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Fixed-point-like theorems on subspaces

Philippe BICH
Bernard CORNET

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Fixed-point-like theorems on subspaces

Philippe Bich and Bernard Cornet

Abstract: In this paper, we prove a fixed-point-like theorem for multi-valued mappings defined on the finite Cartesian product of Grassmannian manifolds and convex sets. Our result generalizes two different kinds of theorems: the fixed-point-like theorem by Hirsh, Magill, Mas-Colell (1990) or Husseiny, Lasry, Magill (1990) and the fixed-point theorem by Gale, Mas-Colell (1975) (which generalizes Kakutani's theorem (1941)).

Key words: fixed-point, Grassmannian manifold, degree theory.

AMS classification: 47H04,47H10,47H11.

1 Introduction

In this paper, we prove a fixed-point-like theorem for multi-valued mappings defined on the finite Cartesian product of Grassmannian manifolds and convex sets. Let k be an integer and let V be a Euclidean space such that $0 \leq k \leq \dim V$, then the k -Grassmannian manifold of V , denoted $G^k(V)$, is the set of all the k -dimensional subspaces of V . The set $G^k(V)$ is a smooth compact manifold but, in general, it does not satisfy properties such as convexity or acyclicity and its Euler characteristic may be null. This prevents from using classical fixed-point theorems as Brouwer's [2], Kakutani's [14], or Eilenberg-Montgomery's theorem [7].

Our main result generalizes two different kinds of theorems: the fixed-point-like theorem by Hirsh, Magill, Mas-Colell [12] or Husseiny, Lasry, Magill [13] and the fixed-point theorem by Gale, Mas-Colell [8] (which generalizes Kakutani's Theorem [14]). As in [12] and [13], we shall mainly use techniques from degree theory. As a consequence of our main result we first deduce the standard fixed-point theorems when the variable is in a convex domain (such as Brouwer's and Kakutani's theorem) and second Borsuk-Ulam's theorem.

The main result of this paper is directly motivated by the existence problem of equilibria in economic models with incomplete markets; in Bich-Cornet [1], it is used to extend the classical existence result by Duffie-Shaffer [6] to the non-transitive case.

The paper is organized as follows. The main result is stated in Section 2 together with some direct consequences of it, namely, the results by Hirsh, Magill, Mas-Colell [12], Gale, Mas-Colell [8] and Borsuk-Ulam's Theorem. The proof of the main result is given in Section 3 and the Appendix recalls the main properties of the Grassmannian manifold, used in this paper.

2 Statement of the results

2.1 Preliminaries

A correspondence Φ from a set X to a set Y is a map from X to the set of all the subsets of Y and the graph of Φ , denoted $G(\Phi)$, is defined by $G(\Phi) = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}$. A mapping $\varphi : X \rightarrow Y$ is said to be a selection of Φ if $\varphi(x) \in \Phi(x)$ for all $x \in X$. If A is a subset of X , we let $\Phi(A) = \cup_{x \in A} \Phi(x)$ and the restriction of Φ to A , denoted $\Phi|_A$, is the correspondence from A to Y defined by $\Phi|_A(x) = \Phi(x)$ if $x \in A$. If X and Y are topological spaces, the correspondence Φ is said to be lower semi-continuous (l.s.c.) [resp. upper semi-continuous (u.s.c.)] if for every open set $U \subset Y$, the set $\{x \in X \mid \Phi(x) \cap U \neq \emptyset\}$ is open in X [resp. the set $\{x \in X \mid \Phi(x) \subset U\}$ is open in X and, for every $x \in X$, $\Phi(x)$ is compact].

If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ belong to R^n , we denote by $x \cdot y = \sum_{i=1}^n x_i y_i$, the scalar product of R^n , $\|x\| = \sqrt{x \cdot x}$, the Euclidian norm. If $x \in R^n$ and $r \in R_+$, we let $B(x, r) = \{y \in R^n \mid \|x - y\| < r\}$ and $\bar{B}(x, r) = \{y \in R^n \mid \|x - y\| \leq r\}$. If E is a vector subspace of R^n , we denote by $E^\perp = \{u \in R^n \mid \forall x \in E, x \cdot u = 0\}$ the orthogonal space to E . If u_1, \dots, u_k belong to E , a vector space, we denote by $\text{span}\{u_1, \dots, u_k\}$ the vector subspace of E spanned by u_1, \dots, u_k .

Let V be a Euclidean space and let k be an integer such that $0 \leq k \leq \dim V$, we denote by $G^k(V)$ the set consisting of all the linear subspaces of V of dimension k , called the (k -)Grassmannian manifold of V . Then it is known that $G^k(V)$ is a smooth manifold of dimension $k(\dim V - k)$ and we refer to the Appendix for the properties we shall use hereafter, together with the precise definition of the manifold structure on $G^k(V)$.

2.2 The main result and some consequences

The aim of this paper is to prove the following result:

Theorem 1 *Let I, J be two finite disjoint sets. For every $i \in I$, let k_i be an integer and let V_i be a Euclidean space such that $0 \leq k_i \leq \dim V_i$. For every $j \in J$, let C_j be a nonempty, convex, compact subset of a Euclidean space V_j , and let $M = \prod_{i \in I} G^{k_i} \times \prod_{j \in J} C_j$.*

For $i \in I$ and $k = 1, \dots, k_i$, let F_i^k be a correspondence from M to V_i with convex values, for $j \in J$, let F_j be a correspondence from M to C_j

with convex values, and suppose that, for every $i \in I$ and $k = 1, \dots, k_i$ [resp. $j \in J$], the correspondence F_i^k [resp. F_j] is either l.s.c or u.s.c.

