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**Equilibrium in Incomplete Markets with Differential
Information: A Basic Model of Generic Existence**

Lionel DE BOISDEFFRE

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EQUILIBRIUM IN INCOMPLETE MARKETS WITH DIFFERENTIAL INFORMATION:

A BASIC MODEL OF GENERIC EXISTENCE

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Abstract

The paper demonstrates the generic existence of general equilibria in incomplete financial markets with asymmetric information. The economy has two periods and an ex ante uncertainty over the state of nature to be revealed at the second period. Securities pay off in cash or commodities at the second period, conditionally on the state of nature to be revealed. They permit transfers across periods and states, which are typically insufficient to span all state contingent claims to value, whatever the spot price to prevail. Under the standard smooth preference and perfect foresight assumptions, the paper shows that equilibria exist, except for a closed set of measure zero of securities and endowments. This theorem generalizes Duffie-Shafer's (1985) to arbitrary financial and information structures. The equilibrium prices are consistent with any collection of state prices and norm values on spot markets. This refinement permits to extend to asymmetric information Cass' (1984) theorem that any collection of state prices supports an equilibrium on purely financial markets.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

This paper demonstrates the generic existence of equilibrium in incomplete financial markets with differential information, in the context of a two-period pure exchange economy where uncertainty prevails at the first period over the state of nature to be revealed at the second period. Asymmetric information is represented by private finite subsets of states of nature, which each agent is correctly informed to contain the true state of the second period. The scope of this specification is discussed in Section 3, jointly with the information that financial markets may reveal. Consumers exchange consumption goods on spot markets, and, unrestrictedly, assets of any kind on typically incomplete financial markets. They are endowed with a bundle of goods in every state, with ordered smooth preferences over consumptions and a perfect foresight of future prices, along Radner (1972).

The paper's generic existence result generalizes a classical theorem of symmetric information with real assets due to Duffie-Shafer (1985). The current proof builds on a fixed arbitrary set of state prices. This device permits to extend to the current model other results of symmetric information, such as Cass' (1984), stating that any collection of state prices supports an equilibrium on purely financial markets.

When assets pay off in goods, equilibrium needs not exist, as shown by Hart (1975) in the symmetric information case. His example is based on the collapse of the span of assets' payoffs, that occurs exceptionnaly at clearing prices. Duffie-Shafer (1985) shows that equilibrium with real assets exists, except for a closed set of measure zero of economies, parametrized by assets' payoffs and agents' endowments.

The current model extends Duffie-Shafer's (1985) in three ways. First, it allows for asymmetric information amongst consumers. Second, its financial structure

may cover any mix of nominal and real assets. Third, but not least, it normalizes to arbitrary values equilibrium prices on every spot market. Normalization to relevant values serves to study subsequently the existence of sequential equilibria when agents lose the perfect foresight of future prices. With no price map, agents typically face an endogenous uncertainty a la Kurz (1994). To be self-fulfilling, their anticipations need therefore focus on sets of relevant values, as argued in Section 7.

The current paper drops Radner's (1979) rational expectations assumption. It prefers a learning process, presented in Section 3, where agents may infer information from markets with no price model. It is a step towards also replacing Radner's (1972) perfect foresight assumption by a milder condition on anticipations, which remains consistent with the definition of sequential equilibrium (see Section 7).

The current proof uses standard differential topology arguments, introduced by Debreu (1970, 1972) for the study of general equilibrium. It defines an auxiliary concept of "*pseudo-equilibrium*" with asymmetric information, shows its full existence from modulo 2 degree theory and derives the generic existence of equilibrium with asymmetric information from Sard's theorem and Grassmannians' properties.

The paper is organized as follows: Section 2 presents the model and the concepts of equilibrium and pseudo-equilibrium. Section 3 describes the information that markets reveal. Section 4 presents Grassmannians and their main properties. Section 5 derives from the latter properties the full existence of pseudo-equilibria. Section 6 proves the existence theorems. Section 7 concludes. An Appendix proves Lemmas.

2 The model

Throughout the paper, we consider a pure-exchange economy with two periods,

$t \in \{0, 1\}$ and an ex ante uncertainty over the state of nature to be revealed at $t = 1$. Consumers exchange goods, on spots markets, and assets of all kinds, on typically incomplete financial markets. The sets, I , S , L and J , respectively, of consumers, states of nature, consumption goods and assets are all finite. Throughout, we let $s = 0$ be the non-random state at $t = 0$, and denote $\Sigma' := \{0\} \cup \Sigma$, for every subset, Σ , of S . Similarly, we denote by $l = 0$ the cash return of assets and let $L' := \{0\} \cup L$.

Uncertainty unfolds as follows. At $t = 0$, each agent, $i \in I$, receives or infers privately a correct information signal, $S_i \subset S$, that tomorrow's true state will be in S_i . At $t = 1$, one state $s \in \underline{\mathbf{S}} := \cap_{i \in I} S_i$ prevails and all uncertainty is removed.

2.1 Markets, prices and information

Agents consume or exchange the consumption goods, $l \in L$, on both periods' spot markets. Admissible prices for commodities are restricted to the common set, $\Delta := \{p \in \mathbb{R}_{++}^L : \|p\| = 1\}$, on every spot market. Normalizing each spot price to one is assumed for convenience, but non restrictive. In any state, $s \in S'$, the unit bound of Δ could be replaced by any positive value without changing the model's results.

Thus, admissible commodity prices, or the collection of spot prices in all realizable states, belong to the set $P := \Delta^{\underline{\mathbf{S}}}$. Such prices are perfectly observed or anticipated at equilibrium, along Radner's (1972) perfect foresight assumption. Following De Boisdeffre (2007), in any unrealizable state, $s \in S_i \setminus \underline{\mathbf{S}}$, the generic unfully informed agent, $i \in I$, has an idiosyncratic anticipation, $p_s^i \in \Delta$, of the spot price to prevail, and we let $p^i := (p_s^i) \in \Delta^{S_i \setminus \underline{\mathbf{S}}}$ be their collection.

Consumers may operate transfers across states by exchanging, at $t = 0$, finitely many assets, $j \in J$ (with $\#J \leq \#\underline{\mathbf{S}}$) whose expected payoffs, at $t = 1$, are conditional on the realization of the state, $s \in S$. These payoffs may be nominal or real or a

mix of both. They define a financial structure, or $(S \times L') \times J$ matrix, denoted by V . Thus, the expected payoffs of a generic asset, $j \in J$, in a state, $s \in S$, are a bundle, $v_j(s) := (v_j^l(s)) \in \mathbb{R}^{L'}$, of the quantities, $v_j^0(s)$, of cash, and $v_j^l(s)$, of each good $l \in L$, which one unit of the asset, j , promises to deliver if state s prevails.

The financial structure, V , identifies (with same notation) to a map, $V : S \times \Delta \rightarrow \mathbb{R}^J$, relating every forecast of a state and spot price, $\omega := (s, p := (p^l)) \in S \times \Delta$, to the row of all assets' payoffs in cash, $V(\omega) := (v_j^0(s) + \sum_{l \in L} p^l v_j^l(s))_{j \in J} \in \mathbb{R}^J$, delivered if both state s and price p obtain. Thus, when the asset price is $q \in \mathbb{R}^J$, agents may buy or sell portfolios of assets, $z = (z_j) \in \mathbb{R}^J$, for $q \cdot z$ units of account at $t = 0$, against the promised delivery of a flow, $V(\omega) \cdot z$, of conditional payoffs across forecasts, $\omega \in S \times \Delta$.

Remark 1 The incompleteness of financial markets is a standard assumption. It states that no agent can insure her risks on markets completely. That is, asset payoffs (in realizable states) cannot span $\mathbb{R}^{\underline{\mathbf{S}}}$, which is written: $\#J < \#\underline{\mathbf{S}}$. The above condition is milder and allows markets to be complete (when $\#J = \#\underline{\mathbf{S}}$) to informed agents only. Asset payoffs show that potentially complete markets to the informed, are incomplete to the uninformed: payoffs can never span \mathbb{R}^{S_i} if $i \in \{j \in I : S_j \neq \underline{\mathbf{S}}\}$.

2.2 Consumers' behaviour and the concept of equilibrium

Each agent, $i \in I$, receives an endowment, $e_i := (e_{is})$, promising the commodity bundles, $e_{i0} \in \mathbb{R}_{++}^L$ at $t = 0$, and $e_{is} \in \mathbb{R}_{++}^L$, in each state, $s \in S_i$, if this state prevails. Her consumption set is $X_i := \mathbb{R}_{++}^{L \times S'_i}$. Given the prices, $p := (p_s) \in P$, for goods, and $q \in \mathbb{R}^J$, for assets, the endowment, $e_i \in \mathbb{R}_{++}^{L \times S'_i}$, and the payoff matrix, $V \in \mathbb{R}^{(S \times L') \times J}$, that she faces at the first period when she elects her strategy, her budget set is:²

² As in Duffie-Shafer (1985), our generic existence proof could not avoid the artificial interior consumptions at equilibrium. Doing without would require to drop the standard modulo 2 degree argument in the existence proof, which is central.

$$B_i(p, q, e_i, V) := \{ (x, z) \in X_i \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leq -q \cdot z \text{ and } p_s \cdot (x_s - e_{is}) \leq V(s, p_s) \cdot z, \forall s \in \underline{\mathbf{S}}, \\ \text{and } p_s^i \cdot (x_s - e_{is}) \leq V(s, p_s^i) \cdot z, \forall s \in S_i \setminus \underline{\mathbf{S}} \}.$$

The budget set consists of consumption and investment plans, which are feasible at the agent's information and anticipations. Preferences are assumed to be ordered, represented by a utility function, $u_i : X_i \rightarrow \mathbb{R}$, for each $i \in I$. Given endowments and payoffs, $(e := (e_i), V) \in (\times_{i \in I} X_i) \times \mathbb{R}^{(S \times L') \times J}$, the above economy is denoted by $\mathcal{E}_{(e, V)} = \{(I, S, L, J), V, (S_i), (p^i), (e_i), (u_i)\}$. Its equilibrium concept is defined as follows:

Definition 1 *Given endowments, $e := (e_i) \in \times_{i \in I} X_i$, and a payoff matrix, $V \in \mathbb{R}^{(S \times L') \times J}$, a collection of prices, $(p, q) \in P \times \mathbb{R}^J$, and strategies, $(x_i, z_i) \in B_i(p, q, e_i, V)$, defined for each $i \in I$, is an equilibrium of the economy, $\mathcal{E}_{(e, V)}$, if the following Conditions hold:*

- (a) $\forall i \in I, x_i \in \arg \max u_i(x)$, for $(x, z) \in B_i(p, q, e_i, V)$;
- (b) $\sum_{i \in I} (x_{is} - e_{is}) = 0, \forall s \in \underline{\mathbf{S}}'$;
- (c) $\sum_{i \in I} z_i = 0$.

