



**HAL**  
open science

# No-Arbitrage Condition and Existence of Equilibrium with Dividends

Cuong Le Van, Nguyen Ba Minh

► **To cite this version:**

Cuong Le Van, Nguyen Ba Minh. No-Arbitrage Condition and Existence of Equilibrium with Dividends. 2004. halshs-03331458

**HAL Id: halshs-03331458**

**<https://shs.hal.science/halshs-03331458v1>**

Submitted on 1 Sep 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License



# CERMSEM

UMR CNRS 8095

C  
a  
h  
i  
e  
r  
s  
de  
la  
M  
S  
E

## No-Arbitrage Condition and Existence of Equilibrium with Dividends

Cuong LE VAN  
Nguyen BA MINH

2004.58



# No-Arbitrage Condition and Existence of Equilibrium with Dividends <sup>1</sup>

Cuong Le Van<sup>2</sup>, Nguyen Ba Minh<sup>3</sup>

June 2004

## Abstract

In this paper we first give an elementary proof of existence of equilibrium with dividends in an economy with possibly satiated consumers. We then introduce a no-arbitrage condition and show that it is equivalent to the existence of equilibrium with dividends.

**Keywords:** Equilibrium with Dividends, Walras Equilibrium, Satiation Points, No-Arbitrage Condition

4

---

<sup>1</sup>The authors are indebted to Jacques Drèze for very valuable remarks and comments. They also thank Enrico Minelli for helpful discussions and comments.

<sup>2</sup>CERMSEM, CORE, levan@core.ucl.ac.be

<sup>3</sup>Hanoi University of Commerce, baminhdhtm@hotmail.com

# 1 Introduction

In the Arrow-Debreu model (1954), the authors impose a nonsatiation assumption which states that for every consumer, whatever the commodity bundle may be, there exists another consumption bundle she/he strictly prefers. It is well-known, that in presence of satiation, a Walras equilibrium may not exist since for every price, there could be a consumer who maximizes her/his preference in the interior of her/his budget set.

The absence of the nonsatiation condition with fixed prices was studied by Drèze and Muller (1980) by introducing the notion of coupons equilibrium, Aumann and Drèze (1986) with the concept of dividends, Mas-Colell (1992) who used the term of slack equilibrium. We can cite other authors who worked on nonsatiation: e.g. Makarov (1981), Florig and Yildiz (2002), and for a continuum of consumers, Cornet, Topuzi and Yildiz (2003).

In this paper we first give an easy proof of existence of equilibria with dividends. For Aumann and Drèze, a dividend is a "cash allowance added to the budget by each trader. Its function is to distribute among the nonsatiated agents the surplus created by the failure of the satiated agents to use their entire budget". Here, we introduce an additional good (e.g. financial asset, or paper money) that the satiated agents will want to have in order to fill up their budget sets. For that, they will buy this additional good from the nonsatiated agents. More precisely, we will introduce an intermediary economy by adding another good that any agent would like to have if she/he meets satiation. In this economy, the nonsatiation condition is satisfied. There thus exists a Walras equilibrium. We show that this equilibrium actually corresponds to an equilibrium with dividends for the initial economy. It is interesting to notice that we show that, at this equilibrium, the satiated agents will buy the additional good from the nonsatiated agents. It is worth noticing that since any equilibrium of the intermediary economy is an equilibrium with dividends for the initial economy, we may know more properties of some equilibria with dividends (e.g.: how will be the changes of equilibrium prices and quantities when the initial endowments change?).

Second, we allow our model to have financial assets. If we assume that the production sets satisfy in particular the inaction and irreversibility conditions (see Debreu, 1959) and the utility functions satisfy the No-Half Line condition (see e.g. Werner, 1986, Page and Wooders 1996, Dana, Le

Van and Magnien, 1999, Allouch, Le Van, Page, 2002), then there exists an equilibrium with dividends iff there exists a no-arbitrage price.

The paper is organized as follows. The model is presented in Section 1. The main result is given in Section 2. Finally, in Section 3, we introduce the no-arbitrage price condition and prove that existence of equilibrium is equivalent to existence of no-arbitrage prices.

