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On Equilibrium Elasticities of Substitution in Simple Overlapping Generations Economies with Heterogenous Goods

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This contribution[1] introduces a *sectoral supply functions* approach of equilibrium dynamics in the context of a simple model of overlapping generations with heterogenous goods. The class of preferences that is here considered hinges upon an endogenous leisure motive and an elementary savings behaviour, that comes as a simpler alternative to the Diamond tradition in the benchmark contributions about the properties of overlapping generations economies with two industries. The presence of some institution making possible intergenerational transfers is shown to influence both the equilibrium aggregate factors shares and elasticity of substitution along a stationary equilibrium. Both Wealth-to-Capital and Golden Rule steady state equilibria being considered, the economies are categorised, either as Samuelsonian or classical, according to the sign of the transfers between generations at the Golden Rule steady state. The local stability properties of the various types of equilibria are successively investigated, the elasticities of substitution between the two inputs being emphasised to play a key-role for that purpose. Interestingly, the smoothing properties of factors substitution on the obtention of local uniqueness may differ between the Samuelsonian and classical economies.

Keywords: Overlapping Generations, Wealth-to-Capital and Golden Rule Equilibria, Heterogeneous Goods, Samuelsonian and classical Economies, Equilibrium Elasticities of Substitution.

JEL Classification: E32, O41.

1 Introduction 1

2 A Constant Returns to Scale Two Sector Overlapping Generations Economy with Simple Savings 3
- 2.1 The Productive Sector . . . 3
- 2.2 A Productive Sector Equilibrium and its Sectoral Supply Functions Reformulation . . . 4
- 2.3 A Productive Sector Equilibrium Elasticity of Substitution . . . 5
- 2.4 The Consumer Sector . . . 7
- 2.5 Competitive Equilibria . . . 9

3 Equilibria: A Characterisation 11
- 3.1 Wealth-to-Capital and Golden Rule Stationary Equilibria . . . 11
- 3.2 Samuelsonian vs Classical Economies . . . 13
- 3.3 Non-Stationary Equilibria . . . 15

[1]This study is dedicated to the loving memory of Carine, a sunny human, a bright spirit and a stellar scholar that will be deeply missed. A very preliminary version by the third author was presented at a conference held at Meiji-Gakuin University. A more advanced one by the current set of authors got presented at a TMM conference held at HEC Montreal where it could benefit from a detailed report and numerous insightful suggestions from T. Seegmuller. The referees, the editor and the co-editors are finally warmly thanked for their incisive remarks and perspicacious hints that led to a significantly improved exposition.

1. Introduction

Dating from the pioneering contributions of Samuelson (1958) in an exchange economy and Diamond (1965) in a production economy, the model of overlapping generations has admittedly become the most convincing *competitive equilibrium alternative* to the mainstream optimal accumulation environment. Such a framework is based upon heterogenous cohorts and emphasises the simplest source of heterogeneity among agents, *i.e.*, their dates of birth. The ensued intergenerational trade will hence capture the differences in agents needs over their life cycle. This, *e.g.*, clearly appears within the benchmark setup with identical agents living for two periods while being solely endowed with positive resources in their first period of life: the young would save for their *retirement* while the old would dissave in order to consume. Now, and for a benchmark production economy, the availability of an intergenerational transfers contrivance on top of physical capital radically changes the equilibrium of the model. Entailed economies can indeed be classified in two types depending on the *sign* of the transfers between successive generations at the Golden Rule steady state, *i.e.*, depending on the relationship between private wealth and the stock of capital. In the Wealth-to-Capital steady state private wealth equals the stock of capital; in the Golden Rule steady state private wealth may be greater or less than the stock of capital. Along the terminology coined by Gale (1973) and emphasised by Weil (2008), the former are called Samuelsonian economies, the latter classical economies. This article will aim at providing an articulation between the signs of the intergenerational transfers, the ensued range of admissible long-run equilibria and the role of the involved *substitutability mechanisms*, everything being allowed by the consideration of a simple *heterogeneous goods* environment.

Indeed, and even though it provided an entirely new framework for analysing inter-temporal economies with heterogeneous goods[2] and significantly enriched the earlier Leontief-type attempts of, *e.g.*, Reichlin (1992), the introduction by Galor (1992) of a two-sector overlapping generations economy with positive factors substitutability only moderately succeeded in the academic area. Notwithstanding the increasing interest into numerical techniques and the ensued disinterest into new and alternative setups that may have hindered its diffusion, key to this misunderstanding was presumably the complexity of his framework and of the local properties of the involved equilibria. As an illustration and in opposition, *e.g.*, to the benchmark homogenous good argument of Diamond (1965) whose dynamical equilibrium is canonically associated with a one-dimensional map, the ensued map was two-dimensional, that complexified any line of comparisons between a homogenous good and a heterogeneous goods structure. It was not before the contributions by Cremers (2006), Drugeon, Nourry & Venditti (2010), Kalray (1996), Li & Lin (2008), Nourry & Venditti (2011a), Nourry & Venditti (2011b) and Venditti (2005) that its structure, and noticeably its efficiency[3] properties, got clarified. Yet, while a recent contribution due to Leriche & Magris (2017) has convincingly considered a two-sectors version of the model of overlapping generations with liquidity constraints, this remains a

[2]Further associated developments have been completed in the recent years and illustrate the endless richness of this environment: as a matter of illustration, Nishimura, Shimomura & Wang (2006) developed a cost function based approach of a two goods structure while Roe, Sirin Saracoglu & Smith (2009) provide a more general purpose integrated picture of multi-sectoral setups.

[3]The classical modern characterisation is due to Abel, Mankiw, Summers & Zeckhauser (1989). *Vide* also Geerolf (2018) for recent challenging views in this area.

highly demanding environment where uniqueness is commonly difficult to obtain and no clear image is still available for distinguishing the sign of the intergenerational transfers that separate Samuelsonian economies from classical economies, not to mention the qualitative behaviour of their equilibria.

This contribution is then aimed at circumventing these remaining difficulties by letting the analysis rest upon a *sectoral supply functions*[4] based approach of equilibrium dynamics[5] in the context of a simple model of overlapping generations with heterogeneous goods. The class of preferences that is considered, based upon Reichlin (1986) and Benhabib & Laroque (1988), hinges upon an endogenous leisure motive and an elementary savings behaviour. It will be argued that this comes as a much simpler alternative to the two periods positive consumptions Diamond (1965) tradition retained by Galor (1992) in his benchmark contribution about the properties of overlapping generations economies with two industries. Enriching earlier approaches of a heterogeneous goods world, this article will also introduce an *equilibrium acceptance* of the measure of factors substitutability for the whole production sector. Considering the associated inter-temporal equilibria, it will then be shown how the contrivance supporting intergenerational transfers—the source of the potential dissociation of savings from the value of the capital good—influences both the equilibrium aggregate factors shares and the equilibrium elasticity of substitution along both the Wealth-to-Capital and the intermediated—Golden Rule—stationary equilibria. To clarify further the issue at stake, the economies will then be categorised according to the sign of the transfers between generations at the Golden Rule steady state. It is, *e.g.*, shown that, for Samuelsonian economies, the equilibrium elasticity at the Golden Rule steady state is, *e.g.*, larger than the one at the over-accumulating Wealth-to-Capital steady state when the sectoral elasticity of substitution is larger in the investment good industry than in the consumption good industry.

An eventual part aims at characterising the stability properties of the Wealth-to-Capital and Golden Rule steady states and to sketch the scope for non-stationary equilibria in the neighbourhood of these steady states. As a simple illustration, the *marginal contributions* of each sectorial elasticity of substitution to the Wealth-to-Capital steady state saddle-point stability condition noticeably differs. A strong discrepancy also surprisingly emerges for the Golden Rule steady state where it it is shown that while positive transfers, and thus Samuelsonian economies, are univocally associated with saddle-point stability and local determinacy, a larger range of properties is allowed for negative transfers and thus classical economies. Depending indeed upon the respective strengths of contradicting properties on the two sectorial elasticities of substitution, the associated economies may, *e.g.*, allow for complex non-stationary equilibria in the neighbourhood of the Golden Rule steady state.

The technological set, an elementary class of preferences and the definition of competitive equilibrium being introduced in Section 2, stationary and non-stationary equilibria are analysed in Section 3 while most of the formal developments are left to the appendix.

[4] *Vide* Drugeon (2010) for a detailed appraisal in an optimal accumulation environment.

[5] In contradistinction with the more standard Production Possibility Frontier approach used, *e.g*, in Venditti (2005), while similarly solving at first the allocation problem and letting the analysis resting upon the aggregate values of the inputs, the current approach is based upon a pair of heterogenous sectoral supply functions and the way their substitutability properties interact with the inputs costs and outputs relative prices ones.

2. A Constant Returns to Scale Two Sector Overlapping Generations Economy with Simple Savings

2.1 The Productive Sector

The economy of a given country encompasses four commodities: two are produced by constant returns no-joint production technologies (sectors); two are factors of production. Produced commodities (outputs) are labeled Y and indexed by $j \in \{0, 1\}$, with the indexes 0 and 1 that respectively feature the consumption and investment outputs; factors of production (inputs) are labeled X and indexed by $i \in \{0, 1\}$ with the indexes 0 and 1 that respectively feature the labour and capital inputs. Within the static context of the current section, the factors of production do correspond to the initial endowment of the economy, the available quantity of factor i being $X_i > 0$. The technological possibilities in sector j are described by the production function F^j:

$$Y^j \leq F^j(X_{0j}, X_{1j}), \tag{2.1}$$

where X_{ij} denotes the amount of input $i, i \in \{0, 1\}$ employed in sector j, $j \in \{0, 1\}$. Capital and labour are freely shifted[6] between the two sectors and feasibility constraints are stated as:

$$\sum_{j=0}^{1} X_{ij} \leq X_i, \quad i \in \{0, 1\}, \tag{2.2}$$

The properties of the production technologies are restricted to the following list of assumptions:

Assumption T1.— Each production function $F^j(\cdot, \cdot)$, $j \in \{0, 1\}$, is homogeneous of degree one[7], increasing in each argument, concave and continuous over $\mathbb{R}_+ \times \mathbb{R}_+$, such that, for any $X_{1j} \in \mathbb{R}_+$, $F^j(0, X_{1j}) = 0$.

Assumption T2.— There exists a level $\bar{X}_1 \in \mathbb{R}_+^*$ such that, for any $X_0 \in \mathbb{R}_+^*$, $X_1 \in \mathbb{R}_+^*$, $F^1(X_0, X_1) > X_1$ for any $X_1 < \bar{X}_1$ and $F^1(X_0, X_1) < X_1$ for any $X_1 > \bar{X}_1$

Assumption T3.— $\forall j \in \{0, 1\}$, $F^j(\cdot, \cdot)$ is of class $\mathscr{C}^3$ over $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

Competitive firms in industry $j \in \{0, 1\}$ take $\{\omega^0, \omega^1, p^j\}$ as given, for ω^0, ω^1 and p^j factor prices and the price of commodity j. They select $\{X_{0j}, X_{1j}\}$, $j \in \{0, 1\}$, in order to maximise their profits:

$$\max_{\{X_{0j}, X_{1j}\}} p^j Y^j - \omega^0 X_{0j} - \omega^1 X_{1j} \text{ s.t. } Y^j \leq F^j(X_{0j}, X_{1j}), X_{0j} \geq 0, X_{1j} \geq 0. \tag{2.3}$$

[6]For example, similar to Ricardo's comparative advantage argument, farmers can shift to work as fishermen without incurring an extra cost.

[7]Constant returns to scale production technologies in the consumption good and investment good sectors implies that outputs increase by the same proportional change as all inputs change.

2.2 A Productive Sector Equilibrium and its Sectoral Supply Functions Reformulation

An interior competitive equilibrium of the productive sector is given by:

$$\omega^i = p^j \frac{\partial F^j}{\partial X_{ij}}(X_{0j}, X_{1j}), i, j \in \{0, 1\}, \tag{2.4a}$$

$$Y^j = F^j(X_{0j}, X_{1j}), j \in \{0, 1\}, \tag{2.4b}$$

$$\sum_{j=0}^{1} X_{ij} = X_i, i, j \in \{0, 1\}. \tag{2.4c}$$

The equilibrium equations (2.4a)-(2.4c) is a system of eight equations determining outputs, factor allocations to each sector, and factor prices for given factor endowments and output prices. As a matter of fact, under assumption T1, the system of equations (2.4a) implies:

$$\frac{\omega^0}{\omega^1} = \frac{\partial F^j(1, X_{1j}/X_{0j})/\partial X_{0j}}{\partial F^j(1, X_{1j}/X_{0j})/\partial X_{1j}}, j \in \{0, 1\}, \tag{2.5a}$$

$$\frac{p^1}{p^0} = \frac{\partial F^0(1, X_{10}/X_{00})/\partial X_{i0}}{\partial F^1(1, X_{11}/X_{01})/\partial X_{i1}}, i \in \{0, 1\}. \tag{2.5b}$$

Under assumption T3, equations (2.5a) define the sectoral input ratios X_{1j}/X_{0j} as a pair of strictly increasing functions $(X_{1j}/X_{0j})(\cdot)$ of the ratio between the price of the two inputs ω^0/ω^1. The argument shall then further be simplified by imposing an extra *no factors-intensity reversal* assumption between the two industries:

Assumption T4.—$\forall\,(\omega^0/\omega^1) > 0$, either $(X_{11}/X_{01})(\omega^0/\omega^1) > (X_{10}/X_{00})(\omega^0/\omega^1)$ or $(X_{10}/X_{00})(\omega^0/\omega^1) > (X_{11}/X_{01})(\omega^0/\omega^1)$.

Under assumption T4, ω^0/ω^1 emerges, from any of the equations (2.5b), as a monotonic function $(\omega^0/\omega^1)(\cdot)$ of p^1/p^0. The same dependency with respect to p^1/p^0 holds for the factor ratios as well, or $(X_{1j}/X_{0j})[(\omega^0/\omega^1)(\cdot)]$.[8] From the balance conditions (2.4c), one gets $X_1/X_0 = \sum_{j=0}^{1}(X_{0j}/X_0)(X_{1j}/X_{0j})$. Thus, under assumption T4, the share of the labour input that is employed in sector j emerges as:

$$X_{0j}/X_0 = \frac{X_1/X_0 - X_{1j'}/X_{0j'}}{X_{1j}/X_{0j} - X_{1j'}/X_{0j'}}, j' \neq j.$$

The sectorial factor ratios being functions of p^1/p^0, it follows that X_{0j}/X_0 emerges as a function $(X_{0j}/X_0)[(\omega^0/\omega^1)(p^1/p^0), X_1/X_0]$ of the two inputs and the relative price of the investment good. Finally, from (2.4b), this eventually equips the analysis with a pair of structures that shall henceforward be labelled *sectoral supply functions*:

$$G^j(X_0, X_1, p^1/p^0) := (X_{0j}/X_0)[(\omega^0/\omega^1)(p^1/p^0), X_1/X_0]X_0F^j\Big(1, (X_{1j}/X_{0j})[(\omega^1/\omega^0)(p^1/p^0)]\Big),$$

where $j \in \{0, 1\}$. These functions are homogenous of degree one and additively separable in X_0 and X_1 and it was established in Drugeon (2010, Prop. 1(ii)a/) that their derivatives directly relate to

[8] Strictly speaking, the values for the relative price p^1/p^0 must be restricted in order for the production to be diversified, that is for both outputs to be strictly positive (*vide* Drugeon (2010)).

competitive prices. Under assumptions T1-4, given the economy input endowment and the (suitable) output prices (X_0, X_1, p^0, p^1), an equilibrium of the productive sector in which no specialisation occurs is a vector of input prices $(\omega^0, \omega^1) \in (\mathbb{R}_+^*)^2$ and an allocation $(Y^0, Y^1, X_{00}, X_{01}, X_{10}, X_{11}) \in (\mathbb{R}_+^*)^6$ such that:

$$\omega^i/p^0 = \sum_{j=0}^{1} (p^j/p^0) \frac{\partial G^j}{\partial X_i}(X_0, X_1, p^1/p^0),\ i \in \{0,1\}, \tag{2.6a}$$

$$\sum_{j=0}^{1} (p^j/p^0) \frac{\partial G^j}{\partial (p^1/p^0)}(X_0, X_1, p^1/p^0) = 0, \tag{2.6b}$$

$$Y^j = G^j(X_0, X_1, p^1/p^0),\ j \in \{0,1\}, \tag{2.6c}$$

where it is noticed that, in (2.6a), neither $\partial G^j/\partial X_0$, nor $\partial G^j/\partial X_1$ depend upon X_0 or X_1. For compactness, the rental rates of the labour and capital units shall henceforth respectively be referred to through $[\sum_{j=0}^{1}(p^j/p^0)(\partial G^j/\partial X_0)](p^1/p^0)$ and $[\sum_{j=0}^{1}(p^j/p^0)(\partial G^j/\partial X_1)](p^1/p^0)$.

