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## Existence of Solutions in Continuous-time Optimal Growth Models

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2004.63



# Existence of Solutions in Continuous-time Optimal Growth Models\*

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**Abstract** This paper studies the existence of solutions in continuous time optimization problems. It provides a theorem whose conditions can be easily checked in most models of the optimal growth theory including those with increasing return and multi-sector economies.

**Keywords** Existence of solutions, optimization, continuous time, optimal growth.

**JEL Classification** C61, D91, E13.

## 1 Introduction

The modern theory of economic growth is based on a behaviorist tradition that analyses both optimal and equilibrium allocations of scarce resources over time. To discuss this issue, infinite dimensional optimization techniques in continuous time have been extensively used over the last forty years. In

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most cases the problem may be re-written as the following variational problem:

$$\begin{aligned} & \max \int_0^{+\infty} G(x_t, \dot{x}_t) e^{-rt} dt \\ & s.t. \quad \left| \begin{array}{l} \forall t, \dot{x}_t \in \Gamma_t(x_t) \\ x_0 \in \mathbb{R}_+^l \text{ given} \end{array} \right. \end{aligned}$$

This problem has a solution provided that  $\int_0^{+\infty} G(x_t, \dot{x}_t) e^{-rt} dt$  exhibits some upper semi-continuity and  $\Gamma$  some compactness. However, in many papers, those properties are not checked and the existence of the discussed solution is simply assumed. The reason is technical and specific to continuous time: to be a feasible path,  $x_t$  should belong to a ball of  $L^1$  which is not compact for the  $L^1$  topology. General results have nevertheless been established. Notably, Magill (1981) provides an existence theorem for problems with a  $G$  concave in its arguments. However, his theorem does not apply to the large body of the literature that consider increasing returns. It is true that if the problem is not concave, the difficulty increases since traditional optimality conditions for the Social Planner problem are necessary but not anymore sufficient. Chichilnisky (1981) and Romer (1986a) hence make an important step by providing an existence theorem for optimal solution that apply to models with increasing returns. While proved in a different way, the conditions they proposed are based on the concavity of  $G$  with respect to the one of its arguments which have the highest derivative. Their demonstrations are very nice but the precise set of conditions they propose is rather technical and it is not simple to know if a given economic problem satisfies it. We hence propose an original existence theorem whose conditions are easy to verify in a given economic model. We show this by confronting some well-known models to our existence conditions.

The demonstration we propose extends an initial result of Askenazy and Le Van (1999) about the regularity of the objective function. Then it uses the Dunford-Pettis Criterion to get some sequential compactness for some weak topology and a multi-dimensional version of the Fatou Lemma to deduce properties for the limit with respect to this topology.

The paper is organized as follows. In section 2, the model is developed while our theorem is presented in section 3. In section 4, we show how our theorem can be easily applied to models of neoclassical growth or endogenous growth.

## 2 The Model

Consider the following problem:

$$\max_{x_t, y_t} \int_0^{+\infty} u(x_t) e^{-rt} dt$$

subject to :

(P)

$$\begin{aligned} \forall t \geq 0, F(y_t) \leq \dot{y}_t \leq G(\bar{y}_t, y_t, x_t), \\ x_t \in \mathbb{R}_+^C, y_t \in \mathbb{R}_+^K, \\ x_0 \in \mathbb{R}_+^C, r > 0 \text{ are given.} \end{aligned}$$

In this setting,  $x_t$  is the control variable,  $y_t$  is the state variable, and  $\bar{y}_t \in \mathbb{R}_+^K$  is the externality. The sets  $C$  and  $K$  are finite (for simplicity,  $(y^1, \dots, y^K)$  denote elements of  $\mathbb{R}^K$  and  $(x^1, \dots, x^C)$  elements of  $\mathbb{R}^C$ ).

Assume:

**A1.** The functions  $F$  and  $G$  are continuous.

**A2.** For any  $j \in K$ , the function  $G^j$  is concave with respect to  $x$ .

**A3.** There exist  $(b_i \geq 0, A'_i > 0, i = 1, \dots, C)$ ,  $(a_i \geq 0, A_i > 0, i = 1, \dots, K)$  such that:

$$\begin{aligned} \forall t, F(y_t) \leq \dot{y}_t \leq G(\bar{y}_t, y_t, x_t), \forall i \in K, \bar{y}_t^i \leq A'_i e^{a_i t} \implies \\ \forall t, \forall i \in K, \forall j \in C, y_t^i \leq A'_i e^{a_i t}, \left| \dot{y}_t^i \right| \leq A'_i e^{a_i t}, x_t^j \leq A_j e^{b_j t}. \end{aligned}$$

