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**Fixed-point-like theorems on subspaces**

Philippe BICH  
Bernard CORNET

**2004.64**



# Fixed-point-like theorems on subspaces

Philippe Bich and Bernard Cornet

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**Abstract:** In this paper, we prove a fixed-point-like theorem for multi-valued mappings defined on the finite Cartesian product of Grassmannian manifolds and convex sets. Our result generalizes two different kinds of theorems: the fixed-point-like theorem by Hirsh, Magill, Mas-Colell (1990) or Husseiny, Lasry, Magill (1990) and the fixed-point theorem by Gale, Mas-Colell (1975) (which generalizes Kakutani's theorem (1941)).

**Key words:** fixed-point, Grassmannian manifold, degree theory.

**AMS classification:** 47H04,47H10,47H11.

## 1 Introduction

In this paper, we prove a fixed-point-like theorem for multi-valued mappings defined on the finite Cartesian product of Grassmannian manifolds and convex sets. Let  $k$  be an integer and let  $V$  be a Euclidean space such that  $0 \leq k \leq \dim V$ , then the  $k$ -Grassmannian manifold of  $V$ , denoted  $G^k(V)$ , is the set of all the  $k$ -dimensional subspaces of  $V$ . The set  $G^k(V)$  is a smooth compact manifold but, in general, it does not satisfy properties such as convexity or acyclicity and its Euler characteristic may be null. This prevents from using classical fixed-point theorems as Brouwer's [2], Kakutani's [14], or Eilenberg-Montgomery's theorem [7].

Our main result generalizes two different kinds of theorems: the fixed-point-like theorem by Hirsh, Magill, Mas-Colell [12] or Husseiny, Lasry, Magill [13] and the fixed-point theorem by Gale, Mas-Colell [8] (which generalizes Kakutani's Theorem [14]). As in [12] and [13], we shall mainly use techniques from degree theory. As a consequence of our main result we first deduce the standard fixed-point theorems when the variable is in a convex domain (such as Brouwer's and Kakutani's theorem) and second Borsuk-Ulam's theorem.

The main result of this paper is directly motivated by the existence problem of equilibria in economic models with incomplete markets; in Bich-Cornet [1], it is used to extend the classical existence result by Duffie-Shaffer [6] to the non-transitive case.

The paper is organized as follows. The main result is stated in Section 2 together with some direct consequences of it, namely, the results by Hirsh, Magill, Mas-Colell [12], Gale, Mas-Colell [8] and Borsuk-Ulam's Theorem. The proof of the main result is given in Section 3 and the Appendix recalls the main properties of the Grassmannian manifold, used in this paper.

## 2 Statement of the results

### 2.1 Preliminaries

A correspondence  $\Phi$  from a set  $X$  to a set  $Y$  is a map from  $X$  to the set of all the subsets of  $Y$  and the graph of  $\Phi$ , denoted  $G(\Phi)$ , is defined by  $G(\Phi) = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}$ . A mapping  $\varphi : X \rightarrow Y$  is said to be a selection of  $\Phi$  if  $\varphi(x) \in \Phi(x)$  for all  $x \in X$ . If  $A$  is a subset of  $X$ , we let  $\Phi(A) = \cup_{x \in A} \Phi(x)$  and the restriction of  $\Phi$  to  $A$ , denoted  $\Phi|_A$ , is the correspondence from  $A$  to  $Y$  defined by  $\Phi|_A(x) = \Phi(x)$  if  $x \in A$ . If  $X$  and  $Y$  are topological spaces, the correspondence  $\Phi$  is said to be lower semi-continuous (l.s.c.) [resp. upper semi-continuous (u.s.c.)] if for every open set  $U \subset Y$ , the set  $\{x \in X \mid \Phi(x) \cap U \neq \emptyset\}$  is open in  $X$  [resp. the set  $\{x \in X \mid \Phi(x) \subset U\}$  is open in  $X$  and, for every  $x \in X$ ,  $\Phi(x)$  is compact].

If  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  belong to  $R^n$ , we denote by  $x \cdot y = \sum_{i=1}^n x_i y_i$ , the scalar product of  $R^n$ ,  $\|x\| = \sqrt{x \cdot x}$ , the Euclidian norm. If  $x \in R^n$  and  $r \in R_+$ , we let  $B(x, r) = \{y \in R^n \mid \|x - y\| < r\}$  and  $\bar{B}(x, r) = \{y \in R^n \mid \|x - y\| \leq r\}$ . If  $E$  is a vector subspace of  $R^n$ , we denote by  $E^\perp = \{u \in R^n \mid \forall x \in E, x \cdot u = 0\}$  the orthogonal space to  $E$ . If  $u_1, \dots, u_k$  belong to  $E$ , a vector space, we denote by  $\text{span}\{u_1, \dots, u_k\}$  the vector subspace of  $E$  spanned by  $u_1, \dots, u_k$ .

Let  $V$  be a Euclidean space and let  $k$  be an integer such that  $0 \leq k \leq \dim V$ , we denote by  $G^k(V)$  the set consisting of all the linear subspaces of  $V$  of dimension  $k$ , called the ( $k$ -)Grassmannian manifold of  $V$ . Then it is known that  $G^k(V)$  is a smooth manifold of dimension  $k(\dim V - k)$  and we refer to the Appendix for the properties we shall use hereafter, together with the precise definition of the manifold structure on  $G^k(V)$ .

### 2.2 The main result and some consequences

The aim of this paper is to prove the following result:

**Theorem 1** *Let  $I, J$  be two finite disjoint sets. For every  $i \in I$ , let  $k_i$  be an integer and let  $V_i$  be a Euclidean space such that  $0 \leq k_i \leq \dim V_i$ . For every  $j \in J$ , let  $C_j$  be a nonempty, convex, compact subset of a Euclidean space  $V_j$ , and let  $M = \prod_{i \in I} G^{k_i} \times \prod_{j \in J} C_j$ .*

*For  $i \in I$  and  $k = 1, \dots, k_i$ , let  $F_i^k$  be a correspondence from  $M$  to  $V_i$  with convex values, for  $j \in J$ , let  $F_j$  be a correspondence from  $M$  to  $C_j$*

with convex values, and suppose that, for every  $i \in I$  and  $k = 1, \dots, k_i$  [resp.  $j \in J$ ], the correspondence  $F_i^k$  [resp.  $F_j$ ] is either l.s.c or u.s.c.

