



**HAL**  
open science

# No-arbitrage equilibria with differential information: a proof of existence

Lionel de Boisdeffre

► **To cite this version:**

Lionel de Boisdeffre. No-arbitrage equilibria with differential information: a proof of existence. 2004. halshs-03322655

**HAL Id: halshs-03322655**

**<https://shs.hal.science/halshs-03322655>**

Submitted on 19 Aug 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**No-arbitrage equilibria with differential information:  
a proof of existence**

Lionel De BOISDEFFRE

2004.35

NO-ARBITRAGE EQUILIBRIA WITH DIFFERENTIAL INFORMATION: A PROOF OF EXISTENCE
--------------------------------------------------------------------------------

*Lionel de Boisdeffre*<sup>1</sup>

(March 2004)

*Abstract*

*On the example of a pure exchange financial economy with two periods, incomplete markets and differential information of the adverse selection's type, Cornet-De Boisdeffre (2002) introduced refined concepts of price, arbitrage and so-called "no-arbitrage equilibria", which extended to the asymmetric information setting the classical concepts of symmetric information. In this paper, we prove that a no-arbitrage equilibrium exists under the same standard conditions with symmetric and asymmetric information.*

*Key words:* general equilibrium, asymmetric information, arbitrage, inference, existence of equilibrium.

*JEL Classification:* D52.

---

<sup>1</sup>INSEE (CREST), Paris, & CERMSEM, Université Paris 1, 106-112, boulevard de l'Hôpital, 75013 Paris, France. Email: [lionel.de.boisdeffre@wanadoo.fr](mailto:lionel.de.boisdeffre@wanadoo.fr)

# 1 Introduction

An enlarged arbitrage theory with differential information, a refined concept of no-arbitrage price and a corresponding notion of “no-arbitrage” equilibrium were introduced in Cornet-De Boisdeffre (2002), on the example of a two-period pure exchange financial economy, with incomplete nominal-asset markets and asymmetric information of the adverse selection’s type. In our setting, agents could infer information from prices without having rational expectations, that is, a model of how equilibrium prices are determined. Our concepts of price and equilibrium were shown to coincide with the classical notions under symmetric information and to extend them to the asymmetric information case.

In this paper, we study the properties of the no-arbitrage equilibrium and show its existence is guaranteed, for any structure of information, which yields no agent an arbitrage, under the same standard conditions, whether agents have symmetric or asymmetric information. This result departs from the classical generic existence property of rational expectations equilibrium models with asymmetric information.

Section 2 presents the main characteristics and definitions of the model: information structures and refinements, markets, consumers’ behavior and the concept of “no-arbitrage equilibrium”. Applying the Gale-Mas-Colell fixed point-like theorem, Section 3 proves that a no-arbitrage equilibrium exists under standard conditions.

## 2 The basic model

We consider a pure-exchange economy  $\mathcal{E}$  with two periods ( $t = 0$  and  $t = 1$ ). At the first period ( $t = 0$ ), there is an a priori uncertainty about which state  $s$  of a finite space  $S$  will prevail at the second period ( $t = 1$ ). There are finitely many agents  $i \in I := \{1, \dots, m\}$ , exchanging  $L$  commodities  $l \in \{1, \dots, L\}$  and  $J$  nominal assets  $j \in \{1, \dots, J\}$ , paying off at the second period. For convenience, we let  $s = 0$  denote the non-random state at  $t = 0$  and  $\Sigma' := \{0\} \cup \Sigma$ , for every (nonempty) subset  $\Sigma \subset S$ .

This Section recalls the specification of risk and information, markets, consumers’ behavior, arbitrage properties of the model and our concept of equilibrium with differential information, based on the no-arbitrage condition.

### 2.1 Information structures and refinements

During the first period, every agent  $i \in I$  receives a private signal in the form of a subset  $S_i$  of  $S$ , called an “information set”, which informs the agent

that an arbitrary state  $s \in S_i$  will prevail at  $t = 1$ . Whenever possible, agents will seek to further reduce their information sets by inferring information from observing market prices. If agents' information is correct, as always assumed in the model, their information sets contain the so-called "true" state, which will be disclosed at  $t = 1$ , and, hence, these sets have a non-empty intersection. This yields the following definitions of information structures and refinements.

**Definition 1** A collection  $(\Sigma_i)$  of  $m$  subsets  $\Sigma_i \subset S$  is said to define an information structure (or structure) if  $\bigcap_{i=1}^m \Sigma_i \neq \emptyset$ . A structure  $(\Sigma_i)$  is said to be symmetric if all the  $\Sigma_i$  are equal. Given two information structures  $(\Sigma_i^1)$  and  $(\Sigma_i^2)$ , we say that  $(\Sigma_i^2)$  is coarser than  $(\Sigma_i^1)$ , or that  $(\Sigma_i^1)$  is finer than (or "refines", or "a refinement of")  $(\Sigma_i^2)$ , if  $\Sigma_i^1 \subset \Sigma_i^2$ , for every  $i \in I$ , and we denote it by  $(\Sigma_i^1) \leq (\Sigma_i^2)$ . The pooled refinement of a given structure  $(\Sigma_i)$  is the symmetric structure  $(\underline{\Sigma}_i)$  defined by  $\underline{\Sigma}_i := \bigcap_{j=1}^m \Sigma_j$  for every  $i \in I$ . A refinement  $(\Sigma_i)$  of  $(S_i)$  is said to be self-attainable if it is coarser than the pooled refinement  $(\underline{S}_i)$ , that is, if  $\bigcap_{i=1}^m S_i = \bigcap_{i=1}^m \Sigma_i$ . We shall denote by  $\Gamma$  the set of all information structures. Given  $(\Sigma_i) \in \Gamma$  and a payoff matrix  $V$ , the couple  $[V, (\Sigma_i)]$  will also be called a (payoff and information) structure.

*Remark 1:* since every information set  $S_i$  (for  $i \in I$ ) contains the true state in the economy  $\mathcal{E}$ , the collection  $(S_i)$  defines a structure and every set  $\Sigma_i$  (for  $i \in I$ ) of a self-attainable refinement  $(\Sigma_i)$  of  $(S_i)$  also contains the true state.

## 2.2 Markets and consumers' behavior

There are two markets: one for assets and one for commodities.

The financial market permits limited income transfers across periods and states, via  $J$  nominal assets  $j \in \{1, \dots, J\}$ , whose payoffs in each state  $s \in S$ , denoted by  $v_j[s]$ , define a  $S \times J$ -matrix  $V$ . The payoff matrix  $V$  is assumed to be of full column-rank and known by all agents. Denoting by  $q \in \mathbb{R}^J$  the asset price, a portfolio is a vector  $z \in \mathbb{R}^J$ , tradable for  $q \cdot z$  monetary units at  $t = 0$ , against the promise of a flow  $V \cdot z$  of contingent payoffs at  $t = 1$ .

There also exist  $\#S'$  spot markets for the  $L$  commodities  $l \in \{1, \dots, L\}$ , which may be consumed at both dates. Given a price  $p \in (\mathbb{R}^L)^{S'}$  and a subset  $\Sigma$  of  $S$ , we denote by  $p(\Sigma')$  truncation of  $p$  on  $(\mathbb{R}^L)^{\Sigma'}$ .<sup>2</sup>

<sup>2</sup>We shall use, throughout the paper, the following notations. The scalar product and Euclidean norm will be denoted by  $\cdot$  and  $\|\cdot\|$ , respectively. For every subset  $\Sigma$  of  $S'$ , every real-valued  $\Sigma \times J$ -matrix  $A$  and vectors  $x \in (\mathbb{R}^L)^{S'}$ ,  $x' \in \mathbb{R}^{S'}$ ,  $y \in \mathbb{R}^\Sigma$ ,  $y' \in \mathbb{R}^{\Sigma}$ ,  $z \in (\mathbb{R}^L)^\Sigma$ ,  $z' \in (\mathbb{R}^L)^\Sigma$  (where  $\Sigma$  and  $S'$  are identified to  $\#\Sigma$  and  $\#S'$ , whenever necessary), every  $(q, s, l) \in \mathbb{R}^J \times \Sigma \times \{1, \dots, L\}$ , we will denote by:

- 1)  $x(\Sigma)$  and  $x'(\Sigma)$ , respectively, the truncations of  $x$  on  $(\mathbb{R}^L)^\Sigma$  and of  $x'$  on  $\mathbb{R}^\Sigma$ ;
- 2)  $A[s]$ ,  $y[s]$ ,  $z[s]$ , respectively, the row, scalar and vector, indexed by  $s$ , of  $A$ ,  $y$ ,  $z$ ;
- 3)  $z^l[s]$  the  $l^{\text{th}}$  component of  $z[s] \in \mathbb{R}^L$  and  $z^l := (z^l[s]) \in \mathbb{R}^\Sigma$ ;
- 4)  $y \leq y'$ , the relationships  $y[s] \leq y'[s]$ , for every  $s \in \Sigma$ ,  $y < y'$  the joint relationships  $y \leq y'$