Then, there exists $\bar{x} = ((\bar{x}_i)_{i \in I}, (\bar{x}_j)_{j \in J}) \in M$ such that:

- (i) [either $F_i^k(\bar{x}) \cap \bar{x}_i \neq \emptyset$ or $F_i^k(\bar{x}) = \emptyset$] for every $i \in I$ and $k = 1, \dots, k_i$;
- (ii) [either $F_j(\bar{x}) \cap \{\bar{x}_j\} \neq \emptyset$ or $F_j(\bar{x}) = \emptyset$] for every $j \in J$.

The proof of Theorem 1 is given in Section 3. A first consequence of Theorem 1 is the following theorem by Hirsch et al. ([12]).

Corollary 1 *Let V_1 be a Euclidean space, let k_1 be an integer such that $0 \leq k_1 \leq \dim V_1$, and for every $k = 1, \dots, k_1$ let $f^k : G^{k_1}(V_1) \rightarrow V_1$ be a continuous mapping. Then, there exists $\bar{x} \in G^{k_1}(V_1)$ such that: for every $k = 1, \dots, k_1$, $f^k(\bar{x}) \in \bar{x}$.*

Proof. Take $I = \{1\}$, $J = \emptyset$ and $F_1^k(x) = \{f^k(x)\}$ for every $x \in G^{k_1}(V_1)$ and for every $k = 1, \dots, k_1$. From Theorem 1, there exists $\bar{x} \in M = G^{k_1}(V_1)$ such that for every $k = 1, \dots, k_1$, $F_1^k(\bar{x}) \cap \bar{x} \neq \emptyset$, i.e., $f^k(\bar{x}) \in \bar{x}$. \square

A second consequence of Theorem 1 is the following generalization of Gale and Mas-Colell Theorem [8], which is also a generalization of Kakutani's Theorem. Hereafter, we use the formulation by Gourdél [9] allowing each correspondence to be either upper semi-continuous or lower semi-continuous.

Corollary 2 *Let J be a finite set, for $j \in J$, let C_j be a nonempty, convex, compact subset of a Euclidean space, let F_j be a correspondence from $M := \prod_{j \in J} C_j$ to C_j with convex values, such that the correspondence F_j is either l.s.c or u.s.c.. Then, there exists $\bar{x} = (\bar{x}_j)_{j \in J} \in M$ such that:*

for every $j \in J$, [either $\bar{x}_j \in F_j(\bar{x})$ or $F_j(\bar{x}) = \emptyset$].

Proof. Take $I = \emptyset$ and apply Theorem 1. \square

Remark 1 According to our definition, an u.s.c. correspondence has compact values and without this requirement, Theorem 1 may not be true, as we can see in the following counterexample. Let $M := G^1(\mathbb{R}^2)$. Each element D of $G^1(\mathbb{R}^2)$ can be written as $D_t = \{\lambda(\cos t, \sin t) \mid \lambda \in \mathbb{R}\}$, for some $t \in [0, \pi[$. Let us define the correspondence F from M to \mathbb{R}^2 by $F(D_0) = \mathbb{R} \times \{1\}$ and $F(D_t) = D_t \cap (\mathbb{R} \times \{1\}) + \{(1, 0)\}$ if $t \in]0, \pi[$. We let the reader check that for every open set $U \subset \mathbb{R}^2$ the set $\{x \in M \mid F(x) \subset U\}$ is open in M and that F has nonempty, convex (and closed) values. Yet, it is straightforward that $F(x) \cap x = \emptyset$ for every $x \in G^1(\mathbb{R}^2)$. \square

Another consequence of our main result is the following multi-valued version of Borsuk-Ulam's theorem. We denote by S^n the unit sphere of \mathbb{R}^{n+1} .

Corollary 3 For $k = 1, \dots, n$, let F^k be a correspondence from S^n to R with nonempty and convex values, such that, for every $k = 1, \dots, n$, F^k is either l.s.c or u.s.c.. Then, there exists $\bar{x} \in S^n$ such that:

$$\forall k \in \{1, \dots, n\}, F^k(\bar{x}) \cap F^k(-\bar{x}) \neq \emptyset.$$

Proof. For every $k = 1, \dots, n$, let \hat{F}^k be the correspondence from S^n to R defined by

$$\hat{F}^k(x) = \{u - v \mid u \in F^k(x), v \in F^k(-x)\}.$$

We let the reader check that for every $k = 1, \dots, n$, the correspondence \hat{F}^k has nonempty, convex values and that it is u.s.c. [resp. l.s.c.] if F^k is u.s.c. [resp. l.s.c.]. So, to prove Corollary 3, it suffices to show the existence of $\bar{x} \in S^n$ such that $0 \in \hat{F}^k(\bar{x})$ for every $k = 1, \dots, n$.

Let us define, for every $k = 1, \dots, n$ the correspondence H^k , from $G^n(R^{n+1})$ to R^{n+1} , as follows: for every $E \in G^n(R^{n+1})$, we let $H^k(E) = \hat{F}^k(x)x$, where x is an arbitrary element of $E^\perp \cap S^n$. The correspondence H^k is well defined since $E^\perp \cap S^n = \{x, -x\}$ for some element $x \in S^n$ and since $\hat{F}^k(x)x = \hat{F}^k(-x)(-x)$.

Take $I = \{1\}$, $V_1 = R^{n+1}$, $k_1 = n$, $J = \emptyset$, and apply Theorem 1 to the correspondences H^k , which clearly satisfy the assumptions of Theorem 1. So there exists $\bar{E} \in G^n(R^{n+1})$ such that $\bar{E} \cap H^k(\bar{E}) \neq \emptyset$ for every $k = 1, \dots, n$.