Then, there exists  $\bar{x} = ((\bar{x}_i)_{i \in I}, (\bar{x}_j)_{j \in J}) \in M$  such that:

- (i) [either  $F_i^k(\bar{x}) \cap \bar{x}_i \neq \emptyset$  or  $F_i^k(\bar{x}) = \emptyset$ ] for every  $i \in I$  and  $k = 1, \dots, k_i$ ;
- (ii) [either  $F_j(\bar{x}) \cap \{\bar{x}_j\} \neq \emptyset$  or  $F_j(\bar{x}) = \emptyset$ ] for every  $j \in J$ .

The proof of Theorem 1 is given in Section 3. A first consequence of Theorem 1 is the following theorem by Hirsch et al. ([12]).

**Corollary 1** *Let  $V_1$  be a Euclidean space, let  $k_1$  be an integer such that  $0 \leq k_1 \leq \dim V_1$ , and for every  $k = 1, \dots, k_1$  let  $f^k : G^{k_1}(V_1) \rightarrow V_1$  be a continuous mapping. Then, there exists  $\bar{x} \in G^{k_1}(V_1)$  such that: for every  $k = 1, \dots, k_1$ ,  $f^k(\bar{x}) \in \bar{x}$ .*

**Proof.** Take  $I = \{1\}$ ,  $J = \emptyset$  and  $F_1^k(x) = \{f^k(x)\}$  for every  $x \in G^{k_1}(V_1)$  and for every  $k = 1, \dots, k_1$ . From Theorem 1, there exists  $\bar{x} \in M = G^{k_1}(V_1)$  such that for every  $k = 1, \dots, k_1$ ,  $F_1^k(\bar{x}) \cap \bar{x} \neq \emptyset$ , i.e.,  $f^k(\bar{x}) \in \bar{x}$ .  $\square$

A second consequence of Theorem 1 is the following generalization of Gale and Mas-Colell Theorem [8], which is also a generalization of Kakutani's Theorem. Hereafter, we use the formulation by Gourdél [9] allowing each correspondence to be either upper semi-continuous or lower semi-continuous.

**Corollary 2** *Let  $J$  be a finite set, for  $j \in J$ , let  $C_j$  be a nonempty, convex, compact subset of a Euclidean space, let  $F_j$  be a correspondence from  $M := \prod_{j \in J} C_j$  to  $C_j$  with convex values, such that the correspondence  $F_j$  is either l.s.c or u.s.c.. Then, there exists  $\bar{x} = (\bar{x}_j)_{j \in J} \in M$  such that:*

*for every  $j \in J$ , [either  $\bar{x}_j \in F_j(\bar{x})$  or  $F_j(\bar{x}) = \emptyset$ ].*

**Proof.** Take  $I = \emptyset$  and apply Theorem 1.  $\square$

**Remark 1** According to our definition, an u.s.c. correspondence has compact values and without this requirement, Theorem 1 may not be true, as we can see in the following counterexample. Let  $M := G^1(\mathbb{R}^2)$ . Each element  $D$  of  $G^1(\mathbb{R}^2)$  can be written as  $D_t = \{\lambda(\cos t, \sin t) \mid \lambda \in \mathbb{R}\}$ , for some  $t \in [0, \pi[$ . Let us define the correspondence  $F$  from  $M$  to  $\mathbb{R}^2$  by  $F(D_0) = \mathbb{R} \times \{1\}$  and  $F(D_t) = D_t \cap (\mathbb{R} \times \{1\}) + \{(1, 0)\}$  if  $t \in ]0, \pi[$ . We let the reader check that for every open set  $U \subset \mathbb{R}^2$  the set  $\{x \in M \mid F(x) \subset U\}$  is open in  $M$  and that  $F$  has nonempty, convex (and closed) values. Yet, it is straightforward that  $F(x) \cap x = \emptyset$  for every  $x \in G^1(\mathbb{R}^2)$ .  $\square$

Another consequence of our main result is the following multi-valued version of Borsuk-Ulam's theorem. We denote by  $S^n$  the unit sphere of  $\mathbb{R}^{n+1}$ .

**Corollary 3** For  $k = 1, \dots, n$ , let  $F^k$  be a correspondence from  $S^n$  to  $R$  with nonempty and convex values, such that, for every  $k = 1, \dots, n$ ,  $F^k$  is either l.s.c or u.s.c.. Then, there exists  $\bar{x} \in S^n$  such that:

$$\forall k \in \{1, \dots, n\}, F^k(\bar{x}) \cap F^k(-\bar{x}) \neq \emptyset.$$

**Proof.** For every  $k = 1, \dots, n$ , let  $\hat{F}^k$  be the correspondence from  $S^n$  to  $R$  defined by

$$\hat{F}^k(x) = \{u - v \mid u \in F^k(x), v \in F^k(-x)\}.$$

We let the reader check that for every  $k = 1, \dots, n$ , the correspondence  $\hat{F}^k$  has nonempty, convex values and that it is u.s.c. [resp. l.s.c.] if  $F^k$  is u.s.c. [resp. l.s.c.]. So, to prove Corollary 3, it suffices to show the existence of  $\bar{x} \in S^n$  such that  $0 \in \hat{F}^k(\bar{x})$  for every  $k = 1, \dots, n$ .

Let us define, for every  $k = 1, \dots, n$  the correspondence  $H^k$ , from  $G^n(R^{n+1})$  to  $R^{n+1}$ , as follows: for every  $E \in G^n(R^{n+1})$ , we let  $H^k(E) = \hat{F}^k(x)x$ , where  $x$  is an arbitrary element of  $E^\perp \cap S^n$ . The correspondence  $H^k$  is well defined since  $E^\perp \cap S^n = \{x, -x\}$  for some element  $x \in S^n$  and since  $\hat{F}^k(x)x = \hat{F}^k(-x)(-x)$ .

Take  $I = \{1\}$ ,  $V_1 = R^{n+1}$ ,  $k_1 = n$ ,  $J = \emptyset$ , and apply Theorem 1 to the correspondences  $H^k$ , which clearly satisfy the assumptions of Theorem 1. So there exists  $\bar{E} \in G^n(R^{n+1})$  such that  $\bar{E} \cap H^k(\bar{E}) \neq \emptyset$  for every  $k = 1, \dots, n$ .