The economy is called standard if it meets the following classical Assumptions:

A1 $\forall i \in I, u_i$ is C^∞ ;

A2 (Inada Conditions) $\forall (i, s, l, x := (x_s^l)) \in I \times S'_i \times L \times X_i, \partial u_i(x) / \partial x_s^l \in \mathbb{R}_{++},$
 $\lim_{x_s^l \rightarrow 0} \partial u_i(x) / \partial x_s^l = \infty$ (where $x_s^l \rightarrow 0$ stands for " x_s^l tends to zero while other components of x are fixed"), $\lim_{x_s^l \rightarrow \infty} \partial u_i(x) / \partial x_s^l = 0$ (in which $x_s^l \rightarrow \infty$ stands for " x_s^l tends to infinity at other components of x fixed"), $\lim_{x_s^l \rightarrow 0} u_i^s(x) = 0$;

A3 (differentiably strictly convex preferences) $\forall i \in I, \forall x \in X_i, h^T D^2 u_i(x) h < 0$, for every $h \in \mathbb{R}^{L \times S'_i} \setminus \{0\}$, such that $\nabla u_i(x) \cdot h = 0$;

A4 at least one agent is fully informed and let $i = 1$ be such that $S_1 = \underline{\mathbf{S}}$.

Assumption A1 guarantees the smoothness of preferences and subsequent maps. The above conditions permit to demonstrate the generic existence of equilibrium. Before proceeding, we need introduce the model's notations.

2.3 The model's notations

First, notations are introduced for anticipation sets and manifold dimensions:

- $\Omega_i := \{(s, p_s^i) : s \in S_i \setminus \underline{\mathbf{S}}\}$ and $\mathcal{S}_i := \Omega_i \cup \underline{\mathbf{S}}$, for each $i \in I$;
- $\Omega := \cup_{i \in I} \Omega_i$ and $\mathcal{S} := \Omega \cup \underline{\mathbf{S}}$, which are exogenously given;
- \mathcal{S} is identified to $\#\mathcal{S}$, Σ to $\#\Sigma$ (for $\Sigma \subset S'$), L to $\#L$, J to $\#J$, whenever needed;
- $v^* := JS(L+1) = \dim \mathbb{R}^{(S \times L') \times J}$;
- $v^{**} := (\mathcal{S} - J)J = \dim \mathbb{R}^{(S-J) \times J}$;
- $e^* := \sum_{i \in I} LS'_i$;
- $l^* := (\underline{\mathbf{S}} + 1)(L - 1) = \dim P$.

Second, the following notations are used for vectors, vector spaces and matrices:

- for every agent, $i \in I$, consumption, $x := (x_s) \in X_i := \mathbb{R}_{++}^{L \times S'_i}$, and subset $\Sigma \subset S'_i$, the $\mathbb{R}^{L \times \Sigma}$ extraction of x is denoted by $x(\Sigma)$, that is, $x(\Sigma) := (x_s)_{s \in \Sigma} \in \mathbb{R}_{++}^{L \times \Sigma}$;
- for all $i \in I$, $p := (p_s) \in P$ and $x := (x_s) \in X_i$, the vector $p \square_i x \in \mathbb{R}^{S_i}$ has for components the scalar products $p_s \cdot x_s$, for each $s \in \underline{\mathbf{S}}$, and $p_s^i \cdot x_s$, for all $(s, p_s^i) \in \Omega_i$.
- $\mathbb{R}^{\Sigma \times \Theta}$, for every finite sets Σ and Θ , stands for the set of $\Sigma \times \Theta$ real matrices;
- $\langle V \rangle$, $V(\omega)$ and $V(\Sigma'')$ denote, for all finite sets, Σ , Θ and $\Sigma'' \subset \Sigma$, element, $\omega \in \Sigma$, and matrix, $V \in \mathbb{R}^{\Sigma \times \Theta}$, respectively, the span of the matrix' columns, in \mathbb{R}^Σ , the matrix' ω row, in \mathbb{R}^Θ , and the extracted $\Sigma'' \times \Theta$ submatrix of V ;
- $V_p \in \mathbb{R}^{S \times J}$, for every $V \in \mathbb{R}^{(S \times L') \times J}$ and every $p := (p_s) \in P$, denotes the matrix defined by $V_p(s) := V(s, p_s)$, for each $s \in \underline{\mathbf{S}}$, and $V_p(\omega) := V(\omega)$, for each $\omega \in \Omega$;
- $\mathcal{G} := \{V \in \mathbb{R}^{S \times J} : \text{rank } V = \#J\}$ and $\mathcal{G}^* := \{\langle V \rangle : V \in \mathcal{G}\}$, called Grassmannian;

- $\underline{\mathcal{G}} := \{ V \in \mathcal{G} : \text{rank } V(\underline{\mathbf{S}}) = \#J \}$ and $\underline{\mathcal{G}}^* := \{ \langle V \rangle : V \in \underline{\mathcal{G}} \}$;
- $G(\Sigma)$ is the \mathbb{R}^Σ truncation of $G \in \mathcal{G}^*$, and $\mathcal{G}^*(\Sigma) := \{ G(\Sigma) : G \in \mathcal{G}^* \}$, for all $\Sigma \subset \mathcal{S}$.

2.4 The concept of pseudo-equilibrium

The concept of pseudo-equilibrium is introduced to circumvent the fall in rank problem a la Hart (1975), which may occur and prevent equilibrium to exist when real assets are traded. In the definition, financial transfers belong to a fixed J -dimensional vector space, which includes the span of asset payoffs valued at market prices. Hereafter, the pseudo-equilibrium is defined with reference to a given vector of state prices, $\lambda \in \mathbb{R}_{++}^{\underline{\mathbf{S}}}$. This will permit to extend Cass' (1984) result to the model.

The two concepts of equilibrium and pseudo-equilibrium coincide when assets are nominal or numeraire and it follows that equilibrium always exists with such assets (see Lemma 1 and Theorem 2, below). An asset, $j \in J$, is said to be nominal if it pays in cash only, that is, $v_j^l(s) = 0$, for every pair, $(s, l) \in S \times L$. It is numeraire if it pays in a bundle of goods, $a \in \mathbb{R}_+^L \setminus \{0\}$, that is, $v_j(s) \in a \mathbb{R}$, for every $s \in S$.

In the general case, however, the concept of pseudo-equilibrium differs and has little economic significance. It is only used to prove the generic existence of equilibrium. Indeed, pseudo-equilibria exist in standard conditions (see Theorem 1, below), and generically coincide with equilibria (Lemma 1, hereafter, and Lemma 3, below).

Definition 2 *Let $\lambda := (\lambda_s) \in \mathbb{R}_{++}^{\underline{\mathbf{S}}}$ be given. The collection of a scalar, $y \in \mathbb{R}_{++}$, matrix, $V \in \mathbb{R}^{(S \times L) \times J}$, vector space, $G \in \mathcal{G}^*$, prices, $p := (p_s) \in P$, endowments, $e_i := (e_{is}) \in X_i$, and consumptions, $x_i := (x_{is}) \in X_i$, defined for each $i \in I$, is a pseudo- λ -equilibrium of the economy, $\mathcal{E}_{((e_i), V)}$, if the following Conditions hold:*

- (a) $x_1 \in \arg \max u_1(x)$, for $x \in \{ x := (x_s) \in X_1 : p_0 \cdot (x_0 - e_{10}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_s p_s \cdot (x_s - e_{1s}) = 0 \}$;

(b) for every $i \in I \setminus \{1\}$, $x_i \in \arg \max u_i(x)$,

for $x \in \{x := (x_s) \in X_i : p_0 \cdot (x_0 - e_{i0}) + \sum_{s \in \underline{\mathbf{S}}} \lambda_s p_s \cdot (x_s - e_{is}) = 0 \text{ and } p \square_i (x - e_i) \in G(\mathcal{S}_i)\}$;

(c) $\langle V_p \rangle \subset G$;

(d) $\forall s \in \underline{\mathbf{S}}', \sum_{i \in I} (x_{is} - e_{is}) = 0$;

(e) $p_0 \cdot e_{10} + \sum_{s \in \underline{\mathbf{S}}} \lambda_s p_s \cdot e_{1s} = y$.

Given $(e := (e_i), V) \in \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$, we say that $(y, p, G) \in \mathbb{R}_{++} \times P \times \mathcal{G}^*$ is a pseudo- λ -equilibrium, if there exists $x \in \times_{i \in I} X_i$, such that (x, y, p, G, e, V) is a pseudo- λ -equilibrium along Conditions (a) to (e), above. We let \mathcal{E}_λ^* be the pseudo- λ -equilibrium manifold, or the set of collections, $(y, p, G, e, V) \in \mathbb{R}_{++} \times P \times \mathcal{G}^* \times \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$, such that (y, p, G) is a pseudo- λ -equilibrium, given $(e, V) \in \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$. We define the projection map, $\pi_\lambda : (y, p, G, e, V) \in \mathcal{E}_\lambda^* \mapsto (e, V) \in \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$.

We end with typical conditions, which turn pseudo-equilibria into equilibria:

Lemma 1 Given $\lambda := (\lambda_s) \in \mathbb{R}_{++}^{\underline{\mathbf{S}}}$, let (x, y, p, G, e, V) be a pseudo- λ -equilibrium of a standard economy, $\mathcal{E}_{(e, V)}$, and let $q := \sum_{s \in \underline{\mathbf{S}}} \lambda_s V_p(s)$ be given. There exists portfolios, $z \in \mathbb{R}^{J \times I}$, such that (p, q, x, z) defines an equilibrium of the economy $\mathcal{E}_{(e, V)}$ whenever one of the following Conditions holds:

(i) assets are nominal or numeraire;

(ii) $\text{rank} V_p(\underline{\mathbf{S}}) = J$, i.e., $G \in \underline{\mathcal{G}}$.

Proof See the Appendix. □

Before studying Grassmanians, we recall what information markets may reveal.

3 The information which financial markets reveal

The paper's learning behaviour, presented in sub-Section 3.2, differs from Radner's (1979) at the rational expectations equilibrium (REE), presented first.

3.1 Agents' inferences from observing prices in the REE model

Radner summarizes his 1979 paper on rational expectations equilibria as follows:

"When traders come to a market with different information about the items to be traded, the resulting market prices may reveal to some traders information originally available only to others. The possibility for such inferences rests upon traders having "models" or "expectations" of how equilibrium prices are related to initial information. This relationship is endogenous, which motivates the term "rational expectations equilibrium." This paper shows that, in a particular model of asset trading, if the number of alternative states of initial information is finite then, generically, REE exist that reveal to all traders all their initial information." Radner establishes the generic existence of a fully revealing REE as follows: he shows that equilibria with symmetric pooled information exist and generically have distinct prices in each state of pooled information. In this approach, each term of the summary matters.

