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**Sufficient conditions for a “simple” decentralization  
with consumption externalities**

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*Version révisée*



# Sufficient conditions for a “simple” decentralization with consumption externalities<sup>★</sup>

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## Abstract

We consider a pure exchange economy with consumption externalities in preferences. Using the notion of competitive equilibrium à la Nash, we point out that a simple condition for restoring the Second Welfare Theorem is that the set of Pareto optima is included in the set of *internal* Pareto optima. We provide the Social Redistribution assumption to ensure such inclusion. This assumption is weaker than other relevant assumptions that have been studied in the literature. We also introduce new conditions in a differential setting, called (Strong) Directional Social Redistribution. Our assumptions entail interesting results on the decentralized implementation of Pareto optima, that relate the competitive supporting price and the *social* shadow price. Finally, we show that, for Bergson-Samuelson utility functions, Strong Directional Social Redistribution is ensured by a specific property of the Jacobian matrix, which has a natural interpretation in terms of externalities.

*JEL classification:* D11, D50, D62.

*Key words:* Consumption externalities, competitive equilibrium à la Nash, decentralization, social redistribution.

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## 1 Introduction

The Second Welfare Theorem is one of the key results in economics. It is the basis of the normative approach to welfare studies in competitive markets. This theorem sets general conditions under which Pareto optima can be decentralized through competitive prices and transfers. Further, under differentiable assumptions, it is used to determine the market prices that implement Pareto optima as competitive equilibria.

In addition to standard technical assumptions, the Second Welfare Theorem relies on another crucial assumption, namely ‘self-interest’, that is individual preferences depend on own consumption only. In absence of the latter, the Second Welfare Theorem may fail. However, externalities in preferences are important and they have been largely recognized in the literature.<sup>1</sup> In the past few years, the normative approach to welfare analysis of competitive equilibrium in economies with externalities has attracted renewed attention, see Geanakoplos and Polemarchakis (2008), Dufwenberg et al. (2011), and Bourlès et al. (2017).<sup>2</sup> Restoring the Second Welfare Theorem in the presence of externalities is an important issue and finds its place in this recent research stream.

We consider an exchange economy with consumption externalities in preferences. We use the notion of competitive equilibrium that combines Arrow–Debreu with Nash. That is, all individuals maximize their preferences by taking as given both the prices and the consumptions of the others and, at equilibrium, markets clear. This definition generalizes the classical notion without externalities. The decentralization of Pareto optima in economies with externalities can be obtained using other equilibrium notions à la Lindahl and personalized prices.<sup>3</sup> As is well known, the decentralization through Lindahl equilibria is more problematic as it requires the opening of personalized markets for externalities. A simple assumption restores the Second Welfare Theorem, without calling for personalized prices and Lindahl equilibria. Such assumption demands  $PO$ , the set of Pareto optima, to be a subset of  $PO^{INT}$ ,

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<sup>1</sup> For a survey, see for example Sobel (2005).

<sup>2</sup> Geanakoplos and Polemarchakis (2008) focus on Pareto improving taxes. The existence of these policies is generic in the space of preferences under strong assumptions of separability. Dufwenberg et al. (2011) study the First and Second Welfare Theorems in economies with consumption and wealth externalities. They obtain the second one under the Social Monotonicity assumption on Bergson-Samuelson utilities. Bourlès et al. (2017) use a network approach in a general equilibrium model with one commodity (income) and Bergson-Samuelson utilities that are additively separable. Under small positive external effects, Nash equilibria with transfers can be seen as solutions of a social planner problem with personalized costs.

<sup>3</sup> See the seminal models in Arrow (1969), and Laffont (1976).

the set of *internal* Pareto optima.  $PO^{INT}$  is the set of feasible allocations that are Pareto optimal whenever individuals take the consumptions of the others as given, see Winter (1969) and Parks (1991). In this paper, we provide new assumptions that make the set inclusion  $PO \subseteq PO^{INT}$  hold and new results for decentralizing Pareto optima via a competitive price.