2.3 A Productive Sector Equilibrium Elasticity of Substitution

At that stage, one may wonder about the nature of the (smooth) dependency of the productive sector equilibrium with respect to the amount of available inputs and the relative price of outputs, *i.e*, a typical comparative statics exercise. In that perspective, let $\pi_{X_i}^j := \omega^i X_{ij}/p^j Y^j, i, j \in \{0,1\}$, feature the factor i income share in sector j; $\pi_{Y^j} := p^j Y^j / \sum_{j=0}^{1} p^j Y^j, j \in \{0,1\}$, denote the share of output j in national income; $\pi_{X_i} := \omega^i X_i / \sum_{i=0}^{1} \omega^i X_{ij} = \sum_{j=0}^{1} \pi_{Y^j} \pi_{X_i}^j, i \in \{0,1\}$, be the share of factor i in national income; and finally let $\Sigma_{X_0X_1}^j := d\ln(X_{1j}/X_{0j})\ /\ d\ln(\omega^1/\omega^0)$ stand for the (Hicksian) elasticity of substitution between the two inputs in sector j. Also let $\mu_{ij} := X_{ij}/X_i$ denote the share of input i used in industry j and $\sigma_i := \sum_{j=0}^{1} \mu_{ij}(1 - \pi_{X_i}^j)\Sigma_{X_0X_1}^j$ a inputs based weighted expression of the sectoral elasticities of substitution. Remarking that this latter coefficient is also implicitly defined through the satisfaction of $\sigma_i = \sum_{j=0}^{1} \pi_{Y^j} \pi_{X_i}^j (1 - \pi_{X_i}^j)\Sigma_{X_0X_1}^j / \pi_{X_i}$, it can also be understood as a given aggregate factor acception of this weighted expression of the sectoral elasticities of substitution, for a weight that emerges as product of the two factors shares of a given industry times the share of that output in national income.

The following statement will now establish the comparative statics properties of costs, prices, outputs and inputs and illustrate the usefulness of this formalism. It is worth emphasising that all of these properties should be understood as being stated in *rates of variation / percentage deviations*.

Lemma 2.1.— *Under Assumptions T1-4, consider a competitive equilibrium of the productive sector:*

(i) *letting* $\Delta_\pi := \pi_{X_0}^0 \pi_{X_1}^1 - \pi_{X_0}^1 \pi_{X_1}^0 = \pi_{X_1}^1 - \pi_{X_1}^0$*—the second equality comes from the constant returns to scale assumption* $\sum_{i=0}^{1} \pi_{X_i}^j = 1$*—denote the difference in factor shares between the two industries,*

a/ *the vector of costs* $\begin{bmatrix}\omega^0 & \omega^1\end{bmatrix}'$ *relates with the vector of prices* $\begin{bmatrix}p^0 & p^1\end{bmatrix}'$ *through the matrix*

$$(\Delta_\pi)^{-1} \begin{bmatrix} \pi_{X_1}^1 & -\pi_{X_1}^0 \\ -\pi_{X_0}^1 & \pi_{X_0}^0 \end{bmatrix};$$

b/ *the relative cost of the capital input* ω^1/ω^0 *relates to the relative price of the capital good* (p^1/p^0) *according to* $(\Delta_\pi)^{-1}$ *;*

(ii) *letting* $\Delta_\mu := \mu_{00}\mu_{11} - \mu_{01}\mu_{01} = \mu_{11} - \mu_{01}$*—the second equality comes from the full employment of the productive factors condition* $\sum_{j=0}^{1} \mu_{ij} = 1$*— denote the difference of inputs shares between the two inputs in a given industry, the vector of the outputs* $\begin{bmatrix} Y^0 & Y^1 \end{bmatrix}'$ *relates with:*

a/ *the vector of inputs* $\begin{bmatrix} X_0 & X_1 \end{bmatrix}'$ *through the matrix*

$$(\Delta_\mu)^{-1} \begin{bmatrix} \mu_{11} & -\mu_{01} \\ -\mu_{01} & \mu_{00} \end{bmatrix};$$

b/ *the relative price of the investment good* p^1/p^0 *through the vector*

$$\begin{bmatrix} -(\mu_{11}\sigma_0 + \mu_{01}\sigma_1) \\ \mu_{10}\sigma_0 + \mu_{00}\sigma_1 \end{bmatrix}.$$

An early remark about Lemma 2.1 could be that, while the first-order derivatives of the sectoral $G^j(\cdot)$ with respect to the stocks of inputs X_0 and X_1 relate to sectoral factor shares and thus the first-order properties of the sectoral production functions F^j, their first-order derivatives with respect to the relative price of the investment good p^1/p^0 relate to sectoral elasticity of substitution and thus the second-order properties of the sectoral production functions $F^j(\cdot,\cdot)$.

A first comment ought also to emphasise the specific features of the coefficients Δ_μ and Δ_π that are introduced through Lemma 2.1. While these are linked on a formal basis[9] and both relate to differences in factor intensity between the two industries, these uncover two different facets of that issue. As this was clarified, *e.g.*, by Jones (1971) and Bond, Wang & Yip (1995), while the first coefficient Δ_μ depicts differences in factor intensities in the *physical sense* through the shares of inputs that are used in a given industry, the second Δ_π relates to differences in factor intensities in the *value sense* through the distributive shares of inputs.

Lemma 2.1(ii)a/ and Lemma 2.1(i) distinctively build upon these two coefficients Δ_μ and Δ_π in order to respectively form the well-known Rybczynski and Stolper-Samuelson theorems[10]. As a basic illustration of the Rybczynski theorem, consider a capital intensive investment output configuration with $\mu_{11} > \mu_{01}$. Then notice that, taking advantage of the full employment of the productive factors holding of $\sum_{j=0}^{1} \mu_{ij} = 1$ for $i \in \{0,1\}$—this is turn implies that $\mu_{11} - \mu_{01} = \mu_{00} - \mu_{10}$—, it derives that the components that respectively relate the consumption output to the labour input and the investment output to the capital output respectively detail as $\mu_{11}/(\mu_{11} - \mu_{01}) > 1$ and $\mu_{00}/(\mu_{00} - \mu_{10}) > 1$. As an equally simple illustration of the latter Stolper-Samuelson theorem, consider a labour intensive consumption output configuration with $\pi^0_{X_0} > \pi^1_{X_0}$. Then notice that, taking advantage of the constant returns to scale assumption $\sum_{i=0}^{1} \pi^j_{X_i} = 1$ for $j \in \{0,1\}$—this is turn implies that $\pi^0_{X_0} - \pi^1_{X_0}$

[9]From the specialisation of a more general argument developed in Drugeon (2009), it can readily be checked that $\prod_{j=0}^{1} \pi_{Y^j}\Delta_\pi = \prod_{i=0}^{1} \pi_{X_i}\Delta_\mu$.

[10]The former theorem states that the increase in the endowment of one factor, for unchanged endowment of the other factor and output good prices, will increase production of the good intensive in this factor, and will reduce production of the good intensive in the other factor. The latter states that an increase in the price of a output good raises the rental rate of the factor used more intensively in its production, and reduces the rental rate of the other factor.

$= \pi^1_{X_1} - \pi^0_{X_1}$—, it derives that the components that respectively relate the cost of the labour input to the price of the consumption output and the cost of the capital input to the price of the investment output respectively detail as $\pi^0_{X_0}/(\pi^0_{X_0} - \pi^1_{X_0}) > 1$ and $\pi^1_{X_1}/(\pi^1_{X_1} - \pi^0_{X_1}) > 1$.

Beyond these famous yet classical results, Lemma 2.1 allows to reach a measure of the substitutability property for the whole productive sector. As a matter of fact, it allows to define an elasticity of substitution relating changes in factor relative scarceness to changes in factors relative price. More specifically, fixing the outputs and letting the relative factor quantity vary:

Proposition 2.1.— *Under Assumptions T1-4, the constant-outputs equilibrium elasticity of substitution between the two aggregate inputs is given by:*

$$\Sigma^A_{X_0X_1} := \sum_{i=0}^{1} \sigma_i = \sum_{j=0}^{1} \frac{\pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}}{\pi_{X_0}\pi_{X_1}} \Sigma^j_{X_0X_1}. \tag{2.7}$$

Unsurprisingly and taking advantage of $\sum_{i=0}^{1} \pi_{X_i} = 1$, this expression can be understood as the direct sum of the earlier coefficients σ_i that featured a given factor i acception of the sectoral elasticities of substitution. It is however worthwhile briefly discussing the specific features of the above derived constant-outputs elasticity of substitution. It indeed literally relates, at the production sector level, changes in the factor endowments to the resulting changes in the *equilibrium* factor prices[11]. In that respect, it does distinguish itself from the (Hicksian) elasticity of substitution[12] that relates, at the individual firm level, changes in the relative factor quantity to changes in the relative factor price. Going from the individual to the *equilibrium* elasticity is, essentially, turning the cause and effect sequence around: in the former exogenous changes pertain to factor prices whereas, in the latter, they are concerned with the endowments of the factors. Further observe that the equilibrium at stake is the one of the productive sector. This derives from the fact that changes in output prices are *exogenous*. As soon as a general equilibrium analysis is contemplated, the response of the relative factor price to the relative output price is not any longer part of a comparative statics exercise. Nor it is consistent, in the current context, to consider that they are fixed; as indeed factor price is determined only by output price (*vide* Lemma 2.1(ii)), a nil variation of the output prices p^j for both j implies a nil variation of the input costs ω^i for both i. Therefore, the substitution relationship (2.7) implicitly supposes that any change in the relative factor price is matched by an appropriate change in the relative output price.

2.4 The Consumer Sector

This section adds a consumer sector to the production sector of the previous section. In a dynamic perspective, the consumer sector is made of an infinite sequence of overlapping cohorts of two-period living agents. In the so obtained overlapping generations production economy, neither the prices of the produced commodities nor the available quantities of factors of production at any date $t \in \mathbb{N}$ are

[11]The equilibrium elasticity of substitution $\Sigma^A_{X_0X_1}$ as it is stated in (2.7) gives, for unchanged outputs, the percentage rise in X_1/X_0 needed to raise ω^0/ω^1 by 1 per cent. It appears as a linear function of the Hicksian elasticities of substitution in each sector, $\Sigma^0_{X_0X_1}$ and $\Sigma^1_{X_0X_1}$.

[12]It is worth recalling that the Hicksian elasticity of substitution is defined, at the firm level, as the elasticity of the relative factor quantities with respect to the marginal rate of substitution holding the output constant. In a competitive economy, profit maximization requires that the marginal rate of substitution equals the factor price ratio.

assumed to be exogenous ; they are rather endogenously determined. The commodities nature has been stipulated in the previous section. It will be assumed that the produced commodities are a pure consumption and a pure investment (capital) good. With regard to the factors of production, while the consumer is born with the first, *i.e.*, labour, the remaining one, *i.e.*, capital, needs to be purchased. Likewise, while the supply of the former is bounded by the endowment, the latter is produced and its stock can therefore be altered through accumulation. Yet capital will not be assumed to be the exclusive savings' support. An institutional apparatus will allow to disconnect the private wealth from the value of the stock of capital. Consequently, the available stock of capital, the factor shares and the equilibrium elasticity of substitution, will rest upon the contrivance supporting the intergenerational transfers.

The economy is populated by generations of agents who live for two periods. At any date $t \in \mathbb{N}$, the total population summarises to a representative young agent of generation t and a representative old agent of generation $t-1$. At date $t=1$, an agent was born old. The labour endowment of an agent of generation $t \in \mathbb{N}$ summarises to $\bar{\ell} > 0$ at the first period of his lifespan. The preferences of an agent of generation t are described by an inter-temporal utility function $u(\ell_t^t, c_{t+1}^t)$, where ℓ_t^t and c_{t+1}^t are the leisure time at period t and consumption at period $t+1$. The properties of the utility function list as:

Assumption P1.— $u \in C^k(]0, \bar{\ell}[\times \mathbb{R}_+, \mathbb{R})$, $k \geq 4$ and is strictly quasi-concave over $]0, \bar{\ell}[\times \mathbb{R}_+^*$.

Assumption P2.— $\forall (\ell_t^t, c_{t+1}^t) \in]0, \bar{\ell}[\times \mathbb{R}_+^*$, (i) $\nabla u(\ell_t^t, c_{t+1}^t) \gg 0$; (ii) $\partial u(\ell_t^t, c_{t+1}^t)/\partial \ell_t^t > 0$, $\partial u(\ell_t^t, c_{t+1}^t) /\partial c_{t+1}^t > 0$; (iii) $\lim_{\ell_t^t \to 0} \partial u(\ell_t^t, c_{t+1}^t)/\partial \ell_t^t = \lim_{c_{t+1}^t \to 0} \partial u(\ell_t^t, c_{t+1}^t) \ /\partial c_{t+1}^t = \infty$.

Assumption P3.— $\forall (\ell_t^t, c_{t+1}^t) \in]0, \bar{\ell}[\times \mathbb{R}_+^*$, (i) $[\partial u(\ell_t^t, c_{t+1}^t)/\partial \ell_t^t][\partial^2 u(\ell_t^t, c_{t+1}^t)/\partial \ell_t^t \partial c_{t+1}^t] - [\partial u(\ell_t^t, c_{t+1}^t) /\partial c_{t+1}^t][\partial^2 u(\ell_t^t, c_{t+1}^t)/\partial (\ell_t^t)^2] > 0$, (ii) $[\partial u(\ell_t^t, c_{t+1}^t)/\partial c_{t+1}^t][\partial^2 u(\ell_t^t, c_{t+1}^t)/\partial \ell_t^t \partial c_{t+1}^t] - [\partial u(\ell_t^t, c_{t+1}^t)/\partial \ell_t^t] \times [\partial^2 u(\ell_t^t, c_{t+1}^t)/\partial (c_{t+1}^t)^2] > 0$.

During the first period of his lifetime a representative agent of generation t supplies all or part of his labour endowment. The resulting wage income $\omega_t^0(\bar{\ell} - \ell_t^t)$ is entirely saved. As for now, the inter-temporal transfer arrangements are not specified. Savings, s_t^t, earn a gross rate of return, $\mathscr{R}_{t+1}$, in the subsequent period and enable the agent to consume. The second period consumption of a representative agent of generation t, c_{t+1}^t, is therefore $c_{t+1}^t = \mathscr{R}_{t+1} s_t^t / p_{t+1}^0$. His feasibility constraints thus summarize in an unique lifespan budget constraint and he selects (ℓ_t^t, c_{t+1}^t) in order to solve:

$$\max_{\{\ell_t^t, c_{t+1}^t\}} u(\ell_t^t, c_{t+1}^t) \ \text{ subject to } \ (p_{t+1}^0/p_t^0)\, c_{t+1}^t/\mathscr{R}_{t+1} \leq (\omega_t^0/p_t^0)\,(\bar{\ell} - \ell_t^t)$$

First-order interior solutions list as:

$$\frac{\partial u}{\partial c_{t+1}^t}(\ell_t^t, c_{t+1}^t)\,(p_{t+1}^0/p_t^0)\,/\,(\omega_t^0/p_t^0) - \mathscr{R}_{t+1}\frac{\partial u}{\partial \ell_t^t}(\ell_t^t, c_{t+1}^t) = 0,$$
$$(p_{t+1}^0/p_t^0)\, c_{t+1}^t/\mathscr{R}_{t+1} - (\omega_t^0/p_t^0)\,(\bar{\ell} - \ell_t^t) = 0.$$

Letting henceforward $\ell_t^t = \mathscr{L}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/\,(p_{t+1}^0/p_t^0)]$ and $c_{t+1}^t = \mathscr{C}'[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/\,(p_{t+1}^0/p_t^0)]$ denote the solution of this system, an extra restriction is required:

Assumption P4.— $\forall\, (\omega_t^0/p_t^0, \mathscr{R}_{t+1}/\,(p_{t+1}^0/p_t^0)) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, *it holds that:*

$$\frac{\partial \mathscr{L}}{\mathscr{R}_{t+1}/\,(p_{t+1}^0/p_t^0)}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/\,(p_{t+1}^0/p_t^0)] \neq 0, \frac{\partial \mathscr{C}'}{\mathscr{R}_{t+1}/\,(p_{t+1}^0/p_t^0)}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/\,(p_{t+1}^0/p_t^0)] \neq 0.$$

The representative consumer of generation $t \in \mathbb{N}$ is equivalently to solve

$$\min_{\{\ell_t^t, c_{t+1}^t\}} (p_{t+1}^0/p_t^0) c_{t+1}^t/\mathscr{R}_{t+1} - (\omega_t^0/p_t^0)(\bar{\ell} - \ell_t^t) \text{ subject to } u(\ell_t^t, c_{t+1}^t) \geq \bar{u},$$

$\bar{u}$ given, that in turns allows for introducing the consumption expense function of the consumer as

$$\begin{aligned} E[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}] :&= (\omega_t^0/p_t^0)\ell_t^{t,H}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}] \\ &\quad + c_{t+1}^{t,H}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}]/\mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \end{aligned}$$

where $\ell_t^{t,H}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}]$ and $c_{t+1}^{t,H}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}]$ denote the dual compensated Hicksian demand functions. Letting $\pi_\ell := (\omega_t^0/p_t^0)\ell_t^t/[(\omega_t^0/p_t^0)\ell_t^t + c_{t+1}^t/\mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0)]$, $\pi_{c'} := [c_{t+1}^t/\mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0)] \;/\; [(\omega_t^0/p_t^0)\ell_t^t + c_{t+1}^t/\mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0)]$ denote the shares of the first period leisure and second-period consumption expenses in the total expenses of the representative agent of generation $t \in \mathbb{N}$, letting $\mathscr{S}$ feature the scale elasticity of $u(\cdot,\cdot)$ and the following indexes describe the ordinal second-order elasticities of the expense function:

$$\Sigma_{\ell c'}^H := E\frac{\partial^2 E}{\partial(\omega_t^0/p_t^0)\partial\left[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}\right]} \Big/ \frac{\partial E}{\partial(\omega_t^0/p_t^0)}\frac{\partial E}{\partial[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]},$$
$$\Sigma_{\ell\bar{u}}^H := E\frac{\partial^2 E}{\partial(\omega_t^0/p_t)\partial\bar{u}} \Big/ \frac{\partial E}{\partial(\omega_t^0/p_t^0)}\frac{\partial E}{\partial\bar{u}}, \Sigma_{YY}^H := E\frac{\partial^2 E}{\partial\bar{u}\partial\bar{u}} \Big/ \frac{\partial E}{\partial\bar{u}}\frac{\partial E}{\partial\bar{u}}.$$

for which any coefficient $\Sigma_{\ell c'}^H$ refers to the dual Hicksian elasticities of substitution. Letting finally $\mathscr{H}_E$ denote the Hessian Elasticities Matrix of the expense function, from the details of Appendix B, it is available as $\mathscr{H}_E = -1\Sigma_{YY}^H\mathscr{S}1' + \Upsilon(\Sigma_{\ell c'}^H\pi_\ell\pi_{c'})^{-1}\Upsilon'$ for $1' = \begin{bmatrix}1 & 1\end{bmatrix}$ and $\Upsilon' = \begin{bmatrix}\Sigma_{\ell c'}^H\pi_{c'} & -\Sigma_{\ell Y}^H\pi_\ell\end{bmatrix}$.