## 2 The Model.

We consider an economy having  $l$  goods,  $J$  producers, and  $I$  consumers. We suppose that the numbers of the producers and the consumers are finite. For each  $i \in I$ , let  $X_i \subset \mathbb{R}^l$  denote the set of consumption goods, let  $u_i : X_i \rightarrow \mathbb{R}$  denote the utility and let  $e_i \in \mathbb{R}^l$  be the initial endowment. Furthermore for each  $j \in J$ , let  $Y_j \subset \mathbb{R}^l$  denote the producing set of the producer  $j$ .

Let  $\theta_{ij}$  be the ratio of the profit that consumer  $i$  can get from the producer  $j$ . We suppose that  $0 \leq \theta_{ij} \leq 1$ ,  $\sum_{i \in I} \theta_{ij} = 1$ . Let  $p \in \mathbb{R}^l$  denote the price of the goods.

In the sequel we will denote this economy by

$$\mathcal{E} = \{(X_i, u_i, e_i)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{i \in I, j \in J}\}.$$

### **Preliminaries:**

We recall that a function  $u_i$  is said to be quasiconcave if its level-set

$$L^\alpha = \{x_i \in X_i : u_i(x_i) \geq \alpha\}$$

is convex for each  $\alpha \in \mathbb{R}$ .

$u_i$  is strictly quasiconcave if and only if  $x_i, x'_i \in X_i$ ,  $u_i(x'_i) > u_i(x_i)$  and  $\lambda \in [0, 1)$ , then

$$u_i(\lambda x_i + (1 - \lambda)x'_i) > u_i(x_i).$$

It means that

$$u_i(\lambda x_i + (1 - \lambda)x'_i) > \min(u_i(x_i), u_i(x'_i)).$$

The function  $u_i$  is upper semicontinuous if and only if  $L^\alpha$  is closed for each  $\alpha$ .

Let  $S_i$  denote the set of satiation points of  $u_i$ . then

$$S_i = \{x'_i \in X_i : u_i(x'_i) \geq u_i(x_i), \text{ for any } x_i \in X_i\}.$$

By this definition, the function  $u_i$  has no satiation point if for all  $x_i \in X_i$  there exists  $x'_i \in X_i$  such that  $u_i(x'_i) > u_i(x_i)$ . It is easy to check that  $S_i$  is convex and closed.

**Definition 1**

A Walras equilibrium of  $\mathcal{E}$  is a list  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*) \in (\mathbb{R}^l)^{|I|} \times (\mathbb{R}^l)^{|J|} \times (\mathbb{R}^l \setminus \{0\})$  which satisfies:

- (a)  $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i + \sum_{j \in J} y_j^*$  ( Market clearing);
- (b) for each  $i$  one has

$$p^* \cdot x_i^* = p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \cdot \sup p^* \cdot Y_j$$

(butget constraint), and for each  $x_i \in X_i$ , with  $u_i(x_i) > u_i(x_i^*)$ , it holds

$$p^* \cdot x_i > p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \cdot \sup p^* \cdot Y_j.$$

- (c) For each  $j \in J, y_j^* \in Y_j$  and  $p^* \cdot y_j^* = \sup p^* \cdot Y_j$ , where  $\sup p \cdot Y_j = \sup_{y_j \in Y_j} p \cdot y_j$ .

**Definition 2**

An equilibrium with dividends  $(d_i^*)_{i \in I} \in \mathbb{R}_+^{|I|}$  of  $\mathcal{E}$  is a list  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*) \in (\mathbb{R}^l)^{|I|} \times (\mathbb{R}^l)^{|J|} \times \mathbb{R}^l$  which satisfies:

- (a)  $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i + \sum_{j \in J} y_j^*$  ( Market clearing);
- (b) for each  $i$  one has

$$p^* \cdot x_i^* \leq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \cdot \sup p^* \cdot Y_j + d_i^*$$

(butget constraint), and for each  $x_i \in X_i$ , with  $u_i(x_i) > u_i(x_i^*)$ , it holds

$$p^* \cdot x_i > p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \cdot \sup p^* \cdot Y_j + d_i^*$$

- (c) For each  $j \in J, y_j^* \in Y_j$  and  $p^* \cdot y_j^* = \sup p^* \cdot Y_j$ , where  $\sup p \cdot Y_j = \sup_{y_j \in Y_j} p \cdot y_j$ .

