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► To cite this version:

| Alexander Afriat. Variations on a theme by Poincaré. 2020. halshs-03168193

HAL Id: halshs-03168193

<https://shs.hal.science/halshs-03168193>

Preprint submitted on 12 Mar 2021

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Variations on a theme by Poincaré

Alexander Afriat

February 17, 2020

Abstract

The theme is Poincaré's conventionalist freedom to do away with something along the lines of 'force' by absorbing it into the background geometry; this can amount to a choice between Newton's first two laws: inertia vs. force. The variations involve Newton-Cartan mechanics, general relativity, the Jacobi metric and symplectic mechanics with magnetism.

1 Introduction

One of Poincaré's main 'conventionalist' themes is that geometry itself (like so much else in physics: the measurement of time at a spot,¹ comparison of times over large distances,² the 'origin' in celestial mechanics³ *etc.*) is conventional inasmuch as experience *on its own* is not enough to choose between the geometries of Euclid, Lobachevsky, Riemann *etc.* Poincaré proposes the themes in *La science et l'hypothèse* (1902); then, to dispel the widespread misunderstanding that science is so conventional as to be mere whim, has to point out in *La valeur de la science* (1905) that the conventions are not arbitrary but natural—dictated by convenience, elegance, simplicity, economy *etc.* Poincaré often refers to light rays: if, for instance, the internal angles of a large optical triangle together exceed a straight angle, one can either reject Euclidean geometry or blame the light itself—which may have been deflected against a Euclidean background. Or

Si la géométrie de Lobatchevsky est vraie, la parallaxe d'une étoile très éloignée sera finie ; si celle de Riemann est vraie, elle sera négative. Ce sont là des résultats qui semblent accessibles à l'expérience et on a espéré que les observations astronomiques pourraient permettre de décider entre les trois géométries. Mais ce qu'on appelle ligne droite en astronomie, c'est simplement la trajectoire du rayon lumineux. Si donc, par impossible, on venait à découvrir des parallaxes négatives, ou à démontrer que toutes les parallaxes sont supérieures à une certaine limite, on aurait le

¹Poincaré (1898, 1905)

²Poincaré (1898, 1905)

³Poincaré (1902, 1905)

choix entre deux conclusions : nous pourrions renoncer à la géométrie euclidienne ou bien modifier les lois de l'optique et admettre que la lumière ne se propage pas rigoureusement en ligne droite.⁴

The choice can be understood in various ways: kinematics *vs.* dynamics (a dynamical influence that deviates paths against a simple kinematical background can often be absorbed into the background, which thereby acquires curvature); in roughly Newtonian terms as a choice between the first two laws; or even in roughly Aristotelian terms as a choice between natural and violent motion. ‘Natural motion’ here would be inertial, unaccelerated, straight, geodesic (with respect to the background kinematical structure); ‘violent motion’ curved, dynamically accelerated by force.

Such themes have been amply discussed in the literature.⁵ I have little to add to the *general* discussion, and only propose to draw attention to relevant but neglected cases; this is not really a philosophical analysis of theoretical equivalence. Too much has been competently written about the ‘Poincaré choice’ in Newton-Cartan theory to justify much further comment in §2; and the choice in general relativity is too similar for me to dwell on it, in §3. The choice in §4, on the Jacobi metric, deserves some attention despite its *fundamental* similarity to the classical-mechanical case already considered in §2. The magnetic choice in §5, between inclusion in the Hamiltonian and in the symplectic background geometry, is worth considering, being interestingly pertinent but somewhat different. Hard to know what to make of the electromagnetic constitutive relations considered in §6, which may ultimately be of little relevance; one can wonder how much sense the ‘Poincaré choice,’ and the distinction on which it rests, make beyond the geometrical limit, with genuine fields. But at least a foray beyond that limit is worth undertaking, however meagre the fruits.

2 Newton-Cartan theory

Before turning to the neglected cases, it is best to begin with a few words about a well-known and indeed paradigmatic example.⁶ Here a particle of unit mass either describes a geodesic $\bar{\gamma}$ of a curved connection $\bar{\nabla} = \bar{\nabla}(\phi)$ which takes account of ϕ , or gets deflected by a force $\nabla\phi$ with potential ϕ . The equation of motion is

$$(1) \quad \dot{\gamma}^n \bar{\nabla}_n \dot{\gamma}^a = 0$$

in the first case,

$$(2) \quad \dot{\gamma}^n \nabla_n \dot{\gamma}^a = -\nabla^a \phi$$

in the second. Repetition of $\dot{\gamma}$ in the *covariant acceleration* $\dot{\gamma}^n \nabla_n \dot{\gamma}^a$ indicates *covariant differentiation of the vector* ($\dot{\gamma}^a$) *with respect to itself* ($\dot{\gamma}^n$), *i.e.* in direction $\dot{\gamma}^n$. So we have a choice between

⁴Poincaré (1891) p. 774, Poincaré (1902) p. 93

⁵See Glymour (1977), Carrier (1990), Weatherall (2016) for instance.