Now, if  $\bar{x}$  is an arbitrary point of  $\bar{E}^\perp \cap S^n$  then we have  $\bar{E} \cap \hat{F}^k(\bar{x})\bar{x} \neq \emptyset$ ; from  $\bar{x} \in \bar{E}^\perp$  and  $\bar{x} \neq 0$ , we finally obtain  $0 \in \hat{F}^k(\bar{x})$  for every  $k = 1, \dots, n$ , which ends the proof of Corollary 3.  $\square$

## 3 Proof of Theorem 1

The proof is given in three steps, corresponding to the following three subsections. The first step gives the proof under the additional assumption that  $J = \emptyset$  and the correspondences  $F_i^k$  are single-valued. The second step only assumes in addition that  $J = \emptyset$ . Finally, the third step gives the proof under the assumptions of Theorem 1.

### 3.1 Proof when $J = \emptyset$ and $F_i^k$ is single-valued

We first prove Theorem 1 under the additional assumptions that  $J = \emptyset$  and the  $F_i^k$  are single-valued. This is exactly the statement below:

**Theorem 2** Let  $I$  be a finite set and for  $i \in I$ , let  $k_i$  be an integer and let  $V_i$  be a Euclidean space such that  $0 \leq k_i \leq \dim V_i$ . Let  $M := \prod_{i \in I} G^{k_i}(V_i)$  and for  $i \in I$ , let  $f_i : M \rightarrow (V_i)^{k_i}$  be a continuous mapping. Then, there exists  $\bar{x} = (\bar{x}_i)_{i \in I} \in M$  such that

$$\forall i \in I, f_i(\bar{x}) \in (\bar{x}_i)^{k_i}.$$

The proof of Theorem 2 is given in two steps. In the first step we additionally assume that the mappings are smooth and the second step gives the proof in the general case.

### 3.1.1 Proof of Theorem 2 when the $f_i$ are smooth

Let  $M := \prod_{i \in I} G^{k_i}(V_i)$  and define  $f : M \rightarrow \prod_{i \in I} V_i^{k_i}$  by

$$f(x) = (\text{proj}_{(x_i^{k_i})^\perp} f_i(x))_{i \in I} \text{ for } x = (x_i)_{i \in I} \in M$$

and the subsets  $Z$ ,  $Z_1$  and  $Z_2$  of  $M \times \prod_{i \in I} V_i^{k_i}$  by

$$Z = \{(x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid \forall i \in I, y_i \in (x_i^{k_i})^\perp\},$$

$$Z_1 = \{(x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = f(x)\},$$

$$Z_2 = \{(x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = 0\}.$$

Proving Theorem 2 amounts to show the existence of  $\bar{x} \in M$  such that  $f(\bar{x}) = 0$ , or equivalently such that  $Z_1 \cap Z_2 \neq \emptyset$ . For this, we will use the following Intersection Theorem, which is a direct consequence of mod2 intersection theory (see, for example, [10] p.79 and [5] p.127).

**Intersection Theorem** *Let  $Z$  be a smooth boundaryless manifold of dimension  $2m$ , let  $Z_1, Z_2$  be two compact boundaryless submanifolds of  $Z$  of dimension  $m$ . If  $\bar{Z}_1$  is a compact boundaryless  $m$ -submanifold of  $Z$  homotopic to  $Z_1$  and if the manifolds  $\bar{Z}_1$  and  $Z_2$  intersect transversally in a unique point  $\bar{z}$  (which means that  $T_{\bar{z}}\bar{Z}_1 + T_{\bar{z}}Z_2 = T_{\bar{z}}Z$ ), then  $Z_1 \cap Z_2 \neq \emptyset$ .*

The proof of Theorem 2 consists to check that the above defined sets  $Z$ ,  $Z_1$  and  $Z_2$  (together with the set  $\bar{Z}_1$  defined below) satisfy the assumptions of the Intersection Theorem.

*The sets  $Z$ ,  $Z_1$  and  $Z_2$  satisfy the assumptions of the Intersection Theorem.* Let us recall that for every  $i \in I$ ,  $G^{k_i}(V_i)$  is a smooth, boundaryless, compact, manifold of dimension  $k_i(\dim V_i - k_i)$  (see Lemma 3 in the Appendix). Thus  $M := \prod_{i \in I} G^{k_i}(V_i)$  is a boundaryless, smooth compact, manifold of dimension  $m = \sum_{i \in I} k_i(\dim V_i - k_i)$ . Clearly  $Z$  is a fiber bundle whose base space is  $M$  and whose fiber at  $x = (x_i)_{i \in I} \in M$  is the vector space  $\prod_{i \in I} (x_i^{k_i})^\perp$  which has the dimension of  $M$ . Hence,  $Z$  is a smooth manifold of dimension  $2m$ .

The mapping  $f : M \rightarrow \prod_{i \in I} V_i^{k_i}$  is a smooth mapping from Parts (c), (d) and (e) of Lemma 3 in the Appendix. Consequently,  $Z_1$  is a smooth

compact boundaryless submanifold of  $Z$  of dimension  $m$ . Finally,  $Z_2$  is clearly a smooth boundaryless compact submanifold of  $Z$  of dimension  $m$ .

The manifold  $Z_1$  is homotopic to the manifold  $\bar{Z}_1$  that we now define. For every  $i \in I$ , let  $\bar{x}_i \in G^{k_i}(V_i)$  and let  $\{\bar{e}_i^1, \dots, \bar{e}_i^{k_i}\}$  be an orthonormal basis of  $\bar{x}_i$ . For every  $i \in I$ , let  $g_i : G^{k_i}(V_i) \rightarrow V_i^{k_i}$  and  $g : M \rightarrow \prod_{i \in I} V_i^{k_i}$  be the mappings defined by:

$$\forall x_i \in G^{k_i}(V_i), g_i(x_i) = (\text{proj}_{x_i^\perp}(\bar{e}_i^1), \dots, \text{proj}_{x_i^\perp}(\bar{e}_i^{k_i})) \in (x_i^\perp)^{k_i} = (x_i^{k_i})^\perp$$

$$\forall x = (x_i)_{i \in I} \in M, g(x) = (g_i(x_i))_{i \in I}.$$

We let

$$\bar{Z}_1 := \{(x, y) \in M \times \prod_{i \in I} V_i^{k_i} \mid y = g(x)\}.$$

To show that the manifold  $Z_1$  is homotopic to  $\bar{Z}_1$ , we let  $H : [0, 1] \times Z_1 \rightarrow Z$  be the continuous mapping defined by  $H(t, (x, f(x))) = (x, (1-t)f(x) + tg(x))$ . Then  $H(0, \cdot)$  is the canonical inclusion from  $Z_1$  to  $Z$ , and  $H(1, \cdot)(Z_1) = \bar{Z}_1$ .