First, Radner's (1979) existence and inference results hold for *"a particular model of asset trading"*. The author points out (p. 677): *"Whether or not rational expectations equilibria exist generically in some fairly general model, is an open question"*. Jordan and Radner (1977) show that the REE model results are not general. Radner's (1979) outcomes build on a specific model with separable expected utility functions and no spot market of various goods, hence, no real asset markets. Moreover, the *"number of alternative states of initial information"* needs to be finite, Radner adds, since prices are typically non-revealing if signal sets are *"too large"*.

Second, reaching equilibrium requires that all traders had models *"or "expectations" of how equilibrium prices are related to initial information"* and used them to infer correct information. This assumption is seen as unrealistic (see Section 7). The study of information that markets reveal with no price model is therefore needed.

3.2 Agents' inferences from trade opportunities with no price model

The information that financial markets may reveal to agents, when they have no price model, is presented hereafter in a simplified setting, where assets are nominal and the state space is finite. These simplifications are not restrictive. The inference principle and properties, presented below, extend to arbitrary financial structures and infinitely many states of nature, events or forecasts (see De Boisdeffre, 2016).

In the simplest two period model that we consider, any collection of information signals may be represented by a structure, (S_i) , consisting of subsets of a finite state space, S , along Section 2. Indeed, any information reduces tomorrow's set of possibilities, which may be represented by a state space. Given a payoff matrix, V , the pair $[V, (S_i)]$ is called the payoff and information structure, or structure. Assume, first, that an equilibrium exists when agents are endowed with the structure $[V, (S_i)]$. Then, the asset price reveals no information. Agents are price-takers, unaware of a relation between prices and signals and witness no specific volatility on financial markets in the absence of arbitrage. Along De Boisdeffre (2007), the existence of equilibrium is characterized by the following condition of absence of future arbitrage opportunity of the structure $[V, (S_i)]$, henceforth referred to as the AFAO Condition:

$$\nexists (z_i) \in \mathbb{R}^{J \times I} : \sum_{i \in I} z_i = 0 \text{ and } V(S_i)z_i \geq 0, \forall i \in I, \text{ with one strict inequality.}$$

Not all structures, $[V, (S_i)]$, meet the AFAO Condition at the outset. If not, asset pricing is impossible, since agents cannot agree on any price. Financial markets are prone to volatility. Arbitragists take advantage of it and of differences in agents' assessments of portfolios. As long as the AFAO Condition fails, they may sell profitably to various buyers a bundle of clearing-market portfolios. However, as arbitragists compete to attract buyers, the prices at which they may sell such clear-

ing portfolios fall to zero. At zero or sufficiently low prices, traders must infer that any state of their prior information sets, upon which they expected to achieve a "free lunch", is actually unrealizable. That is, agents, observing market, learn from arbitrage. This learning process with no price model (or price) features the trader's actual behaviour on financial markets. Call it the "*no-arbitrage principle*".

If the structure, $[V, (S_i)]$, fails to be arbitrage-free at the outset, the no-arbitrage principle leads each agent, $i \in I$, to infer, in finitely many steps, a subset, $S_i \setminus S_i^*$, of unrealizable states, such that $[V, (S_i^*)]$, is the coarsest arbitrage-free refinement of $[V, (S_i)]$: see Cornet-De Boisdeffre (2009), for finite state spaces, and De Boisdeffre (2016), for arbitrary (possibly uncountable) state spaces. Once agents have inferred the refinement (S_i^*) , from the no-arbitrage principle, they cannot learn more information. Either from trade, since the AFAO Condition holds and no arbitrage-state remains, or from prices, since agents agree on prices and have no price model. Erratic market movements vanish. From De Boisdeffre (2007), an equilibrium would obtain, whose no-arbitrage price, for the refinement (S_i^*) , reveals no additional information.

Henceforth, agents are assumed to have reached the coarsest arbitrage-free refinements of their information signals, if needed, that is, $[V, (S_i)]$ is arbitrage-free.

4 Grassmannians with asymmetric information

4.1 A characterization of the set \mathcal{G}^*

Introduced in sub-Section 2.3, the set of J -dimensional subspaces of \mathbb{R}^S , denoted by \mathcal{G}^* , is called the Grassmannian. Claim 1 characterizes \mathcal{G}^* by the following vector spaces, and leads to define a topology, which makes that set compact. First, we let:

$\mathcal{Z}^\circ := \{ W \in \mathbb{R}^{(S-J) \times S} : \text{the rows of } W \text{ are linearly independent} \};$

$\mathcal{Z} := \{ W \in \mathbb{R}^{(S-J) \times S} : \text{the rows of } W \text{ are orthonormal} \};$

$\mathcal{Z}^* := \{ W \in \mathbb{R}^{S \times J} : \text{the columns of } W \text{ are orthonormal} \}.$

Claim 1 *Let G be a sub-vector space of \mathbb{R}^S . The following Assertions hold:*

(i) $(G \in \mathcal{G}^*) \Leftrightarrow (\exists W \in \mathcal{Z}^\circ : G = \{ z \in \mathbb{R}^S : Wz = 0 \});$

(ii) $(G \in \mathcal{G}^*) \Leftrightarrow (\exists W \in \mathcal{Z} : G = \{ z \in \mathbb{R}^S : Wz = 0 \});$

(iii) $\mathcal{G}^* = \{ \langle W \rangle : W \in \mathcal{Z}^* \}.$

Proof Claim 1 is immediate from the definitions. The proof is left to the reader. \square

The manifold, \mathcal{Z}° , is obviously open in the sense that the relation $W \in \mathcal{Z}^\circ$ implies that $W' \in \mathbb{R}^{(S-J) \times S}$ belongs to \mathcal{Z}° in a neighbourhood of W for the Euclidean distance on $\mathbb{R}^{(S-J) \times S}$. To be precise, we say that $U \subset \mathbb{R}^{(S-J) \times S}$ is open in $\mathbb{R}^{(S-J) \times S}$ if it satisfies the following condition: $\forall W \in U, \exists \varepsilon > 0, \{W' \in \mathbb{R}^{(S-J) \times S} : \|W' - W\| < \varepsilon\} \subset U$. This defines a topology, τ° , on $\mathbb{R}^{(S-J) \times S}$, for which \mathcal{Z} is closed, hence, compact.

Two elements, W and W' , of \mathcal{Z}° are said to be equivalent, which we denote by $W \sim W'$, if $\{ z \in \mathbb{R}^S : Wz = 0 \} = \{ z \in \mathbb{R}^S : W'z = 0 \}$. From Claim 1, the Grassmannian, \mathcal{G}^* , identifies to the set of equivalence classes, \mathcal{Z}° / \sim , of \mathcal{Z}° . This permits to define a topology, τ , on $\mathcal{G}^* \equiv \mathcal{Z}^\circ / \sim$. We say that U is open in \mathcal{G}^* for the topology τ , if $p^{-1}(U)$ is open in \mathcal{Z}° for the topology, τ° , where $p : \mathcal{Z}^\circ \rightarrow \mathcal{Z}^\circ / \sim$ is the identification map.

Claim 2 *The Grassmannian, \mathcal{G}^* , is a compact set for the above topology, τ .*

Proof Let $\{G^k\}_{k \in \mathbb{N}}$ be a sequence of elements of \mathcal{G}^* . From Claim 1, there exists a sequence, $\{W^k\}_{k \in \mathbb{N}}$, of elements of \mathcal{Z} , such that $G^k = \{ z \in \mathbb{R}^S : W^k z = 0 \}$, for every $k \in \mathbb{N}$. Since \mathcal{Z} is compact for the topology τ° , the sequence $\{W^k\}_{k \in \mathbb{N}}$ may be assumed to converge, say to $W \in \mathcal{Z}$. From Claim 1, $G := \{ z \in \mathbb{R}^S : Wz = 0 \} \in \mathcal{G}^*$. Let U be an open set in \mathcal{G}^* containing G . From the definition, $p^{-1}(U)$ is open and contains

W . Hence, there exists $N \in \mathbb{N}$, such that $W^k \in p^{-1}(U)$, and $G^k \in U$, for every $k > N$.

Thus, the sequence $\{G^k\}_{k \in \mathbb{N}}$ converges to $G \in \mathcal{G}^*$, which proves that \mathcal{G}^* is compact. \square

4.2 The other main properties of the Grassmannian

Let Σ be the set of permutations between the elements $s \in \mathcal{S}$. For every $\sigma \in \Sigma$, we let $P_\sigma \in \mathbb{R}^{\mathcal{S} \times \mathcal{S}}$ be the corresponding permutation matrix. The elements of \mathcal{S} are ranked, so that the first or "upper" elements (for a matrix) are those of Ω and the lower of $\underline{\mathcal{S}}$. For every $V \in \mathbb{R}^{\mathcal{S} \times J}$, $P_\sigma.V \in \mathbb{R}^{\mathcal{S} \times J}$ is obtained by permuting the matrix' rows along σ . From the definition of \mathcal{G} , for every $V \in \mathcal{G}$, there exists $\sigma \in \Sigma$, which needs not be unique, such that the last J rows of $P_\sigma.V$ are linearly independent.

Thus, for each $\sigma \in \Sigma$, we let:

$$\mathcal{G}_\sigma := \left\{ V \in \mathbb{R}^{\mathcal{S} \times J} : P_\sigma.V = \begin{pmatrix} W \\ V^* \end{pmatrix} \in \mathbb{R}^{\mathcal{S} \times J}, \text{ with } W \in \mathbb{R}^{(\mathcal{S}-J) \times J} \text{ and } \text{rank } V^* = J \right\};$$

$$\mathcal{G}_\sigma^* := \{ \langle V \rangle : V \in \mathcal{G}_\sigma \}.$$

For each $\sigma \in \Sigma$, the generic vector space $G \in \mathcal{G}_\sigma^*$, admits, from above, a unique matrix representation of the form $P_\sigma^{-1} \cdot \begin{pmatrix} -\Phi_\sigma(G) \\ I \end{pmatrix}$, where $\Phi_\sigma(G) \in \mathbb{R}^{(\mathcal{S}-J) \times J}$ takes arbitrary values when G varies, and we let:

- $[I \mid \Phi_\sigma(G)]$ be the $(\mathcal{S}-J) \times \mathcal{S}$ matrix, whose first $(\mathcal{S}-J)$ columns are those of the identity matrix, $I \in \mathbb{R}^{(\mathcal{S}-J) \times (\mathcal{S}-J)}$, followed by the columns of $\Phi_\sigma(G) \in \mathbb{R}^{(\mathcal{S}-J) \times J}$;
- $K_\sigma : P \times \mathcal{G}_\sigma^* \times \mathbb{R}^{(\mathcal{S} \times L') \times J} \rightarrow \mathbb{R}^{(\mathcal{S}-J) \times J}$ be the map defined by $K_\sigma(p, G, V) := [I \mid \Phi_\sigma(G)].P_\sigma.V_p$.