2.5 Competitive Equilibria

The inter-temporal transfer arrangements mentioned in the previous section are now to be clarified. The investment good is an obvious store of value in the current economy. Young agents can therefore save by purchasing the investment good, the latter becoming capital in the subsequent period. Yet further arrangements are indeed conceivable.[13] These arrangements are essentially devices operating *intergenerational* transfers of resources that permit some agents to consume more than their current endowments. The demographic structure ruling out intergenerational exchange, these transfers are *intermediated* by some *social contract* (Esteban & Sakovics (1993, p. 189)). Whatever the arrangement that is considered, the ensuing savings of the young are not exclusively invested in capital. Formally, it is conceivable that $s_t^t - p_t^1 Y_t^1 \neq 0$, *i.e.*, the private wealth could differ from the economy's value of the investment good.

Let $B_t := s_t^t - p_t^1 Y_t^1$. Retaining as a benchmark the financial intermediary institutional arrangement, B_t represents the part of the savings deposits not invested in capital. In equilibrium, deposits and capital investments have the same rate of return:

$$\mathscr{R}_t = w_t^1/p_{t-1}^1.$$

[13] Financial assets could serve as additional means of saving. Agents could save in another intrinsically worthless asset: public debt as in Diamond (1965), fiat money as in Benhabib & Laroque (1988) or a bubble à la Tirole (1985), ownership shares in firms (stock) as in Magill & Quinzii (2003), savings deposits offered by (infinitely-lived) financial intermediaries as briefly discussed in Cass & Yaari (1966).

and in addition the financial intermediary's net earnings are nil:[14]

$$B_t = \mathscr{R}_t B_{t-1}.$$

A positive value would hence correspond to a transfer from the young to the old whereas a negative value would be a transfer from the old to the young.

Definition 2.1.— Under Assumptions T1-4 and P1-4, an inter-temporal competitive equilibrium is a sequence $\{X_{0,t}, X_{1,t}, \omega_t^0/p_t^0, \omega_t^1/p_t^0, p_t^1/p_t^0, B_t/p_t^0, c_t^{t-1}, \ell_t^t\}_{t\geq 1}$ such that :

(i) $c_{t+1}^t = \mathscr{C}'[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/\left(p_{t+1}^0/p_t^0\right)]$ and $\ell_t^t = \mathscr{L}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/\left(p_{t+1}^0/p_t^0\right)]$;

(ii) $\{X_{0,t}, X_{1,t}, p_t^1/p_t^0\}$ maximises $p_t^0 G^0(X_{0,t}, X_{1,t}, p_t^1/p_t^0) + p_t^1 G^1(X_{0,t}, X_{1,t}, p_t^1/p_t^0) - \omega_t^0 X_{0,t} - \omega_t^1 X_{1,t}$ at any $t \in \mathbb{N}$;

(iii) $(p_t^1/p_t^0)\, Y_t^1 + B_t/p_t^0 = (\omega_t^0/p_t^0) X_{0,t}$;

(iv) $X_{1,t+1} = Y_t^1 = G^1(X_{0,t}, X_{1,t}, p_t^1/p_t^0)$;[15]

(v) $X_{0,t} = \bar{\ell} - \ell_t^t$;

(vi) $\omega_t^0/p_t^0 = \Big[\sum_{j=0}^{1} (p^j/p^0) \dfrac{\partial G^j}{\partial X_0}\Big](p_t^1/p_t^0)$, $\omega_t^1/p_t^0 = \Big[\sum_{j=0}^{1} (p^j/p^0) \dfrac{\partial G^j}{\partial X_1}\Big](p_t^1/p_t^0)$;

(vii) $c_t^{t-1} = Y_t^0 = G^0(X_{0,t}, X_{1,t}, p_t^1/p_t^0)$;

(viii) $\mathscr{R}_{t+1}/\left(p_{t+1}^0/p_t^0\right) = \left(\omega_{t+1}^1/p_{t+1}^0\right) / \left(p_t^1/p_t^0\right)$;

(ix) $\left(B_{t+1}/p_{t+1}^0\right) - \mathscr{R}_{t+1}/\left(p_{t+1}^0/p_t^0\right)\left(B_t/p_t^0\right) = 0$.

Letting

$$\ell(p_t^1/p_t^0, p_{t+1}^1/p_{t+1}^0) := \mathscr{L}\Big[\Big[\sum_{j=0}^{1} (p^j/p^0) \frac{\partial G^j}{\partial X_0}\Big](p_t^1/p_t^0), \Big[\sum_{j=0}^{1} (p^j/p^0) \frac{\partial G^j}{\partial X_1}\Big](p_{t+1}^1/p_{t+1}^0)/(p_t^1/p_t^0)\Big],$$

any such equilibrium is canonically associated with a three-dimensional sequence $(X_{1,t}, p_t^1/p_t^0, B_t/p_t^0)_{t\geq 1}$ that satisfies:

$$X_{1,t+1} - G^1(\bar{\ell} - \ell(p_t^1/p_t^0, p_{t+1}^1/p_{t+1}^0), X_{1,t}, p_t^1/p_t^0) = 0; \tag{2.8a}$$

$$\Big[\sum_{j=0}^{1} (p^j/p^0) \frac{\partial G^j}{\partial X_0}\Big](p_t^1/p_t^0)[\bar{\ell} - \ell(p_t^1/p_t^0, p_{t+1}^1/p_{t+1}^0)] - (B_t/p_t^0) - (p_t^1/p_t^0)\, X_{1,t+1} = 0; \tag{2.8b}$$

$$(B_{t+1}/p_{t+1}^0) - \Big[\sum_{j=0}^{1} (p^j/p^0) \frac{\partial G^j}{\partial X_1}\Big](p_{t+1}^1/p_{t+1}^0)(B_t/p_t^0)/(p_t^1/p_t^0) = 0. \tag{2.8c}$$

[14] Observe the similarity with the budget constraint of a government maintaining a zero budget deficit.

[15] The stock of capital is fully depreciated in one period.

3. Equilibria: A Characterisation

There are two kinds of equilibria: the ones for which $(B_t/p_t^0) = 0$, $t \geq 1$, and the ones for which $(B_t/p_t^0) \neq 0$, $t \geq 1$. The former kind, in which the total private wealth is equal to the value of the stock of capital, will be called *Wealth-to-Capital equilibria*[16]. In opposition to this, such an equality won't generically hold in the latter kind for which intergenerational[17] transfers are mediated by, say, a bank: these will be hence be labelled as *intermediated equilibria*. Equilibria in which prices and allocations remain constant over time will more generally be referred to as stationary or steady state equilibria.

3.1 Wealth-to-Capital and Golden Rule Stationary Equilibria

Definition 3.1.— A stationary equilibrium is a triple $\{X_1^\star, (p^1/p^0)^\star, (B/p^0)^\star\}$ that solves the system:

$$0 = X_1^\star - G^1\left(\bar{\ell} - \ell((p^1/p^0)^\star, (p^1/p^0)^\star), X_1^\star, (p^1/p^0)^\star\right); \tag{3.1a}$$

$$0 = \Big[\sum_{j=0}^{1}(p^j/p^0)\frac{\partial G^j}{\partial X_0}\Big](p^1/p^0)^\star[\bar{\ell} - \ell((p^1/p^0)^\star, (p^1/p^0)^\star)] - (B/p^0)^\star - (p^1/p^0)^\star X_1^\star; \tag{3.1b}$$

$$0 = (B/p^0)^\star - \Big[\sum_{j=0}^{1}(p^j/p^0)\frac{\partial G^j}{\partial X_1}\Big](p^1/p^0)^\star(B/p^0)^\star/(p^1/p^0)^\star. \tag{3.1c}$$

Under assumptions T1-3 and P1-4, equations (3.1a)-(3.1c) assume two ranges of solutions: the *Wealth-to-Capital* steady state for which $(B/p^0)^\star = 0$ and the intermediated steady state where $(B/p^0)^\star \neq 0$. Among stationary feasible allocations, the intermediated steady state provides the highest possible utility to the consumer sector, hence the *Golden Rule* denomination that will henceforth be used to refer to the intermediate steady state.[18]

Another facet of these two classes of steady state solutions is clarified in the subsequent statement: beyond their straightforward definition, the key aggregate, *i.e.*, equilibrium, parameters happen to therein assume disaggregated formulations.

Lemma 3.1.— *Consider the set of steady states under assumptions T1-4 and P1-4.*

(i) *An interior Wealth-to-Capital steady state will be characterised by:*

a/ *the holding of* $(B/p^0)^\star = 0$ *and*

$$\Big[\sum_{j=0}^{1}(p^j/p^0)\frac{\partial G^j}{\partial X_0}\Big](p^1/p^0)^\star[\bar{\ell} - \ell((p^1/p^0)^\star, (p^1/p^0)^\star)] = (p^1/p^0)^\star X_1^\star;$$

[16] That terminology was introduced by Gale (1973).

[17] As recalled by Reichlin (1992) in an environment with Leontief-type technologies, in opposition to pure exchange economies of the Gale (1973) type where non-intermediated steady states are autarkic, both types of equilibria are here associated with transfers.

[18] It is actually the unique solution of the first-best steady state problem $\max u(Y^0, \bar{\ell} - X_0)$ subject to the feasibility constraints (2.1), (2.2) and $X_1 = Y^1$. Optimality requires the achievement of $(\partial F^1/\partial X_{11})(1, X_{11}/X_{01}) = [\sum_{j=0}^{1}(p^j/p^0)(\partial G^j/\partial X_1)][(p^1/p^0)]/(p^1/p^0) = 1$ and the transfer $(B/p^0)^\star$ given by (3.1b) is precisely the one that sustains the Golden Rule as a stationary competitive equilibrium.

b/ *aggregate factors and outputs shares that satisfy* $\pi_{X_0} = \pi_{Y^1} = \pi^0_{X_0}/(1+\Delta_\pi)$, $\pi_{X_1} = \pi_{Y^0} = \pi^1_{X_1}/(1+\Delta_\pi)$;

c/ *a steady expression for the elasticity of substitution between the aggregate values of the input* X_0 *and* X_1 *that is given by:*

$$\Sigma^{\mathscr{WC}}_{X_0X_1} = (1+\Delta_\pi)(\pi^0_{X_1}\Sigma^0_{X_0X_1} + \pi^1_{X_0}\Sigma^1_{X_0X_1}).$$

(ii) *An interior Golden Rule steady state will be characterised by:*

a/ *the holding of*

$$\Big[\sum_{j=0}^{1}(p^j/p^0)\frac{\partial G^j}{\partial X_1}\Big](p^1/p^0)^\star/(p^1/p^0)^\star = 1;$$

$$\Big[\sum_{j=0}^{1}(p^j/p^0)\frac{\partial G^j}{\partial X_0}\Big](p^1/p^0)^\star[\bar{\ell} - \ell((p^1/p^0)^\star,(p^1/p^0)^\star)] = (B/p^0)^\star + (p^1/p^0)^\star X_1^\star;$$

$$G^0[\bar{\ell} - \ell((p^1/p^0)^\star,(p^1/p^0)^\star), X_1^\star, (p^1/p^0)^\star)] = \Big[\sum_{j=0}^{1}(p^j/p^0)\frac{\partial G^j}{\partial X_0}\Big](p^1/p^0)^\star[\bar{\ell} - \ell((p^1/p^0)^\star,(p^1/p^0)^\star)];$$

b/ *aggregate factors and outputs shares that satisfy* $\pi_{X_0} = \pi_{Y^0} = \pi^1_{X_0}/(1-\Delta_\pi)$ *and* $\pi_{X_1} = \pi_{Y^1} = \pi^0_{X_1}/(1-\Delta_\pi)$;

c/ *a steady expression for the elasticity of substitution between the aggregate values of the input* X_0 *and* X_1 *that is given by:*

$$\Sigma^{\mathscr{GR}}_{X_0X_1} = (1-\Delta_\pi)\left(\pi^0_{X_0}\Sigma^0_{X_0X_1} + \pi^1_{X_1}\Sigma^1_{X_0X_1}\right).$$

It is worth emphasising that the equalities $\pi_{X_0} = \pi_{Y^1}$ and $\pi_{X_1} = \pi_{Y^0}$ in Lemma 3.1(i) and for the Wealth-to-Capital steady state directly result from the simple class of preferences with no first-period consumption and the ensued savings behaviour that are here considered. Otherwise stated, these result from the assumption that the young do not consume. Consequently, the (young) labour income is wholly accumulated by purchasing the investment good. In opposition to these parametric dimensions of Wealth-to-Capital equilibria, the Golden Rule steady state is fundamentally associated with the equilibrium of the financial sector. Making use of the no arbitrage opportunity condition, this results in the first condition of Lemma 3.1(ii)a/ that in turn leads to the satisfaction of $\pi_{X_1} = \pi_{Y^1}$, a condition that is is directly associated to the *Golden Rule* facet of the intermediated steady state and holds beyond the specifics of the here selected behaviour and budget constraints that allow no first-period consumption.

What is worthy to note is that, by contrast to what is seen at the Wealth-to-Capital steady state, at the Golden Rule steady state, the share of the consumption good equals the share of labour in total income while the share of capital is fully accumulated. Things appear to take place as if there were *workers* who do not save and in each period consume all their labour income, and *capitalists* who do not consume and invest all their capital income.To put these results into perspective, briefly consider the Woodford (1986) finance constrained economy that is populated by heterogeneous infinitely lived agents precisely labelled workers and capitalists. A key assumption is that a financial imperfection

prevents workers from borrowing against their future labour income. In the equilibria considered by this author, labour supply decisions of dynastic workers are actually identical to those that would be made by overlapping generations of two period lived agents that work in their first and consume in their second period of life. Capitalists do not work, hold the whole stock of capital, consume in every period a constant share of their capital income and invest the remaining share[19]. Embedded in a two-sector setup, the Woodford (1986)'s consumer sector would lead to two stationary equilibrium equations, namely $p^0Y^0 = \omega^0 X_0 + (1-\beta)w^1 X_1$ and $p^1Y^1 = \omega^1 X_1$, where $\beta \in]0,1[$ denotes the capitalists' propensity to save. In such an economy, at the steady state, $\pi_{Y^0} = \pi_{X_0} + (1-\beta)\pi_{X_1}$ while $\pi_{Y^1} = \beta\pi_{X_1}$. The Golden Rule relations of Lemma 3.1(ii) are thus recovered as $\beta \to 1$.

3.2 Samuelsonian vs Classical Economies

Lemma 3.1 illustrates how the presence of some institution making possible intergenerational transfers influences both the equilibrium aggregate factors shares and elasticity of substitution along a stationary equilibrium. To clarify the issue at stake, let the economies be categorised according to the sign of the transfers between generations at the Golden Rule steady state. Positive transfers, *i.e.*, the ones directed from the young to the old, indicate that the savings of the young are too high for the capital stock needed at the Golden Rule; the Wealth-to-Capital steady state over-accumulates capital. Oppositely, negative transfers, *i.e.*, the ones directed from the old to the young, indicate that savings are not sufficient to sustain the Golden Rule capital stock; the Wealth-to-Capital steady state under-accumulates capital. Making use of the terminology coined by Gale (1973), economies with positive (resp. negative) transfers at the Golden Rule steady state will be called *Samuelsonian* (resp. *classical*).[20]

Lemma 3.2. *Under Assumptions T1-4, P1-4, the set of classical (Samuelsonian) economies is associated with* $\pi^0_{X_0} < (>) \pi^1_{X_1}$.

In the Gale (1973) pure exchange framework, an economy's type is determined by the agents' preferences and endowment profile. In the current two-sector production framework, it purely rests upon technological factors. Echoing the earlier discussion associated with the classical Rybczynski and Stolper-Samuelson theorems after Lemma 2.1, it is worth emphasising that while the type of the economy depends upon factors shares in *value terms*, namely $\pi^1_{X_1}$ and $\pi^0_{X_0}$—reformulating, the set of classical (Samuelsonian) economies can also be understood as involving the sum of the two sectoral profit shares in being associated with the occurrence of $\pi^0_{X_1} + \pi^1_{X_1} > (<) 1$—, it is not directly linked to the Stolper-Samuelson theorem, which indeed rather involves Δ_π, *i.e.*, the difference between the two profit shares $\pi^1_{X_1}$ and $\pi^0_{X_1}$.

[19] In a related vein, mention is also to be made of the Ramsey model with borrowing constraints of Becker (1980) and Becker & Foias (1987, 1994). This economy is inhabited by heterogeneous infinitely-lived agents, the most patient eventually holding the whole stock of capital, hence receiving the entire capital income, more impatient agents consuming only their wages.