Now, if \bar{x} is an arbitrary point of $\bar{E}^\perp \cap S^n$ then we have $\bar{E} \cap \hat{F}^k(\bar{x})\bar{x} \neq \emptyset$; from $\bar{x} \in \bar{E}^\perp$ and $\bar{x} \neq 0$, we finally obtain $0 \in \hat{F}^k(\bar{x})$ for every $k = 1, \dots, n$, which ends the proof of Corollary 3. \square

3 Proof of Theorem 1

The proof is given in three steps, corresponding to the following three subsections. The first step gives the proof under the additional assumption that $J = \emptyset$ and the correspondences F_i^k are single-valued. The second step only assumes in addition that $J = \emptyset$. Finally, the third step gives the proof under the assumptions of Theorem 1.

3.1 Proof when $J = \emptyset$ and F_i^k is single-valued

We first prove Theorem 1 under the additional assumptions that $J = \emptyset$ and the F_i^k are single-valued. This is exactly the statement below:

Theorem 2 Let I be a finite set and for $i \in I$, let k_i be an integer and let V_i be a Euclidean space such that $0 \leq k_i \leq \dim V_i$. Let $M := \prod_{i \in I} G^{k_i}(V_i)$ and for $i \in I$, let $f_i : M \rightarrow (V_i)^{k_i}$ be a continuous mapping. Then, there exists $\bar{x} = (\bar{x}_i)_{i \in I} \in M$ such that

$$\forall i \in I, f_i(\bar{x}) \in (\bar{x}_i)^{k_i}.$$

The proof of Theorem 2 is given in two steps. In the first step we additionally assume that the mappings are smooth and the second step gives the proof in the general case.

3.1.1 Proof of Theorem 2 when the f_i are smooth

Let $M := \prod_{i \in I} G^{k_i}(V_i)$ and define $f : M \rightarrow \prod_{i \in I} V_i^{k_i}$ by

$$f(x) = (\text{proj}_{(x_i^{k_i})^\perp} f_i(x))_{i \in I} \text{ for } x = (x_i)_{i \in I} \in M$$

and the subsets Z , Z_1 and Z_2 of $M \times \prod_{i \in I} V_i^{k_i}$ by

$$Z = \{(x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid \forall i \in I, y_i \in (x_i^{k_i})^\perp\},$$

$$Z_1 = \{(x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = f(x)\},$$

$$Z_2 = \{(x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = 0\}.$$

Proving Theorem 2 amounts to show the existence of $\bar{x} \in M$ such that $f(\bar{x}) = 0$, or equivalently such that $Z_1 \cap Z_2 \neq \emptyset$. For this, we will use the following Intersection Theorem, which is a direct consequence of mod2 intersection theory (see, for example, [10] p.79 and [5] p.127).

Intersection Theorem *Let Z be a smooth boundaryless manifold of dimension $2m$, let Z_1, Z_2 be two compact boundaryless submanifolds of Z of dimension m . If \bar{Z}_1 is a compact boundaryless m -submanifold of Z homotopic to Z_1 and if the manifolds \bar{Z}_1 and Z_2 intersect transversally in a unique point \bar{z} (which means that $T_{\bar{z}}\bar{Z}_1 + T_{\bar{z}}Z_2 = T_{\bar{z}}Z$), then $Z_1 \cap Z_2 \neq \emptyset$.*

The proof of Theorem 2 consists to check that the above defined sets Z , Z_1 and Z_2 (together with the set \bar{Z}_1 defined below) satisfy the assumptions of the Intersection Theorem.

The sets Z , Z_1 and Z_2 satisfy the assumptions of the Intersection Theorem. Let us recall that for every $i \in I$, $G^{k_i}(V_i)$ is a smooth, boundaryless, compact, manifold of dimension $k_i(\dim V_i - k_i)$ (see Lemma 3 in the Appendix). Thus $M := \prod_{i \in I} G^{k_i}(V_i)$ is a boundaryless, smooth compact, manifold of dimension $m = \sum_{i \in I} k_i(\dim V_i - k_i)$. Clearly Z is a fiber bundle whose base space is M and whose fiber at $x = (x_i)_{i \in I} \in M$ is the vector space $\prod_{i \in I} (x_i^{k_i})^\perp$ which has the dimension of M . Hence, Z is a smooth manifold of dimension $2m$.

The mapping $f : M \rightarrow \prod_{i \in I} V_i^{k_i}$ is a smooth mapping from Parts (c), (d) and (e) of Lemma 3 in the Appendix. Consequently, Z_1 is a smooth

compact boundaryless submanifold of Z of dimension m . Finally, Z_2 is clearly a smooth boundaryless compact submanifold of Z of dimension m .

The manifold Z_1 is homotopic to the manifold \bar{Z}_1 that we now define. For every $i \in I$, let $\bar{x}_i \in G^{k_i}(V_i)$ and let $\{\bar{e}_i^1, \dots, \bar{e}_i^{k_i}\}$ be an orthonormal basis of \bar{x}_i . For every $i \in I$, let $g_i : G^{k_i}(V_i) \rightarrow V_i^{k_i}$ and $g : M \rightarrow \prod_{i \in I} V_i^{k_i}$ be the mappings defined by:

$$\forall x_i \in G^{k_i}(V_i), g_i(x_i) = (\text{proj}_{x_i^\perp}(\bar{e}_i^1), \dots, \text{proj}_{x_i^\perp}(\bar{e}_i^{k_i})) \in (x_i^\perp)^{k_i} = (x_i^{k_i})^\perp$$

$$\forall x = (x_i)_{i \in I} \in M, g(x) = (g_i(x_i))_{i \in I}.$$

We let

$$\bar{Z}_1 := \{(x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = g(x)\}.$$

To show that the manifold Z_1 is homotopic to \bar{Z}_1 , we let $H : [0, 1] \times Z_1 \rightarrow Z$ be the continuous mapping defined by $H(t, (x, f(x))) = (x, (1-t)f(x) + tg(x))$. Then $H(0, \cdot)$ is the canonical inclusion from Z_1 to Z , and $H(1, \cdot)(Z_1) = \bar{Z}_1$.