### Definition 3

A feasible allocation is the list  $((x_i)_{i \in I}, (y_j)_{j \in J}) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j$  which satisfies  $\sum_{i \in I} x_i = \sum_{i \in I} e_i + \sum_{j \in J} y_j$ . We denote by  $A$  the set of feasible allocations.

The main purpose of this paper is to give an easy proof of existence of equilibrium with dividends of economy  $\mathcal{E}$  when satiation points occur in the preferences of the consumers. Our idea is to introduce an intermediary economy with an additional good (think of financial asset or money paper) that the consumers want to possess when they meet satiation. In this new economy, there is no satiation point. Hence, an equilibrium exists under appropriate assumptions. We show that this equilibrium is an equilibrium with dividends for the initial economy. It is worthy to point out that at this equilibrium point, the consumers who meet satiation points will buy the additional good from the consumers who do not meet satiation.

We now list our assumptions.

(H<sub>1</sub>) For each  $i \in I$ , the set  $X_i$  is nonempty closed convex;

(H<sub>2</sub>) For each  $i \in I$ , the function  $u_i$  is strictly quasiconcave and upper semicontinuous;

(H<sub>3</sub>) For each  $j \in J$ , the set  $Y_j$  is nonempty closed convex and  $Y = \sum_{j \in J} Y_j$  is closed.

(H<sub>4</sub>) The feasible set  $A$  is compact.

(H<sub>5</sub>) For every  $i$ ,  $e_i \in \text{int}(X_i - \sum_{j \in J} \theta_{ij} Y_j)$ .

## 3 The Results

We first give an existence of Walras equilibrium theorem when there exists no satiation.

**Theorem 1** *Assume (H<sub>1</sub>) – (H<sub>5</sub>) and*

$$\forall i, \forall x_i \in X_i, \exists x'_i \in X_i \text{ such that } u_i(x'_i) > u_i(x_i).$$

*Then there exists an equilibrium.*

**Proof.** The proof is quite standard. See e.g. Arrow and Debreu (1954) when the consumption sets are bounded from below, or adapt the proof given in Dana, Le Van and Magnien (1999) for an exchange economy. ■

We now come to our main result which is a corollary of the previous theorem.

**Theorem 2** *Assume  $(H_1) - (H_5)$ . Then there exists an equilibrium with dividends.*

**Proof.** Let us introduce the intermediary economy

$$\widehat{\mathcal{E}} = \left\{ (\widehat{X}_i, \widehat{u}_i, \widehat{e}_i)_{i \in I}, (\widehat{Y}_j)_{j \in J}, (\theta_{ij})_{i \in I, j \in J} \right\}$$

where:  $\widehat{X}_i = X_i \times \mathbb{R}_+$ ;  $\widehat{e}_i = (e_i, \delta_i)$  with  $\delta_i > 0$  for any  $i \in I$  and  $\widehat{Y}_j = (Y_j, 0)$  for any  $j \in J$ , and the utilities  $\widehat{u}_i$  are defined as follows (recall that  $S_i$  is the set of satiation points for agent  $i$ ): let  $\mu > 0$ ,  $M_i = \max \{u_i(x) : x \in X_i\}$ .

- If  $x_i \notin S_i$ , then  $\widehat{u}_i(x_i, d_i) = u_i(x_i)$  for any  $d_i \geq 0$ .

- If  $x_i \in S_i$ , then  $\widehat{u}_i(x_i, d_i) = u_i(x_i) + \mu d_i = M_i + \mu d_i$  for any  $d_i \geq 0$ .

We will check that Assumption  $(H_2)$  is satisfied for every  $\widehat{u}_i$ .