The first assumption is Social Redistribution. The idea is that *internal* Pareto improvements can be translated through redistributions into Pareto improvements. This assumption generalizes a condition proposed in Osana (1972), that we call Strong Redistribution.<sup>4</sup> Theorem 1 is one of our key result. It states that, under classical assumptions on utilities, Social Redistribution is equivalent to the set inclusion  $PO \subseteq PO^{INT}$ . Importantly, Social Redistribution is weaker than all the other assumptions used in the literature for decentralization (i.e., Strong Redistribution, Non-malevolence, and Social Monotonicity).<sup>5</sup>

We then extend our analysis to differentiable economies, because the differentiable approach is a powerful tool to characterize supporting prices through marginal utilities, see for instance Mas-Colell (1985), and Balasko (1988). We translate Social (Strong) Redistribution into assumptions on first order effects of externalities, namely (Strong) Directional Social Redistribution.<sup>6</sup> The idea of (Strong) Directional Social Redistribution is that any direction for achieving an *internal* Pareto improvement can be redistributed so that everybody is strictly better-off. In differentiable economies, it is easier to check whether (Strong) Directional Social Redistribution is satisfied or not, rather than checking for Social (Strong) Redistribution. To the best of our knowledge, only the concept of Non-malevolence has been translated into Local Non-malevolence in a differential setting, see Parks (1991). Hence, the differentiable characterizations of (Strong) Social Redistribution make a step forward in this line of research. Further, (Strong) Directional Social Redistribution is weaker than Local Non-malevolence. It is worth noticing that all our assumptions do not require preferences to be separable.<sup>7</sup>

In differentiable economies, our key results study properties of equilibrium prices for decentralization. Under Social Redistribution, at a Pareto optimal allocation, all the gradients with respect to own consumptions are positively proportional. Therefore, the competitive supporting price is determined as in the case without externalities (see Theorem 2). In the presence of externalities, the “social” price integrates social marginal utilities.<sup>8</sup> Theorem 3 states that,

<sup>4</sup> See condition (vi) in Theorem 2 of Osana (1972).

<sup>5</sup> The intuition of all these assumptions is given in Subsection 1.1.

<sup>6</sup> By “first order effects of externalities”, we mean the gradients of utility functions.

<sup>7</sup> The preferences of individual  $i$  are separable if it holds that: if she prefers  $x_i$  to  $x'_i$  for a given externality, then she always prefers  $x_i$  to  $x'_i$  whatever is the externality.

<sup>8</sup> We call “social” price the shadow price that supports a Pareto optimal allocation

under Strong Directional Social Redistribution, the competitive supporting price is positively proportional to the social price. However, this property fails in a simple economy where the externality is generated by the presence of a public good (or “bad”) only. In such economies, Social Redistribution allows us to obtain an interesting comparison between the social price and the competitive supporting price (see Corollary 1). Theorem 4 shows that Local Non-malevolence is equivalent to have that the competitive supporting price is positively proportional to personalized prices for externalities. Furthermore, we illustrate several economic settings where the decentralization through a competitive price can be done using our assumptions and results, while it could not be done under the stronger assumptions adopted in literature, see Examples (a), (d), (e).(i) and (e).(ii).

Finally, we focus on the class of preferences of the Bergson-Samuelson form. This class, firstly introduced in Bergson (1938) and Samuelson (1947), has been largely studied in economic theory, especially in welfare economics. For these preferences, positive externalities entail the set inclusion  $PO \subseteq PO^{INT}$ . Example (e) shows that this is not the case for more general preferences. We further characterize Strong Directional Social Redistribution in terms of properties of the Jacobian matrix of Bergson-Samuelson utilities (see Theorem 5). This is relevant because the entries of this matrix deliver significant information on the nature of externalities. For instance, Strong Directional Social Redistribution is ensured if all individuals positively care about one individual or if the negative external effects are not too strong. These conditions can be easily applied to a larger class of preferences, that we call *proportional marginal utilities*.

The paper is organized as follows. Section 2 presents the economy with consumption externalities, and the notions of competitive equilibria and Pareto optima. In Section 3, we provide the simple version of the Second Welfare Theorem, we introduce Social Redistribution, and we compare it with previous assumptions. Section 4 is devoted to a differentiable approach. We provide (Strong) Directional Social Redistribution assumptions, and we study the relationships with Social (Strong) Redistribution and Local Non-malevolence. In Section 5, we present all our results on the decentralized implementation of Pareto optimal allocations. Section 6 discusses the class of Bergson-Samuelson utility functions. All the results are proved in Section 7.