The manifolds \bar{Z}_1 and Z_2 intersect transversally in a unique point. First, notice that $\bar{Z}_1 \cap Z_2 = \{(x, 0) \in M \times \prod_{i \in I} V_i^{k_i} \mid g(x) = 0\}$ is the singleton $(\bar{x}, 0) = ((\bar{x}_i)_{i \in I}, 0)$. But \bar{Z}_1 and Z_2 intersects each other transversally in Z means that $T_{(\bar{x}, 0)}\bar{Z}_1 + T_{(\bar{x}, 0)}Z_2 = T_{(\bar{x}, 0)}Z$. Recalling that $\dim T_{(\bar{x}, 0)}\bar{Z}_1 + \dim T_{(\bar{x}, 0)}Z_2 = \dim T_{(\bar{x}, 0)}Z = 2m$ we only have to show that $T_{(\bar{x}, 0)}\bar{Z}_1 \cap T_{(\bar{x}, 0)}Z_2 = \{0\}$. Finally, noticing that $T_{(\bar{x}, 0)}\bar{Z}_1 = \{(u, Dg(\bar{x})(u)) \mid u \in T_{\bar{x}}M\}$ and $T_{(\bar{x}, 0)}Z_2 = \{(u, 0) \mid u \in T_{\bar{x}}M\}$ we only have to prove that $Dg(\bar{x})$ is injective which is proved in the following lemma:

Lemma 1 $Dg(\bar{x})$ is injective.

Proof. Recalling that for every $x = (x_i)_{i \in I} \in M, g(x) = (g_i(x_i))_{i \in I}$, the mapping $Dg(\bar{x})$ is injective if and only if for every $i \in I, Dg_i(\bar{x}_i)$ is injective.

So, let $i \in I$, and let (φ, U) be a local chart of $G^{k_i}(V_i)$ at \bar{x}_i and let $\psi : (\bar{x}_i^\perp)^{k_i} \rightarrow G^{k_i}(V_i)$ be the inverse mapping of $\varphi : U \rightarrow (\bar{x}_i^\perp)^{k_i}$. From the Appendix, if $\{\bar{e}_i^1, \dots, \bar{e}_i^{k_i}\}$ is a given orthonormal basis of \bar{x}_i , ψ can be defined by:

$$\psi(u^1, \dots, u^{k_i}) = \text{span} \{\bar{e}_i^1 + u^1, \dots, \bar{e}_i^{k_i} + u^{k_i}\} \text{ for every } (u^1, \dots, u^{k_i}) \in (\bar{x}_i^\perp)^{k_i}.$$

The mapping $g_i \circ \psi$ being the local representation g_i in the chart (φ, U) , proving that $Dg_i(\bar{x}_i)$ is injective amounts to proving that $D(g_i \circ \psi)(0)$ is injective. This is a consequence of the following claim.

Claim 1 $\forall (h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}, D(g_i \circ \psi)(0)(h_1, \dots, h_{k_i}) = -(h_1, \dots, h_{k_i})$.

Proof of Claim 1. Let $p : V_i \times (\bar{x}_i^\perp)^{k_i} \rightarrow V_i$ be defined by

$$p(y, u) := \text{proj}_{\psi(u)} y.$$

If we prove that for every $y \in V_i$, the derivative of the mapping $p_y : u \rightarrow p(y, u)$ is the linear mapping $Dp_y(0) : (\bar{x}_i^\perp)^{k_i} \rightarrow V_i$ defined by

$$Dp_y(0)(h) = \sum_{k=1}^{k_i} (y \cdot \bar{e}_i^k) h_k, \forall h = (h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$$

then Claim 1 will be proved. Indeed, taking $y = \bar{e}_i^k$ for every $k = 1, \dots, k_i$, we would obtain $D_{\bar{e}_i^k} p(0)(h_1, \dots, h_{k_i}) = h_k$. Thus, since $g_i \circ \psi(u) = (\bar{e}_i^1, \dots, \bar{e}_i^{k_i}) - (p_{\bar{e}_i^1}(u), \dots, p_{\bar{e}_i^{k_i}}(u))$, it would entail Claim 1.

Now, for every $u = (u_1, \dots, u_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$, there exists $\lambda(y, u) = (\lambda_k(y, u))_{k=1, \dots, k_i} \in R^{k_i}$ such that

$$p(y, u) = \text{proj}_{\psi(u)} y = \sum_{k=1}^{k_i} \lambda_k(y, u) (\bar{e}_i^k + u^k),$$

with $(\lambda_k(y, u))$ satisfying

$$\left(-y + \sum_{k=1}^{k_i} \lambda_k(y, u) (\bar{e}_i^k + u^k)\right) \cdot (\bar{e}_i^j + u^j) = 0 \text{ for every } j = 1, \dots, k_i.$$

This can be equivalently rewritten as follows

$$(I_{k_i} + G(u))\lambda(y, u) = (y \cdot (\bar{e}_i^1 + u^1), \dots, y \cdot (\bar{e}_i^{k_i} + u^{k_i})),$$

where I_{k_i} is the $k_i \times k_i$ identity matrix, and $G(u)$ is the $k_i \times k_i$ matrix $G(u) = (u^j \cdot u^k)_{j, k=1, \dots, k_i}$. Besides, for u in a neighborhood \mathcal{N} of 0 small enough, the matrix $(I_{k_i} + G(u))$ is invertible. Consequently, the mapping $\lambda(\cdot, \cdot)$ is smooth on $V \times \mathcal{N}$, which implies that the mapping $p(\cdot, \cdot)$ is smooth on $V \times \mathcal{N}$. Differentiating with respect to u , the above equality at $u = 0$, we obtain, for every $h = (h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$:

$$DG(0)(h)\lambda(y, 0) + D_u \lambda(y, 0)(h) = 0.$$

But it is clear that $DG(0) = 0$. Consequently, $D_u \lambda(y, 0) = 0$.