To prove that  $\widehat{u}_i$  is quasi-concave and upper semi-continuous, it suffices to prove that the set  $\widehat{L}_i^\alpha = \{(x_i, d_i) \in X_i \times \mathbb{R}_+ : \widehat{u}_i(x_i, d_i) \geq \alpha\}$  is closed and convex for every  $\alpha$ . We have two cases:

**Case 1:**  $\alpha < M_i$ . We claim that  $\widehat{L}_i^\alpha = L_i^\alpha \times \mathbb{R}_+$ . Indeed, let  $(x_i, d_i) \in \widehat{L}_i^\alpha$ . It follows  $\widehat{u}_i(x_i, d_i) \geq \alpha$  and there are two possibilities for  $x_i$ :

+ If  $x_i \notin S_i$ , then  $\widehat{u}_i(x_i, d_i) = u_i(x_i)$ . It implies  $u_i(x_i) \geq \alpha$  or  $x_i \in L_i^\alpha$  and hence  $(x_i, d_i) \in L_i^\alpha \times \mathbb{R}_+$ .

+ If  $x_i \in S_i$ , then  $u_i(x_i) = M_i > \alpha$ . This follows  $x_i \in L_i^\alpha$  and  $(x_i, d_i) \in L_i^\alpha \times \mathbb{R}_+$ .

So,  $\widehat{L}_i^\alpha \subset L_i^\alpha \times \mathbb{R}_+$ . It is obvious  $L_i^\alpha \times \mathbb{R}_+ \subset \widehat{L}_i^\alpha$ .

**Case 2:**  $\alpha \geq M_i$ . We claim that  $\widehat{L}_i^\alpha = S_i \times \left\{ d_i : d_i \geq \frac{\alpha - M_i}{\mu} \right\}$ . Indeed, if  $\widehat{u}_i(x_i, d_i) \geq \alpha$ , then  $x_i \in S_i$ . In this case,  $\widehat{u}_i(x_i, d_i) = M_i + \mu d_i \geq \alpha$ , and hence  $d_i \geq \frac{\alpha - M_i}{\mu}$ . The converse is obvious.

It is also obvious that  $S_i$  is closed and convex. We have proved that  $\widehat{u}_i$  is u.s.c. and quasi-concave for every  $i$ .

We now prove that  $\widehat{u}_i$  is strictly quasi-concave.

Indeed, take  $M_i = u_i(x)$  with  $x \in S_i$  and  $(x_i, d_i), (x'_i, d'_i) \in X_i \times \mathbb{R}_+$  such that  $\widehat{u}_i(x'_i, d'_i) > \widehat{u}_i(x_i, d_i)$ . For any  $\lambda \in ]0, 1[$ , we verify that

$$\widehat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) > \widehat{u}_i(x_i, d_i).$$



Since  $\hat{u}_i(x'_i, d'_i) > \hat{u}_i(x_i, d_i)$ , we can consider the following cases:

**Case 1:**  $x'_i \in S_i, x_i \in S_i$ . We have

$$\hat{u}_i(x_i, d_i) = M_i + \mu d_i, \hat{u}_i(x'_i, d'_i) = M_i + \mu d'_i.$$

It follows that  $d'_i > d_i$ . Hence

$$\lambda d_i + (1 - \lambda)d'_i > \lambda d_i + (1 - \lambda)d_i = d_i.$$

Since  $\lambda x_i + (1 - \lambda)x'_i \in S_i$ , we deduce

$$\begin{aligned} \hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) = \\ M_i + \mu(\lambda d_i + (1 - \lambda)d'_i) > M_i + \mu d_i = \hat{u}_i(x_i, d_i). \end{aligned}$$

**Case 2:**  $x'_i \in S_i, x_i \notin S_i$ . It implies  $u_i(x'_i) > u_i(x_i)$ . Since  $u_i$  is a strictly quasi-concave function, we obtain

$$u_i(\lambda x_i + (1 - \lambda)x'_i) > u_i(x_i).$$

**2a:** If  $\lambda x_i + (1 - \lambda)x'_i \in S_i$ , then

$$\hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) = M_i + \mu(\lambda d_i + (1 - \lambda)d'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i).$$

**2b:** If  $\lambda x_i + (1 - \lambda)x'_i \notin S_i$ , then

$$\hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) = u_i(\lambda x_i + (1 - \lambda)x'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i).$$