The manifolds  $\bar{Z}_1$  and  $Z_2$  intersect transversally in a unique point. First, notice that  $\bar{Z}_1 \cap Z_2 = \{(x, 0) \in M \times \prod_{i \in I} V_i^{k_i} \mid g(x) = 0\}$  is the singleton  $(\bar{x}, 0) = ((\bar{x}_i)_{i \in I}, 0)$ . But  $\bar{Z}_1$  and  $Z_2$  intersects each other transversally in  $Z$  means that  $T_{(\bar{x}, 0)}\bar{Z}_1 + T_{(\bar{x}, 0)}Z_2 = T_{(\bar{x}, 0)}Z$ . Recalling that  $\dim T_{(\bar{x}, 0)}\bar{Z}_1 + \dim T_{(\bar{x}, 0)}Z_2 = \dim T_{(\bar{x}, 0)}Z = 2m$  we only have to show that  $T_{(\bar{x}, 0)}\bar{Z}_1 \cap T_{(\bar{x}, 0)}Z_2 = \{0\}$ . Finally, noticing that  $T_{(\bar{x}, 0)}\bar{Z}_1 = \{(u, Dg(\bar{x})(u)) \mid u \in T_{\bar{x}}M\}$  and  $T_{(\bar{x}, 0)}Z_2 = \{(u, 0) \mid u \in T_{\bar{x}}M\}$  we only have to prove that  $Dg(\bar{x})$  is injective which is proved in the following lemma:

**Lemma 1**  $Dg(\bar{x})$  is injective.

**Proof.** Recalling that for every  $x = (x_i)_{i \in I} \in M, g(x) = (g_i(x_i))_{i \in I}$ , the mapping  $Dg(\bar{x})$  is injective if and only if for every  $i \in I, Dg_i(\bar{x}_i)$  is injective.

So, let  $i \in I$ , and let  $(\varphi, U)$  be a local chart of  $G^{k_i}(V_i)$  at  $\bar{x}_i$  and let  $\psi : (\bar{x}_i^\perp)^{k_i} \rightarrow G^{k_i}(V_i)$  be the inverse mapping of  $\varphi : U \rightarrow (\bar{x}_i^\perp)^{k_i}$ . From the Appendix, if  $\{\bar{e}_i^1, \dots, \bar{e}_i^{k_i}\}$  is a given orthonormal basis of  $\bar{x}_i$ ,  $\psi$  can be defined by:

$$\psi(u^1, \dots, u^{k_i}) = \text{span} \{\bar{e}_i^1 + u^1, \dots, \bar{e}_i^{k_i} + u^{k_i}\} \text{ for every } (u^1, \dots, u^{k_i}) \in (\bar{x}_i^\perp)^{k_i}.$$

The mapping  $g_i \circ \psi$  being the local representation  $g_i$  in the chart  $(\varphi, U)$ , proving that  $Dg_i(\bar{x}_i)$  is injective amounts to proving that  $D(g_i \circ \psi)(0)$  is injective. This is a consequence of the following claim.



**Claim 1**  $\forall (h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}, D(g_i \circ \psi)(0)(h_1, \dots, h_{k_i}) = -(h_1, \dots, h_{k_i})$ .

**Proof of Claim 1.** Let  $p : V_i \times (\bar{x}_i^\perp)^{k_i} \rightarrow V_i$  be defined by

$$p(y, u) := \text{proj}_{\psi(u)} y.$$

If we prove that for every  $y \in V_i$ , the derivative of the mapping  $p_y : u \rightarrow p(y, u)$  is the linear mapping  $Dp_y(0) : (\bar{x}_i^\perp)^{k_i} \rightarrow V_i$  defined by

$$Dp_y(0)(h) = \sum_{k=1}^{k_i} (y \cdot \bar{e}_i^k) h_k, \forall h = (h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$$

then Claim 1 will be proved. Indeed, taking  $y = \bar{e}_i^k$  for every  $k = 1, \dots, k_i$ , we would obtain  $D_{\bar{e}_i^k} p(0)(h_1, \dots, h_{k_i}) = h_k$ . Thus, since  $g_i \circ \psi(u) = (\bar{e}_i^1, \dots, \bar{e}_i^{k_i}) - (p_{\bar{e}_i^1}(u), \dots, p_{\bar{e}_i^{k_i}}(u))$ , it would entail Claim 1.

Now, for every  $u = (u_1, \dots, u_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$ , there exists  $\lambda(y, u) = (\lambda_k(y, u))_{k=1, \dots, k_i} \in R^{k_i}$  such that

$$p(y, u) = \text{proj}_{\psi(u)} y = \sum_{k=1}^{k_i} \lambda_k(y, u) (\bar{e}_i^k + u^k),$$

with  $(\lambda_k(y, u))$  satisfying

$$\left(-y + \sum_{k=1}^{k_i} \lambda_k(y, u) (\bar{e}_i^k + u^k)\right) \cdot (\bar{e}_i^j + u^j) = 0 \text{ for every } j = 1, \dots, k_i.$$

This can be equivalently rewritten as follows

$$(I_{k_i} + G(u))\lambda(y, u) = (y \cdot (\bar{e}_i^1 + u^1), \dots, y \cdot (\bar{e}_i^{k_i} + u^{k_i})),$$

where  $I_{k_i}$  is the  $k_i \times k_i$  identity matrix, and  $G(u)$  is the  $k_i \times k_i$  matrix  $G(u) = (u^j \cdot u^k)_{j, k=1, \dots, k_i}$ . Besides, for  $u$  in a neighborhood  $\mathcal{N}$  of 0 small enough, the matrix  $(I_{k_i} + G(u))$  is invertible. Consequently, the mapping  $\lambda(\cdot, \cdot)$  is smooth on  $V \times \mathcal{N}$ , which implies that the mapping  $p(\cdot, \cdot)$  is smooth on  $V \times \mathcal{N}$ . Differentiating with respect to  $u$ , the above equality at  $u = 0$ , we obtain, for every  $h = (h_1, \dots, h_{k_i}) \in (\bar{x}_i^\perp)^{k_i}$ :

$$DG(0)(h)\lambda(y, 0) + D_u \lambda(y, 0)(h) = 0.$$

But it is clear that  $DG(0) = 0$ . Consequently,  $D_u \lambda(y, 0) = 0$ .

