\tilde{p}$  to compute her expected lifetime utility. As a consequence, the subjective evaluation of longevity risk does not distort the choice between short- and long-term annuities. The intuition about the comparison of the two savings strategies, which was described in Section 2.5, remains valid.

### 3.3 Endogenous longevity

So far we have ignored the fact that the survival probability can be endogenous. Let us now suppose, similarly to [Dávila and Leroux \(2015\)](#), that it depends on the agent's observable effort. More precisely, suppose that  $p$  depends on longevity-enhancing expenditures  $e_t \geq 0$  (with  $p'(e_t) > 0$  and  $\lim_{e_t \rightarrow 0} p'(e_t) = +\infty$ ) spent by the agent at age 0. We can rewrite (5) and (6) as

$$u(c_{1,t+1}) + \theta p(e_t)u(c_{2,t+2}), \quad (23)$$

$$a_{0,t} + z_t + e_t = w_t. \quad (24)$$

Since  $e_t$  is observed by annuity providers at each date, we can also rewrite (4) as follows:

$$\pi_t^S(p(e_{t-2})) = p(e_{t-2}), \quad \pi_t^L(p(e_{t-2})) = \frac{p(e_{t-2})a_{0,t-2}}{(1+n)a_{0,t-1}}. \quad (25)$$

Correspondingly, the agent's budget constraint at age 2 becomes

$$c_{2,t+2} = a_{0,t} \frac{R_{t+2}}{\pi_{t+2}^L(p(e_t))} + a_{1,t+1} \frac{R_{t+2}}{\pi_{t+2}^S(p(e_t))}, \quad (26)$$

that is, the agent is aware that annuity premiums depend on her choice of  $e_t$  and anticipates its effect. We maintain, however, the assumption that she does not anticipate the effect of  $a_{0,t}$  on the pricing of annuities. In this setting, we have:

**Corollary 3.** *Proposition 1 still holds when longevity is endogenous.*

*Proof.* See Appendix A.

Our previous result holds true for any value of  $e$  chosen in the steady state. Intuitively, as with subjective survival probability, agents apply the same  $p(e_t)$  when deciding between short- and long-term annuities. Annuity premiums are also influenced by the same survival probability for a given observed  $e_t$ , and so there is no distortion in the optimal timing of annuity purchases.

### 3.4 Irreversibility of annuity purchases

Milevsky and Young (2007) have argued that when the purchase of an annuity is irreversible, individuals have an incentive to delay it. Following d’Albis and Etner (2018), we explore this idea by assuming that annuities pay a return without recovering the capital invested until the maximum possible age. To address the issue of timing, we extend the individual’s lifetime to one more period, i.e., the maximum possible age is now 3. The expected lifetime utility of the agent born at date  $t$  takes the form

$$u(c_{1,t+1}) + \theta pu(c_{2,t+2}) + \theta^2 p^2 u(c_{3,t+3}). \quad (27)$$

Here, we make an assumption of equal probabilities of surviving to the next period at age 1 and 2, which allows us to keep the intuition behind the results transparent. We further assume that at age 2, the agent can purchase standard annuities (or life insurance) to smooth consumption during the last two periods of life. Then, the budget constraints at age 2 and 3 are given by

$$c_{2,t+2} + a_{2,t+2} = a_{0,t} \left( \frac{R_{t+2}}{\pi_{t+2}^L} - 1 \right) + a_{1,t+1} \left( \frac{R_{t+2}}{\pi_{t+2}^S} - 1 \right), \quad (28)$$

$$c_{3,t+3} = a_{0,t} \frac{R_{t+3}}{\pi_{t+3}^L} + a_{1,t+1} \frac{R_{t+3}}{\pi_{t+3}^S} + a_{2,t+2} \frac{R_{t+3}}{p}, \quad (29)$$

where  $a_{2,t+2}$  denotes the demand for annuities (or, if negative, the demand for life insurance) at age 2.

As before, annuity premiums  $1/\pi_t^S$  and  $1/\pi_t^L$  are determined such that annuity providers make zero profit. It is readily verified that the respective zero-profit conditions are

$$(N_{1,t-1}a_{1,t-1} + N_{2,t-1}a_{1,t-2})R_t = (N_{2,t}a_{1,t-1} + N_{3,t}a_{1,t-2})\frac{R_t}{\pi_t^S}, \quad (30)$$

$$(N_{0,t-1}a_{0,t-1} + N_{2,t-1}a_{0,t-3})R_t = (N_{2,t}a_{0,t-2} + N_{3,t}a_{0,t-3})\frac{R_t}{\pi_t^L}. \quad (31)$$

Taking into account the population dynamics described by  $N_{3,t+3} = pN_{2,t+2} = p^2N_{1,t+1} = p^2N_{0,t}$  and (1), these conditions lead to

$$\pi_t^S = p, \quad \pi_t^L = \frac{(1+n)pa_{0,t-2} + p^2a_{0,t-3}}{(1+n)^2a_{0,t-1} + pa_{0,t-3}}. \quad (32)$$

Compared to the basic model, the short-term annuity premium is unchanged, whereas the long-term annuity premium at date  $t$  additionally depends on the amount of annuities held by generation  $t-3$ . The amount  $N_{2,t-1}a_{0,t-3}$  is distributed among generation  $t-3$  as well as generation  $t-2$ . Thus, long-term annuities redistribute resources not only from the young to the old but also from the old to the young. The direction of the net transfer is determined in equilibrium. We obtain the following:

**Corollary 4.** *Provided that  $n \geq 0$ , Proposition 1 still holds when annuity purchases are irreversible.*

*Proof.* See Appendix A.

Agents compare the profit from investing a unit of consumption in bonds and short-term annuities with the profit from investing in long-term annuities, computing the expected discounted sum of annuity payments over the last two periods of life. It turns out that a profit comparison between the two savings strategies is still equivalent to a comparison between  $f'(k)$  and  $\delta+n$ . If  $f'(k)$  is smaller (greater) than  $\delta+n$ , redistribution from the young to the old associated with the purchase of long-term annuities makes this strategy more (less) attractive than the strategy with short-term annuities. At the same time, redistribution from the old to the young makes it less (more) attractive. For positive population growth, the first effect dominates, and we replicate previous findings. For negative population growth, we show in the proof that there exists another threshold, in addition to  $\delta+n$ . Then, at least in cases where the steady-state interest rate is positive (i.e., for  $R > 1$ ), agents postpone annuity purchases.