**A4.**  $r > \sup \{b_i, i \in C\}$ .

**A5.** The function  $u$  is concave, non decreasing, continuous on  $\mathbb{R}_+^C$ .

**Lemma 1** *The function  $x \in L_+^1(e^{-rt} dt) \rightarrow \int_0^{+\infty} u(x_t) e^{-rt} dt$  is finite valued and upper semi-continuous for the topology  $\sigma(L^1(e^{-rt} dt), L^\infty)$ .*

**Proof.** See Askenazy and Le Van (1999); note that the assumption  $u(0) = 0$  made in their article is not necessary for the proof of this lemma. Moreover, their proof is still true when the function  $u$  satisfies the following assumption *A5bis*. ■

**Remark 1** Assumption *A5* could be replaced by:

**A5bis.** The function  $u$  is concave and upper semi-continuous from  $\mathbb{R}_+^C$  into  $\mathbb{R} \cup \{-\infty\}$  and finite valued on  $\mathbb{R}_{++}^C$ .

Assumption *A5bis* allows for utility functions unbounded from below, such as  $u(c) = \log(c)$  or  $u(c) = c^\sigma/\sigma$  with  $\sigma < 0$  that are extensively used in economic models.

### 3 Existence of Solutions

This section provides the proof of the existence of a solution for Problem  $(\mathcal{P})$ .

**Theorem 2** *Under assumptions A1, ..., A5, Problem  $(\mathcal{P})$  has a solution.*

**Proof.** First note that the variables  $(x_t, y_t)$  will be said feasible if they satisfy the constraints of Problem  $(\mathcal{P})$ . Observe that assumptions *A3* and *A4* imply that  $\int_0^{+\infty} u(x_t)e^{-rt}dt$  is uniformly bounded above on the set of feasible controls  $x_t$ . Let the sequence  $(x^n)$  of elements in  $L_+^1(e^{-rt}dt)$  satisfy  $\lim_{n \rightarrow \infty} \int_0^{+\infty} u(x_t^n)e^{-rt}dt = M = \sup \int_0^{+\infty} u(x_t)e^{-rt}dt$  over the set of feasible controls. Assumption *A3* implies that the sequence satisfies the Dunford-Pettis criterion<sup>1</sup> and has a subsequence, denoted also by  $(x_t^n)$  for simplicity, which converges to  $x^* \in L_+^1$  for the topology  $\sigma(L^1(e^{-rt}dt), L^\infty)$ . From Lemma 1,  $\int_0^{+\infty} u(x_t^*)e^{-rt}dt \geq M$ . To end the proof, it remains to show that  $x^*$  is feasible.

Observe first that for a defined  $\rho > \max \{a_i : i \in K\}$  then the associated sequences  $(y^n, \dot{y}^n)$  also satisfy the Dunford-Pettis criterion for  $L^1(e^{-\rho t}dt)$ . It can therefore be assumed that they converge respectively to  $y^*, z^*$  for the topology  $\sigma(L^1(e^{-\rho t}dt), L^\infty)$ . Then for all  $t$ :

$$\int_0^t \dot{y}^n(s)ds \rightarrow \varphi(t) = \int_0^t z^*(s)ds.$$

In other words,  $y^n(t) \rightarrow \varphi(t) + y_0$ , for every almost  $t$ . Since  $\forall t, \forall i, 0 \leq y^{n,i}(t) \leq A_i e^{a_i t} \in L^1(e^{-rt}dt)$ , use Lebesgue Theorem to obtain that  $y^n$  converges to  $y^*$  for the strong topology (and consequently for the weak topology) of  $L^1(e^{-\rho t}dt)$ . Thus  $y^* = \varphi + y_0$  and  $\dot{y}^* = z^*$ .

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<sup>1</sup>See Dunford and Schwartz, 1967, corollary 11, p. 294

Using the multidimensional Fatou Lemma (see Appendix), given  $s$ , there exist  $(\theta_{sj})_{j=1,\dots,K+C+1}$  such that  $\sum_{j=1}^{K+C+1} \theta_{sj} = 1, \theta_{sj} \geq 0, \forall j$ , and

$$(\dot{y}_s^*, x_s^*) = \sum_{j=1}^{K+C+1} \theta_{sj} (\varsigma_{sj}^1, \varsigma_{sj}^2),$$

where  $(\varsigma_{sj}^1, \varsigma_{sj}^2) \in ac(\dot{y}_s^n, x_s^n)$  is the set of cluster points of the sequence  $(\dot{y}_s^n, x_s^n)$ . Given  $(j, s)$ , there will be some increasing function  $\phi_{js} : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$(\dot{y}_s^{\phi_{js}(n)}, x_s^{\phi_{js}(n)})_{n \rightarrow \infty} \rightarrow (\varsigma_{sj}^1, \varsigma_{sj}^2).$$