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for the utilitarian social planner whose welfare function is a weighted average of the utilities of the individuals in the economy.

### 1.1 Related literature

In the literature on economies with externalities, the decentralization of Pareto optima via a competitive price has been done under different assumptions. The first one was introduced in Winter (1969), later called Non-malevolence in Rader (1972). The idea is that every internal Pareto improvement, where one individual  $h$  is strictly better off, is also a true Pareto improvement where the same individual  $h$  is still strictly better off. In Strong Redistribution (Osana, 1972), that Pareto improvement is achieved by a redistribution where another individual, not necessarily individual  $h$ , is strictly better off. However, such redistributions are not necessarily feasible. In the context of Bergson-Samuelson utilities, a similar assumption suggested by Dufwenberg et al. (2011) is Social Monotonicity. The intuition is to obtain a Pareto improvement by increasing the consumption of everybody and strictly increasing the total consumption of at least one commodity. Actually, this is a kind of monotonicity at the aggregate level. Local Non-malevolence is a differentiable version of Non-malevolence. This assumption has been introduced in Parks (1991) to ensure that the marginal utility of every individual  $i$  with respect to the consumption of another individual  $j$  is proportional to  $j$ 's own marginal utility (see Theorem 1 in Parks, 1991).

## 2 The economy with consumption externalities

There is a finite number of commodities labeled by the superscript  $\ell \in \mathcal{L} = \{1, \dots, L\}$ , and a finite number of individuals labeled by the subscript  $i \in \mathcal{I} = \{1, \dots, I\}$  with  $I \geq 2$ . The commodity space is  $\mathbb{R}^L$  and  $p = (p^\ell)_{\ell \in \mathcal{L}} \in \mathbb{R}^L$  denotes a price system.

For every  $i \in \mathcal{I}$ , the consumption set of individual  $i$  is the positive orthant of the commodity space. The consumption of individual  $i$  is  $x_i = (x_i^\ell)_{\ell \in \mathcal{L}} \in \mathbb{R}_+^L$ , and the consumption of individuals other than  $i$  is  $x_{-i} = (x_j)_{j \neq i} \in \mathbb{R}_+^{L(I-1)}$ . The bundle  $x = (x_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{LI}$  is an allocation. With innocuous abuse of notation, an allocation  $x$  is also denoted by  $(x_i, x_{-i})$ . The preferences of individual  $i$  are represented by a utility function

$$u_i : x \in \mathbb{R}_+^{LI} \mapsto u_i(x) \in \mathbb{R},$$

where  $u_i(x)$  is the utility level of individual  $i$  associated with the allocation  $x = (x_i, x_{-i}) \in \mathbb{R}_+^{LI}$ . That is,  $u_i$  also describes how individual  $i$ 's preferences are affected by the consumption  $x_{-i}$  of the other individuals.

The initial endowment of individual  $i$  is  $e_i = (e_i^\ell)_{\ell \in \mathcal{L}} \in \mathbb{R}_+^L$ , and the economy

with consumption externalities is denoted by  $\mathcal{E} = (u_i, e_i)_{i \in \mathcal{I}}$ . The set of feasible allocations of the economy  $\mathcal{E}$  is  $F = \{x = (x_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{LI} : \sum_{i \in \mathcal{I}} x_i = r\}$ , where  $r = \sum_{i \in \mathcal{I}} e_i$  is the total endowment.

We first remind the classical definition of Pareto optimality.

**Definition 1 (Pareto optimal allocation)** *An allocation  $x^* \in \mathbb{R}_+^{LI}$  is a Pareto optimal allocation of the economy  $\mathcal{E}$  if  $x^* \in F$ , and there is no feasible allocation  $x \in F$  such that  $u_i(x) \geq u_i(x^*)$  for all  $i \in \mathcal{I}$  with  $u_k(x) > u_k(x^*)$  for some  $k \in \mathcal{I}$ .  $PO$  denotes the set of all Pareto optimal allocations of the economy  $\mathcal{E}$ .*

Next we introduce the notion of *internal* Pareto optimality, see for instance Winter (1969) and Parks (1991). That is,  $x^* = (x_i^*)_{i \in \mathcal{I}}$  is an internal Pareto optimal allocation if it is efficient with respect to the preferences  $(u_i(\cdot, x_{-i}^*))_{i \in \mathcal{I}}$  that keep the externalities constant. Hence, internal Pareto optimality is a less demanding notion of efficiency that keeps the environment as given.