#### 4 Uncertain survival

Following Brugiavini (1993), we now suppose that neither agents nor annuity providers know the future survival probability with certainty when annuities are purchased at age 0. Between ages 0 and 1, the uncertainty is resolved. To simplify the analysis, we assume that the uncertainty is resolved for both parties and that the survival probability can take only two values,  $p^l$  with probability  $\gamma \in (0, 1)$  and  $p^h > p^l$  with probability  $1 - \gamma$ . Let  $p$  denote the expected survival probability:

$$p := \gamma p^l + (1 - \gamma)p^h. \quad (33)$$

Given the information structure, the behavior of agents at age 0 must be the same for both survival types, which can be denoted by upper index  $j \in \{l, h\}$ . It is clear that  $a_{0,t}$  does not vary with  $j$ . Conversely, at age 1, agents may behave differently depending on

their type. Because annuity providers observe  $a_{1,t+1}^j$  for each  $j$ , the return on short-term annuities will also depend on  $j$ . Assuming competitive annuity markets, we then obtain

$$\pi_t^{Sl} = p^l, \quad \pi_t^{Sh} = p^h, \quad \pi_t^L = \frac{pa_{0,t-2}}{(1+n)a_{0,t-1}}, \quad (34)$$

where  $p$  is given by (33) due to the initial uncertainty faced by annuity providers. When deciding on the amount of long-term annuities  $a_{0,t}$  to purchase, the agent of generation  $t$  maximizes the expected utility given by

$$\gamma[u(c_{1,t+1}^l) + \theta p^l u(c_{2,t+2}^l)] + (1-\gamma)[u(c_{1,t+1}^h) + \theta p^h u(c_{2,t+2}^h)]. \quad (35)$$

She does so at age 0, anticipating choosing  $a_{1,t+1}^j$  optimally at age 1 after the uncertainty resolution.

Since the production side of the economy remains unchanged, the process of capital accumulation remains unchanged as well. The following result generalizes Proposition 1 to an uncertain survival setting.

**Proposition 2.** *In the steady-state equilibrium, the demand for annuities satisfies*

$$\begin{cases} a_0 > 0 \quad \text{and} \quad a_1^h = a_1^l = 0 & \text{if } f'(k) \leq \delta + n, \\ a_0 \geq 0 \quad \text{and} \quad a_1^h > a_1^l > 0 & \text{if } \delta + n < f'(k) < \delta + \frac{p^h - p + p^h n}{p}, \\ a_0 = 0 \quad \text{and} \quad a_1^h > a_1^l > 0 & \text{if } f'(k) \geq \delta + \frac{p^h - p + p^h n}{p}. \end{cases} \quad (36)$$

*Proof.* See Appendix A.

Let us first note that if  $p^l = p^h$ , the intermediate case disappears and we are back to Proposition 1, which turns out to be a special case of Proposition 2. We can interpret the latter in terms of dynamic efficiency, as we did in Section 2. In particular, if the economy is dynamically inefficient, that is, if  $f'(k) < \delta + n$ , agents of both survival types do not purchase annuities at age 1. It is only when the economy is efficient that they can delay their annuity purchases. More precisely, if  $f'(k) > \delta + n$ , agents always purchase annuities at age 1 and may also purchase annuities at age 0 as long as the interest rate is not too large. If  $f'(k)$  is sufficiently larger than  $\delta + n$ , annuity purchases are postponed by all agents, regardless of their survival type.

It is interesting to compare our results with those obtained in Brugiavini (1993). According to the prediction of her model, agents should purchase annuities when young and do not undertake further transactions. This is consistent with our results since annuity pricing used by Brugiavini (1993) is a special case of our annuity pricing: it corresponds to the Golden Rule where  $f'(k) = \delta + n$ . We thus show that Brugiavini's result also holds in a more general setting, but given that the economy is dynamically inefficient. Conversely, for a dynamically efficient economy (except for the Golden Rule), her result does not hold.

## 5 Discussion

In this section, we discuss two additional issues that are relevant to our approach and results. We begin with the issue related to the zero-profit conditions (2) and (3). In our basic model, annuity providers operate in two separate competitive markets with free entry. As a consequence, we eliminate the possibility of a positive profit for each contract type and hence eliminate the possibility of cross-subsidization between short- and long-term annuities. Alternatively, we could allow for such cross-subsidization by assuming that both types of annuities are sold in the same (competitive) market. The requirement of zero profit in this case applies to the sum of both annuity contracts. Then, if their premiums may for some reason be different, the profit from one of the two contracts can be permanently negative. In a free-market environment, annuity providers should not sell that contract and focus on the other one generating permanent positive profit, which cannot be sustained in a competitive economy. Our intuition is that we should converge to a situation where both contracts yield zero profits to annuity providers. On the other hand, if premiums for a short- and long-term contract are the same, it is straightforward to show that agents never demand annuities at age 0 unless the economy is characterized by negative interest rates. That said, the fact that the premiums are the same seems to be inconsistent with observations from actual annuity markets: long-term (deferred) annuities are in practice more profitable than short-term (immediate) annuities.

Another issue concerns the dynamic efficiency. Throughout the analysis, we highlight its role in explaining the timing of annuity purchases. However, what if some factors may affect the efficiency? For instance, it is well known (see, e.g., [Blanchard and Fischer \(1989\)](#)) that introducing money or government debt into an OLG framework can lead to a Pareto improvement when the economy is dynamically inefficient. We therefore expect that with such additional features, the postponement of annuity purchases is a generic outcome of the OLG model we propose. To verify this, let us come back to our basic setup and focus on the effects of money. We denote by  $M_{0,t} \geq 0$  and  $M_{1,t} \geq 0$  the amount of money held at date  $t$  by the agent of age 0 and 1, respectively. The price level at date  $t$  is denoted by  $q_t$ . Then, we rewrite the budget constraints (6)-(8) as

$$a_{0,t} + z_t + \frac{M_{0,t}}{q_t} = w_t, \quad (37)$$

$$c_{1,t+1} + a_{1,t+1} + \frac{M_{1,t+1}}{q_{t+1}} = z_t R_{t+1} + \frac{M_{0,t}}{q_{t+1}}, \quad (38)$$

$$c_{2,t+2} = a_{0,t} \frac{R_{t+2}}{\pi_{t+2}^L} + a_{1,t+1} \frac{R_{t+2}}{\pi_{t+2}^S} + \frac{M_{1,t+1}}{q_{t+2}}. \quad (39)$$

We also need to ensure that the following money market-clearing condition holds:

$$N_{0,t}M_{0,t} + N_{1,t}M_{1,t} = N_{1,t}M_{0,t-1} + N_{2,t}M_{1,t-1}. \quad (40)$$

Its derivation can be found in Appendix C, which provides a detailed examination of the model with money. As we show there, adding money has two important implications. First, money can only be demanded by agents when young, so that annuities dominate the intrinsically useless asset; it is, moreover, only demanded at the Golden Rule, in line with the findings of [Dávila and Leroux \(2015\)](#). Second, the postponement of annuity purchases cannot be excluded: it necessarily occurs for  $f'(k) > \delta + n$  and may occur at the Golden Rule. The earlier case for which delaying was not optimal (i.e.,  $f'(k) < \delta + n$ ) no longer exists. This reinforces our results.