[20] Gale (1973) is a pure exchange model. He called the model Samuelsonian or classical depending on whether, at the Golden Rule, the young generations consume less or more than their endowments. In the former case savings (transfers) are then positive, in the latter they are negative.

In line with the earlier comments and making use of Lemma 3.2, one can complete comparisons of steady state equilibria in terms of the associated aggregate technological parameters.

Corollary 3.1.— *Let $\pi_{Y^j}^{\mathcal{SS}}$ and $\pi_{X_i}^{\mathcal{SS}}$ denote the share of output j and the share of input i at the steady state $\mathcal{SS} = \{\mathcal{WC}, \mathcal{GR}\}$ respectively. Then:*

(i) *In Samuelsonian economies,*

a/ $\pi_{Y^1}^{\mathcal{WC}} > \pi_{Y^0}^{\mathcal{WC}}$, $\pi_{X_0}^{\mathcal{WC}} > \pi_{X_1}^{\mathcal{WC}}$;

b/ $\pi_{Y^0}^{\mathcal{GR}} > \pi_{Y^1}^{\mathcal{GR}}$, $\pi_{X_0}^{\mathcal{GR}} > \pi_{X_1}^{\mathcal{GR}}$.

(ii) *In classical economies,*

a/ $\pi_{Y^0}^{\mathcal{WC}} > \pi_{Y^1}^{\mathcal{WC}}$, $\pi_{X_1}^{\mathcal{WC}} > \pi_{X_0}^{\mathcal{WC}}$;

b/ $\pi_{Y^1}^{\mathcal{GR}} > \pi_{Y^0}^{\mathcal{GR}}$, $\pi_{X_1}^{\mathcal{GR}} > \pi_{X_0}^{\mathcal{GR}}$.

Thus, Samuelsonian (resp. classical) economies are associated with a relatively high (resp. low) equilibrium share of labour. Savings being by construction equal to the wage income, abundance (resp. scarcity) of savings reveals, in accordance with the intuition, to be related with a relatively high (resp. low) economy-wide share of labour.

The following statement complements this first line of results by achieving a comparison between the two types of stationary equilibria:

Corollary 3.2.— *Let $\pi_{Y^j}^{\mathcal{SS}}$ and $\pi_{X_i}^{\mathcal{SS}}$ denote the share of output j and the share of input i at the steady state $\mathcal{SS} = \{\mathcal{WC}, \mathcal{GR}\}$ respectively. Then:*

(i) *In Samuelsonian economies,*

a/ $\pi_{Y^1}^{\mathcal{WC}} > \pi_{Y^1}^{\mathcal{GR}}$,

b/ $\pi_{X_1}^{\mathcal{WC}} \gtreqless \pi_{X_1}^{\mathcal{GR}}$ *if* $\pi_{X_1}^{1} \gtreqless \pi_{X_1}^{0}$;

(ii) *In classical economies,*

a/ $\pi_{Y^1}^{\mathcal{GR}} > \pi_{Y^1}^{\mathcal{WC}}$,

b/ *if* $\pi_{X_1}^{\mathcal{GR}} \gtreqless \pi_{X_1}^{\mathcal{WC}}$ *if* $\pi_{X_1}^{1} \gtreqless \pi_{X_1}^{0}$.

Thus, in Samuelsonian (resp. classical) economies the share of the investment good at the Golden Rule is lower (resp. higher) than at the Wealth-to-Capital steady state known to over-accumulate (resp. under-accumulate) capital. As for the economy-wide input shares, observe that the results finally rest upon the relative factor intensity. Noticing that π_{X_1}/π_{Y_1} boils down to $(\omega^1/p^0)^\star/(p^1/p^0)^\star$ and that $(\omega^1/p^0)^\star/(p^1/p^0)^\star$ will be increasing (resp. decreasing) in $(p^1/p^0)^\star$ according to whether $\pi_{X_1}^{1} > (<) \pi_{X_1}^{0}$—*vide* Lemma 2.1(i)—, the aforementioned dependency could be understood as the relative outputs price influence on the production of the investment good, hence upon the stock of capital at the steady state.

An integrated picture is then available from Lemmas 3.1 and 3.2. Consider, *e.g.*, the steady state of an economy where capital is the sole support for savings. Assume, for instance, that $\pi_{X_0}^{0} = \pi_{X_1}^{1}$.

From Lemma 3.1(i)b/, it follows that $\pi^{\mathcal{WC}}_{X_1} < \pi^{\mathcal{WC}}_{Y^1}$ or, equivalently, that $(\omega^1/p^0)^\star/(p^1/p^0)^\star < 1$. Since long run efficiency requires a unitary capital rental rate, one concludes that the Wealth-to-Capital steady state is inefficient; the capital rental rate is too low, signalling that the economy is producing too much investment good (over-accumulates). The introduction of a device allowing for (positive) transfers from the young to the old will ensure the existence of an efficient stationary equilibria in which the production of investment good is lower, *i.e.*, $\pi^{\mathcal{GR}}_{X_1} = \pi^{\mathcal{GR}}_{Y^1}$. In that steady state, by the Stolper-Samuelson effect, the higher capital rental rate in investment good, $(\omega^1/p^0)^\star/(p^1/p^0)^\star$, is matched by a lower relative consumption good price $1/(p^1/p^0)^\star$. These long-run relative price discrepancies explain why the output shares ordering is eventually reversed at the Golden Rule steady state, *i.e.*, $\pi^{\mathcal{GR}}_{Y^0} > \pi^{\mathcal{GR}}_{Y^1}$ (in a Samuelsonian economy). As a last remark, it may be noted that there is a one-to-one correspondence between the type of the economy and the aggregate factors shares. As a matter of fact, and from Lemma 3.2(i), $\pi^{\mathcal{GR}}_{X_0}/\pi^{\mathcal{GR}}_{X_1} = \pi^1_{X_0}/\pi^0_{X_1} = (1-\pi^1_{X_1})/(1-\pi^0_{X_0})$. Thus, in a Samuelsonian (resp. classical) economy, $\pi^{\mathcal{GR}}_{X_0} > \pi^{\mathcal{GR}}_{X_1}$ (resp. $\pi^{\mathcal{GR}}_{X_0} < \pi^{\mathcal{GR}}_{X_1}$). From that perspective, it could be said that an economy is Samuelsonian if the young share of total income (entirely devoted to savings by assumption) is greater than the old share of total income (entirely devoted to consumption).

Lastly, and as regards the equilibrium elasticity of substitution, findings are summarised in the following corollary:

Corollary 3.3.— *Noticing that* $\Sigma^{\mathcal{WC}}_{X_0X_1} - \Sigma^{\mathcal{GR}}_{X_0X_1} = (\pi^0_{X_0} - \pi^1_{X_1})\left(\Sigma^0_{X_0X_1} - \Sigma^1_{X_0X_1}\right)$, *one gets:*

(i) *In Samuelsonian economies,* $\mathrm{Sign}\left(\Sigma^{\mathcal{WC}}_{X_0X_1} - \Sigma^{\mathcal{GR}}_{X_0X_1}\right) = \mathrm{Sign}\left(\Sigma^0_{X_0X_1} - \Sigma^1_{X_0X_1}\right)$;

(ii) *In classical economies,* $\mathrm{Sign}\left(\Sigma^{\mathcal{WC}}_{X_0X_1} - \Sigma^{\mathcal{GR}}_{X_0X_1}\right) = \mathrm{Sign}\left(\Sigma^1_{X_0X_1} - \Sigma^0_{X_0X_1}\right)$.

Otherwise stated, while for Samuelsonian economies, the equilibrium elasticity of substitution between the aggregate inputs at the Golden Rule is larger (resp. higher) than at the Wealth-to-Capital steady state, when the sectoral elasticity of substitution is larger in investment good industry than in the consumption good industry. An opposite configuration prevails for classical economies. Interestingly, *letting both of the sectoral elasticities of substitution share a common value would translate as an identical value for the equilibrium elasticity of substitution for the Golden Rule and Wealth-to-Capital steady states.*

3.3 Non-Stationary Equilibria

A sequence $(X_{1,t}, p^1_t/p^0_t, B_t/p^0_t)_{t\geq 1}$ with a set of initial conditions $(X_{1,0}, p^1_0/p^0_0, B_0/p^0_0)$ that satisfies the system (2.8a)-(2.8c) is an equilibrium; this corresponds to a *Wealth-to-Capital equilibrium* if $B_0/p^0_0 = 0$ and to an *intermediated equilibrium* if $B_0/p^0_0 \neq 0$. Realistically enough, the global properties of the equilibrium trajectories, especially their asymptotic behaviours, although of interest, are unfortunately out of reach. A less ambitious aim, that is the focus of the current section, is to characterize the non-stationary equilibria in the neighbourhood of the steady states. In order to keep it short and concise, it will be assumed in the sequel that the investment good sector is more capital-intensive than the consumption good one, *i.e.*, $\pi^1_{X_1} - \pi^0_{X_1} > 0$.

Proposition 3.1.— *Under Assumptions T1-4, P1-4, assume gross substitutability in preferences,* $\Sigma^H_{\ell c'} > \Sigma^H_{\ell Y}$, *let* $\chi := \pi_\ell(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})$ *and consider an interior Wealth-to-Capital steady state:*

(i) *it is a saddle-point equilibrium if the following condition holds:*

$$\Sigma^0_{X_0X_1} + \frac{\pi^1_{X_0}}{\pi^0_{X_1}}\Sigma^1_{X_0X_1} > 1;$$

(ii) *it is either locally unstable or stable if oppositely:*

$$\Sigma^0_{X_0X_1} + \frac{\pi^1_{X_0}}{\pi^0_{X_1}}\Sigma^1_{X_0X_1} < 1.$$

Specifically:

a/ *it is locally unstable,* $\forall\chi$*, if* $\pi^0_{X_1} + \pi^1_{X_1} > 1$*;*

b/ *if the reverse inequality* $\pi^0_{X_1} + \pi^1_{X_1} < 1$ *prevails, there exists a critical value* $\chi^{\mathcal{PH}}$*, where* $\chi^{\mathcal{PH}} = \pi^1_{X_1}/[1 - (\pi^0_{X_1} + \pi^1_{X_1})]$*, such that the steady state in locally unstable for* $\chi < \chi^{\mathcal{PH}}$ *and locally stable and indeterminate for* $\chi > \chi^{\mathcal{PH}}$*.*

Remark 3.1.— The value of χ labelled $\chi^{\mathcal{PH}}$ is a Poincaré-Hopf bifurcation value. It indicates the potential existence of quasi-periodic trajectories in the neighborhood of the Wealth-to-Capital equilibria in case (iib) (*Vide*, *e.g.*, Guckenheimer & Holmes (2006)).

Remark 3.2.— It is of interest to emphasise that, as a direct byproduct of the class of preferences that are here considered, the results of Proposition 3.1 have their direct counterpart in a one good homogeneous environment with $F = F^0 = F^1$ that could be understood as a *smooth* version of Reichlin (1986) that would allow for factors substitutability. While the results of Proposition 3.1 could be recovered, they would only involve aggregate factors shares and the unique factors substitutability coefficient. It is however to be stressed that this does not extend to the remaining and more demanding configuration with $\Delta_\pi < 0$ that are specific to the current heterogeneous goods structure and the examination of which is left for future research.

Proposition 3.1 introduces a typology of the stability properties of the wealth-to-capital steady state for a regular configuration that involves a gross substitutability assumption on preferences together with a canonical greatest value for the profit share in the investment good industry, *i.e.*, $\Delta_\pi > 0$. Unsurprisingly enough, under sufficiently large values for the sectoral factors substitutability measures $\Sigma^0_{X_0X_1}$ and $\Sigma^1_{X_0X_1}$, the steady state assumes a locally unique convergent trajectory and the saddle-point property is recovered. The remaining configurations that are associated with lower values for the sectoral elasticities of substitution happen to be more involving. A first one, described as case (iia), corresponds to the occurrence of $\pi^0_{X_1} + \pi^1_{X_1} > 1$ and thus to the set of classical economies ; it is unequivocally associated with locally unstable wealth-to-capital steady states. A second one, described as case (iib), that is oppositely associated with $\pi^0_{X_1} + \pi^1_{X_1} < 1$ and thus to Samuelsonian economies, uncovers a more complex scenario. Interestingly, arbitrarily low values of χ and thus a limited discrepancy between substitution and revenue effects in preferences, the environment first recovers the unstable configuration of case (iia). However, and in opposition to this, for larger values of χ and thus a greater discrepancy between substitution and revenue effects in preferences, the wealth-to-capital steady state becomes locally stable and is thus locally indeterminate.

As a matter of fact, these just described smoothness properties of sectoral factors substitutability can be understood from a deeper perspective. Making indeed use of Lemma 3.1(i)c/, it is worth

emphasising that the statements (i) and (ii) of Proposition 3.1 could be reformulated in terms of the equilibrium aggregate elasticity of substitution :

$$\Sigma^{\mathscr{WC}}_{X_0X_1} > (<) \ \ \pi^0_{X_1}(1+\Delta_\pi).$$

Thus, if capital and labour are sufficiently substitutable at the Wealth-to-Capital steady state, the latter is locally saddle-point stable. However, and from a slightly different perspective, recall that Lemma 3.1 also indicates that the availability of an extra asset beyond the sole capital to support savings dramatically influences both the aggregate factors shares and the elasticity of substitution along a stationary equilibrium. One then notices that the *marginal contributions* of each specific sectorial elasticity of substitution, namely $\Sigma^j_{X_0X_1}$, $j \in \{0,1\}$, to the holding of the saddle-point stability condition do differ: the one of the consumption good sector is equal to 1, whereas the one of the investment good sector is equal to $\pi^1_{X_0}/\pi^0_{X_1} \gtreqless 1$. Such an influence upon the aggregate elasticity of substitution can be captured by introducing a substituabilities *assymetry* coefficient $\mathscr{A}$:

$$\mathscr{A} := \frac{\partial\Sigma^A_{X_0X_1}/\partial\Sigma^0_{X_0X_1}}{\partial\Sigma^A_{X_0X_1}/\partial\Sigma^1_{X_0X_1}}. \tag{3.2}$$

That coefficient captures the relative contribution of *sectorial* elasticities of substitution to the aggregate elasticity of substitution. To put it differently, $\mathscr{A}$ could be seen as the slope of an *iso*-substitutability curve. First noticing that the Wealth-to-Capital steady state expression of the asymmetry coefficient (3.2) summarises to $\mathscr{A}^{\mathscr{WC}} = \pi^0_{X_1}/\pi^1_{X_0}$, the border between determinacy and non-determinacy (instability or indeterminacy) in Proposition 3.1 thus reformulates to:

$$\mathscr{A}^{\mathscr{WC}}\Sigma^0_{X_0X_1} + \Sigma^1_{X_0X_1} = \mathscr{A}^{\mathscr{WC}}. \tag{3.3}$$

It is then worth emphasising that if, say, $\mathscr{A}^{\mathscr{WC}} > 1$, then $\Sigma^0_{X_0X_1} > 1$ unambiguously ensures the local determinacy of the Wealth-to-Capital steady state. Put in other words, a technology of production of the consumption good exhibiting a degree of substitutability higher than the unitary Cobb-Douglas one ensures that the Wealth-to-Capital steady state will be a saddle-point *whatever* the degree of substitutability in the capital good sector.

A related appraisal is available for the golden rule steady states:

Proposition 3.2.— *Under Assumptions T1-4, P1-4, assume gross substitutability in preferences,* $\Sigma^H_{\ell c'} > \Sigma^H_{\ell Y}$, *let* $\chi := \pi_\ell(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})$ *and consider an interior Golden Rule steady state:*

(i) *In Samuelsonian economies, the Golden Rule steady state is locally saddle-point stable;*

(ii) *In classical economies, the Golden Rule steady state is locally unstable (locally stable and indeterminate) for:*

$$\frac{\pi^1_{X_1}}{\pi^0_{X_0}} - 1 + \left(\Sigma^0_{X_0X_1} + \frac{\pi^1_{X_1}}{\pi^0_{X_0}}\Sigma^1_{X_0X_1} - 1\right)\frac{1}{\chi} < (>) 0.$$

Remark 3.3.— Local saddle-point stability (resp. local indeterminacy) here means that the local stable manifold has dimension 1 (resp. 2). As for classical economies, the boundary between local instability and indeterminacy corresponds to a Poincaré-Hopf bifurcation locus.

Remark 3.4.— As this was the case for wealth-to-capital steady equilibria, the results of Proposition 3.2 assume a direct counterpart in a one good environment with $F = F^0 = F^1$ where they appear as the components of Propositions IV.1 and IV.2 in Benhabib & Laroque (1988) that are obtained for $\chi := \pi_\ell(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y}) > 0$. Again, the extension of the results of Proposition 3.2 to the alternative sectoral relative factor intensities configuration $\Delta_\pi < 0$ and/or gross complementarity in preferencees along $\Sigma^H_{\ell Y} > \Sigma^H_{\ell c'}$ and $\chi < 0$ is a demanding task that should be the object of future research.

Proposition 3.2 introduces a typology of the stability properties of the golden rule steady state for a regular configuration that involves a gross substitutability assumption on preferences together with a canonical greatest value for the profit share in the investment good industry, *i.e.*, $\Delta_\pi > 0$. A first remarkable result springs from the unconditionality of the saddle point property in (i) and for Samuelsonian economies: in contradistinction with the earlier wealth-to-capital structure, this statement is unrelated to sectoral elasticities of elasticities of substitution. The contents of (ii), as for themselves, may first be clarified by recalling that classical economies are unequivocally associated with the occurrence of $\pi^1_{X_1}/\pi^0_{X_0} - 1 = (\pi^1_{X_1} + \pi^0_{X_1} - 1)/(1 - \pi^0_{X_1}) > 0$. Otherwise stated and under gross substitutability, a configuration with a locally unstable golden rule steady state requires at least one of the sectoral elasticities of substitution to be less than one. In opposition to this and again for classical economies, the emergence of local indeterminacy emerges more naturally for large values of sectoral elasticities of substitution.