⁶My treatment and notation follow Malament (2012) §4.2. See also Glymour (1977), Weatherall (2016), Dürr & Read (2019).

- [i] *Lex prima* (1): the motion $\bar{\gamma}$ is geodesic, inertial, unaccelerated, straight (with respect to the curved background $\bar{\nabla}$).
- [ii] *Lex secunda* (2): acceleration is proportional to force, which deflects the world-line γ from the geodesic (of the flat connection ∇) it would otherwise describe.

The absorption of force into the background geometry can be expressed

$$(\nabla, \nabla\phi) \mapsto (\bar{\nabla}, 0).$$

3 General relativity

This is no more than the relativistic transcription of the Newtonian case just considered. Carrier (1990, p. 376) writes:

Consider the equation of an inertial motion, that is, the equation of a particle moving along the geodesic of an affine connection $\bar{\Gamma}_{jk}^i$. Written in generally covariant manner this equation assumes the form

$$(3) \quad \frac{d^2 x^i}{d\tau^2} + \bar{\Gamma}_{jk}^i \frac{dx^j}{d\tau} \frac{dx^k}{d\tau} = 0.$$

One is now at liberty to introduce a universal force F^i , that is, a force acting on all particles independently of their properties. From this perspective, the same particle would not appear to follow a geodesic (that is, to be free from external influences) but to be subject to the action of F^i . The particle is regarded as accelerated relative to a distinct connection⁷ Γ_{jk}^i . The corresponding equation of motion is

$$(4) \quad \frac{d^2 x^i}{d\tau^2} + \Gamma_{jk}^i \frac{dx^j}{d\tau} \frac{dx^k}{d\tau} = F^i.$$

The expression for F^i is obtained by subtracting (3) from (4). Both equations describe the same motion, that is, it is impossible to tell them apart empirically.

One can also view this as a choice between:

- [i] *Lex prima* (3): The “geodesic” is inertial, unaccelerated, straight, in free fall (with respect to the curved background $\bar{\Gamma}$).
- [ii] *Lex secunda* (4): acceleration is proportional to force, which deflects the world-line from the geodesic (of the connection Γ) it would otherwise describe.

The absorption of force into the background geometry can be expressed

$$(\Gamma, F) \mapsto (\bar{\Gamma}, 0).$$

⁷Carrier distinguishes with an asterisk rather than a bar.

4 The Jacobi metric

With the optico-mechanical analogy in mind one can distinguish between the mechanical *anisotropy* determined by the different masses, and the *inhomogeneity* due to the potential energy $U : Q \rightarrow \mathbb{R}$, taken here to depend only on position in configuration space Q .

Here the ‘Poincaré choice’ is between the anisotropic but homogeneous metric σ which only takes account of the masses, and the more comprehensive *Jacobi metric*

$$\begin{aligned}\bar{\sigma} &= 2(\mathcal{E} - U)\sigma = 2T\sigma \\ q &\mapsto \bar{\sigma}(q) = 2(\mathcal{E} - U(q))\sigma\end{aligned}$$

which includes the potential as well, \mathcal{E} being the total energy. I’m assuming the Lagrangian $\mathcal{L} = T - U$ neatly separates anisotropy and inhomogeneity by confining the first to a quadratic kinetic term T and the second to a potential term U . Since the total energy \mathcal{E} is conserved and the potential depends only on position, the kinetic energy

$$\begin{aligned}T &= \frac{1}{2}\langle \sigma^\flat \dot{q}, \dot{q} \rangle = \frac{1}{2}\sigma(\dot{q}, \dot{q}) \\ &= \frac{1}{2}\langle p, \sigma^\sharp p \rangle = \frac{1}{2}\sigma^{-1}(p, p)\end{aligned}$$

is also a function

$$\begin{aligned}T &: Q_\mathcal{E} \rightarrow \mathbb{R} \\ q &\mapsto T(q) = \mathcal{E} - U(q)\end{aligned}$$