Finally, differentiating the equality $p(y, u) = \sum_{k=1}^{k_i} \lambda_k(y, u) (\bar{e}_i^k + u^k)$ at $(y, 0)$, one obtain, for every $h = (h_k)_{k=1}^{k_i} \in (\bar{x}_i^\perp)^{k_i}$

$$D_u p(y, 0)(h) = \sum_{k=1}^{k_i} \lambda_k(y, 0) h_k = \sum_{k=1}^{k_i} (y \cdot \bar{e}_i^k) h_k,$$

which ends the proof of Claim 1. \square

3.1.2 Proof of Theorem 2 in the general case

Since M is a compact manifold and $V_i^{k_i}$ is a Euclidean space, for every $i \in I$, each continuous mapping $f_i : M \rightarrow V_i^{k_i}$ can be approximated by a sequence of smooth mappings $f_i^n : M \rightarrow V_i^{k_i}$ converging to f_i , in the sense that $\lim_{n \rightarrow \infty} \|f_i^n - f_i\|_\infty = 0$ (see, for example, Hirsch [11]). Applying Step 1 to the smooth mappings f_i^n , we deduce the existence of $(x_i^n)_{i \in I} \in M$ such that

$$\forall i \in I, f_i^n(x_i^n) \in (x_i^n)^{k_i}$$

or equivalently

$$\text{proj}_{(x_i^n)^\perp} f_i(x_i^n) = 0.$$

From the compactness of M , without any loss of generality, one can suppose that the sequence $(x_i^n)_{i \in I}$ converges to some element $(\bar{x}_i)_{i \in I} \in M$. We have:

$$\begin{aligned} \|\text{proj}_{(\bar{x}_i)^\perp} f_i(\bar{x}_i) - \text{proj}_{(\bar{x}_i^n)^\perp} f_i^n(x_i^n)\| &\leq \|\text{proj}_{(\bar{x}_i)^\perp} f_i(\bar{x}_i) - \text{proj}_{(\bar{x}_i^n)^\perp} f_i(x_i^n)\| + \\ &\quad \|f_i^n - f_i\|_\infty. \end{aligned}$$

Consequently, from the convergence of f_i^n to f_i and the continuity of the mapping $(u, v) \rightarrow \text{proj}_{(u)^\perp} v$ (see Lemma 3 in the Appendix), we obtain

$$\text{proj}_{(\bar{x}_i)^\perp} f_i(\bar{x}_i) = 0,$$

or equivalently

$$\forall i \in I, f_i(\bar{x}_i) \in ((\bar{x}_i)^\perp)^{\perp} = (\bar{x}_i)^{k_i},$$

which ends the proof of Theorem 2. \square

3.2 Proof of Theorem 1 when $J = \emptyset$

We now prove Theorem 1 when $J = \emptyset$. The proof rests on the following claim:

Claim 2 *For every $i \in I$ and every $k \in \{1, \dots, k_i\}$, there exists an u.s.c. correspondence \hat{F}_i^k from M to V_i , with nonempty convex values, such that:*

$$\forall x \in M, [F_i^k(x) \neq \emptyset] \Rightarrow [\forall y \in \hat{F}_i^k(x), \exists \lambda \in R, \lambda y \in F_i^k(x)].$$

Proof of Claim 2 Let $i \in I$ and $k \in \{1, \dots, k_i\}$. We distinguish two cases.

Assume first that F_i^k is lower semi-continuous. Let $U_i^k = \{x \in M \mid F_i^k(x) \neq \emptyset\}$. Then U_i^k is an open subset of M and $F_i^k|_{U_i^k}$ is a lower semi-continuous correspondence with non-empty convex values. From Michael [15], there exists a continuous selection f_i^k of $F_i^k|_{U_i^k}$, i.e., $f_i^k : U_i^k \rightarrow V_i$ is a continuous mapping such that $f_i^k(x) \in F_i^k(x)$ for every $x \in U_i^k$. Let B_i be the closed unit ball of V_i , and let us define the correspondence \hat{F}_i^k

from M to B_i by $\hat{F}_i^k(x) = \{f_i^k(x)/\|f_i^k(x)\|\}$ if $[x \in U_i^k$ and $f_i^k(x) \neq 0]$ and $\hat{F}_i^k(x) = B_i$ otherwise. We let the reader check that the correspondence \hat{F}_i^k satisfies the conclusion of Claim 2.

We now consider the case where F_i^k is upper semi-continuous. Let $U_i^k = \{x \in M \mid F_i^k(x) \neq \emptyset\}$. Then U_i^k is a closed subset of M . From Cellina [4], one can extend $F_i^k|_{U_i^k}$ as follows: there exists a correspondence \hat{F}_i^k from M to V_i which is upper semi-continuous, with non-empty, convex and compact values, such that for every $x \in U_i^k$, $F_i^k(x) = \hat{F}_i^k(x)$. \square

We now come back to the proof of Theorem 1 when $J = \emptyset$. For every $i \in I$ and $k = 1, \dots, k_i$, let \hat{F}_i^k be the u.s.c. correspondence from M to V_i with nonempty convex (compact) values defined in Claim 2. From Cellina ([3]), for every integer n there exists a continuous mapping $f_i^{k,n} : M \rightarrow V_i$ such that

$$G(f_i^{k,n}) \subset G(\hat{F}_i^k) + B(0, \frac{1}{n}).$$

Now, for $i \in I$, let $f_i^n : M \rightarrow (V_i)^{k_i}$ be defined by

$$\forall x \in M, f_i^n(x) = (f_i^{1,n}(x), \dots, f_i^{k_i,n}(x)).$$

Applying Theorem 2 to the mappings f_i^n , we deduce the existence of $(\bar{x}_i^n)_{i \in I} \in M$ such that for every $i \in I$, $f_i^n(\bar{x}_i^n) \in (\bar{x}_i^n)^{k_i}$, hence

$$y_i^{k,n} := f_i^{k,n}(\bar{x}_i^n) \in \bar{x}_i^n.$$

Since the correspondence \hat{F}_i^k is bounded (M is compact and \hat{F}_i^k is u.s.c.), the sequence $(y_i^{k,n})$ is bounded. Thus, without any loss of generality, one can suppose that the sequence $(y_i^{k,n})$ converges to some $y_i^k \in V_i$ when n tends to $+\infty$.