Along the earlier substituabilities *assymetry* coefficient $\mathscr{A}$-based discussion of the stability properties of the wealth-to-capital steady state in Proposition 3.1, the (classical) Golden Rule steady state—the Samuelsonian one is always locally determinate—, is associated with the potential existence of a Poincaré-Hopf bifurcation. This delineates a critical border, namely $\pi^0_{X_0}\Sigma^0_{X_0X_1} + \pi^1_{X_1}\Sigma^1_{X_0X_1} = \pi^0_{X_0}$. Noticing however that the Golden Rule steady state expression of this asymmetry coefficient (3.2) summarises to $\mathscr{A}^{\mathscr{GR}} = \pi^0_{X_0}/\pi^1_{X_1}$, this reformulates to:

$$\mathscr{A}^{\mathscr{GR}}\Sigma^0_{X_0X_1} + \Sigma^1_{X_0X_1} = \mathscr{A}^{\mathscr{GR}}. \tag{3.4}$$

The formal similarity between the conditions governing the stability properties of the Wealth-to-Capital steady state (3.3) and the Golden Rule steady state (3.4) conditions, or the condition in Proposition 3.1(i) and the coefficient between parenthesis Proposition 3.2(ii), is remarkable. As a matter of fact, it seems that, while everything sums up, for wealth-to-capital steady states, to the sign of $\pi^1_{X_1}/\pi^0_{X_0} - 1 = (\pi^1_{X_1} + \pi^0_{X_1} - 1)/(1 - \pi^0_{X_1})$, this assumes its counterpart for Golden Rule equilibria with the sign of $\pi^1_{X_0}/\pi^0_{X_1} - 1 = [1 - (\pi^0_{X_1} + \pi^1_{X_1})]/\pi^0_{X_1}$, *i.e.*, closely related formal expressions that have opposite sign requisites on $1 - (\pi^0_{X_1} + \pi^1_{X_1})$. The asymmetric role of sectorial factors substitutability is here clearly underlined once it is remarked that this translates with the substituabilities asymmetry coefficient $\mathscr{A}$ as $\mathscr{A}^{\mathscr{WC}} > 1 \Longrightarrow \mathscr{A}^{\mathscr{GR}} < 1$. More generally, it may be worth emphasising that this article appears to be the first to have emphasised the role of this coefficient that, while building upon an intersectoral comparison in factors shares, does not relate to the classical inter-sectoral differences of a given factor share that is associated with the coefficient Δ_π. Indeed, while the coefficients $\pi^1_{X_1}/\pi^0_{X_0}$ and $\pi^1_{X_0}/\pi^0_{X_1}$ both build from a comparison between a factor share in a given sector and the *other* factor share in the *other* sector, they have been proved to be at the core of the type of economy under consideration—Samuelsonian vs classical—but also to be central to the stability properties of both

wealth-capital and golden rule steady states. At that stage, it remains to wonder in which regards their role could extend to more complex environments.

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A. Proofs of Lemma 2.1 and Proposition 2.1

From the linear homogeneity assumption in T1:

$$Y^j = \sum_{i=0}^{1} \frac{\partial F^j}{\partial X_{ij}}(X_{0j}, X_{1j})X_{ij},\ j \in \{0,1\}.$$

Then, making use of the linear homogeneity assumption on the optimality condition on the inputs (2.4a),

$$p^j Y^j = \sum_{i=0}^{1} \omega^i X_{ij},\ j \in \{0,1\}. \tag{A.1}$$

Letting $\mathscr{X} := dx/x$ and $\Omega^i := d\omega^i/\omega^i$, differentiation of the previous equations gives:

$$\mathscr{P}^j + \mathscr{Y}^j = \sum_{i=0}^{1} \pi^j_{X_i}\left(\Omega^i + \mathscr{X}_{ij}\right),\quad j \in \{0,1\}. \tag{A.2}$$

for $\pi^j_{X_i} := \omega^i X_{ij}/p^j Y^j$ the factor i income share in sector j. In differential terms, the equilibrium conditions on the outputs (2.4b) write as

$$\mathscr{Y}^j = \sum_{i=0}^{1} \pi^j_{X_i}\mathscr{X}_{ij},\ j \in \{0,1\}. \tag{A.3}$$

Plugging (A.3) into (A.2) finally delivers:

$$\sum_{i=0}^{1} \pi^j_{X_i}\Omega^i = \mathscr{P}^j,\ j \in \{0,1\}, \tag{A.4}$$

that show that factor prices depend uniquely upon output prices.

Turning now to the full-employment relations (2.4c), rewrite them as:

$$\sum_{j=0}^{1} Y^j \frac{X_{ij}}{Y^j} = X_i,\ i \in \{0,1\}. \tag{A.5}$$

Totally differentiating these equations, one obtains:

$$\sum_{j=0}^{1} \frac{X_{ij}}{X_i}\left(\mathscr{Y}^j + \mathscr{X}_{ij} - \mathscr{Y}^j\right) = \mathscr{X}_i,\quad i \in \{0,1\}. \tag{A.6}$$

It follows from $X_{ij}/Y^j = X_{ij}/F^j(X_{0j}, X_{1j})$ that:

$$\mathscr{X}_{0j} - \mathscr{Y}^j = \pi^j_{X_1}\left(\mathscr{X}_{0j} - \mathscr{X}_{1j}\right),$$
$$\mathscr{X}_{1j} - \mathscr{Y}^j = \pi^j_{X_0}\left(\mathscr{X}_{1j} - \mathscr{X}_{0j}\right).$$

Plugging the latter expressions into (A.6), and letting $X_{ij}/X_i := \mu_{ij}$ denotes the fraction of the ith factor employed in sector j, yield:

$$\sum_{j=0}^{1} \mu_{0j}\left(\mathscr{Y}^j + \pi^j_{X_1}\left(\mathscr{X}_{0j} - \mathscr{X}_{1j}\right)\right) = \mathscr{X}_0,$$
$$\sum_{j=0}^{1} \mu_{1j}\left(\mathscr{Y}^j + \pi^j_{X_0}\left(\mathscr{X}_{1j} - \mathscr{X}_{0j}\right)\right) = \mathscr{X}_1.$$

Remembering that $\mathscr{X}_{0j} - \mathscr{X}_{1j} = -\Sigma^j_{X_0X_1}(\Omega^0 - \Omega^1)$, and making the appropriate substitutions, one finally obtains:

$$\sum_{j=0}^{1} \mu_{0j}\left(\mathscr{Y}^j - \pi^j_{X_1}\Sigma^j_{X_0X_1}(\Omega^0 - \Omega^1)\right) = \mathscr{X}_0,$$
$$\sum_{j=0}^{1} \mu_{1j}\left(\mathscr{Y}^j + \pi^j_{X_0}\Sigma^j_{X_0X_1}(\Omega^0 - \Omega^1)\right) = \mathscr{X}_1.$$

The total differentiation of the equilibrium equations (2.4a)-(2.4c) now yields:

$$\pi^j_{X_0}\Omega^0 + \pi^j_{X_1}\Omega^1 = \mathscr{P}^j, j \in \{0,1\} \tag{A.7a}$$
$$\mathscr{X}_{1j} - \mathscr{X}_{0j} = \Sigma^j_{X_0X_1}(\Omega^0 - \Omega^1), j \in \{0,1\}, \tag{A.7b}$$
$$\sum_j \mu_{0j}\mathscr{Y}^j = \mathscr{X}_0 + \sigma_0(\Omega^0 - \Omega^1), \tag{A.7c}$$
$$\sum_j \mu_{1j}\mathscr{Y}^j = \mathscr{X}_1 - \sigma_1(\Omega^0 - \Omega^1). \tag{A.7d}$$

for $\mu_{ij} := X_{ij}/X_i$ and $\sigma_i := \sum_{j=0}^{1}\mu_{ij}\left(1 - \pi^j_{X_i}\right)\Sigma^j_{X_0X_1}$.

Now, let $\Delta_\pi := \pi^0_{X_0}\pi^1_{X_1} - \pi^1_{X_0}\pi^0_{X_1} = \pi^1_{X_1} - \pi^0_{X_1}$ where the second equality comes from $\sum_i \pi^j_{X_i} = 1$. Assumption T4 ensures $\Delta_\pi \neq 0$. Thus, solving (A.7a)-(A.7d) delivers:

$$\Omega^0 = \frac{1}{\Delta_\pi}\left(\pi^1_{X_1}\mathscr{P}^0 - \pi^0_{X_1}\mathscr{P}^1\right), \tag{A.8a}$$
$$\Omega^1 = \frac{1}{\Delta_\pi}\left(-\pi^1_{X_0}\mathscr{P}^0 + \pi^0_{X_0}\mathscr{P}^1\right). \tag{A.8b}$$

Hence:

$$\Omega^0 - \Omega^1 = -\frac{1}{\Delta_\pi}\left(\mathscr{P}^1 - \mathscr{P}^0\right), \tag{A.9}$$

that establishes Lemma 2.1(i)a/ and Lemma 2.1(i)b/.

As regards outputs, solving (A.7c)-(A.7d), making use of (A.9) and letting $\Delta_\mu := \mu_{11} - \mu_{01}$, delivers the dependency of the SSF with respect to inputs and the relative price of the investment output:

$$\mathscr{Y}^0 = \frac{1}{\Delta_\mu}\left(\mu_{11}\mathscr{X}_0 - \mu_{01}\mathscr{X}_1 - (\mu_{11}\sigma_0 + \mu_{01}\sigma_1)\frac{1}{\Delta_\pi}\left(\mathscr{P}^1 - \mathscr{P}^0\right)\right), \tag{A.10a}$$
$$\mathscr{Y}^1 = \frac{1}{\Delta_\mu}\left(-\mu_{10}\mathscr{X}_0 + \mu_{00}\mathscr{X}_1 + (\mu_{10}\sigma_0 + \mu_{00}\sigma_1)\frac{1}{\Delta_\pi}\left(\mathscr{P}^1 - \mathscr{P}^0\right)\right). \tag{A.10b}$$

Hence, as $\sum_{j=0}^{1}\mu_{ij} = 1$,

$$\mathscr{Y}^0 - \mathscr{Y}^1 = \frac{1}{\Delta_\mu}\left(\mathscr{X}_0 - \mathscr{X}_1 + (\sigma_0 + \sigma_1)\frac{1}{\Delta_\pi}\left(\mathscr{P}^1 - \mathscr{P}^0\right)\right), \tag{A.11}$$

that establishes Lemma 2.1(ii)a/ and Lemma 2.1(ii)b/.

Fixing the outputs and letting the relative factor quantity vary, equation (A.11) then gives:

$$(\Omega^0 - \Omega^1) = \frac{1}{\sigma_0 + \sigma_1}\left(\mathscr{X}_1 - \mathscr{X}_0\right), \tag{A.12}$$

that establishes Proposition 2.1. QED

B. An Ordinal Representation of Preferences

The representative consumer of generation $t \geq 0$ is equivalently to solve

$$\min_{\{\ell_t^t, c_{t+1}^t\}} (p_{t+1}^0/p_t^0)c_{t+1}^t/\mathscr{R}_{t+1} - (\omega_t^0/p_t^0)(\bar{\ell} - \ell_t^t) \text{ subject to } u(\ell_t^t, c_{t+1}^t) \geqslant \bar{u},$$

$\bar{u}$ given. Letting μ denote the shadow price associated to the constraint, the first-order conditions derive as :

$$\begin{aligned}
&\omega_t^0/p_t^0 = \mu \frac{\partial u}{\partial \ell_t^t}(\ell_t^t, c_{t+1}^t),\\
&(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1} = \mu \frac{\partial u}{\partial c_{t+1}^t}(\ell_t^t, c_{t+1}^t),\\
&\bar{u} = u(\ell_t^t, c_{t+1}^t).
\end{aligned}$$

The consumption expense function of the consumer emerges as

$$\begin{aligned}
E[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}] :&= (\omega_t^0/p_t^0)\ell_t^{t,H}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}]\\
&\quad + c_{t+1}^{t,H}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}]/\mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0),
\end{aligned}$$

where $\ell_t^{t,H}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}]$ and $c_{t+1}^{t,H}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}]$ denote the dual Hicksian demand functions, for

$$\begin{aligned}
\ell_t^{t,H}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}] &= \partial E[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}]/\partial(\omega_t^0/p_t^0),\\
c_{t+1}^{t,H}[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}] &= \partial E[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}]/\partial[\mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0)],\\
\mu^H[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}] &= \partial E[\omega_t^0/p_t^0, \mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0), \bar{u}]/\partial \bar{u}.
\end{aligned}$$

their expression in its turn resulting from the total differentiation of the expense function:

$$\begin{aligned}
\Delta E &= \Big[(\omega_t^0/p_t^0)\frac{\partial \ell_t^{t,H}}{\partial \bar{u}} + [(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]\frac{\partial c_{t+1}^{t,H}}{\partial \bar{u}}\Big]\Delta \bar{u}\\
&\quad + \Big[\ell_t^{t,H} + (\omega_t^0/p_t^0)\frac{\partial \ell_t^{t,H}}{\partial(\omega_t^0/p_t^0)} + [(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]\frac{\partial c_{t+1}^{t,H}}{\partial(\omega_t^0/p_t^0)}\Big]\Delta(\omega_t^0/p_t^0)\\
&\quad + \Big[c_{t+1}^{t,H} + (\omega^0/p_t^0)\frac{\partial \ell_t^{t,H}}{\partial[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]}\\
&\qquad + [(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]\frac{\partial c_{t+1}^{t,H}}{\partial[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]}\Big]\Delta[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]\\
&= \mu\Big(\frac{\partial u}{\partial \ell_t^t}\frac{\partial \ell_t^{t,H}}{\partial \bar{u}} + \frac{\partial u}{\partial c_{t+1}^t}\frac{\partial c_{t+1}^{t,H}}{\partial \bar{u}}\Big)\Delta \bar{u}\\
&\quad + \Big[\ell_t^{t,H} + \mu\Big(\frac{\partial u}{\partial \ell_t^t}\frac{\partial \ell_t^{t,H}}{\partial(\omega_t^0/p_t^0)} + \frac{\partial u}{\partial c_{t+1}^t}\frac{\partial c_{t+1}^{t,H}}{\partial(\omega_t^0/p_t^0)}\Big)\Big]\Delta(\omega_t^0/p_t^0)\\
&\quad + \Big[c_{t+1}^{t,H} + \mu\Big(\frac{\partial u}{\partial \ell_t^t}\frac{\partial \ell_t^{t,H}}{\partial[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]} + \frac{\partial u}{\partial c_{t+1}^t}\frac{\partial c_{t+1}^{t,H}}{\partial[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]}\Big)\Big]\Delta[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}].
\end{aligned}$$

Further noticing that the differentiation of the system of $\bar{u} = u(c_t^t, c_{t+1}^t)$ gives

$$
\begin{aligned}
1 &= \frac{\partial u}{\partial \ell_t^t}\frac{\partial \ell_t^{t,H}}{\partial \bar{u}} + \frac{\partial u}{\partial c_{t+1}^t}\frac{\partial c_{t+1}^{t,H}}{\partial \bar{u}},\\
0 &= \frac{\partial u}{\partial \ell_t^t}\frac{\partial \ell_t^{t,H}}{\partial (\omega_t^0/p_t^0)} + \frac{\partial u}{\partial c_{t+1}^t}\frac{\partial c_{t+1}^{t,H}}{\partial (\omega_t^0/p_t^0)},\\
0 &= \frac{\partial u}{\partial \ell_t^t}\frac{\partial \ell_t^{t,H}}{\partial [(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]} + \frac{\partial u}{\partial c_{t+1}^t}\frac{\partial c_{t+1}^{t,H}}{\partial [(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]},
\end{aligned}
$$

the eventual differentiated expression of the sectoral cost functions simplifies to :

$$
\Delta E = \mu \Delta \bar{u} + \ell_t^{t,H}\Delta(\omega_t^0/p_t^0) + c_{t+1}^{t,H}\Delta[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}],
$$

that establishes the earlier assertion.