**Case 3**  $x'_i \notin S_i, x_i \notin S_i$ . We have

$$\hat{u}_i(x_i, d_i) = u_i(x_i), \hat{u}_i(x'_i, d'_i) = u_i(x'_i).$$

This follows  $u_i(x'_i) > u_i(x_i)$ . Similarly as above we consider

**3a:** If  $\lambda x_i + (1 - \lambda)x'_i \in S_i$ , then

$$\begin{aligned} \hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) = M_i + \mu(\lambda d_i + (1 - \lambda)d'_i) \\ > u_i(x_i) = \hat{u}_i(x_i, d_i). \end{aligned}$$

**3b:** If  $\lambda x_i + (1 - \lambda)x'_i \notin S_i$ , then

$$\hat{u}_i(\lambda x_i + (1 - \lambda)x'_i, \lambda d_i + (1 - \lambda)d'_i) = u_i(\lambda x_i + (1 - \lambda)x'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i).$$

We have proved that the function  $\hat{u}_i$  is strictly quasi-concave.

It remains to prove that the  $\hat{u}_i$  has no satiation point.

Indeed, let  $(x_i, d_i) \in X_i \times \mathbb{R}_+$ . We consider the following cases

**Case 1:**  $x_i \notin S_i$ . Take  $x'_i \in X_i$  such that  $u_i(x'_i) > u_i(x_i)$  and  $d'_i = d_i$ . We have  $\hat{u}_i(x'_i, d_i) \geq u_i(x'_i) > u_i(x_i) = \hat{u}_i(x_i, d_i)$ .

**Case 2:**  $x_i \in S_i$ . Take  $x'_i = x_i$  and  $d'_i > d_i$ . We have

$$\hat{u}_i(x'_i, d'_i) = \hat{u}_i(x'_i) + \mu d'_i > u_i(x_i) + \mu d_i = \hat{u}_i(x_i, d_i).$$

We have proved that the  $\hat{u}_i$  has no satiation point.

Let us consider the feasible set  $\hat{A}$  of  $\hat{\mathcal{E}}$ . We have:

$$\hat{A} = \{((x_i, d_i)_{i \in I}, (y_j, 0)_{j \in J}) : \forall i, x_i \in X_i, d_i \in \mathbb{R}_+, \forall j, y_j \in Y_j \text{ and}$$

$$\sum_{i \in I} x_i = \sum_{i \in I} e_i + \sum_{j \in J} y_j, \sum_{i \in I} d_i = \sum_{i \in I} \delta_i.\}$$

It is obvious that  $\hat{A}$  is compact.

It is also obvious that Assumptions  $(H_1)$ ,  $(H_2)$ ,  $(H_3)$  and  $(H_5)$  are fulfilled in economy  $\hat{\mathcal{E}}$ .

Apply the previous theorem.

There exists an equilibrium  $((x_i^*, d_i^*)_{i \in I}, (y_j^*, 0)_{j \in J}, (p^*, q^*))$  with  $(p^*, q^*) \neq (0, 0)$ . It satisfies:

$$(i) \sum_{i \in I} (x_i^*, d_i^*) = \sum_{i \in I} (e_i, \delta_i) + \sum_{j \in J} (y_j^*, 0),$$

$$(ii) \text{ for any } i \in I, p^* \cdot x_i^* + q^* d_i^* = p^* \cdot e_i + \sum_{j \in J} \theta_{ij} \sup(p^* Y_j + q^* \times 0) + q^* \delta_i,$$

and

$$(iii) \text{ for any } j \in J, p^* y_j^* = \sup(p^* Y_j).$$

Observe that since  $\mu > 0$ , the price  $q^*$  must be nonnegative.

We claim that  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is an equilibrium with dividends  $(q^* \delta_i)_{i \in I}$ .

Indeed, first, we have

$$\forall i \in I, p^* \cdot x_i^* \leq p^* \cdot e_i + \sum_{j \in J} \theta_{ij} p^* y_j^* + q^* \delta_i.$$

Now, let  $x_i \in X_i, u_i(x_i) > u_i(x_i^*)$ . That implies  $x_i^* \notin S_i$  and hence  $\hat{u}_i(x_i^*, d_i^*) = u_i(x_i^*)$ .