Then,

$$\dot{y}_s^{*i} = \sum_{j=1}^{K+C+1} \theta_{sj} \varsigma_{sj}^{1i} = \sum_{j=1}^{K+C+1} \theta_{sj} \lim_{n \rightarrow \infty} \dot{y}_s^{i, \phi_{js}(n)} \leq \sum_{j=1}^{K+C+1} \theta_{sj} \lim_{n \rightarrow \infty} G^i(\bar{y}_s, y_s^{\phi_{js}(n)}, x_s^{\phi_{js}(n)}).$$

Since  $G$  is continuous and  $y^n$  converges pointwise, it yields:

$$\dot{y}_s^{*i} \leq \sum_{j=1}^{K+C+1} \theta_{sj} G^i(\bar{y}_s, \lim_{n \rightarrow \infty} y_s^{\phi_{js}(n)}, \lim_{n \rightarrow \infty} x_s^{\phi_{js}(n)}) = \sum_{j=1}^{K+C+1} \theta_{sj} G^i(\bar{y}_s, y_s^*, \varsigma_{sj}^2).$$

From A2,

$$\dot{y}_s^{*i} \leq G^i(\bar{y}_s, y_s^*, \sum_{j=1}^{K+C+1} \theta_{sj} \varsigma_{sj}^2) = G^i(\bar{y}_s, y_s^*, x_s^*).$$

Recall that

$$\dot{y}_s^{*i} = \sum_{j=1}^{K+C+1} \theta_{sj} \varsigma_{sj}^{1i} = \sum_{j=1}^{K+C+1} \theta_{sj} \lim_{n \rightarrow \infty} \dot{y}_s^{i, \phi_{js}(n)} \geq \sum_{j=1}^{K+C+1} \theta_{sj} \lim_{n \rightarrow \infty} F^i(y_s^{\phi_{js}(n)}).$$

Since  $F$  is continuous and  $y^n$  converges pointwise, it yields:

$$\dot{y}_s^{*i} \geq \sum_{j=1}^{K+C+1} \theta_{sj} F^i(\lim_{n \rightarrow \infty} y_s^{\phi_{js}(n)}) = \sum_{j=1}^{K+C+1} \theta_{sj} F^i(y_s) = F^i(y_s).$$

Thus,  $(y^*, x^*)$  have been proved to be feasible. Hence  $M = \sup \int_0^{+\infty} u(x_t) e^{-rt} dt = \int_0^{+\infty} u(x_t^*) e^{-rt} dt$ . ■

**Remark 2** This proof can be easily adapted in order to study problems with finite horizon:

$$\max_{x_t, y_t} \int_0^T u(x_t) e^{-rt} dt, T < +\infty$$

subject to:

$$\begin{aligned} \forall t \geq 0, F(y_t) \leq \dot{y}_t \leq G(\bar{y}_t, y_t, x_t), \\ x_t \in \mathbb{R}_+^C, y_t \in \mathbb{R}_+^K, \\ x_0 \in \mathbb{R}_+^C, r > 0, T \in ]0, +\infty[ \text{ are given.} \end{aligned}$$

In this setting, Assumption  $A4$  is not necessary.

**Remark 3** The Social Planner Problem is a particular case of Problem  $(\mathcal{P})$ : just consider  $\bar{y}_t = \mathbf{1}^K, \forall t \geq 0$ .

## 4 Some Examples in the Literature

### 4.1 The Ramsey model with discounting

Consider the model developed by Cass (1965) and Koopmans (1965):

$$\max_{c_t, k_t} \int_0^{+\infty} u(c_t) e^{-rt} dt,$$

subject to

$$\begin{aligned} \forall t \geq 0, -\delta k_t \leq \dot{k}_t \leq f(k_t) - \delta k_t - c_t \\ c_t \geq 0, k_t \geq 0, \\ k_0 > 0, r > 0 \text{ are given.} \end{aligned}$$

Variables  $c_t$  and  $k_t$  denote respectively the consumption and the capital stock at time  $t$ . It is supposed that the instantaneous utility function  $u$  satisfies assumption  $A5$ , and that the production function  $f$  is concave, continuous, increasing, differentiable, and satisfies  $f(0) = 0, f'(\infty) = 0$ .