Introducing then $\pi_\ell := (\omega_t^0/p_t^0)\ell_t^t/[(\omega_t^0/p_t^0)\ell_t^t + c_{t+1}^t/\mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0)]$, $\pi_{c'} := c_{t+1}^t/\mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0))\ /\ [(\omega_t^0/p_t^0)\ell_t^t + c_{t+1}^t/\mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0)]$ as the shares of the first period leisure and second-period consumption expenses in the total expenses of the representative agent of generation $t \in \mathbb{N}$, letting $\mathscr{S}$ feature the scale elasticity of $u(\cdot,\cdot)$, the linearised system of the first-order conditions gives :

$$
\begin{bmatrix} \Xi_{\ell\ell}\mathscr{S} & \Xi_{\ell c'}\mathscr{S} & 1 \\ \Xi_{c'\ell}\mathscr{S} & \Xi_{c'c'}\mathscr{S} & 1 \\ 1 & 1 & 0 \end{bmatrix}
\begin{bmatrix} \pi_\ell & 0 & 0 \\ 0 & \pi_{c'} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \mathscr{L}_t^t \\ \mathscr{C}_{t+1}^t \\ \mathscr{M} \end{bmatrix}
=
\begin{bmatrix} \Omega_t^0 - \mathscr{P}_t^0 \\ (\mathscr{P}_{t+1}^0 - \mathscr{P}_t^0) - \mathscr{R}_{t+1} \\ \mathscr{U}/\mathscr{S} \end{bmatrix}
\tag{B.1}
$$

and the following indexes describe the following ordinal second-order elasticities of the expense function:

$$
\begin{aligned}
\Sigma_{\ell c'}^H &:= E\frac{\partial^2 E}{\partial(\omega_t^0/p_t^0)\partial\,[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]} \Big/ \frac{\partial E}{\partial(\omega_t^0/p_t^0)}\frac{\partial E}{\partial[(p_{t+1}^0/p_t^0)/\mathscr{R}_{t+1}]},\\
\Sigma_{\ell Y}^H &:= E\frac{\partial^2 E}{\partial(\omega_t^0/p_t^0)\partial\bar{u}} \Big/ \frac{\partial E}{\partial(\omega_t^0/p_t^0)}\frac{\partial E}{\partial\bar{u}}, \Sigma_{YY}^H := E\frac{\partial^2 E}{\partial\bar{u}\partial\bar{u}} \Big/ \frac{\partial E}{\partial\bar{u}}\frac{\partial E}{\partial\bar{u}}.
\end{aligned}
$$

for which any coefficient $\Sigma_{\ell c'}^H$ refers to the dual Hicksian elasticities of substitution. The linearisation of the defining equations of the system of demand functions then gives :

$$
\begin{bmatrix} \mathscr{L}_{t,H}^t \\ \mathscr{C}_{t+1}^{t,H} \\ \mathscr{M} \end{bmatrix}
=
\begin{bmatrix} \Sigma_{\ell\ell}^H & \Sigma_{\ell c'}^H & \Sigma_{\ell Y}^H \\ \Sigma_{c'\ell}^H & \Sigma_{c'c'}^H & \Sigma_{c'Y}^H \\ \Sigma_{Y\ell}^H & \Sigma_{Yc'}^H & \Sigma_{YY}^H\mathscr{S} \end{bmatrix}
\begin{bmatrix} \pi_\ell & 0 & 0 \\ 0 & \pi_{c'} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \Omega_t^0 - \mathscr{P}_t^0 \\ (\mathscr{P}_{t+1}^0 - \mathscr{P}_t^0) - \mathscr{R}_{t+1} \\ \mathscr{U}/\mathscr{S} \end{bmatrix}.
\tag{B.2}
$$

Taking then advantage of

$$
\begin{aligned}
&\begin{bmatrix} \Sigma_{\ell\ell}^H & \Sigma_{\ell c'}^H \\ \Sigma_{c'\ell}^H & \Sigma_{c'c'}^H \end{bmatrix}
\begin{bmatrix} \pi_\ell & 0 \\ 0 & \pi_{c'} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \end{bmatrix},\\
&\begin{bmatrix} \Sigma_{\ell Y}^H & \Sigma_{c'Y}^H \end{bmatrix}
\begin{bmatrix} \pi_\ell & 0 \\ 0 & \pi_{c'} \end{bmatrix}
\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1,
\end{aligned}
$$

and identifying between (B.1) and (B.2), it derives that the components of the Hessian elasticities

matrix of $u(\cdot,\cdot)$, e.g., $\Xi_{\ell\ell}:=u[\partial^2 u/\partial(\ell_t^t)^2]\ /\ (\partial u/\partial \ell_t^t)^2$, state as:

$$\mathscr{S}\begin{bmatrix}\Xi_{\ell\ell} & \Xi_{\ell c'}\\ \Xi_{c'\ell} & \Xi_{c'c'}\end{bmatrix}=\begin{bmatrix}-\Sigma_{YY}^H\mathscr{S}-\dfrac{(\Sigma_{c'Y}^H)^2\pi_{c'}}{\Sigma_{\ell c'}^H\pi_\ell} & -\Sigma_{YY}^H\mathscr{S}+\dfrac{\Sigma_{\ell Y}^H\Sigma_{c'Y}^H}{\Sigma_{\ell c'}^H}\\ -\Sigma_{YY}^H\mathscr{S}+\dfrac{\Sigma_{\ell Y}^H\Sigma_{c'Y}^H}{\Sigma_{\ell c'}^H} & -\Sigma_{YY}^H\mathscr{S}-\dfrac{(\Sigma_{\ell Y}^H)^2\pi_\ell}{\Sigma_{\ell c'}^H\pi_{c'}}\end{bmatrix},$$
$$=-\Sigma_{YY}^H\mathscr{S}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}-\begin{bmatrix}\Sigma_{c'Y}^H\pi_{c'}\\ -\Sigma_{\ell Y}^H\pi_\ell\end{bmatrix}\begin{bmatrix}\Sigma_{c'Y}^H\pi_{c'} & -\Sigma_{\ell Y}^H\pi_\ell\end{bmatrix}\frac{1}{\Sigma_{\ell c'}^H\pi_\ell\pi_{c'}}$$
$$=-1\Sigma_{YY}^H\mathscr{S}1'+\Upsilon(\Sigma_{\ell c'}^H\pi_\ell\pi_{c'})^{-1}\Upsilon'.$$

for $1'=\begin{bmatrix}1 & 1\end{bmatrix}$ and $\Upsilon'=\begin{bmatrix}\Sigma_{c'Y}^H\pi_{c'} & -\Sigma_{\ell Y}^H\pi_\ell\end{bmatrix}$. Interestingly, the very holding of $\Sigma_{\ell c'}^H>0$ does not preclude the occurrence of $\Xi_{\ell c'}<0$, a conjunction under which the first-period leisure and second-period consumption would simultaneously be p-substitutes and q-complements. QED

C. Proof of Lemma 3.1

(i) The steady state is defined by the holding of $[\sum_{j=0}^1(p^j/p_0)(\partial G^j/\partial X_0)][(p^1/p^0)^*](\bar{\ell}-\ell^*)=(p^1/p^0)^*X_1^*$. Rearranging, this reformulates to: $\pi_{X_0}=\pi_{Y^1}$, that is $\sum_{j=0}^1\pi_{Y^j}\pi_{X_0}^j=\pi_{Y^1}$. Solving in π_{Y^1}, the steady state expression of the latter is obtained as $\pi_{Y^1}^{\mathscr{WC}}=(1-\pi_{X_1}^0)/(1+\pi_{X_1}^1-\pi_{X_1}^0)$, the obtention of the associated expressions of π_{Y^0}, $\pi_{X_0}=\sum_{j=0}^1\pi_{Y^j}\pi_{X_0}^j$, $\pi_{X_1}=\sum_{j=0}^1\pi_{Y^j}\pi_{X_1}^j$ and $\Sigma_{X_0X_1}=\sum_{j=0}^1\pi_{Y^j}\pi_{X_0}^j\pi_{X_1}^j\Sigma_{X_0X_1}^j$ being then immediate.
(ii) Letting the equilibrium rate of return satisfy $[\sum_{j=0}^1(p^j/p_0)(\partial G^j/\partial X_1)][(p^1/p^0)^\star]/(p^1/p^0)^*=1$, this proceeds from an adaptation of the line of argument developed for the Proof of (i). QED

D. Proof of Lemma 3.2

From the Golden Rule expressions of π_{X_0} and π_{Y^1} available from Lemma 3.1(ii):

$$\frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star}\gtrless 0\iff 1-\frac{\pi_{X_0}^0}{\pi_{X_1}^1}\lessgtr 0,$$

that establishes Lemma 3.2. QED

E. Proof of Proposition 3.1

First-order conditions:

$$\frac{\partial u}{\partial \ell_t^t}(\ell_t^t,c_{t+1}^t)-(\omega_t^0/p_t^0)[\mathscr{R}_{t+1}/(p_{t+1}^0/p_t^0)]\frac{\partial u}{\partial c_{t+1}^t}(\ell_t^t,c_{t+1}^t)=0,$$
$$(p_{t+1}^0/p_t^0)c_{t+1}^t/\mathscr{R}_{t+1}-(\omega_t^0/p_t^0)(\bar{\ell}-\ell_t^t)=0.$$

Linearising:

$$\begin{bmatrix}(\Xi_{\ell\ell}-\Xi_{c'\ell})\pi_\ell\mathscr{S} & -(\Xi_{c'c'}-\Xi_{\ell c'})\pi_{c'}\mathscr{S}\\ \pi_\ell & \pi_{c'}\end{bmatrix}\begin{bmatrix}\mathscr{L}\\ \mathscr{C}'\end{bmatrix}$$
$$-\begin{bmatrix}1 & 1\\ \sum_{j=0}^1\pi_{Y^j}\pi_{X_0}^j & \pi_{c'}\end{bmatrix}\begin{bmatrix}\Omega_t^0-\mathscr{P}_t^0\\ \mathscr{R}_{t+1}-(\mathscr{P}_{t+1}^0-\mathscr{P}_t^0)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

The system of demand equations derives as:

$$\begin{bmatrix} \Delta_{\ell(\omega^0/p^0)} & \Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]} \\ \Delta_{c'(\omega^0/p^0)} & \Delta_{c'[\mathscr{R}'/(p^{0\prime}/p^0)]} \end{bmatrix}$$
$$= \{[(\Xi_{\ell\ell} - \Xi_{c'\ell}) + (\Xi_{c'c'} - \Xi_{\ell c'})]\pi_\ell \pi_{c'} \mathscr{S}\}^{-1}$$
$$\times \begin{bmatrix} \pi_{c'}[1 + (\Xi_{c'c'} - \Xi_{\ell c'}) \sum_{j=0}^{1} \pi_{Y^j} \pi_{X_0}^j \mathscr{S}] & \pi_{c'}[1 + (\Xi_{c'c'} - \Xi_{\ell c'})\pi_{c'}\mathscr{S}] \\ -\pi_\ell[1 - (\Xi_{\ell\ell} - \Xi_{c'\ell}) \sum_{j=0}^{1} \pi_{Y^j} \pi_{X_0}^j \mathscr{S}] & -\pi_\ell[1 - (\Xi_{\ell\ell} - \Xi_{c'\ell})\pi_{c'}\mathscr{S}] \end{bmatrix}$$

where a more satisfactory picture in terms of the demand functions is also available, noticing that $\Xi_{\ell\ell} - \Xi_{c'\ell} = -\pi_{c'}\Sigma^H_{c'Y}/\pi_{c'}\pi_\ell\Sigma^H_{\ell c'}$, $\Xi_{c'c'} - \Xi_{\ell c'} = -\pi_\ell\Sigma^H_{\ell Y}/\pi_{c'}\pi_\ell\Sigma^H_{\ell c'}$, as:

$$\begin{bmatrix} \Delta_{\ell(\omega^0/p^0)} & \Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]} \\ \Delta_{c'(\omega^0/p^0)} & \Delta_{c'[\mathscr{R}'/(p^{0\prime}/p^0)]} \end{bmatrix} = \begin{bmatrix} -(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})\pi_{c'} & -(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})\pi_{c'} \\ \Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'Y}\pi_{c'} & \Sigma^H_{\ell c'}\pi_\ell + \Sigma^H_{c'Y}\pi_{c'} \end{bmatrix}.$$

The raw form of the linearised version of this dynamical system in a neighbourhood of the steady state is available, once it is noticed that the components of the dynamical system reformulate to:

$$X_{1,t+2} - G^1[\bar{\ell} - \ell(p^1_{t+1}/p^0_{t+1}, p^1_{t+2}/p^0_{t+2}), X_{1,t+1}, p^1_{t+1}/p^0_{t+1}] = 0,$$
$$X_{1,t+2} = (p^1_{t+1}/p^0_{t+1})^{-1}\Big\{(\omega^0/p^0)(p^1_{t+1}/p^0_{t+1})[\bar{\ell} - \ell(p^1_{t+1}/p^0_{t+1}, p^1_{t+2}/p^0_{t+2})]\Big\},$$
$$\ell(p^1_t/p^0_t, p^1_{t+1}/p^0_{t+1}) = \mathscr{L}\Big[(\omega^0/p^0)(p^1_t/p^0_t), \frac{(\omega^1/p^0)(p^1_{t+1}/p^0_{t+1})}{(p^1_t/p^0_t)}\Big],$$

the linearisation of the dynamical system in the neighbourhood of the Wealth-to-Capital steady state delivers:

$$A'_{11}\Delta(p^1_{t+2}/p^0_{t+2}) + A'_{12}\Delta(p^1_{t+1}/p^0_{t+1}) + A'_{13}\Delta(p^1_t/p^0_t) = 0,$$
$$A'_{11} = \frac{\partial X_{1,t+1}}{\partial(p^1_{t+1}/p^0_{t+1})} + \frac{\partial G^1}{\partial X_0}\frac{\partial \ell}{\partial(p^1_{t+1}/p^0_{t+1})},$$
$$A'_{12} = \frac{\partial X_{1,t+1}}{\partial(p^1_t/p^0_t)} - \frac{\partial G^1}{\partial X_1}\frac{\partial X_{1,t+1}}{\partial(p^1_{t+1}/p^0_{t+1})} + \frac{\partial G^1}{\partial X_0}\frac{\partial \ell}{\partial(p^1_t/p^0_t)} - \frac{\partial G^1}{\partial(p^1_t/p^0_t)},$$
$$A'_{13} = -\frac{\partial G^1}{\partial X_1}\frac{\partial X_{1,t+1}}{\partial(p^1_t/p^0_t)},$$

$$\frac{\partial X_{1,t+1}}{\partial(p^1_t/p^0_t)} = \frac{1}{p^1/p^0}\Big[-\frac{\omega^0/p^0}{p^1/p^0} + \frac{\partial(\omega^0/p^0)}{\partial(p^1/p^0)}\Big](\bar{\ell} - \ell) + \frac{\omega^0/p^0}{p^1/p^0}\Big(-\frac{\partial \ell}{\partial(p^1_t/p^0_t)}\Big),$$
$$\frac{\partial X_{1,t+1}}{\partial(p^1_{t+1}/p^0_{t+1})} = \frac{\omega^0/p^0}{p^1/p^0}\Big(-\frac{\partial \ell}{\partial(p^1_{t+1}/p^0_{t+1})}\Big),$$

$$\frac{\partial \ell}{\partial(p^1_t/p^0_t)} = \frac{\partial \mathscr{L}}{\partial(\omega^0/p^0)}\frac{\partial(\omega^0/p^0)}{\partial(p^1/p^0)} - \frac{\partial \mathscr{L}}{\partial \mathscr{R}'}\frac{(\omega^1/p^0)}{(p^1/p^0)^2},$$
$$\frac{\partial \ell}{\partial(p^1_{t+1}/p^0_{t+1})} = \frac{\partial \mathscr{L}}{\partial \mathscr{R}'}\frac{\partial(\omega^1/p^0)}{\partial(p^1/p^0)}\frac{1}{(p^1/p^0)},$$

whence:

$$
\begin{aligned}
A'_{11} &= \left(-\frac{\omega^0/p^0}{p^1/p^0} + \frac{\partial G^1}{\partial X_0}\right)\left\{\frac{\partial \mathscr{L}}{\partial \mathscr{R}'}\left[\frac{\partial(\omega^1/p^0)}{\partial(p^1/p^0)}\right]\frac{1}{(p^1/p^0)}\right\},\\
A'_{12} &= \frac{1}{p^1/p^0}\left[-\frac{\omega^0/p^0}{p^1/p^0} + \frac{\partial(\omega^0/p^0)}{\partial(p^1/p^0)}\right](\bar{\ell}-\ell) + \left(-\frac{\omega^0/p^0}{p^1/p^0} + \frac{\partial G^1}{\partial X_0}\right)\frac{\partial \ell}{\partial(p^1_t/p^0_t)}\\
&\quad - \left[\frac{\partial G^1}{\partial X_1}\right]\frac{\omega^0/p^0}{p^1/p^0}\left(-\frac{\partial \ell}{\partial(p^1_{t+1}/p^0_{t+1})}\right) - \frac{\partial G^1}{\partial(p^1_t/p^0_t)}\\
&= \frac{1}{p^1/p^0}\left[-\frac{\omega^0/p^0}{p^1/p^0} + \frac{\partial(\omega^0/p^0)}{\partial(p^1/p^0)}\right](\bar{\ell}-\ell) - \frac{\partial G^1}{\partial(p^1_t/p^0_t)}\\
&\quad + \left(-\frac{\omega^0/p^0}{p^1/p^0} + \frac{\partial G^1}{\partial X_0}\right)\left\{\frac{\partial \mathscr{L}}{\partial(\omega^0/p^0)}\frac{\partial(\omega^0/p^0)}{\partial(p^1/p^0)} + \frac{\partial \mathscr{L}}{\partial \mathscr{R}'}\left[-\frac{(\omega^1/p^0)}{(p^1/p^0)^2}\right]\right\}\\
&\quad + \left[\frac{\partial G^1}{\partial X_1}\right]\frac{\omega^0/p^0}{p^1/p^0}\left\{\frac{\partial \mathscr{L}}{\partial \mathscr{R}'}\left[\frac{\partial(\omega^1/p^0)}{\partial(p^1/p^0)}\right]\frac{1}{(p^1/p^0)}\right\},\\
A'_{13} &= -\left[\frac{\partial G^1}{\partial X_1}\right]\left\{\frac{1}{p^1/p^0}\left[-\frac{\omega^0/p^0}{p^1/p^0} + \frac{\partial(\omega^0/p^0)}{\partial(p^1/p^0)}\right](\bar{\ell}-\ell)\right.\\
&\quad \left. - \frac{\omega^0/p^0}{p^1/p^0}\left\{\frac{\partial \mathscr{L}}{\partial(\omega^0/p^0)}\frac{\partial(\omega^0/p^0)}{\partial(p^1/p^0)} + \frac{\partial \mathscr{L}}{\partial \mathscr{R}'}\left[-\frac{(\omega^1/p^0)}{(p^1/p^0)^2}\right]\right\}\right\},
\end{aligned}
$$