Finally, differentiating the equality  $p(y, u) = \sum_{k=1}^{k_i} \lambda_k(y, u) (\bar{e}_i^k + u^k)$  at  $(y, 0)$ , one obtain, for every  $h = (h_k)_{k=1}^{k_i} \in (\bar{x}_i^\perp)^{k_i}$

$$D_u p(y, 0)(h) = \sum_{k=1}^{k_i} \lambda_k(y, 0) h_k = \sum_{k=1}^{k_i} (y \cdot \bar{e}_i^k) h_k,$$

which ends the proof of Claim 1.  $\square$

### 3.1.2 Proof of Theorem 2 in the general case

Since  $M$  is a compact manifold and  $V_i^{k_i}$  is a Euclidean space, for every  $i \in I$ , each continuous mapping  $f_i : M \rightarrow V_i^{k_i}$  can be approximated by a sequence of smooth mappings  $f_i^n : M \rightarrow V_i^{k_i}$  converging to  $f_i$ , in the sense that  $\lim_{n \rightarrow \infty} \|f_i^n - f_i\|_\infty = 0$  (see, for example, Hirsch [11]). Applying Step 1 to the smooth mappings  $f_i^n$ , we deduce the existence of  $(x_i^n)_{i \in I} \in M$  such that

$$\forall i \in I, f_i^n(x_i^n) \in (x_i^n)^{k_i}$$

or equivalently

$$\text{proj}_{(x_i^n)^\perp} f_i(x_i^n) = 0.$$

From the compactness of  $M$ , without any loss of generality, one can suppose that the sequence  $(x_i^n)_{i \in I}$  converges to some element  $(\bar{x}_i)_{i \in I} \in M$ . We have:

$$\begin{aligned} \|\text{proj}_{(\bar{x}_i)^\perp} f_i(\bar{x}_i) - \text{proj}_{(\bar{x}_i^n)^\perp} f_i^n(x_i^n)\| &\leq \|\text{proj}_{(\bar{x}_i)^\perp} f_i(\bar{x}_i) - \text{proj}_{(\bar{x}_i^n)^\perp} f_i(x_i^n)\| + \\ &\quad \|f_i^n - f_i\|_\infty. \end{aligned}$$

Consequently, from the convergence of  $f_i^n$  to  $f_i$  and the continuity of the mapping  $(u, v) \rightarrow \text{proj}_{(u)^\perp} v$  (see Lemma 3 in the Appendix), we obtain

$$\text{proj}_{(\bar{x}_i)^\perp} f_i(\bar{x}_i) = 0,$$

or equivalently

$$\forall i \in I, f_i(\bar{x}_i) \in ((\bar{x}_i)^\perp)^{\perp} = (\bar{x}_i)^{k_i},$$

which ends the proof of Theorem 2.  $\square$

## 3.2 Proof of Theorem 1 when $J = \emptyset$

We now prove Theorem 1 when  $J = \emptyset$ . The proof rests on the following claim:

**Claim 2** *For every  $i \in I$  and every  $k \in \{1, \dots, k_i\}$ , there exists an u.s.c. correspondence  $\hat{F}_i^k$  from  $M$  to  $V_i$ , with nonempty convex values, such that:*

$$\forall x \in M, [F_i^k(x) \neq \emptyset] \Rightarrow [\forall y \in \hat{F}_i^k(x), \exists \lambda \in R, \lambda y \in F_i^k(x)].$$

**Proof of Claim 2** Let  $i \in I$  and  $k \in \{1, \dots, k_i\}$ . We distinguish two cases.

Assume first that  $F_i^k$  is lower semi-continuous. Let  $U_i^k = \{x \in M \mid F_i^k(x) \neq \emptyset\}$ . Then  $U_i^k$  is an open subset of  $M$  and  $F_i^k|_{U_i^k}$  is a lower semi-continuous correspondence with non-empty convex values. From Michael [15], there exists a continuous selection  $f_i^k$  of  $F_i^k|_{U_i^k}$ , i.e.,  $f_i^k : U_i^k \rightarrow V_i$  is a continuous mapping such that  $f_i^k(x) \in F_i^k(x)$  for every  $x \in U_i^k$ . Let  $B_i$  be the closed unit ball of  $V_i$ , and let us define the correspondence  $\hat{F}_i^k$

from  $M$  to  $B_i$  by  $\hat{F}_i^k(x) = \{f_i^k(x)/\|f_i^k(x)\|\}$  if  $[x \in U_i^k$  and  $f_i^k(x) \neq 0]$  and  $\hat{F}_i^k(x) = B_i$  otherwise. We let the reader check that the correspondence  $\hat{F}_i^k$  satisfies the conclusion of Claim 2.

We now consider the case where  $F_i^k$  is upper semi-continuous. Let  $U_i^k = \{x \in M \mid F_i^k(x) \neq \emptyset\}$ . Then  $U_i^k$  is a closed subset of  $M$ . From Cellina [4], one can extend  $F_i^k|_{U_i^k}$  as follows: there exists a correspondence  $\hat{F}_i^k$  from  $M$  to  $V_i$  which is upper semi-continuous, with non-empty, convex and compact values, such that for every  $x \in U_i^k$ ,  $F_i^k(x) = \hat{F}_i^k(x)$ .  $\square$

We now come back to the proof of Theorem 1 when  $J = \emptyset$ . For every  $i \in I$  and  $k = 1, \dots, k_i$ , let  $\hat{F}_i^k$  be the u.s.c. correspondence from  $M$  to  $V_i$  with nonempty convex (compact) values defined in Claim 2. From Cellina ([3]), for every integer  $n$  there exists a continuous mapping  $f_i^{k,n} : M \rightarrow V_i$  such that

$$G(f_i^{k,n}) \subset G(\hat{F}_i^k) + B(0, \frac{1}{n}).$$

Now, for  $i \in I$ , let  $f_i^n : M \rightarrow (V_i)^{k_i}$  be defined by

$$\forall x \in M, f_i^n(x) = (f_i^{1,n}(x), \dots, f_i^{k_i,n}(x)).$$

Applying Theorem 2 to the mappings  $f_i^n$ , we deduce the existence of  $(\bar{x}_i^n)_{i \in I} \in M$  such that for every  $i \in I$ ,  $f_i^n(\bar{x}_i^n) \in (\bar{x}_i^n)^{k_i}$ , hence

$$y_i^{k,n} := f_i^{k,n}(\bar{x}_i^n) \in \bar{x}_i^n.$$

Since the correspondence  $\hat{F}_i^k$  is bounded ( $M$  is compact and  $\hat{F}_i^k$  is u.s.c.), the sequence  $(y_i^{k,n})$  is bounded. Thus, without any loss of generality, one can suppose that the sequence  $(y_i^{k,n})$  converges to some  $y_i^k \in V_i$  when  $n$  tends to  $+\infty$ .