We consider the following cases:

**Case 1:**  $x_i \notin S_i$ . We have  $\hat{u}_i(x_i, 0) = u_i(x_i)$  and hence  $\hat{u}_i(x_i, 0) > \hat{u}_i(x_i^*, d_i^*)$ . Applying the previous theorem, we obtain

$$p^* x_i = p^* .x_i + q^* \times 0 > p^* e_i + \sum_{j \in J} \theta_{ij} . \sup p^* .Y_j + (q^* \delta_i).$$

**Case 2:**  $x_i \in S_i$ . We have

$$\hat{u}_i(x_i, 0) = M_i + \mu \times 0 = M_i = u_i(x_i) > u_i(x_i^*) = \hat{u}_i(x_i^*, d_i^*).$$

Similarly as above we obtain the inequality

$$p^* x_i > p^* e_i + \sum_{j \in J} \theta_{ij} \sup p^* .Y_j + q^* \delta_i.$$

It is clear that  $p^* .y_j^* = \sup p^* .y_j$  with  $y_j \in Y_j$ .

We have proved our claim. ■

### 3. Remarks

3.1. We can replace  $(H_3)$  by  $(H_3 \text{ bis})$ : "The total production set  $\sum_{j \in J} Y_j$  is closed, non-empty and convex" as in Florig and Yildiz (2002), i.e., we do not require every  $Y_j$  be convex.

3.2. If  $x_i^*$  is not a satiation point, then  $q^* d_i^* = 0$ . Indeed, let  $u_i(x_i) = \hat{u}_i(x_i, 0) > u_i(x_i^*) = \hat{u}_i(x_i^*, d_i^*)$ . We then have

$$p^* x_i > p^* e_i + \sum_{j \in J} \theta_{ij} \sup p^* .Y_j + q^* \delta_i = p^* .x_i^* + q^* d_i^*.$$

For any  $\lambda \in ]0, 1[$ , from the strict quasi-concavity of  $u_i$ , we have  $u_i(\lambda x_i + (1 - \lambda)x_i^*) > u_i(x_i^*)$  and hence  $p^* .(\lambda x_i + (1 - \lambda)x_i^*) > p^* .x_i^* + q^* d_i^*$ . Letting  $\lambda$  converge to zero, we obtain  $q^* d_i^* \leq 0$ . Thus  $q^* d_i^* = 0$ . That means that a consumer who does not meet satiation point will sell her/his endowment of the additional good if  $q^* > 0$ . Observe also that if there exists  $i \in I$  such that  $x_i^*$  is not a satiation point, then  $p^* \neq 0$  (if not we have  $0 = q^* \delta_i$ ; this implies  $q^* = 0$ : a contradiction with  $(p^*, q^*) \neq 0$ ).

One deduces from that, if  $x_i^*$  is not a satiation point for every  $i \in I$ , then  $q^* = 0$ , since  $\sum_{i \in I} d_i = \sum_{i \in I} \delta_i > 0$ . In this case,  $p^* \neq 0$ , and  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is a Walras equilibrium.

3.3. Let  $I_1 = \{i \in I : x_i^* \text{ is not a satiation point}\}$ , and  $I_2 = I \setminus I_1$ . Then  $\sum_{i \in I_1} q^* \delta_i = \sum_{i \in I_2} q^* .d_i^* - \sum_{i \in I_2} q^* \delta_i$ . This shows that the group of agents who meet satiation buy the additional good from the group of agents who do not meet satiation.

## 4 No-arbitrage condition and existence of equilibrium with dividends

If we assume that  $0 \in Y_j$  for every  $j$ , and if  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is an equilibrium with dividends, we will have

$$p^* .e_i = p^* .e_i + \sum_j \theta_{ij} p^* .0 \leq p^* .e_i + \sum_{j \in J} \theta_{ij} p^* y_j^* + q^* \delta_i.$$

Hence, for every  $i$ , we have  $u_i(x_i^*) \geq u_i(e_i)$ . We therefore define the set of individually rational feasible allocations  $\tilde{A}$ . More precisely:

$$\tilde{A} = \left\{ ((x_i), (y_j)) \in \prod_{i \in I} X_i \times \prod_{j \in J} Y_j : \sum_{i \in I} x_i = \sum_{i \in I} e_i + \sum_{j \in J} y_j, \forall i, u_i(x_i) \geq u_i(e_i) \right\}.$$

We will replace  $(H_4)$  by

$(H_4bis)$  The set  $\tilde{A}$  is compact.