## 6 Conclusion

In this paper, we address the question of why annuity purchases are postponed to a later age, as suggested by empirical evidence. We develop an overlapping generations model with short- and long-term annuity contracts that agents purchase late and early in life, respectively. Under the assumption of perfectly competitive annuity markets, annuity premiums in our model are determined by the zero-profit condition for each contract type offered by annuity providers. Then, the short-term annuity is the standard annuity that does not redistribute resources between cohorts and whose premium is equal to the inverse of the survival probability. In contrast, the purchase of the long-term annuity involves redistribution from the young to the old, and thus its premium also depends on the rate of population growth.

Our results can be summarized as follows. When the economy is described as dynamically inefficient, agents demand annuities only early in life, whereas for a dynamically efficient economy, annuities are demanded late in life. This result remains robust to the introduction of a bequest motive, a subjective/endogenous survival probability as well as the consideration of irreversible annuity contracts. Furthermore, if the information about longevity is initially uncertain, we show that in a dynamically efficient economy, it is still possible that all agents postpone the purchase of annuities to an older age.

This study can be extended in several ways by relaxing some assumptions of the model. In particular, it would be interesting to consider unknown survival probabilities ([d’Albis and Thibault, 2018](#)) and more sophisticated portfolio decisions with joint purchase of risky assets ([Jang et al., 2016](#)) or long-term care insurance ([Glenzer et al., 2019](#)). Our framework with uncertain survival can be also extended to the asymmetric information case ([Heijdra et al., 2019](#)).

## Appendix A Proofs

### *Proof of Proposition 1*

Applying (4), the first-order conditions (10) at the steady state can be written as

$$\begin{cases} u'(c_1)R - \theta u'(c_2)(1+n)R = \lambda, \\ u'(c_1) - \theta u'(c_2)R = \mu. \end{cases} \quad (\text{A.1})$$



The complementary slackness conditions (11) at the steady state are:

$$\lambda a_0 = 0, \quad \mu a_1 = 0. \quad (\text{A.2})$$

For  $\lambda > 0$  and  $\mu > 0$ , we have  $a_0 = 0$  and  $a_1 = 0$ . By the budget constraint (8) evaluated at the steady state,  $c_2 = 0$ , which cannot be optimal due to  $\lim_{x \rightarrow 0} u'(x) = +\infty$ . Hence, there are three possible cases: (i)  $\lambda = 0$  and  $\mu > 0$ , (ii)  $\lambda > 0$  and  $\mu = 0$ , (iii)  $\lambda = 0$  and  $\mu = 0$ .

Next, combining equations in system (A.1), we obtain

$$\theta u'(c_2)R[R - (1 + n)] - \lambda + \mu R = 0, \quad (\text{A.3})$$

which together with (12) becomes

$$\theta u'(c_2)(1 + f'(k) - \delta)[f'(k) - (\delta + n)] - \lambda + \mu(1 + f'(k) - \delta) = 0. \quad (\text{A.4})$$

Since  $u$  and  $f$  are strictly increasing functions and  $\delta \in [0, 1]$ ,  $f'(k) < \delta + n$  implies  $\mu(1 + f'(k) - \delta) > \lambda$ . It follows from  $\lambda \geq 0$  that  $\mu > 0$  and therefore  $\lambda = 0$ . Conversely, if  $f'(k) > \delta + n$ , then  $\mu(1 + f'(k) - \delta) < \lambda$ . As  $\mu \geq 0$ ,  $\lambda > 0$  and therefore  $\mu = 0$ . Finally, if  $f'(k) = \delta + n$ , then  $\mu(1 + f'(k) - \delta) = \lambda$  implying that  $\lambda = \mu = 0$  is the only possible case. Thus, we have:

$$\begin{cases} \lambda = 0, \quad \mu > 0 & \text{if } f'(k) < \delta + n, \\ \lambda > 0, \quad \mu = 0 & \text{if } f'(k) > \delta + n, \\ \lambda = 0, \quad \mu = 0 & \text{if } f'(k) = \delta + n. \end{cases} \quad (\text{A.5})$$

Taking into account (A.2) and the fact that  $a_0 = a_1 = 0$  cannot be optimal, this proves Proposition 1.  $\square$

### *Proof of Corollary 1*

Let the agent maximize (21) subject to (6), (19), (20), and (9). The first-order conditions are as follows:

$$\begin{cases} u'(c_{1,t+1})R_{t+1} - \theta p u'(c_{2,t+2}) \frac{R_{t+2}}{\pi_{t+2}^L} = \lambda_t, \\ u'(c_{1,t+1}) - \theta p u'(c_{2,t+2}) \frac{R_{t+2}}{\pi_{t+2}^S} = \mu_{t+1}, \\ u'(c_{1,t+1})(1 + n) - \theta p u'(c_{2,t+2})(1 + n)R_{t+2} - \beta(1 - p)v'(z_{1,t+1}R_{t+2})R_{t+2} = 0, \\ \theta u'(c_{2,t+2})(1 + n) - \beta v'(z_{2,t+2}) = 0. \end{cases} \quad (\text{A.6})$$

In the steady state, we have

$$\begin{cases} u'(c_1)R - \theta u'(c_2)(1 + n)R = \lambda, \\ u'(c_1) - \theta u'(c_2)R = \mu, \\ u'(c_1)(1 + n) - \theta p u'(c_2)(1 + n)R - \beta(1 - p)v'(z_1R)R = 0, \\ \theta u'(c_2)(1 + n) - \beta v'(z_2) = 0. \end{cases} \quad (\text{A.7})$$

Furthermore, (A.2) holds. From (A.7), it is clear that (A.3) also holds.