For every (nonempty) subset  $\Sigma \subset S$ , the generic agent  $i \in I$  has, conditionally on the information set  $\Sigma$  she may reach during the first period, a consumption set  $X_i^\Sigma := \{x \in (\mathfrak{R}_+^L)^{S'} : x[s] = 0 \text{ for every } s \in S \setminus \Sigma\}$ , an endowment,  $e_i^\Sigma \in X_i^\Sigma$ , and a preference correspondence  $P_i^\Sigma$  on consumptions. For every  $x \in X_i^\Sigma$ , the set  $P_i^\Sigma(x)$  is defined as the subset of consumptions  $y \in X_i^\Sigma$ , which are *strictly* preferred to  $x$  (hence,  $x \notin P_i^\Sigma(x)$ ). Generalizing Cornet-De Boisdeffre (2002), this definition encompasses the case of preferences represented by a binary relation  $\preccurlyeq_i^\Sigma$  and, in particular, by a conditional utility function  $u_i^\Sigma : X_i^\Sigma \rightarrow \mathfrak{R}$ . Using the notations in footnote, agents' preferences and endowments are assumed to meet the following conditions.

**Assumption A1** (*non-satiation*)

$$\forall i \in I, \forall \Sigma \subset S, \forall (s, x) \in \Sigma' \times X_i^\Sigma, \exists y \in P_i^\Sigma(x), y[s'] = x[s'], \forall s' \in \Sigma' \setminus \{s\}.$$

**Assumption A2** (*strong survival*):

$$\forall i \in I, \forall \Sigma \subset S, \forall s \in \Sigma', e_i^\Sigma[s] > 0.$$

**Assumption A3** (*convex semicontinuous preference correspondences*): for every  $i \in I$ , every  $\Sigma \subset S$  and every triple  $(x, y, \lambda) \in X_i^\Sigma \times P_i^\Sigma(x) \times ]0, 1]$ ,  $P_i^\Sigma$  is lower semicontinuous at  $x$  and  $P_i^\Sigma(x)$  is convex and open in  $X_i^\Sigma$  and such that  $[x + \lambda(y - x)] \in P_i^\Sigma(x)$ .

Given a price system  $(p, q) \in (\mathfrak{R}^L)^{S'} \times \mathfrak{R}^J$ , the generic agent  $i \in I$  detaining the information  $\Sigma$  (for  $\Sigma \subset S$ ) admits the following budget set, whose elements are called “strategies”:

$$B_i^\Sigma(p, q) := \{(x, z) \in X_i^\Sigma \times \mathfrak{R}^J : [p_\square(x - e_i^\Sigma)](\Sigma') \leq W^\Sigma(q)z\},$$

and we define the set of budgetable consumptions as

$$B_i^{*\Sigma}(p, q) := \{x \in X_i^\Sigma : \exists z \in \mathfrak{R}^J, (x, z) \in B_i^\Sigma(p, q)\}.$$

Given an information structure  $(\Sigma_i)$ , a collection  $x := (x_i) \in \prod_{i=1}^m X_i^{\Sigma_i}$  is called an “allocation”. When  $\sum_{i=1}^m (x_i[s] - e_i^{\Sigma_i}[s]) = 0$ , for every  $s \in \cap_{i \in I} \Sigma'_i$ , the allocation is said to be “attainable”. Given a structure  $(\Sigma_i)$ , we denote by

$$\mathcal{A}^{(\Sigma_i)} := \{x := (x_i) \in \prod_{i=1}^m X_i^{\Sigma_i} : \forall s \in \cap_{i \in I} \Sigma'_i, \sum_{i=1}^m (x_i[s] - e_i^{\Sigma_i}[s]) = 0\}$$

and  $y \neq y'$  and  $y \ll y'$  the relationships  $y[s] < y'[s]$ , for every  $s \in \Sigma$  (we define the similar relationships  $<, \leq, \ll$  on  $(\mathfrak{R}^L)^\Sigma$ );

5)  $ImA := \{A.z : z \in \mathfrak{R}^J\} \subset \mathfrak{R}^\Sigma$ ;

6)  $z_\square z'$  the vector  $(z[s] \cdot z'[s]) \in \mathfrak{R}^\Sigma$ ,  $y_\square z$  the vector  $(y[s]z[s]) \in (\mathfrak{R}^L)^\Sigma$ ;

7)  $W^\Sigma(q)$  (when  $0 \notin \Sigma$ ) the  $\Sigma' \times J$ -matrix defined by

$W^\Sigma(q)[0] := -q$  and  $W^\Sigma(q)[s] := V[s]$ , for every  $s \in \Sigma$  and we let  $W(q) := W^S(q)$ ;

8)  $V(\Sigma)$  (when  $0 \notin \Sigma$ ) the  $\Sigma \times J$ -matrix defined by  $V(\Sigma)[s] := V[s]$ , for every  $s \in \Sigma$ .

9)  $(\mathfrak{R}_{++}^L)^\Sigma := \{x \in (\mathfrak{R}^L)^\Sigma : x \gg 0\}$ ,  $\mathfrak{R}_{++}^\Sigma := \{x \in \mathfrak{R}^\Sigma : x \gg 0\}$ .

10)  $(\mathfrak{R}_+^L)^{\Sigma'} := \{x \in (\mathfrak{R}^L)^{\Sigma'} : x \geq 0\}$ ,  $\mathfrak{R}_+^{\Sigma'} := \{x \in \mathfrak{R}^{\Sigma'} : x \geq 0\}$ .

the set of attainable allocations and by

$$\mathcal{A}^{(\Sigma_i)}(p, q) := \{[(x_i, z_i)] \in \prod_{i=1}^m B_i^{\Sigma_i}(p, q) : (x_i) \in \mathcal{A}^{(\Sigma_i)}, \sum_{i=1}^m z_i = 0\},$$

for every price system  $(p, q) \in (\mathfrak{R}^L)^{S'} \times \mathfrak{R}^J$ , the set of attainable strategies.

The generic  $i^{th}$  agent's problem, when her information set is  $\Sigma_i$ , is to elect a strategy that will optimize her welfare. If, during the first period, every agent  $i \in I$  receives a private signal  $S_i \subset S$ , refines her information set up to  $\Sigma_i \subset S_i$  and, then, defines her strategy, the solution (for  $(\mathbf{S}_i) \leq (\Sigma_i) \leq (S_i)$ ) yields the following concept of “no-arbitrage” equilibrium.

**Definition 2** A price system  $(p^*, q^*) \in (\mathfrak{R}^L)^{S'} \times \mathfrak{R}^J$ , a self-attainable refinement  $(\Sigma_i)$  of  $(S_i)$  and a collection of strategies  $(x_i^*, z_i^*) \in B_i^{\Sigma_i}(p^*, q^*)$ , for  $i = 1, \dots, m$ , define an equilibrium of the economy  $\mathcal{E}$ , or “no-arbitrage” equilibrium, if:

- (i)  $\forall i \in I, P_i^{\Sigma_i}(x_i^*) \cap B_i^{\Sigma_i}(p^*, q^*) = \emptyset$ ;
- (ii)  $\forall s \in \cap_{i \in I} \Sigma_i', \sum_{i \in I} x_i^*[s] = \sum_{i \in I} e_i^{\Sigma_i}[s]$ ;
- (iii)  $\sum_{i \in I} z_i^* = 0$ .

*Remark 2* Apart from the use of preference correspondences, the above specification of the economy differs from Cornet-De Boisdeffre's (2002) on one formal point: we had previously defined the consumption set of the generic agent  $i \in I$  detaining the information  $\Sigma_i \subset S$  as  $(\mathfrak{R}_+^L)^{\Sigma_i}$  (and, accordingly, endowments, budget sets and attainable allocations). Now, we let agents' consumption plans belong to a same vector space, namely  $(\mathfrak{R}^L)^{S'}$ , whatever their information. This equivalent specification will be convenient for proving the existence of equilibrium in Section 3. It should be clear, however, that consumptions in unrealizable states have no economic meaning and could be set arbitrary.

We now present some basic properties of the model.

### 2.3 Properties of the no-arbitrage equilibrium

We present two types of properties. First, we recall arbitrage properties of equilibrium with asymmetric information, which permit to tackle the issue of information conveyed by prices in the absence of rational expectations and lead to a refined concept of “price-revealed” equilibrium. Second, we introduce properties of attainable allocations and strategies, which will help bound the economy and apply a fixed-point-like argument in Section 3.

#### 2.3.1 Arbitrage properties

We first recall the following definitions from Cornet-De Boisdeffre (2002).