We have the following result:

**Theorem 3** (i) Assume  $(H_1), (H_2), (H_3), (H_4bis), (H_5)$ , for every  $j, 0 \in Y_j$  and

$$\forall i, \forall x_i \in X_i, \exists x'_i \in X_i \text{ such that } u_i(x'_i) > u_i(x_i).$$

Then there exists a Walras equilibrium.

(ii) Assume  $(H_1), (H_2), (H_3), (H_4bis), (H_5)$  and for every  $j, 0 \in Y_j$ . Then there exists an equilibrium with dividends.

**Proof.** The proof of (i) can be adapted from Dana, Le Van and Magnien (1999). The proof of (ii) is similar to the one of the theorem in section 2. ■

Let  $P_i = \{x_i \in X_i : u_i(x_i) \geq u_i(e_i)\}$ , and  $W_i$  be the recession cone of  $P_i$ . Elements in  $W_i$  which are different from zero will be called useful vectors for agent  $i$  (see Werner, 1987). Let  $Z_j$  denote the recession cone of  $Y_j$ . Take some  $\gamma_j \in Y_j$ . Then  $\gamma_j + \lambda z_j \in Y_j, \forall \lambda \geq 0, \forall z_j \in Z_j$ . We call useful

production vector for firm  $j$  any vector  $z_j \in Z_j \setminus \{0\}$  (the producer can produce an infinitely large quantity  $\gamma_j + \lambda z_j, \lambda \geq 0$ ).

Let  $p \in \mathbb{R}^l$ . We say that there exists an opportunity of arbitrage associated with  $p$  if either there exists  $i \in I, w_i \in W_i \setminus \{0\}$ , such that  $p \cdot w_i \leq 0$ , or there exists  $j \in J, z_j \in Z_j \setminus \{0\}$ , such that  $p \cdot z_j > 0$ .

A price vector  $p \in \mathbb{R}^l$  is a no-arbitrage price for the economy if  $\forall i \in I, w_i \in W_i \setminus \{0\} \implies p \cdot w_i > 0$ , and  $\forall j \in J, z_j \in Z_j \setminus \{0\} \implies p \cdot z_j \leq 0$ .

We introduce the following No-Arbitrage Condition:

(NA) There exists a no-arbitrage price for the economy.

Let us replace  $(H_3)$  by

$(H_{3ter})$  For each  $j \in J$ , the set  $Y_j$  is nonempty closed convex and  $Y = \sum_{j \in J} Y_j$  is closed. Moreover, for every  $j, 0 \in Y_j$  and  $Y \cap -Y = \{0\}$ .

We have the following result

**Theorem 4** (i) Assume  $(H_1), (H_2), (H_{3ter}), (H_5)$  and (NA). Then there exists an equilibrium with dividends.

(ii) Assume the following No-Halfline condition :

(NHL) For  $i \in I$ , if  $w_i \in W_i \setminus \{0\}$ , then for any  $x \in P_i$ , there exists  $\lambda > 0$ , such that  $u_i(x + \lambda w_i) > u_i(x)$ .

Then,  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is an equilibrium with dividends  $\implies p^*$  is a no-arbitrage price.

**Proof.** (i) It suffices to prove that  $\tilde{A}$  is compact. Assume the contrary. Then there is a sequence  $\left( (x_i^n)_i, (y_j^n)_j \right)_{n=1, \dots, \infty} \in \tilde{A}$  such that  $\sigma_n = \sum_i \|x_i^n\| + \sum_j \|y_j^n\| \rightarrow +\infty$  when  $n \rightarrow \infty$ . Since

$$\frac{\sum_i x_i^n}{\sigma_n} = \frac{\sum_i e_i}{\sigma_n} + \frac{\sum_j y_j^n}{\sigma_n}$$

we can assume, without loss of generality, that

$$\left( \left( \frac{x_i^n}{\sigma_n} \right)_i, \left( \frac{y_j^n}{\sigma_n} \right)_j \right) \rightarrow \left( (w_i)_i, (z_j)_j \right) \in \left( \left( \prod_i W_i \right) \times \left( \prod_j Z_j \right) \right) \setminus \{0\}$$