To complete the proof, it suffices to show that the case in which  $\lambda > 0$  and  $\mu > 0$  is not possible. Combining the last three equations in (A.7) and using (12), we obtain

$$\beta(1-p)[v'(z_1(1+f'(k)-\delta)) - v'(z_2)](1+f'(k)-\delta) - \mu(1+n) = 0. \quad (\text{A.8})$$

Note that  $1+f'(k)-\delta > 0$  and  $\mu \geq 0$  imply  $v'(z_1(1+f'(k)-\delta)) \geq v'(z_2)$ . Then  $z_1(1+f'(k)-\delta) \leq z_2$  by the concavity of  $v$ . For  $\lambda > 0$  and  $\mu > 0$ , the demand for annuities is zero and hence  $c_2 = (1+n)[z_1(1+f'(k)-\delta) - z_2]$ . The non-negativity of consumption and  $z_1(1+f'(k)-\delta) \leq z_2$  require  $z_1 = z_2 = 0$ , which cannot be optimal since  $\lim_{x \rightarrow 0} v'(x) = +\infty$ .  $\square$

*Proof of Corollary 2*

Suppose that the expected lifetime utility to maximize is given by (22). Then, the first-order conditions (10) hold with  $p = \hat{p}$ , leading to the following system in the steady state:

$$\begin{cases} u'(c_1)R - \frac{\hat{p}}{p}\theta u'(c_2)(1+n)R = \lambda, \\ u'(c_1) - \frac{\hat{p}}{p}\theta u'(c_2)R = \mu. \end{cases} \quad (\text{A.9})$$

We thus obtain

$$\frac{\hat{p}}{p}\theta u'(c_2)R[R - (1+n)] - \lambda + \mu R = 0. \quad (\text{A.10})$$

The rest of the proof is identical to that of Proposition 1.  $\square$

*Proof of Corollary 3*

Suppose that the agent chooses  $(c_{1,t+1}, c_{2,t+2}, z_t, a_{0,t}, a_{1,t+1}, e_t)$  that maximizes (23) subject to (24), (7), (26), and (9). The first-order conditions (10) can be rewritten as

$$\begin{cases} u'(c_{1,t+1})R_{t+1} - \theta p(e_t)u'(c_{2,t+2})\frac{R_{t+2}}{\pi_{t+2}^L(p(e_t))} = \lambda_t, \\ u'(c_{1,t+1}) - \theta p(e_t)u'(c_{2,t+2})\frac{R_{t+2}}{\pi_{t+2}^S(p(e_t))} = \mu_{t+1}, \end{cases} \quad (\text{A.11})$$

and we also have

$$\begin{aligned} & u'(c_{1,t+1})R_{t+1} - \theta p'(e_t)u(c_{2,t+2}) \\ & + \theta p(e_t) \left[ \frac{\pi_{t+2}^{L'}(p(e_t))p'(e_t)}{[\pi_{t+2}^L(p(e_t))]^2} a_{0,t} + \frac{\pi_{t+2}^{S'}(p(e_t))p'(e_t)}{[\pi_{t+2}^S(p(e_t))]^2} a_{1,t+1} \right] u'(c_{2,t+2})R_{t+2} = 0. \end{aligned} \quad (\text{A.12})$$

Using (25) and considering the steady state, these three conditions become

$$\begin{cases} u'(c_1)R - \theta u'(c_2)(1+n)R = \lambda, \\ u'(c_1) - \theta u'(c_2)R = \mu, \\ u'(c_1)R - \theta p'(e) \left[ u(c_2) - (a_0(1+n) + a_1) u'(c_2) \frac{R}{p(e)} \right] = 0. \end{cases} \quad (\text{A.13})$$

Note that from the first two equations, it follows that condition (A.3) still holds. As a result, Proposition 1 is valid for any chosen  $e$ .  $\square$

*Proof of Corollary 4*

Suppose that the agent maximizes (27) subject to (6), (7), (28), (29), and (9). The first-order conditions of this problem are given by

$$\begin{cases} u'(c_{1,t+1})R_{t+1} - \theta pu'(c_{2,t+2}) \left( \frac{R_{t+2}}{\pi_{t+2}^L} - 1 \right) - \theta^2 p^2 u'(c_{3,t+3}) \frac{R_{t+3}}{\pi_{t+3}^L} = \lambda_t, \\ u'(c_{1,t+1}) - \theta pu'(c_{2,t+2}) \left( \frac{R_{t+2}}{\pi_{t+2}^S} - 1 \right) - \theta^2 p^2 u'(c_{3,t+3}) \frac{R_{t+3}}{\pi_{t+3}^S} = \mu_{t+1}, \\ u'(c_{2,t+2}) - \theta Ru'(c_{3,t+3}) = 0. \end{cases} \quad (\text{A.14})$$

Taking into account (32), the steady-state counterpart of the system is

$$\begin{cases} u'(c_1)R - \theta pu'(c_2) \left( R \frac{(1+n)^2+p}{(1+n+p)p} - 1 \right) - \theta^2 pu'(c_3)R \frac{(1+n)^2+p}{1+n+p} = \lambda, \\ u'(c_1) - \theta pu'(c_2) \left( \frac{R}{p} - 1 \right) - \theta^2 pu'(c_3)R = \mu, \\ u'(c_2) - \theta u'(c_3)R = 0. \end{cases} \quad (\text{A.15})$$

The complementary slackness conditions (A.2) hold as before. Combining equations in (A.15), we obtain

$$\theta pu'(c_2) \left[ \frac{R^2}{p} - \left( R \frac{(1+n)^2+p}{(1+n+p)p} - 1 + \frac{(1+n)^2+p}{1+n+p} \right) \right] - \lambda + \mu R = 0. \quad (\text{A.16})$$

It is straightforward to verify that this equation is equivalent to

$$\theta u'(c_2) \left( R + \frac{np}{1+n+p} \right) [R - (1+n)] - \lambda + \mu R = 0. \quad (\text{A.17})$$

Then, using  $R = 1 + f'(k) - \delta > 0$  and  $n \geq 0$ , we can apply the same reasoning as in the proof of Proposition 1.

To conclude, we show that the case in which  $\lambda > 0$  and  $\mu > 0$  is not possible. Indeed, this would imply  $c_2 = -a_2$  and  $c_3 = a_2(1 + f'(k) - \delta)/p$ . However, by the non-negativity of consumption,  $a_2$  must be equal to 0 and thus  $c_2 = c_3 = 0$ , which cannot be optimal as  $\lim_{x \rightarrow 0} u'(x) = +\infty$ .  $\square$

*Proof of Proposition 2*

The optimal demand for annuities  $(a_{0,t}, a_{1,t+1}^j)$  for a given realized survival type  $j \in \{l, h\}$  is solved by backward induction. At age 1, the agent of each type  $j$  takes  $a_{0,t}$  (and  $z_t$ ) as given and chooses  $a_{1,t+1}^j \geq 0$  that maximizes

$$u(c_{1,t+1}^j) + \theta p^j u(c_{2,t+1}^j) \quad (\text{A.18})$$

subject to the budget constraints given by

$$c_{1,t+1}^j = w_t R_{t+1} - a_{0,t} R_{t+1} - a_{1,t+1}^j, \quad (\text{A.19})$$

$$c_{2,t+2}^j = a_{0,t} \frac{R_{t+2}}{\pi_{t+2}^L} + a_{1,t+1}^j \frac{R_{t+2}}{\pi_{t+2}^S}. \quad (\text{A.20})$$