**Definition 3** Let a structure  $[V, (\Sigma_i)]$  be given. A price  $q \in \mathfrak{R}^J$  is said to be a common no-arbitrage price of the structure  $[V, (\Sigma_i)]$ , or the structure  $[V, (\Sigma_i)]$  to be  $q$ -arbitrage-free, if one of the following equivalent assertions holds:

- (i) there is no agent  $i \in I$  and portfolio  $z_i \in \mathfrak{R}^J$  such that  $-q \cdot z_i \geq 0$  and  $V[s_i] \cdot z_i \geq 0$  for every  $s_i \in \Sigma_i$ , with at least one strict inequality;
- (ii) for every  $i \in I$ , there exists  $\lambda_i = (\lambda_i[s]) \in \mathfrak{R}_{++}^{\Sigma_i}$ , such that  $q = \sum_{s \in \Sigma_i} \lambda_i[s] V[s]$ . We denote by  $Q_c[V, (\Sigma_i)]$  the set of common no-arbitrage prices of  $[V, (\Sigma_i)]$ . The structure  $[V, (\Sigma_i)]$  is said to be arbitrage-free if  $Q_c[V, (\Sigma_i)] \neq \emptyset$ .

We say that  $q \in \mathfrak{R}^J$  is a no-arbitrage price if there exists a refinement  $(\Sigma_i)$  of  $(S_i)$ , such that  $q \in Q_c[V, (\Sigma_i)]$ . Moreover, if the refinement  $(\Sigma_i)$  of  $(S_i)$  is self-attainable,  $q \in Q_c[V, (\Sigma_i)]$  is said to be self-attainable. We denote by  $Q[V, (S_i)]$  the set of no-arbitrage prices and by  $Q[V, (S_i)]$  the subset of self-attainable prices. When no confusion is possible, the reference to  $V$  is omitted in all notations and we use the term of arbitrage-free or  $q$ -arbitrage-free information structure.

We recall that a symmetric structure is always arbitrage-free and, hence, that the set of (self-attainable) no-arbitrage prices, which contains  $Q_c[V, (\underline{S}_i)]$ , is never empty. We can now present the following properties.

**Claim 1** Let a price system  $(p, q) \in (\mathfrak{R}^L)^{S'} \times \mathfrak{R}^J$  and a refinement  $(\Sigma_i)$  of  $(S_i)$  be given. The following Assertions hold:

(i) under Assumptions A1 and A3, if the price system  $(p, q)$ , the refinement  $(\Sigma_i)$  and a collection of strategies  $(x_i, z_i) \in B_i^{\Sigma_i}(p, q)$ , for  $i = 1, \dots, m$ , meet Condition (i) of the definition of the no-arbitrage equilibrium, then  $q$  is a common no-arbitrage price of the structure  $[V, (\Sigma_i)]$  (i.e.,  $q \in Q_c[V, (\Sigma_i)]$ ) and, hence,  $[V, (\Sigma_i)]$  is arbitrage-free;

(ii) the structure  $(\Sigma_i)$  is arbitrage-free if and only if it satisfies the following “AFAO” Condition (“Absence of Future Arbitrage Opportunities”): there is no  $(z_i) \in (\mathfrak{R}^J)^I$  such that  $\sum_{i \in I} z_i = 0$  and  $V[s_i] \cdot z_i \geq 0$  for all  $i \in I$  and all  $s_i \in \Sigma_i$ , with at least one strict inequality;

(iii) if  $q$  is a no-arbitrage price (i.e.,  $q \in Q[V, (S_i)]$ ), there exists a coarsest  $q$ -arbitrage-free refinement of  $(S_i)$ , denoted by  $(\mathbf{S}_i(q))$ .<sup>3</sup> If  $q$  is self-attainable, then  $(\mathbf{S}_i(q))$  is self-attainable. Moreover, if agents have private information represented by  $(S_i)$ , they may always refine their information up to the structure  $(\mathbf{S}_i(q))$  from observing that price  $q$ . Therefore, the structure  $(\mathbf{S}_i(q))$  is said to be “revealed” by price  $q$ ;

(iv) the set of arbitrage-free refinements of  $(S_i)$  contains a coarsest element, denoted by  $(\bar{\mathbf{S}}_i)$ , which is self-attainable, revealed by any price  $\bar{q} \in Q_c[V, (\bar{\mathbf{S}}_i)]$  and equal to  $(S_i)$  if, and only if,  $(S_i)$  is arbitrage-free.<sup>4</sup>

**Proof** See Cornet-De Boisdeffre (2002).<sup>5</sup> ■

<sup>3</sup>“Coarsest” means that any  $q$ -arbitrage-free refinement of  $(S_i)$  is a refinement of  $(\mathbf{S}_i(q))$ .

<sup>4</sup>“Coarsest” means that any arbitrage-free refinement of  $(S_i)$  is a refinement of  $(\bar{\mathbf{S}}_i)$

<sup>5</sup>We had proved Assertion (i) only in the case of utility functions. We let the reader check the proof the same, mutatis mutandis, for preference correspondences.

The concept of no-arbitrage equilibrium does not explain how agents  $i \in I$ , starting from their signal sets  $S_i$  and observing an equilibrium price  $q$  can infer in a decentralized manner from that price the refinement of information  $(\Sigma_i)$  of equilibrium. The properties recalled above permit to tackle this issue and to refine the concept of no-arbitrage equilibrium as follows.

**Definition 4** A no-arbitrage equilibrium  $[(p^*, q^*), (\Sigma_i), [(x_i^*, z_i^*)]] \in ((\mathfrak{R}^L)^{S'} \times \mathfrak{R}^J) \times \Gamma \times \prod_{i=1}^m B_i^{\Sigma_i}(p^*, q^*)$  is called a “coarse information equilibrium” (C.I.E.), if  $(\Sigma_i) = (\bar{\mathbf{S}}_i)$ , and a “price-revealed equilibrium” (P.R.E.), if  $(\Sigma_i) = (\mathbf{S}_i(q^*))$ .

When agents have no model of how equilibrium prices are determined (that is, rational expectations), the two latter concepts of equilibrium extend naturally to the asymmetric information setting the classical notion of financial equilibrium. Indeed, a coarse information equilibrium coincides with the classical notion of financial equilibrium when  $(S_i)$  is symmetric. Moreover, since  $(\bar{\mathbf{S}}_i) = (\mathbf{S}_i(q))$  for every  $q \in Q_c[(\bar{\mathbf{S}}_i)]$ , a coarse information equilibrium is always a price-revealed equilibrium (the converse is not true). Thus, both concepts embed the requirement that the information structure be revealed by prices.

In Section 3, we show that, for any given structure  $[V, (S_i)]$  and any self-attainable arbitrage-free refinement  $(\Sigma_i)$  of  $(S_i)$ , there exists a no-arbitrage equilibrium relative to the refinement  $(\Sigma_i)$ . Hence, taking  $(\Sigma_i) = (\bar{\mathbf{S}}_i)$ , a coarse information equilibrium will always exist. But we need to introduce, first, other useful properties of the model.

### 2.3.2 Bounding attainable strategies

In the remainder of this section, we consider as given an arbitrary self-attainable arbitrage-free refinement  $(\Sigma_i)$  of  $(S_i)$ , assumed to represent agents’ final information at the first period. Our aim is now to show that we can bound agents’ portfolios and consumptions in the equilibrium problem.

First, we consider the vector spaces:

$$Z^{\Sigma_i} := \{z \in \mathfrak{R}^J : V^{\Sigma_i} z = 0\} \text{ and } Z^{\Sigma_i \perp} := (Z^{\Sigma_i})^\perp, \text{ its orthogonal;}$$

$$Z^{(\Sigma_i)} := \sum_{i=1}^m Z^{\Sigma_i} \text{ and } Z^\perp := (Z^{(\Sigma_i)})^\perp, \text{ its orthogonal;}$$

We then let  $\varepsilon > 0$ , be an arbitrary small, but positive, given number and we define the following price sets:

$$\Delta := \{p \in (\mathfrak{R}^L)^{S'} : \forall s \in S', \|p[s]\| \leq 1\};$$

$$\Delta_\varepsilon^{(\Sigma_i)} := \{p \in \Delta : p^l[s] \geq \varepsilon, \forall l \in \{1, \dots, L\}, \forall s \notin \cap_{i \in I} \Sigma_i^l\};$$

$$Q := \{q \in Z^\perp : \|q\| \leq 1\}^6 \text{ and } \Pi_\varepsilon^{(\Sigma_i)} := \Delta_\varepsilon^{(\Sigma_i)} \times Q.$$

For each price  $(p, q) \in \Pi_\varepsilon^{(\Sigma_i)}$  and each  $i \in I$ , we recall that:

---

<sup>6</sup>We notice that  $Q_c[V, (\Sigma_i)] \subset Z^\perp$  and  $Q_c[V, (\Sigma_i)] \cap Q \neq \emptyset$ .

$$B_i^{\Sigma_i}(p, q) := \{(x, z) \in X_i^{\Sigma_i} \times \mathfrak{R}^J : [p_{\square}(x - e_i^{\Sigma_i})](\Sigma'_i) \leq W^{\Sigma_i}(q)z\} \text{ and}$$

$$\mathcal{A}^{(\Sigma_i)} := \{x := (x_i) \in \Pi_{i=1}^m X^{\Sigma_i} : \forall s \in \cap_{i \in I} \Sigma'_i, \sum_{i=1}^m (x_i[s] - e_i^{\Sigma_i}[s]) = 0\}.$$

Denoting  $\mathbf{1}$  the vector of  $\mathfrak{R}^{S'}$  whose components are all equal to one, we let:

$$\begin{aligned} \tilde{B}_i^{\Sigma_i}(p, q) &:= (X_i^{\Sigma_i} \times Z^{\Sigma_i \perp}) \cap B_i^{\Sigma_i}(p, q); \\ \bar{B}_i^{\Sigma_i}(p, q) &:= \{(x, z) \in X_i^{\Sigma_i} \times Z^{\Sigma_i \perp} : [p_{\square}(x - e_i^{\Sigma_i})](\Sigma'_i) \leq W^{\Sigma_i}(q)z + \mathbf{1}(\Sigma'_i)\}; \\ \bar{B}_i^{*\Sigma_i}(p, q) &:= \{x \in X_i^{\Sigma_i} : \exists z \in Z^{\Sigma_i \perp}, (x, z) \in \bar{B}_i^{\Sigma_i}(p, q)\}; \\ \bar{\mathcal{A}}^{(\Sigma_i)}(p, q) &:= \{[(x_i, z_i)] \in \Pi_{i=1}^m \bar{B}_i^{\Sigma_i}(p, q) : (x_i) \in \mathcal{A}^{(\Sigma_i)}, \sum_{i=1}^m z_i \in Z^{(Z_i)}\}. \end{aligned}$$

We can now state a useful property.

**Claim 3** *Let the arbitrage-free refinement  $(\Sigma_i)$  of  $(S_i)$  and a price set  $\Pi_{\varepsilon}^{(\Sigma_i)}$ , as defined above, be given. Then, there exists  $r > 0$ , such that:*

$$[(p, q) \in \Pi_{\varepsilon}^{(\Sigma_i)} \text{ and } (x, z) \in \bar{\mathcal{A}}^{(\Sigma_i)}(p, q)] \implies [\|x\| + \|z\| < r].$$

**Proof**

- First, we let  $\Sigma := \cap_{i \in I} \Sigma_i = \cap_{i \in I} S_i$  and show the existence of  $r^1$  such that:  
 $[(p, q) \in \Pi_{\varepsilon}^{(\Sigma_i)} \text{ and } (x, z) \in \bar{\mathcal{A}}^{(\Sigma_i)}(p, q)] \implies [\|x(\Sigma')\| < r^1].$

Indeed, for each  $i \in I$ , the consumption set  $X_i^{\Sigma_i} \subset (\mathfrak{R}_+^L)^{S'}$  is bounded by zero. Denoting  $\alpha := \max_{(i,s,l) \in I \times S' \times \{1, \dots, L\}} (e^{\Sigma_i})^l[s]$ , we let  $(p, q) \in \Pi_{\varepsilon}^{(\Sigma_i)}$  and  $(x, z) \in \bar{\mathcal{A}}^{(\Sigma_i)}(p, q)$  be given. Then,  $x := (x_i) \in \mathcal{A}^{(\Sigma_i)}$  implies that, for each  $i \in I$ ,  $0 \leq x_i(\Sigma') \leq \sum_{j \in I \setminus \{i\}} e^{\Sigma_j}(\Sigma')$ , hence,  $\|x_i(\Sigma')\| \leq mL(\#S')\alpha$ . This yields:  $[(p, q) \in \Pi_{\varepsilon}^{(\Sigma_i)} \ \& \ (x, z) \in \bar{\mathcal{A}}^{(\Sigma_i)}(p, q)] \implies [\|x(\Sigma')\| < r^1 := m^2L(\#S')\alpha].$

- Second, we show that, for every  $M > 0$ , there exists  $r^M$  such that:  
 $[(p, q) \in \Pi_{\varepsilon}^{(\Sigma_i)}, (x, z) \in \bar{\mathcal{A}}^{(\Sigma_i)}(p, q) \text{ and } \|z\| < M] \implies [\|x\| < r^M].$

Indeed, denoting  $\beta := \max_{(s,j) \in S' \times \{1, \dots, J\}} |v_j[s]|$ , let  $M > 0$ ,  $(p, q) \in \Pi_{\varepsilon}^{(\Sigma_i)}$  and  $(x, z) \in \bar{\mathcal{A}}^{(\Sigma_i)}(p, q)$  be given, such that  $\|z\| < M$ . From the definition of  $\Delta_{\varepsilon}^{(\Sigma_i)}$ ,  $0 \leq x_i^l[s] < \frac{1+\beta M J}{\varepsilon} + \alpha$ , for each  $(i, l) \in I \times \{1, \dots, L\}$  and each  $s \in \Sigma_i \setminus \Sigma$ . Let  $\gamma^M := \frac{1+\beta M J}{\varepsilon} + \alpha$  and  $r^M := r^1 + Lm(\#S')\gamma^M$ . Then, from above,  $\|x\| < r^M$ .

- Third, we prove, by contraposition, that there exists  $r^2 > 0$ , such that:  
 $[(p, q) \in \Pi_{\varepsilon}^{(\Sigma_i)} \text{ and } (x, z) \in \bar{\mathcal{A}}^{(\Sigma_i)}(p, q)] \implies [\|z\| < r^2].$

Assume that the property fails, that is, for every positive integer  $n$ , there exist  $(p^n, q^n) \in \Pi_{\varepsilon}^{(\Sigma_i)}$  and  $(x^n, z^n) \in \bar{\mathcal{A}}^{(\Sigma_i)}(p^n, q^n)$ , such that  $\|z^n\| > n$ . Let  $z'^n := \frac{z^n}{\|z^n\|}$  for every  $n \geq 1$ . Each set  $\bar{\mathcal{A}}^{(\Sigma_i)}(p^n, q^n)$  is closed and convex and contains  $(e, 0) := [(e_i^{\Sigma_i}, 0)]$ . For every  $n \geq 1$ , we let  $x'^n := \frac{1}{\|z^n\|} x^n + (1 - \frac{1}{\|z^n\|})e$ .

Clearly, for every  $n > 1$ ,  $\|z'^n\| = 1$  and  $(x'^n, z'^n) \in \overline{\mathcal{A}}^{(\Sigma_i)}(p^n, q^n)$ . From above, the sequences  $(x'^n, z'^n)$  and  $(p^n, q^n)$ , are bounded in some Euclidean space and in a compact set, and may be assumed to converge, say, towards  $(x, z)$  and  $(p, q) \in \Pi_\varepsilon^{(\Sigma_i)}$ , and, by continuity of the scalar product,  $\|z\| = 1$  and  $(x, z) \in \overline{\mathcal{A}}^{(\Sigma_i)}(p, q)$ .

Given  $N > 1$ , we let, for every  $n > N$ ,  $x_N^n := \frac{N}{\|z^n\|}x^n + (1 - \frac{N}{\|z^n\|})e$ . By the same token, for  $n > N$ ,  $(x_N^n, Nz'^n) \in \overline{\mathcal{A}}^{(\Sigma_i)}(p^n, q^n)$  and the sequence  $[(x_N^n, Nz'^n)]_{n>N}$  may be assumed to converge to some  $(x_N, z_N) \in \overline{\mathcal{A}}^{(\Sigma_i)}(p, q)$ , such that  $z_N = Nz$  (for the same limits  $z := (z_i)$  and  $(p, q)$  as above). Considering the sequence  $[(x_N, z_N)]$ , for  $N > 1$ , one has, for every  $N$ ,  $(x_N, z_N) \in \overline{\mathcal{A}}^{(\Sigma_i)}(p, q)$  and  $z_N = Nz$ , with  $\|z\| = 1$ .