**Definition 2 (Internal Pareto optimal allocation)** *An allocation  $x^* = (x_i^*)_{i \in \mathcal{I}} \in \mathbb{R}_+^{LI}$  is an internal Pareto optimal allocation of the economy  $\mathcal{E}$  if  $x^* \in F$ , and there is no feasible allocation  $x = (x_i)_{i \in \mathcal{I}} \in F$  such that  $u_i(x_i, x_{-i}^*) \geq u_i(x^*)$  for all  $i \in \mathcal{I}$  with  $u_k(x_k, x_{-k}^*) > u_k(x^*)$  for some  $k \in \mathcal{I}$ .  $PO^{INT}$  denotes the set of all internal Pareto optimal allocations of the economy  $\mathcal{E}$ .*

We also remind the classical notions of competitive (quasi-)equilibrium à la Nash.

**Definition 3** *Let  $x^* = (x_i^*)_{i \in \mathcal{I}} \in \mathbb{R}_+^{LI}$  be an allocation and  $p^* \in \mathbb{R}^L \setminus \{0\}$  be a price system.*

- (i) *The bundle  $(x^*, p^*)$  is a competitive equilibrium of the economy  $\mathcal{E}$  if  $x^* \in F$ , and for every  $i \in \mathcal{I}$ ,  $p^* \cdot x_i^* \leq p^* \cdot e_i$  and  $u_i(x^*) \geq u_i(x_i, x_{-i}^*)$  for all  $x_i \in \mathbb{R}_+^L$  such that  $p^* \cdot x_i \leq p^* \cdot e_i$ .*
- (ii) *The bundle  $(x^*, p^*)$  is a quasi-equilibrium of the economy  $\mathcal{E}$  if  $x^* \in F$ , and for every  $i \in \mathcal{I}$ ,  $p^* \cdot x_i^* = p^* \cdot e_i$  and  $p^* \cdot x_i^* \leq p^* \cdot x_i$  for all  $x_i \in \mathbb{R}_+^L$  such that  $u_i(x_i, x_{-i}^*) \geq u_i(x^*)$ .*

### 3 Sufficient conditions for the Second Welfare Theorem

We provide a simple version of the Second Welfare Theorem with consumption externalities, that is Proposition 1. The idea is that the set inclusion  $PO \subseteq$

$PO^{INT}$  ensures that every Pareto optimal allocation  $x^*$  is internally Pareto optimal. This means that  $x^*$  is a Pareto optimal allocation of an economy without externalities, where the individual preferences are

$$v_i^*(x_i) := u_i(x_i, x_{-i}^*), \forall i \in \mathcal{I}.$$

Hence, under standard assumptions that are usually made in economies without externalities,<sup>9</sup> one simply applies the classical separating hyperplane theorem to  $x^*$  and the economy  $(v_i^*)_{i \in \mathcal{I}}$  for proving the Second Welfare Theorem.

In order to state this result, we need some additional notations. For every  $i \in \mathcal{I}$  and for every allocation  $x = (x_i, x_{-i}) \in \mathbb{R}_+^{LI}$ , define the sets of all consumptions  $y_i$  that are (resp., strictly) preferred by individual  $i$  to  $x_i$  when the environment is  $x_{-i}$ , i.e.,  $R_i(x) = \{y_i \in \mathbb{R}_+^L : u_i(y_i, x_{-i}) \geq u_i(x_i, x_{-i})\}$  and  $P_i(x) = \{y_i \in \mathbb{R}_+^L : u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})\}$ .