At age 0, the agent chooses  $a_{0,t} \geq 0$  that maximizes (35) subject to (A.19) and (A.20). The respective first-order conditions can be written as

$$\begin{cases} \gamma \left[ u'(c_{1,t+1}^l) R_{t+1} - \theta p^l u'(c_{2,t+2}^l) \frac{R_{t+2}}{\pi_{t+2}^l} \right] + (1 - \gamma) \left[ u'(c_{1,t+1}^h) R_{t+1} - \theta p^h u'(c_{2,t+2}^h) \frac{R_{t+2}}{\pi_{t+2}^h} \right] = \lambda_t, \\ u'(c_{1,t+1}^j) - \theta p^j u'(c_{2,t+2}^j) \frac{R_{t+2}}{\pi_{t+2}^j} = \mu_{t+1}^j, \quad j = l, h, \end{cases} \quad (\text{A.21})$$

while the complementary slackness conditions associated with  $a_{0,t} \geq 0$  and  $a_{1,t+1}^j \geq 0$  can be written as

$$\lambda_t a_{0,t} = 0, \quad \mu_{t+1}^j a_{1,t+1}^j = 0, \quad j = l, h. \quad (\text{A.22})$$

In the steady state, together with (34), the budget constraints (A.19) and (A.20) become

$$c_1^j = wR - a_0R - a_1^j, \quad j = l, h, \quad (\text{A.23})$$

$$c_2^j = a_0 \frac{R(1+n)}{p} + a_1^j \frac{R}{p^j}, \quad j = l, h, \quad (\text{A.24})$$

the first-order conditions (A.21) become

$$\begin{cases} \gamma \left[ u'(c_1^l) - \theta p^l u'(c_2^l) \frac{1+n}{p} \right] R + (1 - \gamma) \left[ u'(c_1^h) - \theta p^h u'(c_2^h) \frac{1+n}{p} \right] R = \lambda, \\ u'(c_1^j) - \theta u'(c_2^j) R = \mu^j, \quad j = l, h, \end{cases} \quad (\text{A.25})$$

and the complementary slackness conditions (A.22) become

$$\lambda a_0 = 0, \quad \mu^j a_1^j = 0, \quad j = l, h. \quad (\text{A.26})$$

To prove Proposition 2, we use the following lemma.

**Lemma 1.** *There are only three possible cases in the steady state: (i)  $a_0 > 0$  and  $a_1^h = a_1^l = 0$ , (ii)  $a_0 > 0$  and  $a_1^h > a_1^l > 0$ , (iii)  $a_0 = 0$  and  $a_1^h > a_1^l > 0$ .*

*Proof.* We proceed in four steps. First, we show that  $a_1^h \geq a_1^l$ . Suppose, by contradiction, that  $a_1^l > a_1^h$ . It is clear from (A.23) that  $c_1^l < c_1^h$  and hence  $u'(c_1^l) > u'(c_1^h)$ , since the function  $u$  is strictly concave. As  $p^h > p^l$ , we also have  $a_1^l/p^l > a_1^h/p^h$ . Then, using (A.24), we obtain  $c_2^l > c_2^h$  and  $u'(c_2^l) < u'(c_2^h)$ . Applying the second equation of system (A.25) with  $u'(c_1^l) > u'(c_1^h)$  and  $u'(c_2^l) < u'(c_2^h)$  yields  $\mu^l > \mu^h$ . Thus, due to  $\mu^h \geq 0$ , we have  $\mu^l > 0$ , which, by (A.26), implies  $a_1^l = 0$ . This is a contradiction of  $a_1^l > a_1^h$  because  $a_1^h \geq 0$ .

Second, we show that  $a_1^h = a_1^l$  is possible only if  $a_1^h = a_1^l = 0$ . Indeed, if  $a_1^h = a_1^l$ , then  $c_1^h = c_1^l$  and  $c_2^h < c_2^l$  (see (A.23) and (A.24)), implying  $u'(c_1^h) = u'(c_1^l)$  and  $u'(c_2^h) > u'(c_2^l)$ . By the second equation of (A.25), we further have  $\mu^l > \mu^h$ , which leads to  $a_1^l = 0$ .

Third, we show that  $a_1^h > 0$  and  $a_1^l = 0$  is not possible. In this case, by (A.23) and (A.24), we would have  $c_1^h < c_1^l$  and  $c_2^h > c_2^l$ . Therefore  $u'(c_1^h) > u'(c_1^l)$  and  $u'(c_2^h) < u'(c_2^l)$ . Once again, using the second equation of (A.25), we obtain  $\mu^h > \mu^l$ . Then  $a_1^h = 0$ , which is a contradiction.

Fourth,  $a_0 = a_1^j = 0$  for  $j = l, h$  is not possible by assumption  $\lim_{x \rightarrow 0} u'(x) = +\infty$ .  $\square$

By Lemma 1, it is sufficient to consider three cases: (i)  $a_0 > 0$  and  $a_1^h = a_1^l = 0$ , (ii)  $a_0 > 0$  and  $a_1^h > a_1^l > 0$ , (iii)  $a_0 = 0$  and  $a_1^h > a_1^l > 0$ . To do this, let us combine equations in system (A.25),

applying (12):

$$A(f'(k)) + \left( \gamma \frac{p^l}{p} \mu^l + (1 - \gamma) \frac{p^h}{p} \mu^h \right) (1 + n) - \lambda = 0, \quad (\text{A.27})$$

where

$$\begin{aligned} A(f'(k)) := & \gamma u'(c_1^l) \left[ f'(k) - \left( \delta + \frac{p^l}{p} (1 + n) - 1 \right) \right] \\ & + (1 - \gamma) u'(c_1^h) \left[ f'(k) - \left( \delta + \frac{p^h}{p} (1 + n) - 1 \right) \right]. \end{aligned} \quad (\text{A.28})$$

When  $a_0 > 0$  and  $a_1^h = a_1^l = 0$ , one has  $\lambda = 0$  and thus, due to  $\mu^j \geq 0$ ,  $A(f'(k)) \leq 0$ . Furthermore, for  $a_1^h = a_1^l = 0$ , the budget constraint (A.23) implies  $c_1^h = c_1^l$ . Therefore  $A(f'(k)) \leq 0$  holds with  $u'(c_1^h) = u'(c_1^l)$ . From this we obtain  $f'(k) \leq \delta + n$ .