All consumption sets being bounded from below, this latter condition implies  $V[s_i] \cdot z_i \geq 0$ , for every  $i \in I$  and every  $s_i \in \Sigma_i$ , whereas  $\sum_{i=1}^m z_i \in Z^{(\Sigma_i)}$  (since  $(x, z) \in \overline{\mathcal{A}}^{(\Sigma_i)}(p, q)$ ). From the AFAO Condition, this implies  $z_i \in Z^{\Sigma_i}$ , for every  $i \in I$ . However,  $(x, z) \in \overline{\mathcal{A}}^{(\Sigma_i)}(p, q)$  yields  $z := (z_i) \in \prod_{i=1}^m Z^{\Sigma_i \perp}$ . Hence,  $z_i \in Z_i^{\Sigma_i} \cap Z_i^{\Sigma_i \perp} = \{0\}$ , for every  $i \in I$ , that is,  $z = 0$ , which contradicts the condition  $\|z\| = 1$ . This contradiction proves the existence of some  $r^2 > 0$ , such that:  $[(p, q) \in \Pi_\varepsilon^{(\Sigma_i)} \text{ and } (x, z) \in \overline{\mathcal{A}}^{(\Sigma_i)}(p, q)] \implies [\|z\| < r^2]$ .

- Finally, from above, Claim 3 holds for  $r = M + r^M$ , with  $M = r^2$ .

We can now examine existence properties.

### 3 An existence theorem for no-arbitrage equilibria

First, we state the main theorem of the model and define a notion of pseudo-equilibrium, whose existence will insure the desired property. Next, we prove that a pseudo-equilibrium exists, along a standard fixed-point-like argument.

#### 3.1 The existence theorem

We now state the main theorem of the paper.

**Theorem 1** *Under Assumptions A1 to A3, given a structure  $[V, (S_i)]$  and a self-attainable arbitrage-free refinement  $(\Sigma_i)$  of  $(S_i)$ , there exists a price system  $(p, q) \in \Delta \times Q_c[V, (\Sigma_i)]$  and a collection of strategies  $[(x_i, z_i)] \in \prod_{i=1}^m B_i^{\Sigma_i}(p, q)$ , such that  $p[s] \neq 0$  for every  $s \in \cup_{i \in I} \Sigma'_i$  and  $[(p, q), (\Sigma_i), [(x_i, z_i)]]$  defines a no-arbitrage equilibrium. In particular, for any structure  $[V, (S_i)]$ , there exists a coarse information equilibrium.*

To prove this theorem, we henceforth consider as given a self-attainable arbitrage-free refinement  $(\Sigma_i)$  of  $(S_i)$  and we will assume, non restrictively, that  $S = \cup_{i \in I} \Sigma_i$ .<sup>7</sup>

We let a small  $\varepsilon > 0$  be given and, as above, we consider the price sets:

$$\Delta_\varepsilon^{(\Sigma_i)} := \{p \in (\mathfrak{R}^L)^{S'} : \|p[s]\| \leq 1, \forall s \in S', p^l[s] \geq \varepsilon, \forall l \in \{1, \dots, L\}, \forall s \notin \cap_i S'_i\},$$

$$Q := \{q \in Z^\perp := \cap_{i \in I} Z^{\Sigma_i \perp} : \|q\| \leq 1\} \text{ and we simply denote } \Pi := \Delta_\varepsilon^{(\Sigma_i)} \times Q.$$

We set some upper bound  $r$ , satisfying the Condition of Claim 3 relative to  $\Pi$ . We denote by  $B_r$  and  $B_r^J$  the closed centered balls of radius  $r$  of, respectively,  $(\mathfrak{R}^L)^{S'}$  and  $\mathfrak{R}^J$ , and let, for every  $i \in I$  and every  $(p, q) \in \Pi$ :

$$\begin{aligned} X_i &:= X_i^{\Sigma_i} \cap B_r \text{ and } P_i \text{ be the correspondence on } \Pi_{i=1}^m X_i \text{ defined by} \\ P_i(x) &:= P_i^{\Sigma_i}(x_i) \cap X_i \text{ for every } x := (x_i) \in \Pi_{i=1}^m X_i; \\ Z_i &:= Z_i^{\Sigma_i \perp} \cap B_r^J; \\ e_i &:= e^{\Sigma_i}; \\ B_i(p, q) &:= B_i^{\Sigma_i}(p, q) \cap (X_i \times Z_i); \\ B_i^*(p, q) &:= \{x \in X_i : \exists z \in Z_i, (x, z) \in B_i(p, q)\}; \\ \mathcal{A} &:= \{(x_i) \in \Pi_{i=1}^m X_i : \forall s \in \cap_i S'_i, \sum_{i=1}^m (x_i[s] - e_i[s]) = 0\}; \\ \text{and } \mathcal{A}(p, q) &:= \{(x_i, z_i) \in \Pi_{i=1}^m B_i(p, q) : (x_i) \in \mathcal{A}, \sum_{i=1}^m z_i \in Z^{(\Sigma_i)}\}. \end{aligned}$$

We now introduce a notion of “pseudo-equilibrium”, in relation to that of no-arbitrage equilibrium.

**Definition 5** *Given the refinement  $(\Sigma_i)$ , a price system  $(p, q) \in \Pi$  and a collection of strategies  $(x_i, z_i) \in B_i(p, q)$ , for  $i = 1, \dots, m$ , define a pseudo-equilibrium of the economy  $\mathcal{E}$  if:*

- (i)  $\forall i \in I, P_i(x) \cap B_i^*(p, q) = \emptyset$ , where  $x := (x_i) \in \Pi_{i=1}^m X_i$ ;
- (ii)  $\forall s \in \cap_i S'_i, \sum_{i \in I} (x_i[s] - e_i[s]) = 0$ ;
- (iii)  $\sum_{i \in I} z_i \in Z^{(\Sigma_i)}$ .

For every  $i \in I$  and every  $z_i \in \mathfrak{R}^J$ , we shall henceforth denote by  $z_i^{i \perp}$  the orthogonal projection of  $z_i$  on  $Z^{\Sigma_i \perp}$ . The relationships between the two concepts of pseudo-equilibrium and no-arbitrage equilibrium can now be presented.

**Claim 4** *Let a price system  $(p, q) \in \Pi$  and a collection of strategies  $[(x_i, z_i)] \in \Pi_{i=1}^m B_i^{\Sigma_i}(p, q)$  be given. Then,*

- (i) *if  $((p, q), (\Sigma_i), [(x_i, z_i)]) \in \Pi \times \Gamma \times \Pi_{i=1}^m B_i^{\Sigma_i}(p, q)$  defines a no-arbitrage equilibrium, then  $((p, q), [(x_i, z_i^{i \perp})])$  is a pseudo-equilibrium;*
- (ii) *if  $((p, q), [(x_i, z_i)])$  is a pseudo-equilibrium, then, for every  $i \in I$ , there exists  $z_i^* \in \mathfrak{R}^J$ , such that  $z_i = z_i^{*i \perp}$  and  $((p, q), (\Sigma_i), [(x_i, z_i^*)])$  defines a no-arbitrage equilibrium.*

<sup>7</sup>Indeed, we may replace, if necessary, the economy  $\mathcal{E}$  by one having  $\Omega := \cup_{i \in I} \Sigma_i$  for state space and  $V^\Omega$  (or a full-rank extraction of  $V^\Omega$ ) for payoff matrix.

**Proof**

(i) Assume that  $((p, q), (\Sigma_i), [(x_i, z_i)]) \in \Pi \times \Gamma \times \prod_{i=1}^m B_i^{\Sigma_i}(p, q)$  defines a no-arbitrage equilibrium.

First, Conditions (ii) – (iii) of the Definition of a no-arbitrage equilibrium yield  $[(x_i, z_i)] \in \mathcal{A}_i^{(\Sigma_i)}(p, q)$  and  $\sum_{i \in I} z_i^{\perp} = \sum_{i \in I} (z_i^{\perp} - z_i) \in Z^{(\Sigma_i)}$ . Referring to the above definitions, we then notice that  $[(x_i, z_i^{\perp})] \in \overline{\mathcal{A}}^{(\Sigma_i)}(p, q) := \{[(x_i, z_i)] \in \prod_{i=1}^m \overline{B}_i^{\Sigma_i}(p, q) : (x_i) \in \mathcal{A}^{(\Sigma_i)}, \sum_{i=1}^m z_i \in Z^{(Z_i)}\}$ .

Hence, from Claim 3,  $\|(x_i)\| + \|(z_i^{\perp})\| < r$ . Consequently, the collection  $[(x_i, z_i^{\perp})]$  belongs to  $\prod_{i=1}^m B_i(p, q)$  and satisfies  $(x_i) \in \mathcal{A}$ ,  $\sum_{i=1}^m z_i^{\perp} \in Z^{(Z_i)}$ , that is, Conditions (ii) and (iii) of the Definition of a pseudo-equilibrium.