Besides, from the compactness of  $M$ , without any loss of generality, one can suppose that  $(\bar{x}_i^n)_{i \in I}$  converges to  $\bar{x} = (\bar{x}_i)_{i \in I} \in M$  when  $n$  tends to  $+\infty$ .

Moreover, from Part (d) of Lemma 3 in the Appendix and from  $y_i^{k,n} \in \bar{x}_i^{k,n}$ , at the limit we have

$$\forall i \in I, \forall k = 1, \dots, k_i, y_i^k \in \bar{x}_i.$$

Since the graph of  $\hat{F}_i^k$  is closed (it is upper semi-continuous with compact values) and from  $G(f_i^{k,n}) \subset G(\hat{F}_i^k) + B(0, \frac{1}{n})$ , one obtain

$$y_i^k \in \hat{F}_i^k(\bar{x}).$$

To end the proof we assume that  $F_i^k(\bar{x}) \neq \emptyset$ . Since  $y_i^k \in \hat{F}_i^k(\bar{x})$ , by Claim 2, there exists  $\lambda \in R$  such that  $\lambda y_i^k \in F_i^k(\bar{x})$ . Hence  $\lambda y_i^k \in F_i^k(\bar{x}) \cap \bar{x}_i \neq \emptyset$  (since  $y_i^k \in \bar{x}_i$ ). This ends the proof of Theorem 1.  $\square$

### 3.3 Proof of Theorem 1 in the general case

We first prove the following lemma:

**Lemma 2** *Let  $C$  be a nonempty, convex, compact subset of a Euclidean space  $V$ . Then, there exists a continuous mapping  $\rho : G^1(V \times R) \rightarrow C$  such that:*

$$\forall x \in G^1(V \times R), x \cap [C \times \{1\}] \neq \emptyset \Rightarrow x \cap [C \times \{1\}] = \{(\rho(x), 1)\}.$$

**Proof.** Since  $C$  is compact, it is included in a closed ball  $\overline{B}(0, k)$  of  $V$ . We let  $r : V \rightarrow \overline{B}(0, k+1)$  be defined by  $r(u) = \alpha(\|u\|)u$  where  $\alpha : R_+ \rightarrow R_+$  is defined by

$$\alpha(t) = 1 \text{ if } t \in [0, k], \alpha(t) = k+1-t \text{ if } t \in [k, k+1], \alpha(t) = 0 \text{ if } t \geq k+1.$$

Let  $\pi_1 : V \times R \rightarrow V$  and  $\rho : G^1(V \times R) \rightarrow C$  be defined by  $\pi_1(x, t) = x$  and

$$\rho(x) = \begin{cases} \text{proj}_C \circ r \circ \pi_1(x \cap [V \times \{1\}]) & \text{if } x \cap [V \times \{1\}] \neq \emptyset, \\ \text{proj}_C(0) & \text{if } x \cap [V \times \{1\}] = \emptyset. \end{cases}$$

where  $\text{proj}_C : V \rightarrow C$  denotes the projection from  $V$  to  $C$ . Then, one easily sees that  $\rho$  satisfies the conclusion of Lemma 2.  $\square$

**Proof of Theorem 1.** Using Lemma 2, we first modify the correspondences  $F_j$  for every  $j \in J$  and replace each nonempty compact convex set  $C_j \subset V_j$  by the Grassmannian manifold  $G^1(V_j \times R)$ . For every  $j \in J$  let  $\rho_j : G^1(V_j \times R) \rightarrow C_j$  be the mapping associated to  $C_j \subset V_j$  by Lemma 2. Let

$$\rho : \tilde{M} := \prod_{i \in I} G^{k_i}(V_i) \times \prod_{j \in J} G^1(V_j \times R) \rightarrow M := \prod_{i \in I} G^{k_i}(V_i) \times \prod_{j \in J} C_j$$

defined by

$$\rho(x) = ((x_i)_{i \in I}, \rho_j(x_j)_{j \in J}), \text{ for } x = ((x_i)_{i \in I}, (x_j)_{j \in J})$$

For  $i \in I$  and  $k = 1, \dots, k_i$ , let  $\tilde{F}_i^k$  be the correspondence from  $\tilde{M}$  to  $V_i$ , defined by

$$\tilde{F}_i^k(x) = F_i^k(\rho(x)).$$

For  $j \in J$ , let  $\tilde{F}_j$  be the correspondence, from  $\tilde{M}$  to  $V_j \times R$ , defined by

$$\tilde{F}_j(x) = F_j(\rho(x)) \times \{1\}.$$

Now, applying the result proved in Section 3.2 (that is, Theorem 1 when  $J = \emptyset$ ) to the correspondences  $\tilde{F}_i^k$  and  $\tilde{F}_j$ , there exists  $x = ((x_i)_{i \in I}, (x_j)_{j \in J}) \in \tilde{M}$  such that:

- (i) [either  $\tilde{F}_i^k(x) \cap x_i \neq \emptyset$  or  $\tilde{F}_i^k(x) = \emptyset$ ] for every  $i \in I$  and  $i = 1, \dots, k_i$ ,

(ii) [either  $\tilde{F}_j(x) \cap x_j \neq \emptyset$  or  $\tilde{F}_j(x) = \emptyset$ ] for every  $j \in J$ .

Let  $\bar{x} = \rho(x) \in M$ , we end the proof by showing that it satisfies the conclusion of Theorem 1. From above, it is clearly the case for  $i \in I$  and  $k = 1, \dots, k_i$ , i.e., we have:

(i) [either  $F_i^k(x) \cap x_i \neq \emptyset$  or  $F_i^k(x) = \emptyset$ ] for every  $i \in I$  and  $i = 1, \dots, k_i$ .