Claim 3 *Let $\sigma \in \Sigma$ and $G \in \mathcal{G}_\sigma^*$ be given. The following Assertions hold:*

- (i) $\{\mathcal{G}_\sigma\}_{\sigma \in \Sigma}$ is an open cover of \mathcal{G} ;
- (ii) $G = \{z \in \mathbb{R}^{\mathcal{S}} : [I \mid \Phi_\sigma(G)].P_\sigma z = 0\}$;
- (iii) Φ_σ is a homeomorphism;

- (iv) $\{\mathcal{G}_\sigma^*\}_{\sigma \in \Sigma}$ is an open cover of \mathcal{G}^* ;
- (v) \mathcal{G}^* is a manifold without boundary;
- (vi) the map $(p, V) \in P \times \mathbb{R}^{(S \times L') \times J} \rightarrow K_\sigma(p, G, V) \in \mathbb{R}^{(S-J) \times J}$ is C^∞ ;
- (vii) the sets $\text{Im } K_\sigma$, \mathcal{G}_σ , \mathcal{G}_σ^* and \mathcal{G}^* are manifolds of dimension $v^{**} := (S - J).J$; the derivative $D_V K_\sigma(p, G, V)$ has full rank, v^{**} .

Proof Throughout, $\sigma \in \Sigma$, is given and we typically assume, w.l.o.g., that $\sigma = Id$.

Assertion (i) results from the definitions. □

Assertion (ii) Let $G \in \mathcal{G}_{Id}^*$ be given. The relation $[I \mid \Phi_{Id}(G)] \begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix} = 0$ holds from the definition. Let $z \in G$ be given. From above, there exists $z' \in \mathbb{R}^J$, such that $z = \begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix} z'$. Hence, the relation $[I \mid \Phi_{Id}(G)] z = [I \mid \Phi_{Id}(G)] \begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix} z' = 0$ holds and the relation $G \subset \{z \in \mathbb{R}^S : [I \mid \Phi_{Id}(G)] z = 0\}$ follows.

Conversely, let $z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^S$ be such that $[I \mid \Phi_{Id}(G)] z = 0$, where $z_1 \in \mathbb{R}^{S-J}$ and $z_2 \in \mathbb{R}^J$. From the above definitions, the relation $[I \mid \Phi_{Id}(G)] z = 0$ is written:

$$z_1 = -\Phi_{Id}(G) z_2, \text{ that is, } z = \begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix} z_2 \in \left\langle \begin{pmatrix} -\Phi_{Id}(G) \\ I \end{pmatrix} \right\rangle = G. \quad \square$$

Assertion (iii) From above, the map Φ_{Id} is one-to-one and onto. We show that Φ_{Id} is bicontinuous, that is, $\Psi_{Id} : G \in \mathcal{G}_{Id}^* \mapsto [I \mid \Phi_{Id}(G)] \in \mathcal{Z}^\circ$ is bicontinuous.

Let $G \in \mathcal{G}_{Id}^*$, $\varepsilon \in]0, 1[$, $W := \Psi_{Id}(G)$ and $\mathcal{W}^n := \{W' \in \mathcal{Z}^\circ : \|W' - W\| < \varepsilon/n\}$ be given, for every $n \in \mathbb{N}$. From the definitions, the relation $p(\mathcal{W}^1) \subset \mathcal{G}_{Id}^*$ holds and the reader will easily check that $\Psi_{Id}(p(\mathcal{W}^N)) \subset \mathcal{W}^1$, for $N \in \mathbb{N}$ large enough (which we set as given). By construction, $U := p(\mathcal{W}^N)$, is a neighbourhood of G , such that $\Psi_{Id}(U) := \Psi_{Id}(p(\mathcal{W}^N)) \subset \mathcal{W}^1$. Hence, Ψ_{Id} is continuous at $G \in \mathcal{G}_{Id}^*$ and Φ_{Id} is continuous.

Conversely, let $W \in \Psi_{Id}(\mathcal{G}_{Id}^*)$, $G := \Psi_{Id}^{-1}(W) \in \mathcal{G}_{Id}^*$ and a neighbourhood, U , of G , be given. From the definition of τ , there exists $\varepsilon > 0$, such that $\mathcal{W}^\circ := \{W' \in \mathcal{Z}^\circ : \|W' - W\| < \varepsilon\} \subset p^{-1}(U)$. Then, $\mathcal{W} := \mathcal{W}^\circ \cap \Psi_{Id}(\mathcal{G}_{Id}^*)$ is a neighbourhood of W in $\Psi_{Id}(\mathcal{G}_{Id}^*)$, such that $\Psi_{Id}^{-1}(\mathcal{W}) \subset p(\mathcal{W}^\circ) \subset U$. Thus, Ψ_{Id}^{-1} is continuous at W , hence, continuous. \square

Assertions (iv) and (v) result from Assertions (i)-(iii) and the definition of \mathcal{G}^* . \square

Assertion (vi) results from the definition of K_σ . \square

Assertion (vii) Let $\sigma \in \Sigma$ be given. From Assertion (iii) and above, \mathcal{G}_σ^* is homeomorphic to $\{P_\sigma^{-1} \cdot \begin{pmatrix} W \\ I \end{pmatrix}, W \in \mathbb{R}^{(\mathcal{S}-J) \times J}\}$, whose dimension is $v^{**} := (\mathcal{S} - J).J$.

Hence, from the definitions, Assertions (i) – (iv) and above, \mathcal{G}^* , \mathcal{G}_σ , \mathcal{G} and $\text{Im } K_\sigma$ are all manifolds of dimension v^{**} . Let $\mathcal{J} \subset \underline{\mathcal{S}}$ be the set of last J states and notice: $K_{Id}(p, G, V) := [I \mid \Phi_{Id}(G)].V_p = V_p(\mathcal{S} \setminus \mathcal{J}) + \Phi_{Id}(G).V_p(\mathcal{J})$, for $(p, G, V) \in P \times \mathcal{G}_{Id}^* \times \mathbb{R}^{(\mathcal{S} \times L') \times J}$.

The derivatives of $K_{Id}(p, G, V)$ with respect to payoffs, for $s \in \mathcal{S} \setminus \mathcal{J}$, are of the form of a $(\mathcal{S} - J) \times (\mathcal{S} - J)$ block diagonal matrix, P , of diagonal elements:

$$\text{the } J \times J(L+1) \text{ matrices } P(\omega) = \begin{pmatrix} (1, p_s^{iT}) & 0 & 0 & \dots & 0 \\ 0 & (1, p_s^{iT}) & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & (1, p_s^{iT}) \end{pmatrix}, \text{ for every}$$

$$(i, \omega := (s, p_s^i)) \in I \times \Omega_i,$$

$$\text{and } J \times J(L+1) \text{ matrices } P(s) = \begin{pmatrix} (1, p_s^T) & 0 & 0 & \dots & 0 \\ 0 & (1, p_s^T) & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & (1, p_s^T) \end{pmatrix}, \text{ for every } s \in \underline{\mathcal{S}} \setminus \mathcal{J}.$$

The matrix P , therefore, has rank $(\mathcal{S} - J).J$. It follows from above that the derivative $D_V K_{Id}(p, G, V)$ has maximal rank, $v^{**} = (\mathcal{S} - J).J$. \square

5 The pseudo-equilibrium manifold and existence theorem

We now define agents' demands, characterize the pseudo-equilibrium manifold and prove the full existence of the pseudo-equilibrium. Throughout, $\lambda := (\lambda_s) \in \mathbb{R}_{++}^{\underline{\mathbf{S}}}$ and $\lambda_0 := 1$ are set as given and we let $\tilde{x} := (\tilde{x}_s) := (\lambda_s \cdot x_s)$, for every $x := (x_s) \in \mathbb{R}_{++}^{L \times \underline{\mathbf{S}}}$.

5.1 The demand and excess demand correspondences

The first agent's ($i = 1$) demand is defined as: $D_1 : (y, p) \in \mathbb{R}_{++} \times P \mapsto D_1(y, p) := \arg \max u_1(x)$, for $x \in \{x \in X_1 : \tilde{p} \cdot x = y\}$, where $y > 0$ is given. As classical results, in a standard economy, D_1 is a C^∞ map, such that $\lim_{p \rightarrow \bar{p}} \|D_1(y, p)\| = +\infty$, if $\bar{p} \in \partial P \setminus \{0\}$.

Similarly, for each $i \in I \setminus \{1\}$, the agent's demand correspondence, $D_i : (p, G, e_i) \in P \times \mathcal{G}^* \times X_i \mapsto \arg \max u_i(x)$ for $x \in \{x \in X_i : \tilde{p} \cdot (x - e_i)(\underline{\mathbf{S}}') = 0 \text{ and } p \square_i (x - e_i) \in G(\mathcal{S}_i)\}$, is a continuous map in a standard economy, which is C^∞ with respect to $(p, e_i) \in P \times X_i$.

Using Walras' law, we select one good, say $l = 1$. We recall that $\dim P = l^* := (\underline{\mathbf{S}} + 1)(L - 1)$. For every $i \in I$, and every consumption $x_i \in X_i$, we denote by $x_i^* := (x_{is}^*) \in \mathbb{R}_{++}^{l^*}$, the extracted vector of x_i , which drops all consumptions in states $s \notin \underline{\mathbf{S}}'$ and in good $l = 1$. We denote similarly (with stars) the extracted demands in \mathbb{R}^{l^*} and define the excess demand correspondence, $Z : \mathbb{R}_{++}^{l^*+1} \times \mathcal{G}^* \times \mathbb{R}_{++}^{e^*} \rightarrow \mathbb{R}^{l^*}$, namely:

$$(y, p, G, e := (e_i)) \mapsto Z(y, p, G, e) := D_1^*(y, p) + \sum_{i \in I \setminus \{1\}} D_i^*(p, G, e_i) - \sum_{i \in I} e_i^*.$$

From above, Z is a same C^∞ map, with partial derivative: $D_{e_1^*} Z(y, p, W, (e_i)) = -I$.

5.2 The pseudo-equilibrium manifold's characterization and properties

For each $\sigma \in \Sigma$, we now consider the maps:

- $h : (y, p, e_1) \in \mathbb{R}_{++} \times P \times \mathbb{R}_{++}^{l^*} \mapsto h(y, p, e_1) := (\tilde{p} \cdot e_1 - y) \in \mathbb{R}$;

- $(y, p, G, e) \in \mathbb{R}_{++} \times P \times \mathcal{G}_\sigma^* \times \mathbb{R}^{e^*} \mapsto Z(y, p, G, e) \in \mathbb{R}^{l^*}$, as defined above;
- $K_\sigma : (p, G, V) \in P \times \mathcal{G}_\sigma^* \times \mathbb{R}^{(S \times L') \times J} \mapsto [I \mid \Phi_\sigma(G)] \cdot P_\sigma \cdot V_p \in \mathbb{R}^{v^{**}}$, along Section 4;
- $H_\sigma : (y, p, G, e, V) \in \mathbb{R}_{++} \times P \times \mathcal{G}_\sigma^* \times \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J} \mapsto (h(y, p), Z(y, p, G, e), K_\sigma(p, G, V))$.