Applying Condition (i) of the Definition of the no-arbitrage equilibrium to  $x := (x_i) \in \mathcal{A}$  yields  $P_i^{\Sigma_i}(x_i) \cap B_i^{*\Sigma_i}(p, q) = \emptyset$ , for  $i = 1, \dots, m$ . Since  $P_i(x) \subset P_i^{\Sigma_i}(x_i)$  and  $B_i^*(p, q) \subset B_i^{*\Sigma_i}(p, q)$ , for every  $i \in I$ , the collection  $[(x_i, z_i^{\perp})] \in \prod_{i=1}^m B_i(p, q)$  also meets Condition (i) of the Definition of the pseudo-equilibrium and Claim 4-(i) holds.

(ii) Assume, now, that  $((p, q), [(x_i, z_i)]) \in \Pi \times \prod_{i=1}^m B_i(p, q)$  defines a pseudo-equilibrium.

Then, from Condition (iii) of the above Definition,  $\sum_{i \in I} z_i \in Z^{(\Sigma_i)}$ , that is,  $\sum_{i \in I} z_i = \sum_{i \in I} z'_i$  for some collection  $(z'_i) \in \prod_{i=1}^m Z^{\Sigma_i}$ . Letting  $(z_i^*) := (z_i - z'_i) \in (\mathbb{R}^J)^I$ , the collection  $(z_i^*)$  satisfies  $(z_i) = (z_i^{*\perp})$  and meets Condition (iii) of the definition of the no-arbitrage equilibrium, by construction. Moreover, the collection  $[(x_i, z_i^*)]$  belongs to  $\prod_{i=1}^m B_i^{\Sigma_i}(p, q)$  and satisfies Condition (ii) of the definition of the no-arbitrage equilibrium, since  $x := (x_i) \in \mathcal{A}$ .

Claim 4-(ii) will hold if we show that  $[(x_i, z_i^*)]$  satisfies Condition (i) of that definition.

Assume, by contraposition, that this is not the case, that is, there exists  $i \in I$  and  $(x'_i, z'_i) \in B_i^{\Sigma_i}(p, q)$ , such that  $x'_i \in P_i^{\Sigma_i}(x_i)$ . It is clear that  $(x'_i, z_i^{\perp}) \in \tilde{B}_i^{\Sigma_i}(p, q) := (X_i^{\Sigma_i} \times Z^{\Sigma_i \perp}) \cap B_i^{\Sigma_i}(p, q)$ . Then, for every positive integer  $n$ , we define  $x_i^n := [x_i + \frac{1}{n}(x'_i - x_i)]$  and  $z_i^n := [z_i + \frac{1}{n}(z_i^{\perp} - z_i)] \in Z^{\Sigma_i \perp}$ . Since  $\tilde{B}_i(p, q)$  is convex and contains  $(x_i, z_i)$  and  $(x'_i, z_i^{\perp})$ , one has:  $(x_i^n, z_i^n) \in \tilde{B}_i(p, q)$ , for every  $n > 0$ . Yet, from Assumption A3,  $x_i^n \in P_i^{\Sigma_i}(x_i)$ , for every  $n > 0$ . Recalling the definition of a pseudo-equilibrium and of the set  $\overline{\mathcal{A}}^{(\Sigma_i)}(p, q)$ , we notice that  $[(x_i, z_i)]$  belongs to  $\overline{\mathcal{A}}^{(\Sigma_i)}(p, q)$ . Hence, from Claim 3  $\|x\| + \|z\| < r$ . Therefore, there exists  $N > 0$ , such that  $\|x_i^N\| < r$  and  $\|z_i^N\| < r$ , that is,  $(x_i^N, z_i^N) \in B_i(p, q)$  and  $x_i^N \in P_i(x) := X_i \cap P_i^{\Sigma_i}(x_i)$ . Hence,  $x_i^N \in B_i^*(p, q) \cap P_i(x)$ , which contradicts Condition (i) of the definition of the pseudo-equilibrium for  $[(p, q), (x, z)]$ . This completes the proof. ■

From Claim 4 and the fact that the self-attainable arbitrage-free refinement  $(\Sigma_i)$  of  $(S_i)$  has been chosen arbitrary, Theorem 1 will be proven if we display a pseudo-equilibrium along Definition 5. This is the final step of the paper.

### 3.2 The existence proof

To prove the existence of a pseudo-equilibrium, we introduce correspondences on the bounded sets  $(X_i, Z_i, \Pi\dots)$ , which will yield an equilibrium, along successive Claims, derived from standard proofs of the symmetric information literature (e.g., Florenzano's, 1999). Following Bergstrom (1976), we thus define, for each  $i = 1, \dots, m$  and each  $(p, q) \in \Pi$ , the modified budget sets:

$$\begin{aligned} B'_i(p, q) &:= \{(x, z) \in X_i \times Z_i : [p_{\square}(x - e_i)](\Sigma'_i) \leq W^{\Sigma_i}(q)z + \gamma_i(p, q)\}; \\ B''_i(p, q) &:= \{(x, z) \in B'_i(p, q) : [p_{\square}(x - e_i)](\Sigma'_i) << W^{\Sigma_i}(q)z + \gamma_i(p, q)\}, \end{aligned}$$

where  $\gamma_i(p, q) \in \mathfrak{R}^{\Sigma_i}$  is defined by:

$$\begin{aligned} \gamma_i(p, q)[0] &:= 1 - \min(1, \|p[0]\| + \|q\|), \\ \gamma_i(p, q)[s] &:= 1 - \|p[s]\| \text{ for every } s \in \cap_i \Sigma_i = \cap_i S_i \\ \text{and } \gamma_i(p, q)[s] &:= 0 \text{ for every } s \in \Sigma_i \setminus (\cap_i S_i). \end{aligned}$$

**Claim 5** For every  $i \in I$ , and every  $(p, q) \in \Pi$ ,  $B''_i(p, q) \neq \emptyset$ .

**Proof** Let  $i \in I$  and  $(p, q) \in \Pi$  be given. From Assumption A2, we can choose  $x \in X_i$ , such that  $[p_{\square}(x - e_i)](\Sigma'_i) \leq 0$ , with a strict inequality in any state  $s \in \Sigma'_i$  such that  $p[s] \neq 0$  (in particular, for every  $s \in \Sigma_i \setminus (\cap_i S_i)$ ). In either cases,  $p[0] \neq 0$  or  $[p[0] = 0 \text{ and } q = 0]$ , one has  $(x, 0) \in B''_i(p, q)$ . Alternatively, if  $p[0] = 0$  and  $q \neq 0$  (recalling  $q \in (\sum_{i \in I} Z^{\Sigma_i})^{\perp} = \cap_{i \in I} Z^{\Sigma_i \perp}$ ), the nonemptiness of  $B''_i(p, q)$  is straightforward from the fact that  $q \in Z^{\Sigma_i \perp}$ : for a sufficiently small  $z \in Z_i$ , such that  $q \cdot z < 0$ ,  $(x, z) \in B''_i(p, q)$ . ■

**Claim 6** For every  $i \in I$ , the convex valued correspondence  $B''_i$  is lower semicontinuous.

**Proof** Let  $i \in I$  and  $(p, q) \in \Pi$  be given. The convexity and the fact that  $B''_i(p, q)$  is non-empty insures that  $B'_i(p, q) = \overline{B''_i(p, q)}$ . The correspondence:  $(p, q) \mapsto \{(x, z) \in X_i \times Z_i : [p_{\square}(x - e_i)](\Sigma'_i) << W^{\Sigma_i}(q)z + \gamma_i(p, q)\}$  has an open graph. By a standard corollary, it is lower semicontinuous. ■

**Claim 7** For every  $i \in I$ , the closed convex non-empty valued correspondence  $B'_i$  is upper semicontinuous.

**Proof** Let  $i \in I$  be given. The correspondence  $B'_i$  has a closed graph, with non-empty convex values in the compact set  $X_i \times Z_i$ . By a standard corollary, it is, hence, upper semicontinuous. ■

We now introduce an additional agent  $i = 0$  representing the market and, following Gale-Mas-Colell (1975-1979), a reaction correspondence for each agent, defined on the convex compact set  $\Omega := \Pi \times (\prod_{i=1}^m X_i \times Z_i)$ . We thus let, for every  $i \in I$  and every  $((p, q), (x, z)) \in \Omega$  (where  $(x, z) := (x_i, z_i) \in \prod_{i=1}^m X_i \times Z_i$ ):

$$\Psi_i((p, q), (x, z)) := \left\{ \begin{array}{ll} B'_i(p, q) & \text{if } (x_i, z_i) \notin B'_i(p, q) \\ B''_i(p, q) \cap (P_i(x) \times Z_i) & \text{if } (x_i, z_i) \in B'_i(p, q) \end{array} \right\}.$$

We let the price correspondence satisfy, for every  $((p, q), (x, z)) \in \Omega$ :

$$\begin{aligned} \Psi_0((p, q), (x, z)) := & \{(p', q') \in \Pi : p \cdot \sum_{i=1}^m (x_i - e_i) \\ & + q \cdot \sum_{i=1}^m z_i < p' \cdot \sum_{i=1}^m (x_i - e_i) + q' \cdot \sum_{i=1}^m z_i\}. \end{aligned}$$

**Claim 8** For each  $i \in \{0, 1, \dots, m\}$ ,  $\Psi_i$  is lower semicontinuous.

**Proof** First, the correspondence  $\Psi_0$  is lower semicontinuous for having an open graph. Second, we let  $i \in I$  and  $\omega := ((p, q), (x, z)) \in \Omega$  be given and consider separately the two alternatives  $(x_i, z_i) \notin B'_i(p, q)$  and  $(x_i, z_i) \in B'_i(p, q)$  and show that, in both cases,  $\Psi_i$  is lower semicontinuous at  $\omega$ .