Besides, from the compactness of M , without any loss of generality, one can suppose that $(\bar{x}_i^n)_{i \in I}$ converges to $\bar{x} = (\bar{x}_i)_{i \in I} \in M$ when n tends to $+\infty$.

Moreover, from Part (d) of Lemma 3 in the Appendix and from $y_i^{k,n} \in \bar{x}_i^{k,n}$, at the limit we have

$$\forall i \in I, \forall k = 1, \dots, k_i, y_i^k \in \bar{x}_i.$$

Since the graph of \hat{F}_i^k is closed (it is upper semi-continuous with compact values) and from $G(f_i^{k,n}) \subset G(\hat{F}_i^k) + B(0, \frac{1}{n})$, one obtain

$$y_i^k \in \hat{F}_i^k(\bar{x}).$$

To end the proof we assume that $F_i^k(\bar{x}) \neq \emptyset$. Since $y_i^k \in \hat{F}_i^k(\bar{x})$, by Claim 2, there exists $\lambda \in R$ such that $\lambda y_i^k \in F_i^k(\bar{x})$. Hence $\lambda y_i^k \in F_i^k(\bar{x}) \cap \bar{x}_i \neq \emptyset$ (since $y_i^k \in \bar{x}_i$). This ends the proof of Theorem 1. \square

3.3 Proof of Theorem 1 in the general case

We first prove the following lemma:

Lemma 2 *Let C be a nonempty, convex, compact subset of a Euclidean space V . Then, there exists a continuous mapping $\rho : G^1(V \times R) \rightarrow C$ such that:*

$$\forall x \in G^1(V \times R), x \cap [C \times \{1\}] \neq \emptyset \Rightarrow x \cap [C \times \{1\}] = \{(\rho(x), 1)\}.$$

Proof. Since C is compact, it is included in a closed ball $\overline{B}(0, k)$ of V . We let $r : V \rightarrow \overline{B}(0, k+1)$ be defined by $r(u) = \alpha(\|u\|)u$ where $\alpha : R_+ \rightarrow R_+$ is defined by

$$\alpha(t) = 1 \text{ if } t \in [0, k], \alpha(t) = k+1-t \text{ if } t \in [k, k+1], \alpha(t) = 0 \text{ if } t \geq k+1.$$

Let $\pi_1 : V \times R \rightarrow V$ and $\rho : G^1(V \times R) \rightarrow C$ be defined by $\pi_1(x, t) = x$ and

$$\rho(x) = \begin{cases} \text{proj}_C \circ r \circ \pi_1(x \cap [V \times \{1\}]) & \text{if } x \cap [V \times \{1\}] \neq \emptyset, \\ \text{proj}_C(0) & \text{if } x \cap [V \times \{1\}] = \emptyset. \end{cases}$$

where $\text{proj}_C : V \rightarrow C$ denotes the projection from V to C . Then, one easily sees that ρ satisfies the conclusion of Lemma 2. \square

Proof of Theorem 1. Using Lemma 2, we first modify the correspondences F_j for every $j \in J$ and replace each nonempty compact convex set $C_j \subset V_j$ by the Grassmannian manifold $G^1(V_j \times R)$. For every $j \in J$ let $\rho_j : G^1(V_j \times R) \rightarrow C_j$ be the mapping associated to $C_j \subset V_j$ by Lemma 2. Let

$$\rho : \tilde{M} := \prod_{i \in I} G^{k_i}(V_i) \times \prod_{j \in J} G^1(V_j \times R) \rightarrow M := \prod_{i \in I} G^{k_i}(V_i) \times \prod_{j \in J} C_j$$

defined by

$$\rho(x) = ((x_i)_{i \in I}, \rho_j(x_j)_{j \in J}), \text{ for } x = ((x_i)_{i \in I}, (x_j)_{j \in J})$$

For $i \in I$ and $k = 1, \dots, k_i$, let \tilde{F}_i^k be the correspondence from \tilde{M} to V_i , defined by

$$\tilde{F}_i^k(x) = F_i^k(\rho(x)).$$

For $j \in J$, let \tilde{F}_j be the correspondence, from \tilde{M} to $V_j \times R$, defined by

$$\tilde{F}_j(x) = F_j(\rho(x)) \times \{1\}.$$

Now, applying the result proved in Section 3.2 (that is, Theorem 1 when $J = \emptyset$) to the correspondences \tilde{F}_i^k and \tilde{F}_j , there exists $x = ((x_i)_{i \in I}, (x_j)_{j \in J}) \in \tilde{M}$ such that:

- (i) [either $\tilde{F}_i^k(x) \cap x_i \neq \emptyset$ or $\tilde{F}_i^k(x) = \emptyset$] for every $i \in I$ and $i = 1, \dots, k_i$,

(ii) [either $\tilde{F}_j(x) \cap x_j \neq \emptyset$ or $\tilde{F}_j(x) = \emptyset$] for every $j \in J$.

Let $\bar{x} = \rho(x) \in M$, we end the proof by showing that it satisfies the conclusion of Theorem 1. From above, it is clearly the case for $i \in I$ and $k = 1, \dots, k_i$, i.e., we have:

(i) [either $F_i^k(x) \cap x_i \neq \emptyset$ or $F_i^k(x) = \emptyset$] for every $i \in I$ and $i = 1, \dots, k_i$.