From Claim 3, the pseudo- λ -equilibrium manifold, \mathcal{E}_λ^* , coincides with the union of inverse images, $\cup_{\sigma \in \Sigma} H_\sigma(0)^{-1}$. Its properties therefore stem from the Claims below.

Claim 4 *Given $\sigma \in \Sigma$, the image 0 is a regular value of the map H_σ , which is continuous, and class C^∞ with respect to the (y, p, e, V) derivatives.*

Proof Let $\sigma \in \Sigma$ be given. From Claim 3, the proof that H_σ is continuous and C^∞ with respect to (y, p, e, V) is standard and akin to Duffie-Shafer (1985, pp. 292-293). To show that 0 is regular, consider the derivative of H_σ with respect to y, e_1^* and V :

$$D_{(y, e_1^*, V)} H_\sigma(y, p, G, e, V) := \begin{pmatrix} D_y h(y, p) = -1 & D_y D_1^*(y, p) & 0 \\ D_{e_1^*} h(y, p) & D_{e_1^*} Z(y, p, G, e) = -I & 0 \\ 0 & 0 & D_V K_\sigma(p, G, V) \end{pmatrix}.$$

We need only show the above matrix has full rank, $1+l^*+v^{**}$ (for all (y, p, G, e, V)) or,

from Claim 3, that the matrix $A := \begin{pmatrix} D_y h(y, p) & D_y D_1^*(y, p) \\ D_{e_1^*} h(y, p) & D_{e_1^*} Z(y, p, G, e) \end{pmatrix}$ has rank l^*+1 .

The relations $D_{e_1^*} h(y, p) = \tilde{p}^* = -D_{e_1^*} Z(y, p, G, e)\tilde{p}^*$ and $D_y D_1(y, p) \gg 0$ hold from the definitions and Assumption A2, whereas the relation $D_y D_1(y, p) \cdot \tilde{p} = 1$ follows from differentiating the condition, $D_1(y, p) \cdot \tilde{p} = y$, on demand. Hence, the matrices A and

$$\begin{pmatrix} D_y D_1^*(y, p) \cdot \tilde{p}^* - D_y D_1(y, p) \cdot \tilde{p} & D_y D_1^*(y, p) \\ 0 & -I \end{pmatrix},$$

in which $D_y D_1^*(y, p) \cdot \tilde{p}^* - D_y D_1(y, p) \cdot \tilde{p} < 0$,

have the same rank. The relation $\text{rank} A = l^* + 1$ follows. \square

Claim 5 \mathcal{E}_λ^* is a submanifold of $\mathbb{R}_{++}^{l^*+1} \times \mathcal{G}^* \times \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{v^*}$ without boundary of dimension $e^* + v^*$. Hence, π_λ is a map between manifolds of the same dimension.

Proof From Claims 3 and 4 and the pre-image theorem, the pseudo- λ -equilibrium set, $\mathcal{E}_\lambda^* = \cup_{\sigma \in \Sigma} H_\sigma(0)^{-1}$, is a boundaryless submanifold of $\mathbb{R}_{++}^{l^*+1} \times \mathcal{G}^* \times \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$ of dimension $(l^* + 1 + v^{**} + e^* + v^*) - (1 + l^* + v^{**}) = e^* + v^*$. \square

Claim 6 The map $\pi_\lambda : \mathcal{E}_\lambda^* \rightarrow \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$ is smooth and proper, that is, the inverse image by π_λ of a compact set is compact.

Proof The map π_λ is smooth from the definitions. To show that it is proper, let $Y \subset \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$ be a compact set and $\{ \mathcal{C}^k := (y^k, p^k, G^k, (e_i^k), V^k) \}_{k \in \mathbb{N}}$ be a given sequence of elements of $\pi_\lambda^{-1}(Y)$. Since Y is compact, the sequence $\{(e_i^k), V^k\}_{k \in \mathbb{N}}$ may be assumed to converge, say to $((e_i), V) \in Y$. From the above limit relation on the demand map D_1 and relation $\mathcal{E}_\lambda^* = \cup_{\sigma \in \Sigma} H_\sigma(0)^{-1}$, the price sequence has a positive lower bound. Hence, $\{ (y^k, p^k) \}_{k \in \mathbb{N}}$ is assumed to converge, say to $(y, p) \in \mathbb{R}_{++} \times P$. From Claim 2, the sequence $\{ G^k \}_{k \in \mathbb{N}}$ may be assumed to converge, say to $G \in \mathcal{G}^*$.

Let $\mathcal{C} := (y, p, G, (e_i), V) := \lim_{k \rightarrow \infty} \mathcal{C}^k$ be given from above. From Claims 3-4 and the relation $\mathcal{E}_\lambda^* = \cup_{\sigma \in \Sigma} H_\sigma(0)^{-1}$, there exists $\sigma \in \Sigma$, such that the relations $H_\sigma(\mathcal{C}^k) = 0$ hold, for $k \in \mathbb{N}$ big enough. From Claim 4, the latter relations pass to the limit and yield: $H_\sigma(\mathcal{C}) = 0$ and, therefore, $\lim_{k \rightarrow \infty} \mathcal{C}^k = \mathcal{C} \in \mathcal{E}_\lambda^*$. Thus, $\pi_\lambda^{-1}(Y)$ is compact. \square

Lemma 2 There is a regular value, $((e_i^*), V^*)$, of π_λ , such that $\#\pi_\lambda^{-1}((e_i^*), V^*) = 1$.

Proof See the Appendix. \square

The full existence of pseudo- λ -equilibria follows from the above properties:

Theorem 1 For every $\lambda \in \mathbb{R}_{++}^S$ and every collection of endowments and payoffs, $(e, V) \in \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{(S \times L') \times J}$, a standard economy, $\mathcal{E}_{(e, V)}$, admits a pseudo- λ -equilibrium. Moreover, the set of regular values, $\mathcal{R}_{\pi_\lambda}$, of π_λ , is open and of null complement.

Proof As standard from modulo 2 degree theory, if $f : X \rightarrow Y$ is a smooth proper map between two boundaryless manifolds of same dimension, Y being connected, the number, $\#f^{-1}(y)$, of elements $x \in X$, such that $y = f(x)$, is the same, mod. 2, for every regular value $y \in Y$. In particular, if one regular value, y , of f , satisfies $\#f^{-1}(y) = 1$, then, $f^{-1}(y)$ is non-empty for every $y \in Y$. Indeed, $y \in Y$ is regular by definition if $f^{-1}(y) = \emptyset$. From Claims 5-6 and Lemma 2, the map, π_λ , meets all the desired conditions above, for $X := \mathcal{E}_\lambda^*$ and $Y := \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{v^*}$. It follows that, for every pair $(e, V) \in \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{v^*}$, a standard economy, $\mathcal{E}_{(e,V)}$, admits a pseudo- λ -equilibrium. Since π_λ is proper, its sets of singular values, $\mathcal{R}_{\pi_\lambda}^c$, is closed (i.e., $\mathcal{R}_{\pi_\lambda}$ is open). From Sard's Theorem (see Milnor, 1997, p. 10), $\mathcal{R}_{\pi_\lambda}^c$ is of zero measure. \square

6 The existence of equilibrium

6.1 The full existence of equilibrium with nominal or numeraire assets

Nominal and numeraire assets are defined in Section 2. The definition of a numeraire differs from Geanakoplos-Polemarchakis' (1986), where it is one good. From Lemma 1, pseudo-equilibria and equilibria coincide with such assets, which yields:

Theorem 2 *Let $\lambda := (\lambda_s) \in \mathbb{R}_{++}^{\mathbb{S}}$ and $e := (e_i) \in \mathbb{R}_{++}^{e^*}$ be given, and let $V \in \mathbb{R}^{(S \times L') \times J}$ be the payoff matrix of either nominal, or numeraire, assets. In a standard economy, $\mathcal{E}_{(e,V)}$, there exist prices, $(p, q) \in P \times \mathbb{R}^J$, consumptions, $x := (x_i) \in \mathbb{R}_{++}^{e^*}$, and portfolios, $z := (z_i) \in \mathbb{R}^{J \times I}$, such that $q := \sum_{s \in \mathbb{S}} \lambda_s V_p(s)$ and (p, q, x, z) defines an equilibrium.*

Proof Theorem 2 is a direct consequence of Theorem 1 and Lemma 1, above. \square

The existence of equilibrium on arbitrage-free purely financial markets was proved in De Boisdeffre (2007) in a broader setting, where no agent needed to have ordered

preferences or full information. When the economy is standard along Section 2, Theorem 2 shows the additional result that equilibria can be supported by any state prices, $\lambda \in \mathbb{R}_{++}^{\mathbb{S}}$. With symmetric information, this result is known as Cass' (1984).

To extend Cass' theorem, Definition 2 introduced so-called "*pseudo- λ -equilibria*". With nominal assets and symmetric information, it follows from Assumption A2, Lemma 1 and the no-arbitrage condition, that the budget constraints in Cass (1984) and in Definition 2 are equivalent. Moreover, one agent ($i = 1$) has a single budget constraint, *a la Debreu*, while the other agents are constrained *a la Radner*, i.e., in every state of nature. This so-called "*Cass trick*" is the device, which both this paper and Cass' (1984) use to prove that any collection of state prices supports equilibria on purely financial markets. But the two papers' techniques of proof differ.

With standard assumptions and a slightly different financial structure, Geanakoplos-Polemarchakis (1986) also demonstrates the full existence of financial equilibria with numeraire assets and symmetric information, and shows their generic local uniqueness. The issue of supporting equilibrium state prices is not considered. Thus, Theorem 2 extends Cass' to a larger set of assets and information signals.

With reference to the REE model, Theorem 2 provides a stronger existence result. As recalled in Section 3, the Radner (1979) model has no spot markets, for goods, nor real asset markets. Whereas generic existence holds in the latter model, full existence holds from Theorem 2 in a broader setting, which includes spot markets.

6.2 The generic existence of equilibrium with arbitrary assets

The proof proceeds as follows: given $\lambda := (\lambda_s) \in \mathbb{R}_{++}^{\mathbb{S}}$, a pseudo- λ -equilibrium of a standard economy, $\mathcal{E}_{(e,V)}$, exists from Theorem 1 and coincides with equilibrium for an open set of null complement of payoffs and endowments. The latter is supported

by the state prices, λ , i.e., equilibrium prices, $(p, q) \in P \times \mathbb{R}^J$, satisfy $q = \sum_{s \in \underline{\mathbf{S}}} \lambda_s V_p(s)$.