In elasticities terms and for

$$
A'_{11}(\mathscr{P}^1_{t+2} - \mathscr{P}^0_{t+1}) + A'_{12}(\mathscr{P}^1_{t+1} - \mathscr{P}^0_{t+1}) + A_{12}(\mathscr{P}^1_t - \mathscr{P}^0_t) = 0,
$$

this gives:

$$
\begin{aligned}
A'_{11} &= -\frac{\pi_\ell}{1-\pi_\ell}\left[1 - \Delta_{(\omega^0/p^0)(p^1/p^0)}\right]\\
&\quad \times \left\{\Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]}\left[\frac{\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}{1-\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}\right]^{-1}\left[\Delta_{(\omega^1/p^0)(p^1/p^0)}\frac{\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}{1-\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
A'_{12} &= -1 + \Delta_{(\omega^0/p^0)(p^1/p^0)} - \frac{\pi_\ell}{1-\pi_\ell}\left[\Delta_{\ell(\omega^0/p^0)}\Delta_{(\omega^0/p^0)(p^1/p^0)} - \Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]}\right]\\
&\quad - \left[\Delta_{(\omega^1/p^0)(p^1/p^0)}\frac{\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}{1-\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}\right]\left\{-\Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]}\left[\frac{\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}{1-\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}\right]^{-1}\right.\\
&\qquad \left. \times \left[\Delta_{(\omega^1/p^0)(p^1/p^0)}\frac{\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}{1-\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}\right]\right\} - \frac{\sum_{j=0}^1 \pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}}{\left(\pi^1_{X_1} - \pi^0_{X_1}\right)^2\left(1-\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}\right)}\\
&\quad + \Delta_{(\omega^0/p^0)(p^1/p^0)}\frac{\pi_\ell}{1-\pi_\ell}\left[\Delta_{\ell(\omega^0/p^0)}\Delta_{(\omega^0/p^0)(p^1/p^0)} - \Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]}\right],\\
A_{12} &= -\left[\Delta_{(\omega^1/p^0)(p^1/p^0)}\frac{\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}{1-\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}\right]\\
&\quad \times \left\{-1 + \Delta_{(\omega^0/p^0)(p^1/p^0)} - \frac{\pi_\ell}{1-\pi_\ell}\left[\Delta_{\ell(\omega^0/p^0)}\Delta_{(\omega^0/p^0)(p^1/p^0)} - \Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]}\right]\right\}.
\end{aligned}
$$

Remembering the parametrical features of the Wealth-to-Capital steady state assessed through Lemma

3.1, the expressions of the first coefficients of the characteristic polynomial derive as:

$$\begin{aligned}
\mathrm{tr}(\mathcal{J}) &= -\frac{A'_{12}}{A'_{11}} \\
&= 1 + \left\{ -\frac{\pi_\ell}{1-\pi_\ell}\left(1+\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right)^2 \left[\frac{\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}{\pi_{Y^1}}\right]^{-1} \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]} \right\}^{-1} \\
&\quad \times \left\{ 1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}} - \frac{\pi_\ell}{1-\pi_\ell}\left(1+\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right)^2 \Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]} + \frac{\sum_{j=0}^1 \pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2\pi_{Y^1}} \right\} \\
&= 1 + \left[\frac{\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}{1-\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}}\right] \left\{ 1 + \left\{ -\frac{\pi_\ell}{1-\pi_\ell}\pi^1_{X_1}\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]}\pi^1_{X_1} \right\}^{-1} \right. \\
&\qquad\qquad \left. \times \left\{ \pi^1_{X_1}\left(\pi^1_{X_1}-\pi^0_{X_1}\right) + \frac{\sum_{j=0}^1 \pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}}{\left(1-\sum_{j=0}^1 \pi^j_{X_1}\pi_{Y^j}\right)} \right\} \right\}.
\end{aligned}$$

where the last row takes advantage of the earlier arguments about the dual Hicksian representations of the demand functions and where it should further be noticed that:

$$\begin{aligned}
&\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_1}}{\pi_{Y^1}} \\
&\quad = \frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left(\pi^0_{X_1}\pi_{Y^0}+\pi^1_{X_1}\pi_{Y^1}\right)\frac{1}{\pi_{Y^1}} \\
&\quad = \left[\frac{\left(1-\pi^0_{X_1}\right)\pi^0_{X_1}\pi_{Y^0}}{\pi^1_{X_1}-\pi^0_{X_1}} + \left(1-\pi^0_{X_1}\right)\left(\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}+1\right)\pi_{Y^1}\right]\frac{1}{\pi_{Y^1}} \\
&\quad = \left[\frac{\pi^0_{X_1}\left(1-\pi^0_{X_1}\right)\pi_{Y^0}}{\pi^1_{X_1}-\pi^0_{X_1}} + \pi^0_{X_1}\left(\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}-1\right)\pi_{Y^1} + \pi_{Y^1}\right]\frac{1}{\pi_{Y^1}} \\
&\quad = \left[\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\left[\left(1-\pi^0_{X_1}\right)\pi_{Y^0}+\left(1-\pi^1_{X_1}\right)\pi_{Y^1}\right] + \pi_{Y^1}\right]\frac{1}{\pi_{Y^1}} \\
&\quad = \left[\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}\right]\frac{1}{\pi_{Y^1}} + 1 \\
&\quad = \frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}} + 1
\end{aligned}$$

or $[(1-\pi^0_{X_1})/(\pi^1_{X_1}-\pi^0_{X_1})]\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_1}/\pi_{Y^1} = \pi^1_{X_1}$ at the Wealth-to-Capital steady state that is also characterised, for future reference, by the prevalence of $\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]} = \Delta_{\ell(\omega^o/p^o)}$. It is also noticed that

$$\frac{\sum_{j=0}^1 \pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}}{\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2}\frac{1}{\pi_{Y^1}} = \sum_{j=0}^1 \pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1} \Big/ \sum_{j=0}^1 \pi^j_{X_0}\pi_{Y^j}\left(\pi^1_{X_1}-\pi^0_{X_1}\right)^2$$

The trace thence reformulates to:

$$\begin{aligned}
\mathrm{tr}([\mathcal{J}]) &= 1 + \left[\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_1}/\pi_{Y^1}\right] \\
&\quad \times \left\{ 1 + \frac{\sum_{j=0}^1 \pi^j_{X_0}\left[\sum_{j=0}^1 \pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}/\sum_{j=0}^1 \pi^j_{X_0}\pi_{Y^j} + \pi^1_{X_1}\left(\pi^1_{X_1}-\pi^0_{X_1}\right)\right]}{-\pi^1_{X_1}\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]}\left\{\left(1-\pi^0_{X_1}\right)\left(\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_1}/\pi_{Y^1}\right)\right\}} \right\} \\
&= 1 + \left[\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_1}/\pi_{Y^1}\right] \\
&\qquad \times \left\{ 1 + \frac{\sum_{j=0}^1 \pi^j_{X_0}\pi_{Y^j}\left[\sum_{j=0}^1 \pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}/\sum_{j=0}^1 \pi^j_{X_0}\pi_{Y^j}\pi^1_{X_1} + \left(\pi^1_{X_1}-\pi^0_{X_1}\right)\right]}{-\pi^1_{X_1}\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{o\prime}/p^o)]}} \right\}.
\end{aligned}$$

In ordinal terms, its expression emerges as:

$$\begin{aligned}
\mathrm{tr}([\mathscr{J}]) &= 1 + [\sum_{j=0}^{1} \pi_{Y^j}\pi^j_{X_1}/\pi_{Y^1}] \\
&\quad \times \left\{1 + \frac{\sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}\left[\sum_{j=0}^{1}\pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}/\sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}\pi^1_{X_1} + (\pi^1_{X_1} - \pi^0_{X_1})\right]}{\pi^1_{X_1}\pi_\ell\pi_{c'}(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})}\right\} \\
&= 1 + \left[\frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})}\right]\left\{1 + \frac{\left[\pi^0_{X_1}\Sigma^0_{X_0X_1} + (1-\pi^1_{X_1})\Sigma^1_{X_0X_1}\right] + (\pi^1_{X_1} - \pi^0_{X_1})}{\pi^1_{X_1}\pi_\ell(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})}\right\},
\end{aligned}$$

once the definition of the steady state has been taken into account.

Similarly, the remaining coefficient of the characteristic polynomial derives as:

$$\begin{aligned}
\det(\mathscr{J}) &= \frac{A_{12}}{A'_{11}} \\
&= \left\{-\frac{\pi_\ell}{1-\pi_\ell}\left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right)\left[\left[\frac{\sum_{j=0}^{1}\pi^j_{X_1}\pi_{Y^j}}{1 - \sum_{j=0}^{1}\pi^j_{X_1}\pi_{Y^j}}\right]^{-1}\Delta_{\ell[\mathscr{R}'/(p^{o\prime}/p^o)]}\left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right)\right]\right\}^{-1} \\
&\quad \times \left\{-\left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right)\left[\left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right)\left(-1 + \frac{\pi_\ell}{1-\pi_\ell}\Delta_{\ell[\mathscr{R}'/(p^{o\prime}/p^o)]}\right)\right]\right\} \\
&= \left[\frac{\sum_{j=0}^{1}\pi^j_{X_1}\pi_{Y^j}}{1 - \sum_{j=0}^{1}\pi^j_{X_1}\pi_{Y^j}}\right]\left\{1 + \left\{-\frac{\pi_\ell}{1-\pi_\ell}\Delta_{\ell[\mathscr{R}'/(p^{o\prime}/p^o)]}\right\}^{-1}\right\} \\
&= \frac{\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0} - \pi_\ell\Delta_{\ell[\mathscr{R}'/(p^{o\prime}/p^o)]}}{-\pi_\ell\Delta_{\ell[\mathscr{R}'/(p^{o\prime}/p^o)]}}\left[\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_1}/\pi_{Y^1}\right],
\end{aligned}$$

where the steady state holding of $\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0} = \pi_{Y^1}$ was used for the obtention of the eventual line. Alternatively:

$$\det(\mathscr{J}) = \left[\frac{\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}}{-\pi_\ell\Delta_{\ell[\mathscr{R}'/(p^{o\prime}/p^o)]}} + 1\right][\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_1}/\pi_{Y^1}],$$

In ordinal terms, noticing that

$$\begin{aligned}
\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0} - \pi_\ell\Delta_{\ell(\omega^o/p^o)} &= \sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0} - \pi_\ell(-\Sigma^H_{\ell c'}\pi_{c'} + \Sigma^H_{\ell Y}\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}) \\
&= \sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}(1 - \Sigma^H_{\ell Y}\pi_\ell) - \Sigma^H_{\ell c'}\pi_\ell\pi_{c'} \\
&= \pi_{c'}\left(\Sigma^H_{c'Y}\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0} + \Sigma_{\ell c'}\pi_\ell\right),
\end{aligned}$$

or

$$\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0} - \pi_\ell\Delta_{\ell(\omega^o/p^o)} = \sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0} - \pi_\ell\Delta_{\ell[\mathscr{R}'/(p^{o\prime}/p^o)]} = \sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}[1 + (\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})\pi_\ell]$$

it reformulates to :

$$\begin{aligned}
\det(\mathscr{J}) &= \frac{\Sigma^H_{c'Y}\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0} + \Sigma_{\ell c'}\pi_\ell}{\pi_\ell(\Sigma_{\ell c'} - \Sigma^H_{\ell Y})}[\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_1}/\pi_{Y^1}] \\
&= \left[\frac{1}{(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})\pi_\ell} + 1\right][\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_1}/\pi_{Y^1}] \\
&= \left[\frac{1}{(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})\pi_\ell} + 1\right]\frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})}
\end{aligned}$$

Finally, the stability argument shall build from the critical values undergone by the characteristic polynomial at $+1$ and -1. Then computing $Z(-1) = 1 + \mathrm{tr}(\mathcal{J}) + \det(\mathcal{J})$:

$$\begin{aligned}
Z(-1) &= 2\Big[1 + \sum_{j=0}^{1} \pi^j_{X_1}\pi_{Y^j}/\pi_{Y^1}\Big] \\
&\quad - \frac{\sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}\big[\sum_{j=0}^{1}\pi^j_{X_1}\pi_{Y^j}/\pi_{Y^1}\big]}{\pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}} \\
&\qquad \times \Big[\frac{\sum_{j=0}^{1}\pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}/\sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}\pi^1_{X_1} + (\pi^1_{X_1} - \pi^0_{X_1})}{\pi^1_{X_1}} + 1\Big] \\
&= 2\Big[1 + \sum_{j=0}^{1} \pi^j_{X_1}\pi_{Y^j}/\pi_{Y^1}\Big] \\
&\quad + \frac{\sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}\big[\sum_{j=0}^{1}\pi^j_{X_1}\pi_{Y^j}/\pi_{Y^1}\big]}{\pi_\ell\pi_{c'}(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})} \\
&\qquad \times \Big[\frac{\sum_{j=0}^{1}\pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}\sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}\pi^1_{X_1} + (\pi^1_{X_1} - \pi^0_{X_1})}{\pi^1_{X_1}} + 1\Big] \\
&= 2\Big[1 + \frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})}\Big] \\
&\quad + \frac{1}{\pi_\ell(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})}\Big[\frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})}\Big]\Big[\frac{[\pi^0_{X_1}\Sigma^0_{X_0X_1} + (1-\pi^1_{X_1})\Sigma^1_{X_0X_1}] + (\pi^1_{X_1} - \pi^0_{X_1})}{\pi^1_{X_1}} + 1\Big].
\end{aligned}$$

As for $Z(+1) = 1 - \mathrm{tr}(\mathcal{J}) + \det(\mathcal{J})$, its expression is available as:

$$\begin{aligned}
Z(+1) &= -\frac{\sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}\big[\sum_{j=0}^{1}\pi^j_{X_1}\pi_{Y^j}/\pi_{Y^1}\big]}{\pi_\ell \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}} \\
&\quad \times \Big[-\frac{\sum_{j=0}^{1}\pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}/\sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}\pi^1_{X_1} + (\pi^1_{X_1} - \pi^0_{X_1})}{\pi^1_{X_1}} + 1\Big] \\
&= \frac{\sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}\big[\sum_{j=0}^{1}\pi^j_{X_1}\pi_{Y^j}/\pi_{Y^1}\big]}{\pi_\ell\pi_{c'}(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})} \\
&\quad \times \Big[-\frac{\sum_{j=0}^{1}\pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1}/\sum_{j=0}^{1}\pi^j_{X_0}\pi_{Y^j}\pi^1_{X_1} + (\pi^1_{X_1} - \pi^0_{X_1})}{\pi^1_{X_1}} + 1\Big] \\
&= \frac{1}{\pi_\ell(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})}\Big[\frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})}\Big]\Big[-\frac{[\pi^0_{X_1}\Sigma^0_{X_0X_1} + (1-\pi^1_{X_1})\Sigma^1_{X_0X_1}] + (\pi^1_{X_1} - \pi^0_{X_1})}{\pi^1_{X_1}} + 1\Big].
\end{aligned}$$

As $Z(-1) > 0$ for $\Sigma^H_{\ell c'} > \Sigma^H_{\ell Y}$ and $\pi^1_{X_1} > \pi^0_{X_1}$, the saddle-point parameter configurations are associated with the ones that then underlie the holding of $Z(+1) < 0$. Noticing that such a conjunction reformulates to the one of:

$$\frac{\pi^0_{X_1} - [\pi^0_{X_1}\Sigma^0_{X_0X_1} + (1-\pi^1_{X_1})\Sigma^1_{X_0X_1}]}{\pi^1_{X_1}} < 0 \Longleftrightarrow 1 - \Big[\Sigma^0_{X_0X_1} + \Big(\frac{1-\pi^1_{X_1}}{\pi^0_{X_1}}\Big)\Sigma^1_{X_0X_1}\Big] < 0,$$

the statements (i) and (ii) follow. As for the statements (ii)a/-b/, the point to note is that the boundary is associated with the holding of $\det(\mathcal{J}) = 1$. QED

F. Proof of Proposition 3.2

Building upon the expressions of A'_{11}, A'_{12} and A_{12} listed in Appendix E but specialising them to let