- Assume, first, that  $(x_i, z_i) \notin B'_i(p, q)$ . Then,  $\Psi_i(\omega) = B'_i(p, q)$ .

Let  $V$  be an open set in  $X_i \times Z_i$ , such that  $V \cap B'_i(p, q) \neq \emptyset$ . It follows from the convexity of  $B'_i(p, q)$  and the non-emptiness of  $B''_i(p, q)$  that  $V \cap B''_i(p, q) \neq \emptyset$ . From Claim 6, there exists a neighborhood  $U$  of  $(p, q)$  in  $\Pi$ , such that  $V \cap B'_i(p', q') \supset V \cap B''_i(p', q') \neq \emptyset$ , for every  $(p', q') \in U$ .

Since  $B'_i(p, q)$  is nonempty and closed in the compact set  $X_i \times Z_i$ , there exist two open sets  $V_1$  and  $V_2$  in  $X_i \times Z_i$ , such that  $(x_i, z_i) \in V_1$ ,  $B'_i(p, q) \subset V_2$  and  $V_1 \cap V_2 = \emptyset$ .<sup>8</sup> From Claim 7, there exists a neighborhood  $U_1 \subset U$  of  $(p, q)$ , such that  $B'_i(p', q') \subset V_2$ , for every  $(p', q') \in U_1$ . Let  $W = V_1 \times U_1$ . Then,  $W$  is a neighborhood of  $\omega$ , such that  $\Psi_i(\omega') = B'_i(p', q')$ , and, from above,  $V \cap \Psi_i(\omega') \neq \emptyset$ , for every  $\omega' := ((x', z'), (p', q')) \in W$ . This proves the lower semicontinuity of  $\Psi_i$  at  $\omega$ .

- Assume, now, that  $(x_i, z_i) \in B'_i(p, q)$ , i.e.,  $\Psi_i(\omega) = B''_i(p, q) \cap (P_i(x) \times Z_i)$ .

The lower semicontinuity of  $\Psi_i$  at  $\omega$  is immediate if  $\Psi_i(\omega) = \emptyset$ . Assume  $\Psi_i(\omega) \neq \emptyset$ . We recall that  $P_i$  (from Assumption A3) is lower semicontinuous with open values and that  $B''_i$  is non-empty valued and has an open graph. By a standard corollary (see Florenzano, 1999, page 34), this insures the lower semicontinuity of the correspondence  $((p', q'), (x', z')) \in \Omega \mapsto B''_i(p', q') \cap (P_i(x') \times Z_i) \subset B'_i(p', q')$ . Hence,  $\Psi_i$  is lower semicontinuous at  $\omega$ . ■

**Claim 9** The correspondences  $\Psi_i$  (for  $i \in \{0, 1, \dots, m\}$ ) admit an element  $((p^*, q^*), (x^*, z^*))$  of  $\Omega$ , such that:

$$\begin{aligned} (i), \forall (p, q) \in \Pi, & p \cdot \sum_{i=1}^m (x_i^* - e_i) + q \cdot \sum_{i=1}^m z_i^* \\ & \leq p^* \cdot \sum_{i=1}^m (x_i^* - e_i) + q^* \cdot \sum_{i=1}^m z_i^*; \\ (ii), \forall i \in I, & (x_i^*, z_i^*) \in B'_i(p^*, q^*) \text{ and } B''_i(p^*, q^*) \cap (P_i(x^*) \times Z_i) = \emptyset. \end{aligned}$$

<sup>8</sup>e.g., let  $V_1$  be the open ball centered on  $(x_i, z_i)$  of radius  $\frac{\alpha}{2}$ , where  $\alpha > 0$  is the distance from  $(x_i, z_i)$  to  $B'_i(p, q)$ , and  $V_2$  be a finite union of open balls of radius  $\frac{\alpha}{2}$ , centered on elements of  $B'_i(p, q)$ , containing  $B'_i(p, q)$ .

**Proof** We recall the following fixed-point-like theorem, due to Gale and Mas-Colell (1975, 1979):

“Given  $X = \prod_{i=1}^m X_i$ , where  $X_i$  is a non-empty compact convex subset of  $\mathbb{R}^n$ , let  $\varphi_i : X \rightarrow X_i$  be  $m$  convex (possibly empty) valued correspondences, which are lower semicontinuous. Then there exists  $x$  in  $X$  such that for each  $i$  either  $x_i \in \varphi_i(x)$  or  $\varphi_i(x) = \emptyset$ ”.

From above, the correspondences  $\Psi_0 : \Omega \rightarrow \Pi$  and  $\Psi_i : \Omega \rightarrow X_i \times Z_i$  (for  $i = 1, \dots, m$ ) satisfy the conditions of the theorem and, hence, admit an element  $((p^*, q^*), (x^*, z^*)) \in \Omega$ , such that either  $(p^*, q^*) \in \Psi_0((p^*, q^*), (x^*, z^*))$  or  $\Psi_0((p^*, q^*), (x^*, z^*)) = \emptyset$ , and, for each  $i \in I$ , either  $(x_i^*, z_i^*) \in \Psi_i((p^*, q^*), (x^*, z^*))$  or  $\Psi_i((p^*, q^*), (x^*, z^*)) = \emptyset$ . By construction,  $(p^*, q^*) \notin \Psi_0((p^*, q^*), (x^*, z^*))$  and, for each  $i \in I$ ,  $(x_i^*, z_i^*) \notin \Psi_i((p^*, q^*), (x^*, z^*))$ , since  $x_i^* \notin P_i(x^*)$ . Hence,  $\Psi_0((p^*, q^*), (x^*, z^*)) = \emptyset$ , which yields the above Assertion (i) and, for every  $i \in I$ ,  $\Psi_i((p^*, q^*), (x^*, z^*)) = \emptyset$ , which yields the above Assertion (ii). ■

**Claim 10**  $\sum_{i \in I} z_i^* \in Z^{(\Sigma_i)}$ .

**Proof** Assume, by contraposition,  $\sum_{i \in I} z_i^* \notin Z^{(\Sigma_i)}$ . Then, from Claim 9-(i),  $q \cdot \sum_{i=1}^m z_i^* \leq q^* \cdot \sum_{i=1}^m z_i^*$ , for every  $q \in Q := \{q \in (Z^{(\Sigma_i)})^\perp : \|q\| \leq 1\}$ , which implies  $q^* \cdot \sum_{i \in I} z_i^* > 0$  and  $\|q^*\| = 1$ , hence,  $\gamma_i(p^*, q^*)[0] = 0$  for every  $i \in I$ .

From Claim 9-(ii), for every  $i \in I$ ,  $(x_i^*, z_i^*) \in B_i'(p^*, q^*)$  and the budget constraint in state  $s = 0$  is written:  $p^*[0] \cdot (x_i^*[0] - e_i[0]) \leq -q^* \cdot z_i^*$ . Summing up these inequalities for  $i \in I$  yields:  $p^*[0] \cdot \sum_{i \in I} (x_i^*[0] - e_i[0]) \leq -q^* \cdot \sum_{i \in I} z_i^* < 0$ , in contradiction to Claim 9-(i), which insures:  $p^*[0] \cdot \sum_{i \in I} (x_i^*[0] - e_i[0]) \geq 0$ . ■

**Claim 11**  $x^* = (x_i^*) \in \mathcal{A}^{(\Sigma_i)}$ , i.e.,  $\sum_{i \in I} (x_i^*[s] - e_i[s]) = 0$ , for every  $s \in \cap_{i \in I} S_i'$ .

**Proof** (by contraposition). From Claim 10, there exists  $(z_i') \in \prod_{i=1}^m Z^{\Sigma_i}$  such that  $\sum_{i=1}^m z_i^* = \sum_{i=1}^m z_i'$ . Then,  $(z_i) := ((z_i^* - z_i')) \in (\mathbb{R}^J)^I$  satisfies  $\sum_{i \in I} z_i = 0$  and  $W(q^*)[s] \cdot z_i^* = W(q^*)[s] \cdot z_i$ , for every  $i \in I$  and every  $s \in \cap_{i \in I} S_i'$ .