Now, let $j \in J$. We first notice that $\tilde{F}_j(x) = \emptyset$ if and only if $F_j(x) = \emptyset$. Assume now that $\tilde{F}_j(x) \cap x_j \neq \emptyset$ and recall that $x_j \cap \tilde{F}_j(x) = x_j \cap (F_j(\bar{x}) \times \{1\})$ and $F_j(\bar{x}) \subset C_j$. Consequently, $x_j \cap (C_j \times \{1\}) \neq \emptyset$ and from Lemma 2 we get

$$\emptyset \neq x_j \cap (F_j(\bar{x}) \times \{1\}) \subset x_j \cap (C_j \times \{1\}) = \{(\rho_j(x_j), 1)\}.$$

Hence, the equality holds and $\bar{x}_j = \rho_j(x_j) \in F_j(\bar{x})$. This ends the proof of Theorem 1. \square

4 Appendix: the Grassmannian manifold $G^k(V)$

Let V be a Euclidean space and let k be an integer such that $0 \leq k \leq \dim V$. In this section, we recall the properties of $G^k(V)$ which are used in this paper.

First, we recall that $G^k(V)$ is a smooth boundaryless manifold of dimension $k(\dim V - k)$ (see, for example, Hirsh [11] and Lemma 3 below). The local charts can be defined as follows. Let $\bar{E} \in G^k(V)$ and let $\{\bar{e}_1, \dots, \bar{e}_k\}$ be some given orthonormal basis of \bar{E} , we define the mapping $\psi_{\bar{E}} : (\bar{E}^\perp)^k \rightarrow G^k(V)$ by

$$\psi_{\bar{E}}(u) = \text{span} \{\bar{e}_1 + u_1, \dots, \bar{e}_k + u_k\}, \text{ for } u = (u_1, \dots, u_k) \in (\bar{E}^\perp)^k.$$

Then it is easy to check that the mapping $\psi_{\bar{E}}$ is injective (see Claim 3 below), so $\psi_{\bar{E}}$ is a bijection from $(\bar{E}^\perp)^k$ onto $U_{\bar{E}} = \psi_{\bar{E}}((\bar{E}^\perp)^k)$. We can now consider the inverse mapping $\varphi_{\bar{E}} : U_{\bar{E}} \rightarrow (\bar{E}^\perp)^k$ defined by $\varphi_{\bar{E}}(E) = \psi_{\bar{E}}^{-1}(E)$, which is clearly a bijection.

Lemma 3 (a) $G^k(V)$ is a smooth boundaryless (i.e., C^∞) manifold without boundary of dimension $k(\dim V - k)$ and $(U_{\bar{E}}, \varphi_{\bar{E}})_{\bar{E} \in G^k(V)}$ defines a C^∞ atlas of $G^k(V)$.

(b) The set $G^k(V)$ is compact.

(c) The mapping $E \rightarrow E^\perp$ from $G^k(V)$ to $G^\ell(V)$ ($\ell = \dim V - k$) is a smooth diffeomorphism.

(d) The mapping $p : V \times G^k(V) \rightarrow V$ defined by $p(x, E) = \text{proj}_E(x)$ is smooth. Hence, the set $\{(x, E) \in V \times G^k(V) \mid x \in E\}$ is a closed subset of $V \times G^k(V)$.

(e) The mapping $x \rightarrow x^p$ from $G^k(V)$ to $(G^k(V))^p$ is smooth.

We prepare the proof of the lemma with a claim.

Claim 3 Let $\bar{E} \in G^k(V)$ and let $\{\bar{e}_1, \dots, \bar{e}_k\}$ be an orthonormal basis of \bar{E} .

(a) The mapping $\psi_{\bar{E}}$ is injective.

(b) For every $u \in (\bar{E}^\perp)^k$ $\psi(0) \cap \psi(u)^\perp = \psi(0)^\perp \cap \psi(u) = \{0\}$.

Proof of Claim 3. Part (a). Let $u = (u_1, \dots, u_k)$ and $v = (v_1, \dots, v_k)$ in $(\bar{E}^\perp)^k$ such that

$$\psi_{\bar{E}}(u) = \text{span} \{\bar{e}_1 + u_1, \dots, \bar{e}_k + u_k\} = \psi_{\bar{E}}(v) = \text{span} \{\bar{e}_1 + v_1, \dots, \bar{e}_k + v_k\}.$$

Then, there exist some real numbers λ_{ij} ($i = 1, \dots, k, j = 1, \dots, k$) such that

$$\bar{e}_i + u_i = \sum_{j=1}^k \lambda_{ij}(\bar{e}_j + v_j) \text{ for every } i = 1, \dots, k.$$

Taking, for each inequality, the scalar product with \bar{e}_l , where $l = 1, \dots, k$, we obtain $\lambda_{il} = 0$ if $i \neq l$ and $\lambda_{il} = 1$. Hence $u_l = v_l$ for every $l = 1, \dots, k$, and finally $u = v$.

Part (b). Let $x \in \bar{E} \cap \psi(u)^\perp$. Then there exists some real numbers λ_i ($i = 1, \dots, k$) such that $x = \sum_{i=1}^k \lambda_i \bar{e}_i$. Taking the scalar product with $\bar{e}_j + u_j$ ($j = 1, \dots, k$), we obtain: $\lambda_j = 0$ for every $j = 1, \dots, k$ which proves $\psi(0) \cap \psi(u)^\perp = \{0\}$. Similarly, let $x \in \bar{E}^\perp \cap \psi(u)$. Then there exists some real numbers λ_i ($i = 1, \dots, k$) such that $x = \sum_{i=1}^k \lambda_i(\bar{e}_i + u_i)$. Taking the scalar product with \bar{e}_j ($j = 1, \dots, k$), we obtain: $\lambda_j = 0$ for every $j = 1, \dots, k$ which proves $\psi(0) \cap \psi(u)^\perp = \{0\}$ and ends the proof of the claim. \square