. Moreover, we have

$$\sum_i w_i = \sum_j z_j.$$

Let  $p$  be a no-arbitrage price. If  $(w_i)_i \neq 0$ , we have a contradiction:  $0 < p \cdot \sum_i w_i = p \cdot \sum_j z_j \leq 0$ . If  $(w_i)_i = 0$ , then  $\sum_j z_j = 0$ . We have:  $\sum_{k \neq j} z_k = -z_j$ . From *(H<sub>3ter</sub>)*,  $\sum_{k \neq j} z_k \in Y$  and  $z_j \in Y$ . Hence  $z_j \in Y \cap -Y$ . This implies  $z_j = 0$ . We have shown that, in this case, we have  $(z_j)_j = 0$  and a contradiction with  $((w_i)_i, (z_j)_j) \neq 0$ .

We have proved that  $\tilde{A}$  is compact.

(ii) Let  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  be an equilibrium with dividends. It is obvious that  $p^* \cdot z_j \leq 0$ , for every  $z_j \in Z_j$  since  $y_j^* + z_j \in Y_j$  and  $p^* \cdot y_j^* = \max p^* Y_j$ .

We have two cases.

**Case 1.** There exists some  $i \in I$  such that  $x_i^*$  is not a satiation point. From Remark 3.2.,  $p^* \neq 0$ . If  $w_i \in W_i \setminus \{0\}$ , then condition *(NHL)* implies  $u_i(x_i^* + \lambda w_i) > u_i(x_i^*)$ , for some  $\lambda > 0$ . Since  $((x_i^*)_{i \in I}, (y_j^*)_{j \in J}, p^*)$  is an equilibrium, we have  $p^* \cdot w_i > 0$ .

**Case 2.** For any  $i \in I$ ,  $x_i^*$  is a satiation point. Condition *(NHL)* implies that  $W_i = \{0\}$ , for every  $i$ . No-arbitrage condition is satisfied in this case with  $p^*$ . ■

## References

- [1] Allouch, N., Le Van, C., Page Jr., F.H., 2002, The geometry of arbitrage and the existence of competitive equilibrium, *Journal of mathematical economics*, 34, Nb04, 373 – 391.
- [2] Arrow, K.J., Debreu, G., 1954, Existence of equilibrium for a competitive economy, *Econometrica*, 22, N°3, 265-290.
- [3] Aumann, R.J., Drèze, J.H., 1986, Values of markets with satiation or fixed prices, *Econometrica*, 54, 1271-1318.
- [4] Cornet, B., Topuzu, M., Yildiz, A., 2003, Equilibrium theory with a measure space of possibly satiated consumers, *Journal of mathematical economics*, 39, Nb03 – 4, 175 – 196.
- [5] Dana, R.A., Le Van, C., Magnien, F., 1999, On the different notions of arbitrage and existence of equilibrium, *Journal of economic theory*, 87, 169-193.

- [6] Debreu, G., 1959, *Theory of value*, New York: Wiley.
- [7] Drèze, J.H., Muller, H., 1980, Optimality properties of rationing schemes, *Journal of economic theory*, 23, 131-149.
- [8] Florig, M., Yildiz, A., 2002, Equilibrium without nonsatiation, *Cahiers de la MSE, série bleue*, 2002-107.
- [9] Makarov, V.L., 1981, Some results on general assumptions about the existence of economic equilibrium, *Journal of mathematical economics*, 8, 87-100.
- [10] Mas-Collel, A., 1992, Equilibrium theory with possibly satiated preferences, in Majumdar, M. (Ed.): *Equilibrium and dynamics: proceedings of the essays in honour of David Gale*, MacMillan, London, 201-213.
- [11] Page Jr., F.H., Wooders, M., 1996, A necessary and sufficient condition for compactness of individually rational and feasible outcomes and existence of an equilibrium, *Economics letters*, 52, 153-162.
- [12] Werner, J., 1987, Arbitrage and the existence of competitive equilibrium, *Econometrica*, 55, 1403-1418.