of position in the accessible part

$$Q_\mathcal{E} = \{q \in Q : U(q) \leq \mathcal{E}\} \subset Q$$

of configuration space, where momentum

$$p = \dot{q}^\flat = \sigma^\flat \dot{q}$$

is dual to velocity

$$\dot{q} = p^\sharp = \sigma^\sharp p,$$

and the metrics

$$\begin{aligned}\sigma^\flat &= \sigma^\flat(\cdot) : \mathbb{T}_q Q \rightarrow \mathbb{T}_q^* Q \\ \sigma^\sharp &= \sigma^\sharp(\cdot) : \mathbb{T}_q^* Q \rightarrow \mathbb{T}_q Q\end{aligned}$$

are respectively obtained from

$$\begin{aligned}\sigma &= \sigma(\cdot, \cdot) : \mathbb{T}_q Q \times \mathbb{T}_q Q \rightarrow \mathbb{R} \\ \sigma^{-1} &= \sigma^{-1}(\cdot, \cdot) : \mathbb{T}_q^* Q \times \mathbb{T}_q^* Q \rightarrow \mathbb{R}\end{aligned}$$

by partial evaluation.

The Levi-Civita connection $\bar{\mathcal{D}} = \mathcal{D}^{\bar{\sigma}}$ of the Riemannian metric $\bar{\sigma}$ is curved; we can suppose for definiteness that the Levi-Civita connection $\mathcal{D} = \mathcal{D}^{\sigma}$ of the metric σ is flat. A geodesic γ satisfying $\mathcal{D}_{\dot{\gamma}}\dot{\gamma} = 0$ gets bent by a force

$$(5) \quad -\sigma^b(dU) = \mathcal{D}_{\dot{\gamma}}\dot{\gamma}$$

into a geodesic $\bar{\gamma}$ satisfying

$$(6) \quad 0 = \bar{\mathcal{D}}_{\dot{\bar{\gamma}}}\dot{\bar{\gamma}},$$

where the potential U provides the inhomogeneity that curves the background $\bar{\sigma}$. We again have a choice between

- [i] *Lex prima* (6): natural, straight, unaccelerated, geodesic motion (with respect to a curved background $\bar{\sigma}$).
- [ii] *Lex secunda* (5): ‘violent’ motion, accelerated by force dU (against a flat background σ).

The geodesic $\bar{\gamma}$ is also determined by an appropriate variational principle, which takes account of the anisotropy and inhomogeneity of the medium. The generality added by the Hamiltonian action

$$W = \int_{t_1}^{t_2} \mathcal{L} dt = \int_{t_1}^{t_2} (\langle p, p^\sharp \rangle - \mathcal{H}) dt$$

to the Maupertuis action

$$S = \int_{t_1}^{t_2} \langle p, \dot{q} \rangle dt = \int_{q_1}^{q_2} p$$

is only interesting if the Hamiltonian $\mathcal{H} : \Xi \rightarrow \mathbb{R}$ depends explicitly on time, Ξ being the phase space. But the assumptions adopted here confine us to a surface $\Xi_{\mathcal{E}} \subset \Xi$ where the Hamiltonian is equal to a constant

$$\mathcal{E} = \mathcal{H}|_{\Xi_{\mathcal{E}}},$$

and the Lagrangian

$$\mathcal{L}_{\mathcal{E}} = \langle p, \dot{q} \rangle - \mathcal{E}.$$

The variational principles $\delta S = 0$ and

$$\delta W = \delta(S - \mathcal{E}t) = 0$$

are essentially equivalent inasmuch as both give the pre-geodesic, in other words the one-dimensional image of the geodesic $\bar{\gamma}$; Hamilton’s principle $\delta W = 0$ only adds the trivial parametrization $-\mathcal{E}t$ along the geodesic.