When  $a_0 > 0$  and  $a_1^h > a_1^l > 0$ , one has  $\lambda = 0$  and  $\mu^j = 0$  for all  $j = l, h$ . Then  $A(f'(k)) = 0$ , which can only be satisfied if  $f'(k) = \delta + n$ . Moreover, it follows from (A.23) with  $a_1^h > a_1^l$  that  $c_1^h < c_1^l$  and hence  $u'(c_1^h) > u'(c_1^l)$  (by the concavity of  $u$ ). For  $f'(k) < \delta + (1 + n)p^h/p - 1$ , this implies

$$\begin{aligned} \frac{A(f'(k))}{u'(c_1^l)} & < \gamma \left[ f'(k) - \left( \delta + \frac{p^l}{p} (1 + n) - 1 \right) \right] \\ & + (1 - \gamma) \left[ f'(k) - \left( \delta + \frac{p^h}{p} (1 + n) - 1 \right) \right], \end{aligned} \quad (\text{A.29})$$

which is equivalent to  $f'(k) > \delta + n$ , if  $A(f'(k)) = 0$ . As a result, the case in which  $a_0 > 0$  and  $a_1^h > a_1^l > 0$  is possible only if  $\delta + n < f'(k) < \delta + (1 + n)p^h/p - 1$ .

Finally, when  $a_0 = 0$  and  $a_1^h > a_1^l > 0$ , one has  $\mu^j = 0$  for all  $j = l, h$ . Then, as  $\lambda \geq 0$ ,  $A(f'(k)) \geq 0$ , so that  $f'(k) > \delta + (1 + n)p^l/p - 1$  necessarily holds. Since  $a_1^h > a_1^l$ , we again have  $u'(c_1^h) > u'(c_1^l)$ , which for  $f'(k) > \delta + (1 + n)p^l/p - 1$  leads to

$$\begin{aligned} \frac{A(f'(k))}{u'(c_1^h)} & < \gamma \left[ f'(k) - \left( \delta + \frac{p^l}{p} (1 + n) - 1 \right) \right] \\ & + (1 - \gamma) \left[ f'(k) - \left( \delta + \frac{p^h}{p} (1 + n) - 1 \right) \right]. \end{aligned} \quad (\text{A.30})$$

Using  $A(f'(k)) \geq 0$ , we obtain that this inequality holds only if  $f'(k) > \delta + n$ .

To sum up, we have:

$$\begin{cases} a_0 > 0, a_1^h = a_1^l = 0 & \text{only if } f'(k) \leq \delta + n, \\ a_0 > 0, a_1^h > a_1^l > 0 & \text{only if } \delta + n < f'(k) < \delta + \frac{p^h - p + p^h n}{p}, \\ a_0 = 0, a_1^h > a_1^l > 0 & \text{only if } f'(k) > \delta + n, \end{cases} \quad (\text{A.31})$$

which proves (36). □

## Appendix B Derivation of equation (17)

Consider the steady-state equilibrium characterized by (A.1), (A.2), and

$$\begin{cases} z = w - a_0, \\ c_1 = zR - a_1, \\ c_2 = a_0 \frac{R(1+n)}{p} + a_1 \frac{R}{p}, \\ k = \frac{z+a_0}{1+n} + \frac{a_1}{(1+n)^2}. \end{cases} \quad (\text{B.1})$$

Applying (12) and (16), we get

$$\begin{cases} z = (1 - \alpha)k^\alpha - a_0, \\ k = \frac{(1-\alpha)k^\alpha}{1+n} + \frac{a_1}{(1+n)^2}, \\ \frac{1+\alpha k^{\alpha-1}-\delta}{z(1+\alpha k^{\alpha-1}-\delta)-a_1} - \frac{\theta p(1+n)}{a_0(1+n)+a_1} = \lambda, \\ \frac{1}{z(1+\alpha k^{\alpha-1}-\delta)-a_1} - \frac{\theta p}{a_0(1+n)+a_1} = \mu. \end{cases} \quad (\text{B.2})$$

From Proposition 1, we know that there are three possible cases: (i)  $a_0 > 0$  and  $a_1 = 0$ , (ii)  $a_0 > 0$  and  $a_1 > 0$ , (iii)  $a_0 = 0$  and  $a_1 > 0$ .

For  $a_0 > 0$  and  $a_1 = 0$ , one has  $\lambda = 0$  and  $\mu \geq 0$ . Then, using the first three equations of (B.2), we obtain

$$\begin{cases} z = (1 - \alpha)k^\alpha - a_0, \\ k = \frac{(1-\alpha)k^\alpha}{1+n}, \\ a_1 = 0, \\ \frac{1}{z} - \frac{\theta p}{a_0} = 0, \end{cases} \quad (\text{B.3})$$

which leads to

$$\begin{cases} z = \frac{1}{1+\theta p}(1 - \alpha)k^\alpha, \\ a_0 = \frac{\theta p}{1+\theta p}(1 - \alpha)k^\alpha, \\ a_1 = 0, \\ k = \left(\frac{1-\alpha}{1+n}\right)^{\frac{1}{1-\alpha}}. \end{cases} \quad (\text{B.4})$$

Combining this with the last equation of (B.2) gives

$$\begin{aligned} \mu &= \frac{1 + \theta p}{(1 + \alpha k^{\alpha-1} - \delta)(1 - \alpha)k^\alpha} - \frac{1 + \theta p}{(1 + n)(1 - \alpha)k^\alpha} \\ &= \frac{1 + \theta p}{(1 - \alpha)^2 k^\alpha} \frac{n(1 - 2\alpha) - \alpha + \delta(1 - \alpha)}{(1 + \alpha \frac{1+n}{1-\alpha} - \delta)(1 + n)}. \end{aligned} \quad (\text{B.5})$$

Clearly, the condition  $\mu \geq 0$  holds if and only if  $n \geq \bar{n}$ .

For  $a_0 > 0$  and  $a_1 > 0$ , one has  $\lambda = 0$  and  $\mu = 0$ . Then, system (B.2) becomes

$$\begin{cases} z = (1 - \alpha)k^\alpha - a_0, \\ k = \frac{(1-\alpha)k^\alpha}{1+n} + \frac{a_1}{(1+n)^2}, \\ \frac{1}{z(1+\alpha k^{\alpha-1}-\delta)-a_1} - \frac{\theta p}{a_0(1+n)+a_1} = 0, \\ 1 + \alpha k^{\alpha-1} - \delta = 1 + n. \end{cases} \quad (\text{B.6})$$

Rearranging equations yields

$$\begin{cases} z = \frac{\alpha - \delta\sigma(1-\alpha) - n[\sigma(1-\alpha) - \alpha]}{\delta+n} k^\alpha, \\ a_0 = \frac{-\alpha + \delta(1+\sigma)(1-\alpha) + n[(1+\sigma)(1-\alpha) - \alpha]}{\delta+n} k^\alpha, \\ a_1 = \frac{1+n}{\delta+n} [\alpha - \delta(1-\alpha) - n(1-2\alpha)] k^\alpha, \\ k = \left(\frac{\alpha}{\delta+n}\right)^{\frac{1}{1-\alpha}}. \end{cases} \quad (\text{B.7})$$

Since  $a_0 > 0$  and  $a_1 > 0$ , we must also have

$$\frac{\alpha - \delta(1+\sigma)(1-\alpha)}{(1+\sigma)(1-\alpha) - \alpha} < n < \frac{\alpha - \delta(1-\alpha)}{1-2\alpha}, \quad (\text{B.8})$$

which corresponds to  $\underline{n} < n < \bar{n}$ .