Now, let  $j \in J$ . We first notice that  $\tilde{F}_j(x) = \emptyset$  if and only if  $F_j(x) = \emptyset$ . Assume now that  $\tilde{F}_j(x) \cap x_j \neq \emptyset$  and recall that  $x_j \cap \tilde{F}_j(x) = x_j \cap (F_j(\bar{x}) \times \{1\})$  and  $F_j(\bar{x}) \subset C_j$ . Consequently,  $x_j \cap (C_j \times \{1\}) \neq \emptyset$  and from Lemma 2 we get

$$\emptyset \neq x_j \cap (F_j(\bar{x}) \times \{1\}) \subset x_j \cap (C_j \times \{1\}) = \{(\rho_j(x_j), 1)\}.$$

Hence, the equality holds and  $\bar{x}_j = \rho_j(x_j) \in F_j(\bar{x})$ . This ends the proof of Theorem 1.  $\square$

## 4 Appendix: the Grassmannian manifold $G^k(V)$

Let  $V$  be a Euclidean space and let  $k$  be an integer such that  $0 \leq k \leq \dim V$ . In this section, we recall the properties of  $G^k(V)$  which are used in this paper.

First, we recall that  $G^k(V)$  is a smooth boundaryless manifold of dimension  $k(\dim V - k)$  (see, for example, Hirsh [11] and Lemma 3 below). The local charts can be defined as follows. Let  $\bar{E} \in G^k(V)$  and let  $\{\bar{e}_1, \dots, \bar{e}_k\}$  be some given orthonormal basis of  $\bar{E}$ , we define the mapping  $\psi_{\bar{E}} : (\bar{E}^\perp)^k \rightarrow G^k(V)$  by

$$\psi_{\bar{E}}(u) = \text{span} \{\bar{e}_1 + u_1, \dots, \bar{e}_k + u_k\}, \text{ for } u = (u_1, \dots, u_k) \in (\bar{E}^\perp)^k.$$

Then it is easy to check that the mapping  $\psi_{\bar{E}}$  is injective (see Claim 3 below), so  $\psi_{\bar{E}}$  is a bijection from  $(\bar{E}^\perp)^k$  onto  $U_{\bar{E}} = \psi_{\bar{E}}((\bar{E}^\perp)^k)$ . We can now consider the inverse mapping  $\varphi_{\bar{E}} : U_{\bar{E}} \rightarrow (\bar{E}^\perp)^k$  defined by  $\varphi_{\bar{E}}(E) = \psi_{\bar{E}}^{-1}(E)$ , which is clearly a bijection.

**Lemma 3** (a)  $G^k(V)$  is a smooth boundaryless (i.e.,  $C^\infty$ ) manifold without boundary of dimension  $k(\dim V - k)$  and  $(U_{\bar{E}}, \varphi_{\bar{E}})_{\bar{E} \in G^k(V)}$  defines a  $C^\infty$  atlas of  $G^k(V)$ .

(b) The set  $G^k(V)$  is compact.

(c) The mapping  $E \rightarrow E^\perp$  from  $G^k(V)$  to  $G^\ell(V)$  ( $\ell = \dim V - k$ ) is a smooth diffeomorphism.

(d) The mapping  $p : V \times G^k(V) \rightarrow V$  defined by  $p(x, E) = \text{proj}_E(x)$  is smooth. Hence, the set  $\{(x, E) \in V \times G^k(V) \mid x \in E\}$  is a closed subset of  $V \times G^k(V)$ .

(e) The mapping  $x \rightarrow x^p$  from  $G^k(V)$  to  $(G^k(V))^p$  is smooth.

We prepare the proof of the lemma with a claim.

**Claim 3** Let  $\bar{E} \in G^k(V)$  and let  $\{\bar{e}_1, \dots, \bar{e}_k\}$  be an orthonormal basis of  $\bar{E}$ .

(a) The mapping  $\psi_{\bar{E}}$  is injective.

(b) For every  $u \in (\bar{E}^\perp)^k$   $\psi(0) \cap \psi(u)^\perp = \psi(0)^\perp \cap \psi(u) = \{0\}$ .

**Proof of Claim 3.** Part (a). Let  $u = (u_1, \dots, u_k)$  and  $v = (v_1, \dots, v_k)$  in  $(\bar{E}^\perp)^k$  such that

$$\psi_{\bar{E}}(u) = \text{span} \{\bar{e}_1 + u_1, \dots, \bar{e}_k + u_k\} = \psi_{\bar{E}}(v) = \text{span} \{\bar{e}_1 + v_1, \dots, \bar{e}_k + v_k\}.$$

Then, there exist some real numbers  $\lambda_{ij}$  ( $i = 1, \dots, k, j = 1, \dots, k$ ) such that

$$\bar{e}_i + u_i = \sum_{j=1}^k \lambda_{ij}(\bar{e}_j + v_j) \text{ for every } i = 1, \dots, k.$$

Taking, for each inequality, the scalar product with  $\bar{e}_l$ , where  $l = 1, \dots, k$ , we obtain  $\lambda_{il} = 0$  if  $i \neq l$  and  $\lambda_{il} = 1$ . Hence  $u_l = v_l$  for every  $l = 1, \dots, k$ , and finally  $u = v$ .

Part (b). Let  $x \in \bar{E} \cap \psi(u)^\perp$ . Then there exists some real numbers  $\lambda_i$  ( $i = 1, \dots, k$ ) such that  $x = \sum_{i=1}^k \lambda_i \bar{e}_i$ . Taking the scalar product with  $\bar{e}_j + u_j$  ( $j = 1, \dots, k$ ), we obtain:  $\lambda_j = 0$  for every  $j = 1, \dots, k$  which proves  $\psi(0) \cap \psi(u)^\perp = \{0\}$ . Similarly, let  $x \in \bar{E}^\perp \cap \psi(u)$ . Then there exists some real numbers  $\lambda_i$  ( $i = 1, \dots, k$ ) such that  $x = \sum_{i=1}^k \lambda_i(\bar{e}_i + u_i)$ . Taking the scalar product with  $\bar{e}_j$  ( $j = 1, \dots, k$ ), we obtain:  $\lambda_j = 0$  for every  $j = 1, \dots, k$  which proves  $\psi(0) \cap \psi(u)^\perp = \{0\}$  and ends the proof of the claim.  $\square$