The absorption of force into the background geometry can be expressed

$$(\mathcal{D}, dU) \mapsto (\bar{\mathcal{D}}, 0).$$

5 Symplectic mechanics with magnetism

The symplectic two-form ω gives the dynamics

$$X_H = \omega^\sharp(dH)$$

generated by the Hamiltonian H . The Lorentz force exerted by a magnetic field

$$(7) \quad \mathbf{B} = d\mathbf{A}$$

on a charge e can be included in the Hamiltonian itself, or absorbed into the background symplectic geometry represented by ω , the one-form \mathbf{A} being the magnetic potential.⁸ To include it in the Hamiltonian we can write

$$\begin{aligned} \bar{H} = \bar{T} + U &= \frac{1}{2} \langle \bar{p}, \sigma^\sharp \bar{p} \rangle + e\varphi \\ &= \frac{1}{2} \langle (p + e\mathbf{A}), (\sigma^\sharp p + e\sigma^\sharp \mathbf{A}) \rangle + e\varphi \end{aligned}$$

and

$$(8) \quad X_{\bar{H}} = \omega^\sharp(d\bar{H}),$$

where

$$p \mapsto \bar{p} = p + e\mathbf{A}$$

and φ is the electric potential. To absorb the magnetic field into the background geometry we can instead write

$$\bar{X}_H = \bar{\omega}^\sharp(dH)$$

and

$$(\omega, \mathbf{A}) \mapsto (\bar{\omega}, 0),$$

where

$$\omega \mapsto \bar{\omega} = \omega + e\mathbf{B}$$

and $\bar{X}_H = X_{\bar{H}}$. Differentiation is involved in the passage from the dynamics (or the momentum or the Hamiltonian) to the symplectic background; $\omega = d\theta$ is after all the curl of the ‘momentum’ one-form θ . When magnetism goes over from the Hamiltonian to the symplectic background, and gets moved from the (‘differentiated’) right to the (‘undifferentiated’) left of the derivative d in (8), the curl (7) is needed to compensate. The conventional choice between symplectic (in other words ‘kinematical,’ ‘background’) and dynamical magnetism is a choice of order—between the two-form \mathbf{B} (background magnetism) and the one-form \mathbf{A} (dynamical magnetism).

A peculiarity of this case is that the kinematical object \mathbf{B} is the derivative $d\mathbf{A}$ of the dynamical object \mathbf{A} ; whereas in §4 the dynamical object dU was the derivative of the kinematical object U (and similarly in §2).

⁸My treatment is based on Sternberg & Guillemin (1984).

6 Constitutive relations in electromagnetism

One can wonder whether these variations can be extended, beyond the geometrical limit considered so far, to field theories such as electromagnetism. It could be that the choice, as it figures in the geometrical limit, ceases to make sense beyond it.

The curved background into which force dU got absorbed was characterised in §4 by the Jacobi metric $\bar{\sigma}$, the mechanical analog of the (tensorial) index of refraction⁹ v expressing the anisotropy and inhomogeneity of the optical medium. In geometrical optics there's just one 'element'—light—and hence a single v ; in electromagnetism there's electricity *and* magnetism, each of which has its indicatrix and corresponding anisotropy, so v gets split into the dielectric tensor ε and magnetic permeability μ . The two can be combined into a single constitutive tensor density¹⁰ $\chi = \chi(\varepsilon, \mu)$, which will soon reappear. The electric Hodge duality¹¹ ε turns the electric field one-form \mathbf{E} into the dielectric displacement two-form $\mathbf{D} = \varepsilon\mathbf{E}$; the magnetic Hodge duality μ turns the magnetic field one-form \mathbf{H} into the magnetic induction two-form $\mathbf{B} = \mu\mathbf{H}$. The two (homogeneous¹²) equations

$$\begin{aligned} d\mathbf{D} &= 0 \\ d\mathbf{B} &= 0 \end{aligned}$$

differentiate two-forms, whereas the two equations

$$\begin{aligned} d\mathbf{E} + \partial_t \mathbf{B} &= 0 \\ d\mathbf{H} - \partial_t \mathbf{D} &= 0 \end{aligned}$$

require the one-forms \mathbf{E} and \mathbf{H} .

The electric and magnetic fields get combined into the Faraday two-form

$$\begin{aligned} F_\mu &= F(\mathbf{H}, \mathbf{E}; \mu) = \mathbf{B} + \mathbf{E} \wedge dt \\ &= \mu\mathbf{H} + \mathbf{E} \wedge dt \end{aligned}$$

with Hodge dual

$$\begin{aligned} G_\varepsilon &= G(\mathbf{E}, \mathbf{H}; \varepsilon) = \mathbf{D} - \mathbf{H} \wedge dt \\ &= \varepsilon\mathbf{E} - \mathbf{H} \wedge dt. \end{aligned}$$

Here both F_μ and G_ε take account of the material properties of the medium. But it will be convenient to treat the Faraday tensor F as 'constitutively neutral,' concentrating all the material properties in G ; in other words $F = F_{\mu_0}$ is characterised by the multiple μ_0 of the identity corresponding to the isotropic propagation of light at maximum

⁹The rank two symmetric tensor v , an anisotropic generalisation of the merely scalar 'slowness' (or inverse speed) that usually figures in Fermat's principle, assigns an ellipsoidal indicatrix to every point. In Finsler optics the index of refraction is no longer a tensor, the indicatrix no longer ellipsoidal.