Define  $x_t = c_t, y_t = k_t, \bar{y}_t = 1, G(\bar{y}_t, y_t, x_t) = \bar{y}_t (f(y_t) - \delta y_t - x_t), F(y_t) = -\delta y_t$ . Now, check that Assumptions  $A1, \dots, A4$  are satisfied:

$A1$ : It is obvious that  $F$  and  $G$  are continuous.



*A2*: It is also obviously satisfied.

*A3*: Since  $f'(\infty) = 0$ , for any  $\varepsilon > 0$ , it exists  $B$  such that  $f'(y) \leq B + \varepsilon y, \forall y \geq 0$ . It is easy to see that it exists  $A'' > 0$ , such that  $y_t \leq A'' e^{\varepsilon t}, \forall t \geq 0$ . This implies that  $\forall t \geq 0, c_t \leq A e^{\varepsilon t}$ , with some  $A > 0$ . Since  $-\delta y_t \leq \dot{y}_t$ , then  $|\dot{y}_t| \leq A' = \max\{\delta k_0, A''\}$ .

*A4*: Choose  $\varepsilon < r$ .

## 4.2 The Ramsey model with leisure

Consider now

$$\max_{c_t, l_t, k_t} \int_0^{+\infty} u(c_t, l_t) e^{-rt} dt,$$

subject to

$$\begin{aligned} \forall t \geq 0, & -\delta k_t \leq \dot{k}_t \leq f(k_t, 1 - l_t) - \delta k_t - c_t, \\ c_t \geq 0, & k_t \geq 0, 0 \leq l_t \leq 1, \\ k_0 > 0, & r > 0 \text{ are given.} \end{aligned}$$

The variable  $l_t$  denotes the leisure at time  $t$ . It is supposed that the utility function  $u$  satisfies *A5* and that the production function  $f$  is concave, continuously differentiable, and satisfies  $f(0, l) = f(k, 0) = 0$ , and  $f'(\infty, l) = 0$ .

As in the previous example, one can find some  $\varepsilon$  in  $]0, r[$  such that for any  $t \geq 0, c_t \leq A e^{\varepsilon t}, k_t \leq A' e^{\varepsilon t}, |\dot{k}_t| \leq A' e^{\varepsilon t}$ . Put  $x_t = (c_t, l_t), y_t = k_t, \bar{y}_t = 1, G(\bar{y}_t, y_t, x_t) = \bar{y}_t (f(y_t) - \delta y_t - x_t), F(y_t) = -\delta y_t$ . It is then easy to check that Assumptions *A1, ..., A4* are satisfied.

## 4.3 The Romer Model (1986b)

For a given path  $\bar{k}_t \geq 0$ , solve:

$$\max_{c_t, k_t} \int_0^{+\infty} u(c_t) e^{-rt} dt,$$

subject to

$$\begin{aligned} \forall t \geq 0, & c_t + i_t \leq F(\bar{k}_t, k_t), \\ \frac{\dot{k}_t}{k_t} & = g\left(\frac{i_t}{k_t}\right), \end{aligned}$$

$$\begin{aligned} c_t &\geq 0, i_t \geq 0, k_t \geq 0, \\ k_0 &> 0, r > 0 \text{ are given.} \end{aligned}$$

Variables  $i_t, k_t, c_t$  respectively denote the investment, the stock of knowledge and the consumption at time  $t$ . It is supposed that the function  $g$  is concave, increasing, and satisfies  $g(0) = 0, g(\infty) = \lambda > 0$ . Assume, moreover, that  $F$  is concave, nonnegative, and satisfies  $F(y, k) \leq y^\alpha k^\beta$  for every  $(y, k) \geq 0$ , and  $\alpha > 0, \beta > 0$ . The function  $u$ , as previously, satisfies Assumption A5.

Define

$$\begin{aligned} G(\bar{y}, y, x) &= yg\left(\frac{F(\bar{y}, y) - x}{y}\right) \text{ when } y > 0, \\ &F(\bar{y}, y) - x \geq 0, \end{aligned}$$

and

$$G(\bar{y}, 0, 0) = 0.$$

When  $x_t = c_t, y_t = k_t$ , the problem is then equivalent to:

$$\max_{c_t, k_t} \int_0^{+\infty} u(c_t) e^{-rt} dt,$$

subject to

$$\begin{aligned} \forall t &\geq 0, 0 \leq \dot{k}_t \leq G(\bar{k}_t, k_t, c_t), k_t \geq 0, \\ k_0 &> 0, r > 0 \text{ are given.} \end{aligned}$$

Now, check that Assumptions A1, ..., A4 are fulfilled.