Assume that  $\sum_{i \in I} (x_i^*[s] - e_i[s]) \neq 0$ , for some  $s \in \cap_{i \in I} S_i'$ . Then, from Claim 9-(i),  $p^*[s] = \frac{\sum_{i \in I} (x_i^*[s] - e_i[s])}{\|\sum_{i \in I} (x_i^*[s] - e_i[s])\|}$ , which implies  $p^*[s] \cdot \sum_{i \in I} (x_i^*[s] - e_i[s]) > 0$ ,  $\|p^*[s]\| = 1$  and  $\gamma_i(p^*, q^*)[s] = 0$  for every  $i \in I$ .

From Claim 9-(ii), for every  $i \in I$ ,  $(x_i^*, z_i^*) \in B_i'(p^*, q^*)$  and the budget constraint in state  $s$  is written  $p^*[s] \cdot (x_i^*[s] - e_i[s]) \leq W(q^*)[s] \cdot z_i^*$ . Summing up these inequalities for  $i = 1, \dots, m$  yields, from above:

$p^*[s] \cdot \sum_{i \in I} (x_i^*[s] - e_i[s]) \leq \sum_{i \in I} (W(q^*)[s] \cdot z_i^*) = W(q^*)[s] \cdot \sum_{i \in I(s)} z_i = 0$ , hence the desired contradiction (with  $p^*[s] \cdot \sum_{i \in I} (x_i^*[s] - e_i[s]) > 0$ ). ■

**Claim 12**  $(x^*, z^*) := ((x_i^*, z_i^*)) \in \overline{\mathcal{A}}(p^*, q^*)$ , hence,  $\|x^*\| + \|z^*\| < r$ .

**Proof** Recalling the above definitions (see page 8), Claim 9-(ii) yields  $(x_i^*, z_i^*) \in B_i'(p^*, q^*) \subset \overline{B}_i^{\Sigma_i}(p^*, q^*)$ , for every  $i \in I$ . Hence, from Claims 10 and 11,  $(x^*, z^*) \in \overline{\mathcal{A}}^{(\Sigma_i)}(p^*, q^*)$  and, from Claim 3,  $\|x^*\| + \|z^*\| < r$ . ■

**Claim 13** For each  $i \in I$ ,  $(x_i^*, z_i^*)$  is optimal in  $B_i'(p^*, q^*)$ .

**Proof** Let  $i \in I$  be given. From Claim 9-(ii),  $(x_i^*, z_i^*) \in B_i'(p^*, q^*)$  and  $B_i''(p^*, q^*) \cap (P_i(x^*) \times Z_i) = \emptyset$ .

By contraposition, assume there exists  $(x_i, z_i) \in B_i'(p^*, q^*) \cap (P_i(x^*) \times Z_i)$ . From Claim 12, the strict inequalities  $\|x_i^*\| < r$  and  $\|z_i^*\| < r$  hold and, from Assumption A3, the better strategy  $(x_i, z_i)$  may be chosen “sufficiently close” to  $(x_i^*, z_i^*)$  so that  $\|x_i\| < r$  and  $\|z_i\| < r$ .

From Claim 5, there exists  $(x_i', z_i') \in B_i''(p^*, q^*) \subset B_i'(p^*, q^*)$ . Since  $B_i'(p^*, q^*)$  is convex,  $(x_i^n, z_i^n) := [\frac{1}{n}(x_i', z_i') + (1 - \frac{1}{n})(x_i, z_i)] \in B_i'(p^*, q^*)$ , whereas, by construction,  $(x_i^n, z_i^n) \in B_i''(p^*, q^*)$ , for every  $n > 0$ . From Assumption A3,  $P_i$  is open valued and there exists  $N > 0$ , such that  $(x_i^N, z_i^N) \in P_i(x^*) \times Z_i$ . Hence,  $(x_i^N, z_i^N) \in B_i''(p^*, q^*) \cap (P_i(x^*) \times Z_i)$ , which contradicts Claim 9-(ii). ■

**Claim 14** For every  $i \in I$ ,  $\gamma_i(p^*, q^*) = 0$ , that is,  $B_i'(p^*, q^*) = B_i(p^*, q^*)$ .

**Proof** Let  $\Sigma := \cap_{i \in I} \Sigma_i = \cap_{i \in I} S_i$  and consider a given state  $s \in \Sigma'$ . From Claim 10, there exists  $z' := (z_i') \in \prod_{i=1}^m Z^{\Sigma_i}$  such that  $\sum_{i=1}^m z_i^* = \sum_{i=1}^m z_i'$  and the collection  $(z_i) := ((z_i^* - z_i'))$  satisfies  $\sum_{i=1}^m z_i = 0$  and  $W(q^*)[s] \cdot z_i^* = W(q^*)[s] \cdot z_i$ , for every  $i \in I$ .

Let  $i \in I$  be given.

We show first that  $p^*[s] \cdot (x_i^*[s] - e_i[s]) = W(q^*)[s] \cdot z_i^* + \gamma_i(p^*, q^*)[s]$ .

Indeed, from Claim 9-(ii),  $p^*[s] \cdot (x_i^*[s] - e_i[s]) \leq W(q^*)[s] \cdot z_i^* + \gamma_i(p^*, q^*)[s]$ , whereas, from Claim 12,  $\|x_i^*\| < r$  and, from Assumptions A1-A3, there exists  $x_i \in P_i(x^*)$ , such that  $x_i[s'] = x_i^*[s']$  for every  $s' \neq s$ . From Assumption A3, if we had  $p^*[s] \cdot (x_i^*[s] - e_i[s]) < W(q^*)[s] \cdot z_i^* + \gamma_i(p^*, q^*)[s]$ , then,  $x_i$  could be chosen “sufficiently close” to  $x_i^*$  so that  $p^*[s] \cdot (x_i[s] - e_i[s]) < W(q^*)[s] \cdot z_i^* + \gamma_i(p^*, q^*)[s]$  and this would contradict Claim 13.

Hence, from above,  $p^*[s] \cdot (x_i^*[s] - e_i[s]) = W(q^*)[s] \cdot z_i^* + \gamma_i(p^*, q^*)[s]$ , for every  $i \in I$ . Summing up these relations for  $i = 1, \dots, m$  yields, from Claim 11 and from above:

$$0 = \sum_{i \in I} \gamma_i(p^*, q^*)[s] := m(1 - \|p^*[s]\|), \text{ for every } s \in \Sigma'.$$

From the definitions, this yields  $\gamma_i(p^*, q^*) = 0$  and  $B_i'(p^*, q^*) = B_i(p^*, q^*)$ , for every  $i \in I$ . ■

**Claim 15** *The collection  $[(p^*, q^*), (x^*, z^*) := [(x_i^*, z_i^*)]] \in \Omega$  defines a pseudo-equilibrium of the economy  $\mathcal{E}$ , such that  $\|p^*[s]\| = 1$ , for every  $s \in \cap_{i \in I} S'_i$  and, hence, Theorem 1 holds.*

**Proof** The collection  $[(p^*, q^*), (x^*, z^*)]$  satisfies Conditions (i) of the definition of a pseudo-equilibrium, jointly with  $\|p^*[s]\| = 1$  for every  $s \in \cap_{i \in I} S'_i$ , from Claims 13 and 14, and Conditions (ii) – (iii) from Claims 10 and 11. Hence, Claim 4 insures the existence of a no-arbitrage equilibrium relative to the refinement  $(\Sigma_i)$  and the price system  $(p^*, q^*)$ , which, indeed, satisfies  $p^*[s] \neq 0$  for every  $s \in \cup_{i \in I} \Sigma'_i$ . The other properties of Theorem 1, namely  $q^* \in Q_c[V, (\Sigma_i)]$  and the existence of a C.I.E., are straightforward from Claim 1. ■

#### REFERENCES

- Bergstrom, T.C., 1976: “How to discard ‘free disposability’ at no cost”, JOURNAL OF MATHEMATICAL ECONOMICS 3, 131-134.
- Cornet, B., De Boisdeffre, L., 2002: “Arbitrage and price revelation with asymmetric information and incomplete markets”, JOURNAL OF MATHEMATICAL ECONOMICS 38, 393-410.
- Florenzano, M., 1999: “General equilibrium of financial markets”, Summer lecture at the National Economics University of Hanoi, mimeo.
- Gale, D., Mas-Colell, A., 1975: “An equilibrium existence theorem for a general model without ordered preferences”, JOURNAL OF MATHEMATICAL ECONOMICS 2, 9-15.
- Gale, D., Mas-Colell, A., 1979: “Corrections to an equilibrium existence theorem for a general model without ordered preferences”, JOURNAL OF MATHEMATICAL ECONOMICS 6, 297-298.