$\sum_{j=0}^{1} \pi_{Y^j} \pi^j_{X_1} \;/\; \pi_{Y^1} = 1$ uniformly prevail, namely

$$
\begin{aligned}
A'_{11} &= -\frac{\pi_\ell}{1-\pi_\ell}\Big[1 - \Delta_{(\omega^0/p^0)(p^1/p^0)}\Big]\Big\{\Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]}\Delta_{(\omega^1/p^0)(p^1/p^0)}\Big\},\\
A'_{12} &= -1 + \Delta_{(\omega^0/p^0)(p^1/p^0)} - \frac{\pi_\ell}{1-\pi_\ell}\big[\Delta_{\ell(\omega^0/p^0)}\Delta_{(\omega^0/p^0)(p^1/p^0)} - \Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]}\big]\\
&\quad - \Big[\Delta_{(\omega^1/p^0)(p^1/p^0)}\Big]\Big\{-\Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]}\Delta_{(\omega^1/p^0)(p^1/p^0)}\Big\},\\
&\quad - \sum_{j=0}^{1} \pi^j_{X_0}\pi^j_{X_1}\pi_{Y^j}\Sigma^j_{X_0X_1} \Big/ \big(\pi^1_{X_1} - \pi^0_{X_1}\big)^2\pi_{Y^1},\\
A_{12} &= -\Delta_{(\omega^1/p^0)(p^1/p^0)}\Big\{-1 + \Delta_{(\omega^0/p^0)(p^1/p^0)} - \frac{\pi_\ell}{1-\pi_\ell}\big[\Delta_{\ell(\omega^0/p^0)}\Delta_{(\omega^0/p^0)(p^1/p^0)} - \Delta_{\ell[\mathscr{R}'/(p^{0\prime}/p^0)]}\big]\Big\},
\end{aligned}
$$

the linearisation of the dynamical system in the neighbourhood of the Golden Rule steady state delivers:

$$
\begin{bmatrix} A'_{11} & A'_{12} & -\dfrac{(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1}\\ 0 & 1 & 0\\ 0 & -\Delta_{(\omega^1/p^0)(p^1/p^0)} & 1 \end{bmatrix}
\begin{bmatrix} \mathscr{Q}_{t+1}\\ \mathscr{P}^1_{t+1} - \mathscr{P}^0_{t+1}\\ \mathscr{B}_{t+1} - \mathscr{P}^0_{t+1} \end{bmatrix}
+ \begin{bmatrix} 0 & A_{12} & \Delta_{(\omega^1/p^0)(p^1/p^0)}\dfrac{(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1}\\ -1 & 0 & 0\\ 0 & 1 & -1 \end{bmatrix}
\begin{bmatrix} \mathscr{Q}_t\\ \mathscr{P}^1_t - \mathscr{P}^0_t\\ \mathscr{B}_t - \mathscr{P}^0_t \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}.
$$

Letting $D_{A'} = A'_{11}$, the components of the Jacobian Matrix list as:

$$
\begin{aligned}
\mathscr{J}_{11} &= -\Big(D_{A'}\Big)^{-1}\Big\{-\Big[A'_{12} - \frac{(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1}\Delta_{(\omega^1/p^0)(p^1/p^0)}\Big](-1)\Big\};\\
\mathscr{J}_{12} &= -(\mathscr{D}_{A'})^{-1}\Big\{A_{12} + \frac{(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1}\Big\};\\
\mathscr{J}_{13} &= -(\mathscr{D}_{A'})^{-1}\Big\{\Delta_{(\omega^1/p^0)(p^1/p^0)}\frac{(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1} + \frac{(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1}(-1)\Big\};\\
\mathscr{J}_{21} &= -(\mathscr{D}_{A'})^{-1}\Big\{A'_{11}(-1)\Big\};\\
\mathscr{J}_{22} &= 0;\\
\mathscr{J}_{23} &= 0;\\
\mathscr{J}_{31} &= -(\mathscr{D}_{A'})^{-1}\Big\{\Delta_{(\omega^1/p^0)(p^1/p^0)}A'_{11}(-1)\Big\};\\
\mathscr{J}_{32} &= -(\mathscr{D}_{A'})^{-1}\Big\{A'_{11}\Big\};\\
\mathscr{J}_{33} &= -(\mathscr{D}_{A'})^{-1}\Big\{A'_{11}(-1)\Big\}.
\end{aligned}
$$

The critical coefficients of the Jacobian Matrix can then be derived. First consider its trace:

$$\begin{aligned}
\mathrm{tr}(\mathcal{J}) &= \mathcal{J}_{11} + \mathcal{J}_{33} \\
&= -(\mathcal{D}_{A'})^{-1}\left\{\left[A'_{12} - \frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star}\Delta_{(\omega^1/p^0)(p^1/p^0)}\right] - A'_{11}\right\} \\
&= 1 + (\mathcal{D}_{A'})^{-1}\left\{-\left[A'_{12} - \frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star}\Delta_{(\omega^1/p^0)(p^1/p^0)}\right]\right\} \\
&= 1 + \frac{(1-\pi^0_{X_1})(\pi^1_{X_1} - \pi^0_{X_1})}{\pi^1_{X_1}} \\
&\quad + \left\{-\frac{\pi_\ell}{1-\pi_\ell}\left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right)\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\right\}^{-1} \\
&\quad \times \left\{-\frac{\pi_\ell}{1-\pi_\ell}\left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right)^2\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]} + \Delta_{Y^1(p^1/p^0)}\right. \\
&\qquad \left. + 1 + \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star}\right\}
\end{aligned}$$

Then noticing the holding of

$$1 + \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + \frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star} = \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right],$$

the expression of the trace is hence eventually available along:

$$\begin{aligned}
\mathrm{tr}(\mathcal{J}) = 1 &+ \frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}}\left[1 + \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}(\pi^1_{X_1} - \pi^0_{X_1})}{-\pi_\ell[\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})]}\right] \quad + \frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})} \\
&+ \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}\pi^j_{X_1}\Sigma^j_{X_0X_1}}{-\pi_\ell\pi^1_{X_1}[\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})]}.
\end{aligned}$$

Relying anew to the ordinal-like parametric formulations of the Hicksian demand functions but also to the Golden Rule expressions of the aggregate values of the inputs and of the elasticity of substitution between the latter, this reformulates to:

$$\mathrm{tr}(\mathcal{J}) = 1 + \frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})} + \frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}} + \frac{\pi^0_{X_1}\left[(1-\pi^0_{X_1})\Sigma^0_{X_0X_1} + \pi^1_{X_1}\Sigma^1_{X_0X_1}\right]}{\pi_\ell(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})\pi^1_{X_1}(1-\pi^0_{X_1})} + \frac{\pi^1_{X_1} - \pi^0_{X_1}}{\pi_\ell(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})\pi^1_{X_1}},$$

where use was made of the occurrence of

$$\begin{aligned}
\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} &= \left[\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\pi_{\bar{X}_0}\right]\frac{1}{\pi_{Y^1}} + 1 \\
&= \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + 1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star},
\end{aligned}$$

that also implies the one of

$$\begin{aligned}
\frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} &+ 1 + \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star} \\
&= \left[1 + \frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star}\right]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right], \\
&= \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{(p^1/p^0)^\star X_1^\star}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right],
\end{aligned}$$

for the obtention of the eventual expression. Then consider the sum of the principal minors of order two:

$$
\begin{aligned}
\operatorname{spm}(\mathcal{J}) &= -\mathcal{J}_{12}\mathcal{J}_{21} + \mathcal{J}_{11}\mathcal{J}_{33} - \mathcal{J}_{13}\mathcal{J}_{31} \\
&= -(\mathcal{D}_{A'})^{-1}\Bigg\{-\left[A_{12} + \frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star}\right] + \left[A'_{12} - \frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star}\Delta_{(\omega^1/p^0)(p^1/p^0)}\right] \\
&\qquad - \left\{\Delta_{(\omega^1/p^0)(p^1/p^0)} - 1\right\}\frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star}\left[\Delta_{(\omega^1/p^0)(p^1/p^0)}\right]\Bigg\}\Bigg\} \\
&= \left\{-\frac{\pi_\ell}{1-\pi_\ell}\frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right)\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\right\}^{-1} \\
&\qquad \times \Bigg\{-\Bigg\{-\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\Bigg[-1 - \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} \\
&\qquad\qquad + \frac{\pi_\ell}{\pi_{Y^1}}\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right)\Bigg] - \frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star}\Bigg\} \\
&\qquad + \Bigg\{-\left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right)\left(\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + \Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right) \\
&\qquad - \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\Bigg\} \\
&\qquad + \Delta_{Y^1(p^1/p^0)} + 1 + \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]^2\frac{(B/p^0)^\star}{(p^1/p^0)^\star X_1^\star}\Bigg\}.
\end{aligned}
$$

Completing the same approach as for the coefficient $\operatorname{tr}(\mathcal{J})$, it is obtained that:

$$
\begin{aligned}
\operatorname{spm}(\mathcal{J}) &= 1 + \frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})} + \frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}} \\
&\qquad + \left\{-\frac{\pi_\ell}{1-\pi_\ell}\frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\left(1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right)\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\right\}^{-1} \\
&\qquad\quad \times \Bigg\{\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\left\{\frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + 1 + \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\frac{(B/p^0)^\star}{\pi_{Y^1}}\right\} \\
&\qquad\qquad + \Delta_{Y^1(p^1/p^0)} + \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + 1 + \frac{(B/p^0)^\star}{\pi_{Y^1}}\Bigg\} \\
&= 1 + \frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}} + \frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})} + \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}\pi^j_{X_1}\Sigma^j_{X_0X_1}}{-\pi_\ell\pi^1_{X_1}\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})} \\
&\qquad + \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}(\pi^1_{X_1} - \pi^0_{X_1})}{-\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})} + \frac{\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_0}(1-\pi^0_{X_1})^2}{-\pi_\ell\pi^1_{X_1}\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})}.
\end{aligned}
$$

This reformulates to:

$$
\begin{aligned}
\operatorname{spm}(\mathcal{J}) &= 1 + \frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})} + \frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}} + \frac{\pi^0_{X_1}\left[(1-\pi^0_{X_1})\Sigma^0_{X_0X_1} + \pi^1_{X_1}\Sigma^1_{X_0X_1}\right]}{\pi_\ell(\Sigma^H_{\ell c^{0\prime}} - \Sigma^H_{\ell Y})\pi^1_{X_1}(1-\pi^0_{X_1})} \\
&\qquad + \frac{\pi^1_{X_1} - \pi^0_{X_1}}{\pi_\ell(\Sigma^H_{\ell c^{0\prime}} - \Sigma^H_{\ell Y})(1-\pi^0_{X_1})} + \frac{(1-\pi^0_{X_1})}{\pi_\ell(\Sigma^H_{\ell c^{0\prime}} - \Sigma^H_{\ell Y})\pi^1_{X_1}}.
\end{aligned}
$$

It is noticed that the Golden Rule steady state being defined from the satisfaction of $\sum_{j=0}^1 \pi_{Y^j}\pi^j_{X_1}/\pi_{Y^1} = 1$, from the definition of π_{Y^1}, this gives $\pi_{Y^1} = \pi^0_{X_1}/(\pi^0_{X_1} + \pi^1_{X_0})$ and $\pi_{Y^0} = \pi^1_{X_0}/(\pi^0_{X_1} + \pi^1_{X_0})$, those expressions having being incorporated in the eventual forms of the above coefficients.

Finally consider the determinant:

$$\begin{aligned}
\det(\mathcal{J}) &= \mathcal{J}_{11}(-\mathcal{J}_{21}\mathcal{J}_{33}) + \mathcal{J}_{13}(\mathcal{J}_{21}\mathcal{J}_{32}) \\
&= -\left\{-\frac{\pi_\ell}{\pi_{Y^1}}\left(1+\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right)\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right]\right\}^{-1} \\
&\quad\times\left\{\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right]\left[-1-\frac{\sum_{j=0}^1\pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}-\frac{(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1}\right.\right. \\
&\qquad\left.\left.-\frac{\pi_\ell}{1-\pi_\ell}\frac{\sum_{j=0}^1\pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\left(-\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}-\Delta_{\ell(\omega^0/p^0)}\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right)\right]\right\}.
\end{aligned}$$

Rearranging along the approach followed for $\mathrm{tr}(\mathcal{J})$ and $\mathrm{spm}(\mathcal{J})$, it is obtained that:

$$\begin{aligned}
\det(\mathcal{J}) &= 1+\frac{\sum_{j=0}^1\pi_{Y^j}\pi^j_{X_0}(1-\pi^0_{X_1})}{-\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})} \\
&= \left\{\mathcal{D}_{A'}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right]^{-1}\right\}^{-1}\left\{\left[1+\frac{\pi^0_{X_1}}{\pi^1_{X_1}-\pi^0_{X_1}}\right]\left[\frac{\sum_{j=0}^1\pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}-\frac{\pi_\ell}{\pi_{Y^1}}\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}\right]\right\} \\
&= \frac{\sum_{j=0}^1\pi_{Y^j}\pi^j_{X_0}-\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}{-\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}} \\
&= 1-\frac{\sum_{j=0}^1\pi^j_{X_0}\pi_{Y^j}}{\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}}.
\end{aligned}$$

Alternatively, the determinant is available as :

$$\det(\mathcal{J}) = 1+\frac{1}{\pi_\ell(\Sigma^H_{\ell c'}-\Sigma^H_{\ell Y})}.$$

Finally computing

$$\begin{aligned}
&-1+\mathrm{tr}(\mathcal{J})-\mathrm{spm}(\mathcal{J})+\det(\mathcal{J}) = \\
&\quad= \frac{\sum_{j=0}^1\pi_{Y^j}\pi^j_{X_0}[(1-\pi^0_{X_1})](\pi^1_{X_1}-\pi^0_{X_1})}{-\pi_\ell\pi^1_{X_1}\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})}-\frac{\sum_{j=0}^1\pi_{Y^j}\pi^j_{X_0}(\pi^1_{X_1}-\pi^0_{X_1})}{-\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})} \\
&\qquad+\frac{\sum_{j=0}^1\pi_{Y^j}\pi^j_{X_0}(1-\pi^0_{X_1})}{-\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})}-\frac{\sum_{j=0}^1\pi_{Y^j}\pi^j_{X_0}(1-\pi^0_{X_1})^2}{-\pi_\ell\pi^1_{X_1}\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})} \\
&= \frac{\sum_{j=0}^1\pi_{Y^j}\pi^j_{X_0}(1-\pi^1_{X_1})}{-\pi_\ell\Delta_{\ell[\mathcal{R}'/(p^{0\prime}/p^0)]}(1-\pi^0_{X_1})}\left[1-\frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}}\right],
\end{aligned}$$

where

$$\begin{aligned}
1-\frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}} &= 1-\frac{(1-\pi^1_{X_1})}{\pi^1_{X_1}} \\
&= \frac{\pi^0_{X_1}-(1-\pi^1_{X_1})}{\pi^1_{X_1}}.
\end{aligned}$$

Then similarly computing:

$$\begin{aligned}
Z(-1) &= 1+\mathrm{tr}(\mathcal{J})+\mathrm{spm}(\mathcal{J})+\det(\mathcal{J}) \\
&= 2\left\{1+\frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})}+\frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}}\right\}+\frac{\pi^0_{X_1}[(1-\pi^0_{X_1})\Sigma^0_{X_0X_1}+\pi^1_{X_1}\Sigma^1_{X_0X_1}]}{\pi_\ell(\Sigma^H_{\ell c'}-\Sigma^H_{\ell Y})\pi^1_{X_1}(1-\pi^0_{X_1})} \\
&\quad+\frac{1}{\pi_\ell(\Sigma^H_{\ell c'}-\Sigma^H_{\ell Y})}\left[1+\frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}}\right]+\frac{1}{\pi_\ell(\Sigma^H_{\ell c'}-\Sigma^H_{\ell Y})}\left[1+\frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})}\right]\frac{\pi^1_{X_1}-\pi^0_{X_1}}{\pi^1_{X_1}},
\end{aligned}$$

$$
\begin{aligned}
Z(+1) &= -1 + \mathrm{tr}(\mathcal{J}) - \mathrm{spm}(\mathcal{J}) + \det(\mathcal{J}) \\
&= \frac{1}{\pi_\ell(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})}\left[1 - \frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}}\right]\left[1 - \frac{\pi^1_{X_1} - \pi^0_{X_1}}{\pi^1_{X_1}}\frac{\pi^1_{X_1}}{(1-\pi^0_{X_1})}\right] \\
&= \frac{1}{\pi_\ell(\Sigma^H_{\ell c'} - \Sigma^H_{\ell Y})}\left[1 - \frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}}\right]\frac{(1-\pi^1_{X_1})}{(1-\pi^0_{X_1})}.
\end{aligned}
$$

Use was made in the foregoing computations of the occurrence of

$$
\begin{aligned}
\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} &= \left[\frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}\right]\frac{1}{\pi_{Y^1}} + 1 \\
&= \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}} + 1 + \frac{\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\frac{(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1},
\end{aligned}
$$

that also implies the one of

$$
\begin{aligned}
\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}\frac{\pi^0_{X_1}/\pi_{Y^1}}{\pi^1_{X_1} - \pi^0_{X_1}} + 1 + \left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right]\frac{(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1} \\
&= \left[1 + \frac{(B/p^0)^\star}{(p^1/p^0)^\star X^\star_1}\right]\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right], \\
&= \frac{\sum_{j=0}^{1}\pi_{Y^j}\pi^j_{X_0}}{\pi_{Y^1}}\left[\frac{1-\pi^0_{X_1}}{\pi^1_{X_1} - \pi^0_{X_1}}\right].
\end{aligned}
$$

Adapting then the line of argument developed by Barinci & Drugeon (2017), as the focus on a configuration $\Sigma^H_{\ell c'} > \Sigma^H_{\ell Y}$ specialises the analysis to $\det(\mathcal{J}) > 1$, a saddle-point stable Golden Rule will be available for $Z(+1)Z(-1) < 0$.

For $\pi^1_{X_1} > \pi^0_{X_1}$, it is readily observed that $Z(-1) > 0$ whilst:

$$
Z(+1) \lesseqgtr 0 \Longleftrightarrow 1 - \frac{(1-\pi^0_{X_1})}{\pi^1_{X_1}} \lesseqgtr 0.
$$

The statement about the Samuelson economies immediately follows. As for classical economies and from Barinci & Drugeon (2017), their properties will be ruled by the sign of $\mathrm{spm}(\mathcal{J}) - 1 - [\mathrm{tr}(\mathcal{J}) - \det(\mathcal{J})]\det(\mathcal{J})$ for $|\mathrm{tr}(\mathcal{J}) - \det(\mathcal{J})| < 2$. This latter restriction being satisfied for classical economies with $Z(-1) > 0$, the expression of the main text follows from a direct computation of the first. QED