¹⁰The corresponding tensor is κ , see Hehl & Obukhov (2003) pp. 245ff.

¹¹I'll use the same symbol (ε or μ) for the tensor and its constitutive Hodge duality.

¹²Here sources just confuse matters by upsetting symmetries and are best left out.

speed. Alongside the constitutively neutral

$$\begin{aligned} F = F_{\mu_0} = F(\mathbf{H}, \mathbf{E}) &= \mathbf{B} + \mathbf{E} \wedge dt \\ &= \mu_0 \mathbf{H} + \mathbf{E} \wedge dt \end{aligned}$$

we would then have its constitutive dual

$$\begin{aligned} \chi(F) = G_\chi = G(\mathbf{E}, \mathbf{B}; \varepsilon, \mu^{-1}) &= \mathbf{D} - \mathbf{H} \wedge dt \\ &= \varepsilon \mathbf{E} - \mu^{-1} \mathbf{B} \wedge dt, \end{aligned}$$

which alone takes account of the material properties of the medium.

Here we have an operation

$$\chi : G = G_0 \mapsto G_\chi$$

analogous to

$$\begin{aligned} \nabla &\mapsto \bar{\nabla} \text{ in } \S 2 \\ \Gamma &\mapsto \bar{\Gamma} \text{ in } \S 3 \\ \mathcal{D} &\mapsto \bar{\mathcal{D}} \text{ in } \S 4 \\ \omega &\mapsto \bar{\omega} \text{ in } \S 5, \end{aligned}$$

where the constitutively neutral $G_0 = \chi_0(F)$ corresponds to the vacuum value

$$\chi_0 = \chi(\varepsilon_0, \mu_0).$$

So if there were an appropriate ‘force’ or ‘deflecting influence’ we might have a way of absorbing it into the background. But is there such a force?

What deflects an electromagnetic wave? A lens, for instance—but its properties get encoded into χ right away, hence precluding any choice. How about space-time curvature? Surely it also bends an electromagnetic wave. We can begin with an electromagnetic field F satisfying equations

$$(9) \quad \begin{aligned} dF &= 0 \\ dG_0 &= 0 \end{aligned}$$

on a constitutively neutral flat spacetime with metric η , then deform in two different ways:

[1] *gravitationally*: $\eta \mapsto g$

[2] *constitutively*: $\eta \mapsto \eta_\chi$.

If we assume that the metric and constitutive tensor density determine each other,¹³ the gravitational deformation $\eta \mapsto g_\chi$ produces an electromagnetic deformation which gives meaning to the notion of ‘space-time curvature bending an electromagnetic wave’; the constitutive deformation likewise gives meaning to the notion of ‘material properties perturbing an electromagnetic wave’:

¹³The skewon and axion parts of χ have to vanish, see Hehl & Obukhov (2003) pp. 248ff.

$$[1] \ G_0 \mapsto G_{\bar{\chi}}$$

$$[2] \ G_0 \mapsto G_{\chi}.$$

We already have a framework in which we can at least speak of equivalence, perhaps even of ‘absorbing an influence into the background’; there is already a general Poincaré choice between the gravitational deformation [1] and the constitutive deformation [2], but we would like to be more specific. Of course we cannot *directly* establish a meaningful equivalence between the gravitationally deformed electromagnetic field¹⁴ $G_{\bar{\chi}}$ and the materially deformed G_{χ} , so we will have to proceed more indirectly. All we can do is set up a one-one correspondence between the gravitational and material deformations, thus establishing a conventional choice between *this* gravitational influence and *that* constitutive background. We can view the gravitational deformations as being indexed by a first parameter π_1 , the material deformations by a second π_2 ,

$$[1] \ G_0 \mapsto G_{\bar{\chi}}(\pi_1)$$

$$[2] \ G_0 \mapsto G_{\chi}(\pi_2),$$

and then set up an arbitrary correspondence between the two paths of deformation with a single parameter π :

$$[1] \ G_0 \mapsto G_{\bar{\chi}}(\pi) = G_{\bar{\chi}}(\pi_1(\pi))$$

$$[2] \ G_0 \mapsto G_{\chi}(\pi) = G_{\chi}(\pi_2(\pi)).$$

The correspondence $G_{\bar{\chi}}(\pi) \leftrightarrow G_{\chi}(\pi)$ is a start, and already allows us to make a conventional choice between the gravitational deformation $G_{\bar{\chi}}(\pi)$ and the material deformation $G_{\chi}(\pi)$. But how exactly is $G_{\bar{\chi}}(\pi)$ related to $G_{\chi}(\pi)$? What do they have in common? Maybe nothing.