**Proof of Lemma 3.** Part (a). We prove that  $(U_E, \varphi_E)_{E \in G^k(V)}$  is a smooth (i.e.,  $C^\infty$ ) atlas of  $G^k(V)$  and it is then clear that  $\dim M = \dim(E^\perp)^k = k(\dim V - k)$ . Let  $(U_E, \varphi_E)$  and  $(U_F, \varphi_F)$  be two local charts at  $E$  and  $F$ , respectively, such that  $U_E \cap U_F \neq \emptyset$ . We shall prove that  $\varphi_F \circ \varphi_E^{-1}$  is smooth (i.e.,  $C^\infty$ ). We let  $\{e_1, \dots, e_k\}$ , and  $\{f_1, \dots, f_k\}$ , be two orthonormal basis, respectively of  $E$  and  $F$ . Let  $(v_1, \dots, v_k) = \varphi_F \circ \varphi_E^{-1}(u_1, \dots, u_k)$  and  $u = (u_1, \dots, u_k)$ , then there exist real numbers  $\lambda_{ij}(u)$  ( $i, j = 1, \dots, k$ ) such that

$$f_i + v_i = \sum_{j=1}^k \lambda_{ij}(u)(e_j + u_j) \quad (i = 1, \dots, k).$$

The proof will be complete by showing that the real-valued functions  $\lambda_{ij}(u)$  are differentiable with respect to  $u$ . Taking the scalar product with  $f_l$  ( $l = 1, \dots, k$ ) we obtain

$$f_l \cdot f_l = \sum_{j=1}^k \lambda_{lj}(u)(e_j + u_j) \cdot f_l \quad (l = 1, \dots, k).$$

Thus for every  $i = 1, \dots, k$ , the vector  $\lambda_i(u) = (\lambda_{ij}(u))_{j=1}^k$  is the solution of a linear system whose matrix  $G(u) = ((e_j + u_j) \cdot f_l)_{j,l=1,\dots,k}$  is now shown to be invertible (which clearly implies the differentiability of  $\lambda_i(u)$ ). Indeed, if  $G(u)\lambda = 0$  for some  $\lambda \in R^k$  then  $\sum_{j=1}^k \lambda_j(e_j + u_j) \cdot f_l = 0$  (for  $l = 1, \dots, k$ ), thus  $\sum_{j=1}^k \lambda_j(e_j + u_j) \in F^\perp$ . Besides, since  $\sum_{j=1}^k \lambda_j(e_j + u_j) \in \psi_E(u_1, \dots, u_j) = \psi_F(v_1, \dots, v_j)$ , we finally obtain  $\sum_{j=1}^k \lambda_j(e_j + u_j) \in F^\perp \cap \psi_F(v_1, \dots, v_k) = \{0\}$  [from Claim 3]. Now, since  $(e_j + u_j)_{j=1,\dots,k}$  is a basis we obtain  $\lambda_j = 0$  for every  $j = 1, \dots, k$ .  $\square$

*Part (b).* Let  $(E^\nu)$  be a sequence in  $G^k(V)$  and, for every  $\nu$ , let  $\{e_1^\nu, \dots, e_k^\nu\}$  an orthonormal basis of  $E^\nu$ . Without any loss of generality, we can assume that, for every  $i = 1, \dots, k$  the sequence  $(e_i^\nu)$  converges to some element  $e_i$  in  $V$ . Clearly,  $\{e_1, \dots, e_k\}$  is an orthonormal family in  $V$  and we let  $E = \text{span}\{e_1, \dots, e_k\}$ . We shall now prove that the sequence  $(E^\nu)$  converges to  $E$ . Indeed, for  $\nu$  large enough, there exists  $u^\nu = (u_1^\nu, \dots, u_k^\nu) \in (E^\perp)^\perp$  such that  $E^\nu = \psi_E(u_1^\nu, \dots, u_k^\nu)$ . It can be written  $e_i + u_i^\nu = \sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu$  ( $i = 1, \dots, k$ ). Multiplying these equalities by  $e_l$  ( $l = 1, \dots, k$ ) we obtain:  $\sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu \cdot e_l = 0$  if  $i \neq l$  and  $\sum_{j=1}^k \lambda_{ij}^\nu e_j^\nu \cdot e_i = 1$ . This can be written (for every  $i = 1, \dots, k$ )  $(e_j^\nu \cdot e_l)_{j,l=1,\dots,k} (\lambda_{ij}^\nu)_{j=1}^k = (e_i \cdot e_l)_{l=1,\dots,k}$ . If  $\nu$  is large enough then  $(e_j^\nu \cdot e_l)_{j,l=1,\dots,k}$  is invertible and converges to Id, which prove that the sequence  $(\lambda_{ij}^\nu)_{j=1}^k$  converges to  $(e_i \cdot e_l)_{l=1,\dots,k}$  for every  $i = 1, \dots, k$ , i.e  $(\lambda_{ii}^\nu)$  converges to 1 and  $(\lambda_{ij}^\nu)$  converges to 0 for  $i \neq j$ . We finally obtain that  $(u_i^\nu)$  converges to 0, which proves that  $E^\nu$  converges to  $E$ .  $\square$

*Part (c).* Let  $\bar{E} \in G^k(V)$ , and let  $(\bar{e}_1, \dots, \bar{e}_k)$ , and  $(\bar{f}_1, \dots, \bar{f}_\ell)$  be orthonormal basis, respectively, of  $\bar{E}$  and  $\bar{E}^\perp$ . Let  $(u_1, \dots, u_k) \in (\bar{E}^\perp)^\perp$  and let  $E = \psi(u)$ . Then it is easy to see that  $E^\perp = \text{span}\{f_1 + v_1, \dots, f_\ell + v_\ell\}$  where  $v_i = -\sum_{j=1}^k (f_i \cdot u_j) \bar{e}_j$ . So each  $v_i$  is a smooth mapping with respect to the  $u_i$  and conversely, from  $(E^\perp)^\perp = E$ , each  $u_i$  is a smooth mapping with respect to the  $v_i$ . This ends the proof of Part (c).  $\square$

*Part (d).* The differentiability of the mapping  $p$  is left to the reader. Then notice that  $\{(x, E) \in V \times G^k(V) \mid x \in E\} = \{(x, E) \in V \times G^k(V) \mid x = \text{proj}_E(x)\}$ , which is clearly closed since the mapping  $p : (x, E) \rightarrow \text{proj}_E(x)$  is continuous.  $\square$

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