Finally, for  $a_0 = 0$  and  $a_1 > 0$ , one has  $\lambda \geq 0$  and  $\mu = 0$ . Using the first two and the last equations of (B.2), we obtain

$$\begin{cases} z = (1 - \alpha)k^\alpha, \\ k = \frac{(1-\alpha)k^\alpha}{1+n} + \frac{a_1}{(1+n)^2}, \\ a_0 = 0, \\ \frac{1}{z(1+\alpha k^{\alpha-1}-\delta)-a_1} - \frac{\theta p}{a_1} = 0. \end{cases} \quad (\text{B.9})$$

This system can be written as

$$\begin{cases} z = (1 - \alpha)k^\alpha, \\ a_0 = 0, \\ a_1 = \sigma(1 - \alpha)(1 + \alpha k^{\alpha-1} - \delta)k^\alpha, \\ k = \left(\frac{1 + \sigma \frac{1-\delta}{1+n} + \sqrt{(1 + \sigma \frac{1-\delta}{1+n})^2 + \sigma \frac{4\alpha}{1-\alpha}}}{2(1+n)(1-\alpha)^{-1}}\right)^{\frac{1}{1-\alpha}}, \end{cases} \quad (\text{B.10})$$

where the expression for  $k$  is a unique positive solution to

$$\sigma\alpha(1 - \alpha)k^{2(\alpha-1)} + [1 + n + \sigma(1 - \delta)](1 - \alpha)k^{\alpha-1} - (1 + n)^2 = 0. \quad (\text{B.11})$$

Let us now show that the condition  $\lambda \geq 0$  holds if and only if  $n \leq \underline{n}$ . Indeed, applying the third equation of (B.2) with (B.10) and using the fact that  $\theta p = \sigma/(1 - \sigma)$ , we have

$$\begin{aligned} \lambda &= \frac{1}{(1 - \sigma)(1 - \alpha)k^\alpha} - \frac{\theta p(1 + n)}{\sigma(1 + \alpha k^{\alpha-1} - \delta)(1 - \alpha)k^\alpha} \\ &= \frac{2\alpha(1+n)(1-\alpha)^{-1}}{1 + \sigma \frac{1-\delta}{1+n} + \sqrt{(1 + \sigma \frac{1-\delta}{1+n})^2 + \sigma \frac{4\alpha}{1-\alpha}}} - \delta - n \\ &= \frac{2\alpha(1+n)(1-\alpha)^{-1}}{(1 - \sigma)(1 + \alpha k^{\alpha-1} - \delta)(1 - \alpha)k^\alpha}. \end{aligned} \tag{B.12}$$

Then, it is not difficult to check that

$$\frac{2\alpha(1+n)(1-\alpha)^{-1}}{1 + \sigma \frac{1-\delta}{1+n} + \sqrt{(1 + \sigma \frac{1-\delta}{1+n})^2 + \sigma \frac{4\alpha}{1-\alpha}}} - \delta - n \geq 0 \Leftrightarrow n \leq \underline{n}. \tag{B.13}$$

We conclude by noting that if  $n = \underline{n}$ , (B.7) is identical to (B.10), and if  $n = \bar{n}$ , (B.7) is identical to (B.4).

## Appendix C Analysis of the model with money

Suppose that the agent chooses  $(c_{1,t+1}, c_{2,t+2}, z_t, a_{0,t}, a_{1,t+1}, M_{0,t}, M_{1,t+1})$  that maximizes (5) subject to (37), (38), (39), (9), and

$$M_{0,t} \geq 0, M_{1,t+1} \geq 0. \tag{C.1}$$

The first-order conditions of this problem are

$$\begin{cases} u'(c_{1,t+1})R_{t+1} - \theta pu'(c_{2,t+2})\frac{R_{t+2}}{\pi_{t+2}^L} = \lambda_t, \\ u'(c_{1,t+1}) - \theta pu'(c_{2,t+2})\frac{R_{t+2}}{\pi_{t+2}^S} = \mu_{t+1}, \\ u'(c_{1,t+1})\left(\frac{R_{t+1}}{q_t} - \frac{1}{q_{t+1}}\right) = \tau_t, \\ u'(c_{1,t+1})\frac{1}{q_{t+1}} - \theta pu'(c_{2,t+2})\frac{1}{q_{t+2}} = \xi_{t+1}, \end{cases} \tag{C.2}$$

where  $\tau_t \geq 0$  and  $\xi_{t+1} \geq 0$  denote the Kuhn-Tucker multipliers associated with the non-negativity constraints (C.1), so that we must have

$$\tau_t M_{0,t} = 0, \xi_{t+1} M_{1,t+1} = 0. \tag{C.3}$$

To derive the money market-clearing condition (40), we use the resource constraint of the economy, written as

$$N_{1,t}c_{1,t} + N_{2,t}c_{2,t} + K_{t+1} = F(K_t, N_{0,t}) + (1 - \delta)K_t, \tag{C.4}$$



or, dividing both sides by  $N_{0,t}$ ,

$$\frac{c_{1,t}}{1+n} + \frac{pc_{2,t}}{(1+n)^2} + (1+n)k_{t+1} = f(k_t) + (1-\delta)k_t. \quad (\text{C.5})$$

Taking into account (4), let us apply the budget constraints (38) and (39). After rearranging terms, we have

$$\begin{aligned} f(k_t) + (1-\delta)k_t &= \left[ \frac{z_{t-1} + a_{0,t-1}}{1+n} + \frac{a_{1,t-1}}{(1+n)^2} \right] R_t - \frac{a_{1,t}}{1+n} + (1+n)k_{t+1} \\ &+ \frac{1}{1+n} \frac{M_{0,t-1}}{q_t} - \frac{1}{1+n} \frac{M_{1,t}}{q_t} + \frac{p}{(1+n)^2} \frac{M_{1,t-1}}{q_t}. \end{aligned} \quad (\text{C.6})$$

By (13), the expression in square brackets is equal to  $k_t$ . Then, applying (12) and (37) yields

$$a_{0,t} + z_t + \frac{M_{0,t}}{q_t} = -\frac{a_{1,t}}{1+n} + (1+n)k_{t+1} + \frac{1}{1+n} \frac{M_{0,t-1}}{q_t} - \frac{1}{1+n} \frac{M_{1,t}}{q_t} + \frac{p}{(1+n)^2} \frac{M_{1,t-1}}{q_t}. \quad (\text{C.7})$$

Once again, using (13), we eventually obtain

$$(1+n)^2 M_{0,t} + (1+n)M_{1,t} = (1+n)M_{0,t-1} + pM_{1,t-1}. \quad (\text{C.8})$$

Note that when multiplied by  $N_{0,t-2}$ , this corresponds to (40).