Lemma 3 *Given $\lambda \in \mathbb{R}_{++}^{\underline{\mathbf{S}}}$, there exists an open set, Ω , of null complement in the set of regular values, $\mathcal{R}_{\pi_\lambda}$, of π_λ , such that, for every $(e, V) \in \Omega$, a standard economy, $\mathcal{E}_{(e, V)}$, admits a pseudo- λ -equilibrium, (x, y, p, G, e, V) , such that $\text{rank} V_p(\underline{\mathbf{S}}) = J$.*

Proof See the Appendix. □

Theorem 3 *Given $\lambda := (\lambda_s) \in \mathbb{R}_{++}^{\underline{\mathbf{S}}}$, there exists an open set, $\Omega \subset \mathcal{R}_{\pi_\lambda}$, of null complement in $\mathbb{R}_{++}^{e^*} \times \mathbb{R}^{v^*}$, such that, for every $(e, V) \in \Omega$, a standard economy, $\mathcal{E}_{(e, V)}$, admits an equilibrium supported by the state prices λ .*

Proof Theorem 3 results immediately from Theorem 1 and Lemmas 1 and 3. □

Moreover, we notice from the proof of Lemmas 2 and 3 that equilibria in $\pi_\lambda^{-1}(e, V)$ are in odd number, and are continuous functions of (e, V) , for every $(e, V) \in \Omega$.

7 Concluding remarks

The current paper drops Radner's (1979) inferences, in an attempt to meet the main criticisms faced by REE models. This led existence results to be stronger and to apply to a broader setting with spot markets. Under close assumptions, the paper's results also extend classical theorems of symmetric information, such as Cass' (1984), or Duffie-Shafer's (1985). Yet, the paper's sequential equilibrium concept retains the perfect foresight hypothesis. This restriction needs be explained.

The paper's additional existence results, in relation to the arbitrary choice of state prices, are presented in sub-Section 7.1. Sub-Sections 7.2 and 7.3 are devoted to a discussion of the perfect foresight restriction.

7.1 The existence of equilibrium for arbitrary state prices

On a purely financial market with symmetric information, Cass (1984) shows that any collection of state prices is consistent with the existence of a financial equilibrium, whether markets be complete or not. The current paper extends this result to other financial and information structures than Cass' (see Section 6). It also proves the generic existence of equilibrium for arbitrary assets, signals and state prices. Beyond Cass' paper, the fact that the existence of equilibria, or pseudo-equilibria, is consistent with arbitrary state prices and assets is little emphasized in the literature, as shown by the example of Geanakoplos-Polemarchakis' (1986) paper. The fact that state prices may be set as given is yet easily understood from the definition of equilibrium. In Definition 1 above, exogenously given state prices, $\lambda := (\lambda_s) \in \mathbb{R}_{++}^{\mathbf{S}}$ (using the model's notations) do not affect any separate budget constraint of the second period. Moreover, the equilibrium asset price, when it exists, is $q := \sum_{s \in \mathbf{S}} \lambda_s V_p(s)$, where $p := (p_s)_{s \in \mathbf{S}}$ is the collection of spot prices. The variables, which adapt, are agents' consumptions and portfolios and the spot prices. There is no need to let λ vary to reach equilibrium. It may be fixed.

In complete markets, the argument of this property follows immediately from the standard equivalence between the Arrow-Debreu and Radner equilibria. Indeed, let a consumption price, $p^* = (p_s^*) \in \mathbb{R}_+^{S'L}$, and an allocation, $(x_i) \in \mathbb{R}_+^{S'LI}$, define an equilibrium of some Arrow-Debreu economy, which exists in standard conditions.³ Let \mathcal{E} be the equivalent Radner economy, with $\#S$ random states and Arrow-securities, whose payoffs define the $S \times S$ identity matrix. Let state prices, $\lambda := (\lambda_s) \in \mathbb{R}_{++}^S$, and $\lambda_0 := 1$, be given. From the equivalence between the Arrow-Debreu and Radner

³ We recall that agents have the same information, $s = \mathbf{s}$, and face no financial market, in the Arrow-Debreu economy, but a complete market of contingent goods.

equilibria, the asset price, $\lambda \in \mathbb{R}_{++}^S$, and commodity price, $p = (p_s) \in \mathbb{R}_+^{S'L}$, where $p_s = \frac{p_s^*}{\lambda_s^*}$, for every $s \in S'$, are joint equilibrium prices in the Radner economy, \mathcal{E} .

Thus, any collection of state prices supports an equilibrium if financial markets are complete. The same result holds in incomplete markets, from the Cass trick, if assets are nominal and agents have symmetric information, or from Theorem 2 above, when a broader class of assets and information signals is considered.

7.2 The two sides of rational expectations

In the literature, the rational expectations' assumption refers either to Radner's (1979) treatment of asymmetric information, or to the Arrow (1953) - Radner (1972) perfect foresight hypothesis, which characterizes the classical sequential equilibrium.

As seen in Section 3, in Radner (1979), equilibrium prices are typically distinct in agents' joint information signals. This outcome holds in a specific model and does not prove, by itself, the generic existence result. The latter requires that agents be aware of a map relating signals to equilibrium prices, so as to typically infer joint signals from prices. This map is endogenous. It depends on all agents' characteristics.

Such characteristics are typically private. If so, the construction of a forecast function (relating unobservable signals to observable prices), would normally require agents to truthfully disclose their private information, in relation to the observed price, at every occurrence of the joint signals. This requirement is more demanding than sharing information at just one period. Alternatively, if all individual characteristics were commonly known by every agent, inferring a price function from them would require from traders a reckoning skill, which is seen as unrealistic.

On same grounds, the perfect foresight hypothesis, faces similar criticisms as Kurz and Wu's (1996): "*agents need to know the maps from states at future dates*

to prices in the future and it is entirely unrealistic to assume that agents can find out what this sequence of maps is." Radner (1982) himself admits it "seems to require of the traders a capacity for imagination and computation far beyond what is realistic". Yet, the classical concept of sequential equilibrium relies on perfect foresight.

Probably the first, best known and most radical escape to the assumption comes from the temporary equilibrium framework, introduced by J. Hicks and developed by J.-M. Grandmont later. At a temporary equilibrium, agents have exogenous anticipations and their demands clear on all current markets. Agents typically revise their beliefs and their plans ex post, at each period, after the realized state of nature and the actual spot prices (if uncorrectly anticipated) are observed. Moreover, temporary equilibrium allocations need no longer clear tomorrow on spot markets, where consumers may also face bankruptcy, due to mistaken anticipations.

A less radical approach is referred to as bounded rationality. In this line of research, Kurz' (1994) equilibrium allows agents to lack the "*structural knowledge*" of how equilibrium prices obtain. This unawareness may be due to uncertainty about the beliefs, characteristics and actions of other agents. Agents' behaviour consists in ruling out anticipations, which observation contradicts. It typically leads to an additional uncertainty over future prices, which Kurz calls "*endogenous uncertainty*", describes as the major cause of economic fluctuations, and shows to be consistent with heterogenous beliefs. Bounded rationality may require of agents less awareness than perfect foresight to compute a price map, but requires no less reckoning skill.

The current paper stems from a dissatisfaction with both rationality assumptions, but presents a hybrid model. Borrowing to the temporary equilibrium, anticipations are exogenous in unrealizable states, $S \setminus \underline{S}$. Borrowing to the standard

sequential equilibrium, anticipations on realizable spot markets are endogenous and perfect. This setting is a step towards dropping both sides of rational expectations.

7.3 From perfect to correct foresight: revisiting sequential equilibrium

The paths of the classical models of temporary and sequential equilibrium diverge, as recalled by Grandmont (1982). The two concepts might yet be reconciled.

To see this, an equilibrium could be defined as "*sequential*" whenever agents make self-fulfilling - if not perfect - anticipations, at other conditions of equilibrium unchanged. That is, agents would anticipate sets of admissible prices, containing (but not necessarily limited to) future equilibrium prices, on each spot market. Call the latter sequential equilibrium a "*correct foresight*" equilibrium (CFE). With symmetric information, classical sequential equilibria are CFE, and the two concepts coincide when agents anticipate one price on each spot market with certainty.

If markets clear and agents keep their plans unchanged, face no bankruptcy tomorrow and, yet, have no price model to refer to, their anticipations should typically be uncountable. Ex ante, a continuum of anticipations would not be ruled out completely by cautious rational agents. Call "*forecast*" a joint expectation, $(s, p) \in S \times \Delta$, of a state of nature, s , and a spot price, p , in state s . With no price model, the generic i^{th} agent's anticipation set, say Ω_i , consists in her plausible forecasts, given her idiosyncratic information and beliefs. Forecasts are exogenous and typically differ across agents. Thus, the CFE shares a characteristic of temporary equilibria: anticipations are exogenously given and reflect beliefs. Contrarily to the standard sequential equilibrium, they result from no calculation and require no price map. So the argument that the CFE may reconcile sequential and temporary equilibria.

At a CFE, the collection of agents' anticipation sets, (Ω_i) , replaces the previous

Sections' information structure, (S_i) . Anticipations play the role of random states. So, dropping perfect foresight makes the study of agents' inferences, and of the existence of equilibrium, more complex. It implies dealing with infinite dimensional models. On existence issues, conditions need be studied, which make exogenous anticipations self-fulfilling. Keeping perfect foresight, in a first step, is easy to explain.

First, perfect foresight permits to consider a finite economy, where exposition and proofs are simpler. In a finite setting, for instance, the no-arbitrage principle of Section 3 relies on simple induction arguments (Cornet-De Boisdeffre, 2009). This inference behaviour leads to the same results when agents' anticipation sets are infinite. But the proof relies on infinite dimensional topology (De Boisdeffre, 2016).

Second, it is standard to prove a property in a finite economy, first, in order to extend it to the infinite setting. The typical method applies an asymptotic argument to a sequence of finite economies with the desired property, which approximate the infinite economy more and more closely. This method can be used to study correct foresight equilibria, building on existence results of finite economies with perfect foresight (see De Boisdeffre, 2007 & 2015). So the usefulness of the above theorems.

Finally, assuming perfect foresight circumvents a difficulty in finding alternative conditions, which make anticipations self-fulfilling ex post. To our best knowledge, no such alternative exists in the literature. Yet, CFE prices, which depend on agents' private, possibly changing, beliefs, describe a "*minimum uncertainty set*". Without *structural knowledge*, agents cannot infer which particular price of that limited set will prevail tomorrow (any of which can prevail). Yet, agents' anticipations, if they embed the minimum uncertainty set, are always self-fulfilling, along De Boisdeffre (2015). The issue is a topic of research by itself and goes beyond the paper's scope. So the focus on finite anticipation sets, which led to assume perfect foresight.

Appendix

Lemma 1 Given $\lambda := (\lambda_s) \in \mathbb{R}_{++}^{\mathbf{S}}$, let (x, y, p, G, e, V) be a pseudo- λ -equilibrium of a standard economy, $\mathcal{E}_{(e, V)}$, and let $q := \sum_{s \in \mathbf{S}} \lambda_s V_p(s)$ be given. There exists portfolios, $z \in \mathbb{R}^{J \times I}$, such that (p, q, x, z) defines an equilibrium of the economy $\mathcal{E}_{(e, V)}$ whenever one of the following Conditions holds:

- (i) assets are nominal or numeraire;
- (ii) $\text{rank} V_p(\mathbf{S}) = J$, i.e., $G \in \underline{\mathcal{G}}$.