A1: It is easy to see that  $G$  is continuous.

A2: It is obvious that  $G$  is concave in  $x$ .

A3 : Since  $0 \leq \dot{y}_t \leq \lambda y_t$ , it implies that  $\forall t, y_t \leq k_0 e^{\lambda t}, \left| \dot{y}_t \right| \leq \lambda k_0 e^{\lambda t}$  and  $x_t \leq \bar{k}_0 k_0 e^{\lambda(\alpha+\beta)t}$ .

A4: It is satisfied if  $r > \lambda(\alpha + \beta)$ .

#### 4.4 The Lucas Model (1988)

For a given human capital path  $(\bar{h}_t)$ , solve:

$$\max_{c_t, \theta_t, k_t, h_t} \int_0^{+\infty} u(c_t) e^{-rt} dt,$$

subject to:

$$\begin{aligned}
\forall t &\geq 0, -\delta k_t \leq \dot{k}_t \leq k_t^\alpha (\theta_t h_t)^{1-\alpha} \bar{h}_t^\gamma - \delta k_t - c_t, \\
0 \leq \dot{h}_t &\leq \phi(1 - \theta_t) h_t, \\
c_t &\geq 0, k_t \geq 0, 0 \leq h_t, 0 \leq \theta_t \leq 1, \\
k_0 &> 0, h_0 > 0 \text{ are given,} \\
r &> 0, 0 < \alpha < 1, \gamma > 0.
\end{aligned}$$

Variables  $c_t$ ,  $\theta_t$ ,  $k_t$  and  $h_t$  denote respectively the consumption, the working time, the physical capital and the human capital at time  $t$ . Moreover, it is supposed that function  $\phi$  is increasing, concave and satisfies  $\phi(0) = 0$ ,  $\phi(1) = \mu > 0$  while function  $u$  is concave and increasing.

Define  $x_t = (c_t, \theta_t)$ ,  $y_t = (k_t, h_t)$ ,  $\bar{y}_t = (1, \bar{h}_t)$ ,  $u(x) = v(c)$ ,  $F(y) = (-\delta k, 0)$ ,  $G(\bar{y}_t, y_t, x_t) = k_t^\alpha (\theta_t h_t)^{1-\alpha} \bar{h}_t^\gamma - \delta k_t - c_t$ .

Obviously, Assumption A5 is satisfied. Now, check A1,...,A4.

A1: It is obviously true.

A2: It is immediate since  $0 < \alpha < 1$

A3: It follows from the dynamic constraint on human capital that  $|\dot{h}_t| \leq \mu h_0 e^{\mu t}$ ,  $h_t \leq h_0 e^{\mu t}$ . From the constraints on physical capital, one obtains that  $k_t \leq \left( \frac{h_0(1-\alpha)}{\mu(1-\alpha+\gamma)} \right)^{\frac{1}{1-\alpha}} e^{\frac{\mu(1-\alpha+\gamma)t}{(1-\alpha)}}$ ,  $c_t \leq \left( \frac{h_0(1-\alpha)}{\mu(1-\alpha+\gamma)} \right)^{\frac{1}{1-\alpha}} h_0^{1-\alpha+\mu} e^{\frac{\mu(1-\alpha+\gamma)t}{(1-\alpha)}}$ ,  $|\dot{k}_t| \leq \max \left\{ h_0^{\frac{1}{1-\alpha}} \left( \frac{(1-\alpha)}{\mu(1-\alpha+\gamma)} \right)^{\frac{\alpha}{1-\alpha}}, \delta k_0 \right\} e^{\frac{\mu(1-\alpha+\gamma)t}{(1-\alpha)}}$ .

A4: It is satisfied under the additional condition  $r > \frac{\mu(1-\alpha+\gamma)}{(1-\alpha)}$ .

## 5 Appendix

The original Fatou's lemma has been generalized to this proposition (See Hildenbrand and Mertens, 1971).

**Proposition 3** *Let  $(h_n)$  be an integrably bounded sequence of integrable mappings from  $\Omega \subset \mathbb{R}$  to  $\mathbb{R}^k$ , which converges weakly to a integrable mapping  $f : \Omega$  to  $\mathbb{R}^k$ . Then, we have for a.e.*

$$\text{for a.e. } a \in \Omega, f(a) \in \text{co}Ls_n\{f_n(a)\},$$

where  $Ls_n\{x_n\}$  denotes the set of limit points of converging subsequences of  $(x_n)$ , and  $\text{co}(Z)$  the convex hull of the set  $Z$ .

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