De Sitter spacetimes \mathcal{S}_{λ} , conveniently indexed by the cosmological constant λ , lend themselves to the electromagnetic parametrisation $G_0 \mapsto G(\lambda(\pi))$; where the electromagnetic $G_{\bar{\chi}}$ field is a function of the original F and of $\bar{\chi}$, which is assumed to be a function of g , a function of λ , which is in turn a function of π . But to what $\chi(\pi)$ on Minkowski spacetime would the De Sitter tensor density $\bar{\chi}(\lambda(\pi))$, which is generally a function of position, correspond? How can the points of the curved spacetime \mathcal{S}_{λ} be brought into correspondence with the points of flat Minkowski spacetime? The problem can be avoided by making variable $\bar{\chi}$ correspond to constant χ , but that would amount to absorbing a curved gravitational influence into a flat material background. Only an unsatisfactory coordinate-dependent solution comes to mind: one can

¹⁴A single equation $dG = 0$ (whatever the subscript) on its own is no more than a ‘topological’ condition, being subject to the freedom $G \mapsto G' = G + d\beta$, where β is a one-form; on its own, $dF = 0$ is subject to the similar freedom $F \mapsto F' = F + d\alpha$. But if the two conditions (9) are taken together, the duality connecting F and G heavily constrains the relationship between α and β , which can no longer be chosen independently. One can also impose gauge conditions like $d*A_{\xi} = 0$ (Lorenz), where

$$[A_{\xi}] = [A + d\xi]_{\xi} = d^{-1}F \quad A \mapsto A_{\xi} = A + d\xi,$$

ξ is a zero-form and $*$ represents the relevant Hodge duality.

assign the tensor density $\bar{\chi}(x, \pi)$ at point $x \in \mathcal{S}_{\lambda(\pi)}$ with coordinates (x_0, x_1, x_2, x_3) to the point with the same coordinates in Minkowski spacetime. But we seem to have reached a dead end. Perhaps we can do no better than establish an admittedly arbitrary equivalence, of doubtful significance, between the electromagnetic fields $G_{\bar{\chi}}(\pi)$ and $G_{\chi}(\pi)$, and leave it at that. This variation could be too unpromising to deserve further pursual.

For details—which would be out of place here—on the relationship between the constitutive tensor density χ and the conformal structure or even the metric, see Kottler (1922), Dantzig (1934), Post (1962), Nguyen (1992), Hehl & Obukhov (2003).

In the geometrical limit one has a choice between a bent curve against a flat background and a straight curve with respect to a curved background; beyond the geometrical limit there seems to be a more confusing choice between different backgrounds: here, between a gravitational and a material background. The notion of equivalence adopted here, beyond the geometrical limit, is no more than an abstract correspondence established by a common parameter π ; equivalence in the geometrical limit can be much stronger and more meaningful.

7 Final remarks

Since experiment, for Poincaré, underdetermines geometry, since different geometries are compatible with experience, the choice of geometry has a conventional character. He then has to specify that the convention adopted is not arbitrary, dictated by convenience or simplicity or elegance or economy—but here that's beside the point.

I have looked at a handful of mechanical cases which illustrate the conventionalist freedom in different ways. The first three cases, considered in §§2-4, correspond almost exactly to the choice Poincaré had in mind; the Jacobi metric even has an 'optical' flavour which lends itself to Poincaré's optical analysis. I have also devoted some attention to the neglected case of a charge in a magnetic field against a symplectic background.

Poincaré's geometrical conventionalism may make more sense with paths γ , in the geometrical limit, than with fields F . A field F can admittedly be deflected, like a path—but who knows whether the influence, which already corresponds to one 'background,' can then be absorbed into another. It may boil down to a matter of *dimensions*: In the geometrical limit there is a clear choice, a clear difference between curving a one-dimensional path and a many-dimensional background; but with a field F already defined on the many-dimensional background, the choice and the distinction on which it rests may make little sense.

I thank Ermenegildo Caccese for many interesting discussions.

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