Proof of Lemma 3. Part (a). We prove that $(U_E, \varphi_E)_{E \in G^k(V)}$ is a smooth (i.e., C^∞) atlas of $G^k(V)$ and it is then clear that $\dim M = \dim(E^\perp)^k = k(\dim V - k)$. Let (U_E, φ_E) and (U_F, φ_F) be two local charts at E and F , respectively, such that $U_E \cap U_F \neq \emptyset$. We shall prove that $\varphi_F \circ \varphi_E^{-1}$ is smooth (i.e., C^∞). We let $\{e_1, \dots, e_k\}$, and $\{f_1, \dots, f_k\}$, be two orthonormal basis, respectively of E and F . Let $(v_1, \dots, v_k) = \varphi_F \circ \varphi_E^{-1}(u_1, \dots, u_k)$ and $u = (u_1, \dots, u_k)$, then there exist real numbers $\lambda_{ij}(u)$ ($i, j = 1, \dots, k$) such that

$$f_i + v_i = \sum_{j=1}^k \lambda_{ij}(u)(e_j + u_j) \quad (i = 1, \dots, k).$$

The proof will be complete by showing that the real-valued functions $\lambda_{ij}(u)$ are differentiable with respect to u . Taking the scalar product with f_l ($l = 1, \dots, k$) we obtain

$$f_l \cdot f_l = \sum_{j=1}^k \lambda_{lj}(u)(e_j + u_j) \cdot f_l \quad (l = 1, \dots, k).$$

Thus for every $i = 1, \dots, k$, the vector $\lambda_i(u) = (\lambda_{ij}(u))_{j=1}^k$ is the solution of a linear system whose matrix $G(u) = ((e_j + u_j) \cdot f_l)_{j,l=1,\dots,k}$ is now shown to be invertible (which clearly implies the differentiability of $\lambda_i(u)$). Indeed, if $G(u)\lambda = 0$ for some $\lambda \in R^k$ then $\sum_{j=1}^k \lambda_j(e_j + u_j) \cdot f_l = 0$ (for $l = 1, \dots, k$), thus $\sum_{j=1}^k \lambda_j(e_j + u_j) \in F^\perp$. Besides, since $\sum_{j=1}^k \lambda_j(e_j + u_j) \in \psi_E(u_1, \dots, u_j) = \psi_F(v_1, \dots, v_j)$, we finally obtain $\sum_{j=1}^k \lambda_j(e_j + u_j) \in F^\perp \cap \psi_F(v_1, \dots, v_k) = \{0\}$ [from Claim 3]. Now, since $(e_j + u_j)_{j=1,\dots,k}$ is a basis we obtain $\lambda_j = 0$ for every $j = 1, \dots, k$. \square

Part (b). Let (E^ν) be a sequence in $G^k(V)$ and, for every ν , let $\{e_1^\nu, \dots, e_k^\nu\}$ an orthonormal basis of E^ν . Without any loss of generality, we can assume that, for every $i = 1, \dots, k$ the sequence (e_i^ν) converges to some element e_i in V . Clearly, $\{e_1, \dots, e_k\}$ is an orthonormal family in V and we let $E = \text{span}\{e_1, \dots, e_k\}$. We shall now prove that the sequence (E^ν) converges to E . Indeed, for ν large enough, there exists $u^\nu = (u_1^\nu, \dots, u_k^\nu) \in (E^\perp)^\perp$ such that $E^\nu = \psi_E(u_1^\nu, \dots, u_k^\nu)$. It can be written $e_i + u_i^\nu = \sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu$ ($i = 1, \dots, k$). Multiplying these equalities by e_l ($l = 1, \dots, k$) we obtain: $\sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu \cdot e_l = 0$ if $i \neq l$ and $\sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu \cdot e_i = 1$. This can be written (for every $i = 1, \dots, k$) $(e_j^\nu \cdot e_l)_{j,l=1,\dots,k} (\lambda_{ij}^\nu)_{j=1}^k = (e_i \cdot e_l)_{l=1,\dots,k}$. If ν is large enough then $(e_j^\nu \cdot e_l)_{j,l=1,\dots,k}$ is invertible and converges to Id, which prove that the sequence $(\lambda_{ij}^\nu)_{j=1}^k$ converges to $(e_i \cdot e_l)_{l=1,\dots,k}$ for every $i = 1, \dots, k$, i.e (λ_{ii}^ν) converges to 1 and (λ_{ij}^ν) converges to 0 for $i \neq j$. We finally obtain that (u_i^ν) converges to 0, which proves that E^ν converges to E . \square

Part (c). Let $\bar{E} \in G^k(V)$, and let $(\bar{e}_1, \dots, \bar{e}_k)$, and $(\bar{f}_1, \dots, \bar{f}_\ell)$ be orthonormal basis, respectively, of \bar{E} and \bar{E}^\perp . Let $(u_1, \dots, u_k) \in (\bar{E}^\perp)^\perp$ and let $E = \psi(u)$. Then it is easy to see that $E^\perp = \text{span}\{f_1 + v_1, \dots, f_\ell + v_\ell\}$ where $v_i = -\sum_{j=1}^k (f_i \cdot u_j) \bar{e}_j$. So each v_i is a smooth mapping with respect to the u_i and conversely, from $(E^\perp)^\perp = E$, each u_i is a smooth mapping with respect to the v_i . This ends the proof of Part (c). \square

Part (d). The differentiability of the mapping p is left to the reader. Then notice that $\{(x, E) \in V \times G^k(V) \mid x \in E\} = \{(x, E) \in V \times G^k(V) \mid x = \text{proj}_E(x)\}$, which is clearly closed since the mapping $p : (x, E) \rightarrow \text{proj}_E(x)$ is continuous. \square

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