Proof Let $(\lambda := (\lambda_s), e, V) \in \mathbb{R}_{++}^{\mathbf{S}} \times \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{v^*}$ be given. Let $(x := (x_i), y, p, G, e, V)$ be a pseudo- λ -equilibrium of a standard economy, $\mathcal{E}_{(e, V)}$, and denote $q := \sum_{s \in \mathbf{S}} \lambda_s V_p(s)$.

Assume that V is the payoff matrix of either nominal or numeraire assets, or that the condition $\text{rank} V_p(\mathbf{S}) = J$ holds. If assets are numeraire, there exists a vector, $a \in \mathbb{R}_+^L \setminus \{0\}$, and matrix, $W \in \mathbb{R}^{S \times J}$, such that $V(s, p) = p \cdot a W(s)$, for every forecast $(s, p) \in S \times \Delta$ (hence, $p \cdot a > 0$). If assets are nominal, there exists a fixed matrix, $W \in \mathbb{R}^{S \times J}$, such that $V_{p'} = W$, for every $p' \in P$. In both cases, the matrix W has full rank (redundant assets are eliminated).

From the definitions and above, the relation $G = \langle V_p \rangle$ holds, under Condition (i) or (ii) of Lemma 1, and implies that $G(\mathcal{S}_i) = \langle V_p(\mathcal{S}_i) \rangle$, for every $i \in I$, from Definition 2-(c). From Definition 2-(b), there exists $z_i \in \mathbb{R}^J$ (for each $i \in I \setminus \{1\}$) such that $p \square_i (x_i - e_i) = V_p(\mathcal{S}_i) z_i$. Let $z_1 := -\sum_{i \in I \setminus \{1\}} z_i$ and $z := (z_i) \in \mathbb{R}^{J \times I}$ be given. The relation $\sum_{i \in I} z_i = 0$ holds. From Definition 2-(d) and Assumption A4, the following relations also hold: $p \square_1 (x_1 - e_1) = -(\sum_{i \in I \setminus \{1\}} p_s \cdot (x_{is} - e_{is}))_{s \in \mathbf{S}} = -\sum_{i \in I \setminus \{1\}} V_p(\mathcal{S}_1) z_i = V_p(\mathcal{S}_1) z_1$.

From above, the collection, $\mathcal{C} := (p, q, x, z)$, is such that $(x_i, z_i) \in B_i(p, q, e_i, V)$, for every $i \in I$, and it meets Conditions (b)-(c) of Definition 1 of equilibrium. Let $i \in I \setminus \{1\}$

be given. From Assumption $A\mathcal{Q}$, the budget set $B_i(p, q, e_i, V)$ may be replaced by $B'_i(p, q, e_i, V) := \{ (x, z) \in X_i \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) = -q \cdot z \text{ and } p \square_i (x - e_i) = V_p(S_i)z \}$ in Definition 1 at no cost. From the definition of q and above, the i^{th} agent's pseudo- λ -equilibrium budget set coincides with $B_i^* := \{ x \in X_i : \exists z \in \mathbb{R}^J, (x, z) \in B'_i(p, q, e_i, V) \}$. Since x_i is optimal in B_i^* , the strategy (x_i, z_i) is optimal in $B_i(p, q, e_i, V)$ from above. By similar arguments, we show that (x_1, z_1) is optimal in $B_1(p, q, e_1, V)$. Thus, \mathcal{C} also meets Condition (a) of Definition 1 and is an equilibrium. \square

Lemma 2 *There is a regular value, $((e_i^*), V^*)$, of π_λ , such that $\#\pi_\lambda^{-1}((e_i^*), V^*) = 1$.*

Proof We order the set $\mathcal{S} := \Omega \cup \underline{\mathbf{S}}$, so that its "last" J elements be in $\underline{\mathbf{S}}$. We denote by s_j , for $j \in J$, these last states, and by $\mathcal{J} := \{s_j\}_{j \in J} \subset \underline{\mathbf{S}}$ their set. We set as given $\lambda_0 := 1$ and $\lambda := (\lambda_s) \in \mathbb{R}_{++}^{\underline{\mathbf{S}}}$ and let $\tilde{x} := (\tilde{x}_s) := (\lambda_s x_s)_{s \in \underline{\mathbf{S}}}$, for every $x := (x_s) \in \mathbb{R}^{\underline{\mathbf{S}}}$.

We let $a := (0, 1, 0, \dots, 0) \in \mathbb{R}^{L'}$ be the payoff bundle made of one unit of the first good and $V^* \in \mathbb{R}^{(S \times L') \times J}$ be the numeraire asset matrix defined (for each $j \in J$) by the payoffs $v_j^*(s) = a$, if $s = s_j$, and $v_j^*(s) = 0$ otherwise (N.B. choosing real assets is not restrictive, since a could be replaced by $a' := (1, 0, \dots, 0) \in \mathbb{R}^{L'}$ costlessly). We let $p^* := (p_s^{*l}) \in P$, such that $p_s^{*1} = \frac{1}{2}$ for all $s \in \mathcal{J}$, $G^* := \langle V_{p^*}^* \rangle$ and $q^* := \sum_{s \in \underline{\mathbf{S}}} \lambda_s V_{p^*}^*(s)$ be given. Then, $V_{p^*}^*(\mathcal{J}) = \frac{1}{2} I$ is half the identity matrix and $V_{p^*}^*(\mathcal{S} \setminus \mathcal{J}) = 0$.

From Assumption $A\mathcal{Q}$, we may choose agents' endowments, $(e_i^*) \in \times_{i \in I} X_i$, such that, for each $i \in I$, the relation $\nabla u_i(e_i^*) = p^{i*} := (p_s^{i*}) \in \mathbb{R}_{++}^{L \times S_i}$ holds, where $p_s^{i*} = \lambda_s p_s^*$, for every $s \in \underline{\mathbf{S}}'$, and $p^{i*}(S_i \setminus \underline{\mathbf{S}}) = p^i$ (if $S_i \neq \underline{\mathbf{S}}$). A standard economy, $\mathcal{E}_{((e_i^*), V^*)}$, obtains.

The relations $e_1^* \in D_1(y^*, p^*)$ and $e_i^* \in D_i(p^*, G^*, e_i^*)$ hold, for each $i \in I \setminus \{1\}$, and $\mathcal{C}^* := (y^* := p^* \cdot e_1^*, p^*, G^*, (e_i^*), V^*)$ is a pseudo- λ -equilibrium and defines a tradeless equilibrium, from Lemma 1. Indeed, the allocation (e_i^*) is interior, meets gradients'

conditions $(\nabla u_i(e_i^*) = p^{i*}, \text{ for each } i \in I)$ and budget constraints. By construction, the allocation, (e_i^*) , is Pareto-optimal in the following sense: "*there is no other attainable allocation, which preserves agents' wealth in unrealizable states and strictly increases the utility of one agent without decreasing the utility of another agent*". This follows from the optimality of e_i^* , for every $i \in I$, in the set $B_i(p^*) := \{x \in X_i : p^{i*} \cdot (x - e_i^*) \leq 0\}$.

Let $\mathcal{C} := (y, p := (p_s), G, (e_i^*), V^*) \in \pi_\lambda^{-1}((e_i^*), V^*)$ be a pseudo- λ -equilibrium, and $x := (x_i)$ be a related allocation. From Lemma 1, \mathcal{C} defines an equilibrium for the allocation x and price $q = \sum_{s \in \underline{\mathbf{S}}} \lambda_s V_p^*(s)$, and $G = \langle V_p^* \rangle = G^*$ holds, by construction. From Definition 2 and the identity $G = G^*$, the relations $u_i(x_i) \geq u_i(e_i^*)$ and $p_s^i \cdot x_{is} = p_s^i \cdot e_{is}^*$ hold for every $i \in I$ and every $s \in S_i \setminus \underline{\mathbf{S}}$. It follows from the Pareto-optimality of (e_i^*) that $x = (e_i^*)$ and, hence, that $\#\pi_\lambda^{-1}((e_i^*), V^*) = 1$. \square

The last part is to show that $((e_i^*), V^*)$ is regular, relying on Lemmata 1:

Lemmata 1 For every $i \in I \setminus \{1\}$ and every $p \in P$, let $D_i(p) \in \mathbb{R}_{++}^{L \times \mathbf{S}'}$ be the vector of the i^{th} agent's demands in realizable states, that is, $D_i(p) = D_i(p, G^*, e_i^*)(\underline{\mathbf{S}}')$. Let

$D_1(p) = D_1(y^*, p)$ and $Z(p) := \sum_{i \in I} D_i(p)$. For all $(i, p) \in I \setminus \{1\} \times P$, the following holds:

- (i) $\tilde{p}^T D_p(D_1(p)) = p^T D_p(\tilde{D}_1(p)) = -\tilde{D}_1(p)^T$;
- (ii) $h^T D_p(\tilde{D}_1(p)) h < 0, \forall h \in \mathbb{R}^{L \times \mathbf{S}' \setminus \{0\}}$, such that $\tilde{h} \cdot D_1(p) = h \cdot \tilde{D}_1(p) = 0$;
- (iii) $\tilde{p}^{*T} D_p(D_i(p^*)) = p^{*T} D_p(\tilde{D}_i(p^*)) = 0$;
- (iv) $D_p(\tilde{D}_i(p^*))$, is negative semi-definite.

Proof of Lemmata 1 Assertion (i) results from differentiating the budget constraint, $\tilde{p} \cdot D_1(p) = p \cdot \tilde{D}_1(p) = y^*$, which holds from the definition of demands. \square

Assertion (ii) For every $p \in P$, let $D(\tilde{p}) := D_1(p)$. From Assumptions A2-A3, it is standard that the map $\tilde{p} \mapsto D(\tilde{p})$ meets the relation, $\tilde{h}^T D_{\tilde{p}}(D(\tilde{p})) \tilde{h} < 0$, for every $h \in \mathbb{R}^{L \times \mathbf{S}' \setminus \{0\}}$, such that $\tilde{h} \cdot D(\tilde{p}) = 0$ (see, e.g., Dufie-Shafer, 1985, fact 4, p. 292).