As before, we are interested in the steady state, which now additionally requires  $M_{0,t}/q_t = M_{0,t+1}/q_{t+1}$  and  $M_{1,t}/q_t = M_{1,t+1}/q_{t+1}$ . Together with (C.8), these two conditions imply

$$\frac{q_t}{q_{t+1}} = \frac{(1+n)[(1+n)M_{0,t+1} + M_{1,t+1}]}{(1+n)M_{0,t+1} + pM_{1,t+1}}. \quad (\text{C.9})$$

Furthermore, because of  $\tau_t \geq 0$  and  $\xi_{t+1} \geq 0$ , the following inequalities hold:

$$\begin{cases} u'(c_{1,t+1}) \left( \frac{R_{t+1}}{q_t} - \frac{1}{q_{t+1}} \right) \geq 0, \\ u'(c_{1,t+1}) \frac{1}{q_{t+1}} - \theta p u'(c_{2,t+2}) \frac{1}{q_{t+2}} \geq 0. \end{cases} \quad (\text{C.10})$$

From the first inequality, we deduce that  $R_{t+1} \geq q_t/q_{t+1}$ . In the steady-state equilibrium, substituting the expression for  $q_t/q_{t+1}$  and dividing both sides by  $1+n$ , we get

$$\frac{1 + f'(k) - \delta}{1+n} \geq \frac{(1+n)M_{0,t+1} + M_{1,t+1}}{(1+n)M_{0,t+1} + pM_{1,t+1}}. \quad (\text{C.11})$$

Since  $p \in (0, 1)$ , the right-hand side is greater or equal to 1 and thus the case where  $f'(k) < \delta + n$  no longer exists.

Next, let us show that having both  $\lambda > 0$  and  $\mu > 0$  is also not possible. In this case, to guarantee that  $c_2 > 0$ , we need  $M_{1,t+1} > 0$  and therefore  $\xi_{t+1} = 0$  implying, by the last equation in (C.2) evaluated at the steady state and (C.9),

$$\frac{u'(c_1)}{\theta p u'(c_2)} = \frac{q_{t+1}}{q_{t+2}} = \frac{(1+n)[(1+n)M_{0,t+2} + M_{1,t+2}]}{(1+n)M_{0,t+1} + pM_{1,t+2}}. \quad (\text{C.12})$$

Then, the condition  $\lambda = (1 + f'(k) - \delta)[u'(c_1) - \theta u'(c_2)(1 + n)] > 0$  is satisfied if and only if  $pM_{0,t+2} > M_{0,t+2}$ . This cannot be true for  $p \in (0, 1)$  and  $M_{0,t} \geq 0$ ,  $t \geq 0$ .

Finally, we consider two possible cases:  $f'(k) > \delta + n$  and  $f'(k) = \delta + n$ . In the first case, using the same argument as in the proof of Proposition 1, we obtain  $\lambda > 0$  and  $\mu = 0$ . Hence,  $\mu = u'(c_1) - \theta u'(c_2)(1 + f'(k) - \delta) = 0$ , which allows us to express  $\tau_t$  and  $\xi_t$  in the steady state as

$$\tau_t = \frac{u'(c_1)}{q_t}(1 + n) \left[ \frac{1 + f'(k) - \delta}{1 + n} - \frac{(1 + n)M_{0,t+1} + M_{1,t+1}}{(1 + n)M_{0,t+1} + pM_{1,t+1}} \right], \quad (\text{C.13})$$

$$\xi_t = \frac{\theta u'(c_2)}{q_t}(1 + n) \left[ \frac{1 + f'(k) - \delta}{1 + n} - \frac{p[(1 + n)M_{0,t+1} + M_{1,t+1}]}{(1 + n)M_{0,t+1} + pM_{1,t+1}} \right]. \quad (\text{C.14})$$

Note that

$$\frac{p((1 + n)M_{0,t+1} + M_{1,t+1})}{(1 + n)M_{0,t+1} + pM_{1,t+1}} \leq 1. \quad (\text{C.15})$$

Hence,  $\xi_t > 0$  when  $f'(k) > \delta + n$  and therefore  $M_{1,t} = 0$  for each  $t$ . Furthermore, for  $M_{1,t+1} = 0$ , it is clear that  $\tau_t > 0$  when  $f'(k) > \delta + n$  and therefore  $M_{0,t} = 0$  for each  $t$ .

In the case where  $f'(k) = \delta + n$ , we know from the proof of Proposition 1 that  $\lambda = \mu = 0$ . At the same time, for  $\mu = 0$  and  $f'(k) = \delta + n$ , the steady-state values of  $\tau_t$  and  $\xi_t$  are given by

$$\tau_t = \frac{u'(c_1)}{q_t}(1 + n) \left[ 1 - \frac{(1 + n)M_{0,t+1} + M_{1,t+1}}{(1 + n)M_{0,t+1} + pM_{1,t+1}} \right], \quad (\text{C.16})$$

$$\xi_t = \frac{\theta u'(c_2)}{q_t}(1 + n) \left[ 1 - \frac{p[(1 + n)M_{0,t+1} + M_{1,t+1}]}{(1 + n)M_{0,t+1} + pM_{1,t+1}} \right]. \quad (\text{C.17})$$

Then,  $M_{1,t+1} = 0$  for all  $t$  is the only possibility (due to  $\tau_t \geq 0$  for all  $t$ ). We thus have  $\tau_t = 0$  and  $\xi_t \geq 0$ .

Taking into account the complementary slackness conditions (A.2) and (C.3), we conclude that the steady-state demand for annuities and money should be such that

$$\begin{cases} a_0 = 0, a_1 > 0, M_{0,t} = 0, M_{1,t} = 0 & \text{if } f'(k) > \delta + n, \\ a_0 \geq 0, a_1 \geq 0, M_{0,t} \geq 0, M_{1,t} = 0 & \text{if } f'(k) = \delta + n. \end{cases} \quad (\text{C.18})$$

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