Then, Assertion (ii) follows from the relation $h^T D_p(\widetilde{D}_1(p)) h = \widetilde{h}^T D_{\widetilde{p}}(D(\widetilde{p})) \widetilde{h}$, which holds from the definitions, for every $h \in \mathbb{R}^{L \times \underline{S}}$ and every $p \in P$. \square

Assertion (iii) Let $i \in I \setminus \{1\}$ be given. For every $p \in P$, the first budget constraint in the i^{th} agent's demand, $D_i(p, G^*, e_i^*)$, is written: $\widetilde{p} \cdot D_i(p) = \widetilde{p} \cdot e_i^* = \widetilde{p} \cdot D_i(p^*)$. Differentiating the latter at $p = p^*$ yields Assertion (iii). \square

Assertion (iv) Let $i \in I \setminus \{1\}$ be given. From the the satiated budget constraints at the agent's demand, the relations $\widetilde{p}^* \cdot D_i(p) \geq \widetilde{p}^* \cdot D_i(p^*)$ and $\widetilde{p} \cdot D_i(p) = \widetilde{p} \cdot D_i(p^*)$ hold and imply the following relations: $(\widetilde{p} - \widetilde{p}^*) \cdot (D_i(p) - D_i(p^*)) = (p - p^*) \cdot (\widetilde{D}_i(p) - \widetilde{D}_i(p^*)) \leq 0$, for every $p \in P$. Assertion (iv) follows as a standard corollary. \square

To complete the proof of Lemma 2, following Duffie-Shafer (1985, pp. 296-297), we let $E^* := \Phi_{Id}(G^*) = 0$ and $E := \Phi_{Id}(G)$, for every $G \in \mathcal{G}_{Id}^*$, and define the maps:

$$\begin{aligned} (y, p) \in \mathbb{R}_{++} \times P &\mapsto h^*(y, p) := h(y, p, e_1^*) = (\widetilde{p} \cdot e_1^* - y) \in \mathbb{R}; \\ (p, E) \in P \times \mathbb{R}^{v^{**}} &\mapsto Z^*(p, E) = Z(y^*, p, \Phi_{Id}^{-1}(E), (e_i^*)); \\ (p, E) \in P \times \mathbb{R}^{v^{**}} &\mapsto K^*(p, E) = K_{Id}(p, \Phi_{Id}^{-1}(E), V^*); \\ (y, p, E) \in \mathbb{R}_{++} \times P \times \Phi_{Id}(\mathcal{G}_{Id}^*) &\mapsto H^*(y, p, E) = (h^*(y, p), Z^*(p, E), K^*(p, E)) \text{ and derivative,} \end{aligned}$$

$$(y, p, E) \mapsto D H^*(y, p, E) := \begin{pmatrix} D_y h^*(y, p) = -1 & D_p h^*(y, p) & 0 \\ 0 & D_p Z^*(p, E) & D_E Z^*(p, E) \\ 0 & D_p K^*(p, E) & D_E K^*(p, E) \end{pmatrix}.$$

Showing that \mathcal{C}^* is a regular point of π_λ , is equivalent to showing that the derivative, $D H^*(y^*, p^*, E^*)$, has full rank, $1 + l^* + v^{**}$. The relations $K^*(p, E^*) = 0$, for every $p \in P$, hence, $D_p K^*(p^*, E^*) = 0$ hold from the definitions. Moreover, $\text{rank } D_E K^*(p^*, E^*) = v^{**} := (S - J)J$. Indeed, $V_{p^*}^*(\mathcal{J}) := \frac{1}{2}I$ and E^* is a $(S - J) \times J$ matrix. From the definitions, the relation $K^*(p^*, E) = \frac{1}{2}E$ holds for every $E \in \Phi_{Id}(\mathcal{G}_{Id}^*)$ and the derivative, $D_E K^*(p^*, E^*)$, has rank $(S - J)J = v^{**}$, no matter the value of E^* .

Therefore, it suffices to show that $D_p Z^*(p^*, E^*)$, has full rank l^* . We may write $D_p Z^*(p^*, E^*)$ as a $L\underline{\mathbf{S}}' \times (L-1)\underline{\mathbf{S}}'$ real matrix. If it fails to have full rank, there exists $h \in \mathbb{R}^{(L-1) \times \underline{\mathbf{S}}'} \setminus \{0\}$, such that $D_p Z^*(p^*, E^*) h = 0$. Let $h^\circ \in \mathbb{R}^{L \times \underline{\mathbf{S}}'}$ be the vector whose $\mathbb{R}^{(L-1) \times \underline{\mathbf{S}}'}$ extraction is h and whose components in good $l = 1$ are all zeros. Let $h^* \in \mathbb{R}^{L\underline{\mathbf{S}}'} \setminus \{0\}$ be such that $\widetilde{h}^* = h^\circ$. By construction, $D_p Z^*(p^*, E^*) h = 0$ implies $D_p Z(p^*) h^\circ := D_p Z(p^*) \widetilde{h}^* = 0$, and, hence:

$$\begin{aligned} 0 &= D_p \widetilde{Z}(p^*) h^* := D_p(\widetilde{D}_1(p^*)) h^* + \sum_{i \in I \setminus \{1\}} D_p(\widetilde{D}_i(p^*)) h^* \text{ and, from Lemmata 1,} \\ 0 &= p^{*T} D_p(\widetilde{D}_1(p^*)) h^* + \sum_{i \in I \setminus \{1\}} p^{*T} D_p(\widetilde{D}_i(p^*)) h^* = -\widetilde{D}_1(p^*) \cdot h^*. \end{aligned}$$

From Lemmata 1, the above relation, $\widetilde{D}_1(p^*) \cdot h^* = 0$, implies the following ones: $h^{*T} D_p(\widetilde{D}_1(p^*)) h^* < 0$ and $h^{*T} D_p \widetilde{Z}(p^*) h^* < 0$, contradicting the above, $D_p \widetilde{Z}(p^*) h^* = 0$. This contradiction proves that $D_p Z^*(p^*, E^*)$ and, from above, $D H^*(y^*, p^*, E^*)$ have full rank, i.e., $((e_i^*), V^*)$ is a regular value of π_λ . The proof of Lemma 2 is complete. \square

Lemma 3 *Given $\lambda \in \mathbb{R}_{++}^{\underline{\mathbf{S}}}$, there exists an open set, Ω , of null complement in the set of regular values, $\mathcal{R}_{\pi_\lambda}$, of π_λ , such that, for every $(e, V) \in \Omega$, a standard economy, $\mathcal{E}_{(e, V)}$, admits a pseudo- λ -equilibrium, (x, y, p, G, e, V) , such that $\text{rank} V_p(\underline{\mathbf{S}}) = J$.*

Proof Given $\lambda \in \mathbb{R}_{++}^{\underline{\mathbf{S}}}$, let $(e', V') \in \mathcal{R}_{\pi_\lambda}$ be a regular value of π_λ , which exists from Lemma 2. From Claims 3 and 4, Theorem 1, the implicit function theorem and the definition of a regular value, there exists a pseudo-equilibrium, $\mathcal{C}' := (y', p', G', e', V') \in \pi_\lambda^{-1}(e', V')$, and two (relatively) open sets, $W \subset \mathcal{E}_\lambda^*$ and $U \subset \mathcal{R}_{\pi_\lambda}$, containing \mathcal{C}' and (e', V') , respectively, which are mapped homeomorphically by π_λ .

We denote by $\mathcal{C} := (y, p, G, e, V)$ the generic element of W . We assume w.l.o.g. that $G \in \mathcal{G}_{Id}^*$ whenever $(y, p, G, e, V) \in W$. The maps, $(e, V) \in U \mapsto f_1(e, V) \in P$, and $(e, V) \in U \mapsto f_2(e, V) \in \mathcal{G}_{Id}^*$, defined by $(f_1(e, V) \cdot e_1, f_1(e, V), f_2(e, V), e, V) \in W$, for every $(e, V) \in U$,

are homeomorphisms from above. The map $\theta : (y, p, G, e, V) \in W \mapsto (p, \Phi_{Id}(G), e, V)$ is a homeomorphism and we let $W^\circ := \theta(W)$ be its image set. Moreover, the map,

$$H : \mathcal{C}^\circ := (p, E, e, V) \in W^\circ \mapsto (Z(p \cdot e_1, p, \Phi_{Id}^{-1}(E), e), K_{Id}(p, \Phi_{Id}^{-1}(E), V)) \in \mathbb{R}^{l^* + v^{**}},$$

is differentiable and zero-valued, from the definition and Claims 3 and 4 (we can identify $\Phi_{Id}^{-1}(E)$ to E for the derivations). From above, $DH = 0$ holds, which implies:

$$[D_{(p,E)} H(\mathcal{C}^\circ)] [D_{(e_1^*, V)} (f_1, f_2)(e, V)] + D_{(e_1^*, V)} H(\mathcal{C}^\circ) = 0 \text{ for } \mathcal{C}^\circ = (p, E, e, V) \in W^\circ.$$

The latter equation states that the rows of $D_{(e_1^*, V)} H(\mathcal{C}^\circ)$ are linear combinations of those of $D_{(e_1^*, V)} (f_1, f_2)(e, V)$. From the definitions and Claim 4, $D_{(e_1^*, V)} H(\mathcal{C}^\circ)$ has full rank, with $l^* + (S - J) \cdot J$ independent rows, and so does $D(f_1, f_2)(e, V)$. Thus, $rank Df_1(e, V) = l^*$ holds for every $(e, V) \in U$. Consider now maps and sets as follows:

$$\Psi : (e, V) \in U \mapsto (f_1(e, V), V) \in P \times \mathbb{R}^{(S \times L') \times J};$$

$$\Theta : (p, V) \in P \times \mathbb{R}^{(S \times L') \times J} \mapsto V_p \in \mathbb{R}^{S \times J};$$

$$Q := \Theta^{-1}(\underline{\mathcal{G}}), \text{ where } \underline{\mathcal{G}} := \{V \in \mathbb{R}^{S \times J}, rank V(\underline{\mathbf{S}}) = J\}.$$

The set $\underline{\mathcal{G}}$ is open, and of null complement in $\mathbb{R}^{S \times J}$, from Sard's theorem. From above, the derivatives $D\Psi$ and $D_V \Theta$ clearly have maximal rank, respectively, $l^* + v^*$ and $S \cdot J$, so that Ψ and Θ are submersions. Since Θ is a submersion and $\underline{\mathcal{G}}$ is open and of null complement, so is $Q := \Theta^{-1}(\underline{\mathcal{G}})$ in $P \times \mathbb{R}^{(S \times L') \times J}$. Let $\Psi(U)$ be the image set of U by Ψ . Then, $Q' := Q \cap \Psi(U)$ is open and of null complement in $\Psi(U)$, which is open. By the same token, $\Omega_U := \Psi^{-1}(Q')$ is open and of null complement in U .

From a classical local to global argument, there exists an open set, $\Omega \subset \mathcal{R}_\pi$, with null complement in $\mathcal{R}_\pi \subset \mathbb{R}_{++}^{e^*} \times \mathbb{R}^{v^*}$, such that, for every $(e, V) \in \Omega$, a standard economy, $\mathcal{E}_{(e, V)}$, has a pseudo- λ -equilibrium, (x, y, p, G, e, V) , such that $rank V_p(\underline{\mathbf{S}}) = J$. \square

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