**Proposition 1 (The Second Welfare Theorem)** *Assume that  $PO \subseteq PO^{INT}$ . Let  $x^* = (x_i^*)_{i \in \mathcal{I}} \in PO$  such that*

- (a) *there is at least one individual  $\bar{h} \in \mathcal{I}$  with  $P_{\bar{h}}(x^*) \neq \emptyset$  (i.e.,  $x_{\bar{h}}^*$  is not a satiation consumption for individual  $\bar{h}$  given  $x_{-\bar{h}}^*$ ),*
- (b) *for all  $i \in \mathcal{I}$  and for all  $(y_i, x_i) \in \mathbb{R}_+^L \times \mathbb{R}_+^L$  with  $y_i \in P_i(x_i, x_{-i}^*)$ , it holds that  $ty_i + (1-t)x_i \in P_i(x_i, x_{-i}^*)$  for every  $t \in ]0, 1[$ ,*
- (c) *for all  $i \in \mathcal{I}$ ,  $R_i(x^*)$  is closed in  $\mathbb{R}_+^L$ .*

*Then, there exists  $p^* \in \mathbb{R}^L \setminus \{0\}$  such that  $(x^*, p^*)$  is a quasi-equilibrium of the economy  $\mathcal{E}^* = (u_i, x_i^*)_{i \in \mathcal{I}}$ .*

**Remark 1** *If, in addition to the assumptions of Proposition 1, for every  $i \in \mathcal{I}$ , there exists  $x_i \in \mathbb{R}_+^L$  such that  $p^* \cdot x_i < p^* \cdot x_i^*$  and the set  $P_i(x^*)$  is open in  $\mathbb{R}_+^L$ , then the Pareto optimal allocation  $x^*$  can be achieved as a competitive equilibrium of the economy  $\mathcal{E}$  by using suitable lump-sum transfers.*

Therefore, the key assumption of Proposition 1 is the inclusion  $PO \subseteq PO^{INT}$ . The aim of this section is to study under what conditions this set inclusion is satisfied. In Subsection 3.1, we introduce a novel assumption, called Social Redistribution, in the spirit of Strong Redistribution.<sup>10</sup> In Subsection 3.2, we compare these assumptions with (Weak) Non-malevolence and Social Monotonicity.

<sup>9</sup> Assumptions (a), (b), and (c) in Proposition 1 are obviously satisfied if  $u_i(\cdot, x_{-i}^*)$  is monotone, quasi-concave and continuous in  $x_i$ . We recall that  $u_i(\cdot, x_{-i}^*)$  is monotone in  $x_i$  if for all  $(x'_i, x_i) \in \mathbb{R}_+^{2L}$  with  $x'_i \gg x_i$ , it holds that  $u_i(x'_i, x_{-i}^*) > u_i(x_i, x_{-i}^*)$ .

<sup>10</sup> That is, condition (vi) in Theorem 2 of Osana (1972).

### 3.1 Social Redistribution

We provide the Social Redistribution assumption. The intuition is that, starting from any internal Pareto improvement  $x'$ , where a specific individual  $h$  is strictly better off, there is a redistribution of  $x'$  that leads to a Pareto improvement  $y$ .

**Assumption 1 (Social Redistribution)** *For all feasible allocations  $x \in F$ , there is at least one individual  $h$  such that for all feasible allocations  $x' = (x'_i)_{i \in \mathcal{I}} \in F$  with  $u_i(x'_i, x_{-i}) \geq u_i(x)$  for all  $i \in \mathcal{I}$  and  $u_h(x'_h, x_{-h}) > u_h(x)$ , there exists an allocation  $y = (y_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{L\mathcal{I}}$  such that*

- (1)  $\sum_{i \in \mathcal{I}} y_i = \sum_{i \in \mathcal{I}} x'_i$ , and
- (2)  $u_i(y) \geq u_i(x)$  for all  $i \in \mathcal{I}$  with  $u_k(y) > u_k(x)$  for some individual  $k$ .

It is worth noting that Social Redistribution is weaker than Strong Redistribution, because in Strong Redistribution the internal Pareto improvement  $x'$  is not required to be feasible. Consequently, the suitable redistribution  $y$  of  $x'$  is not feasible as well. Focusing on feasible redistributions allows us for a complete characterization of the set inclusion  $PO \subseteq PO^{INT}$ . This key result is given below.

**Theorem 1** *Assume that for all  $i \in \mathcal{I}$ ,  $u_i$  is continuous and strongly monotone in  $x_i$ .<sup>11</sup> Social Redistribution is equivalent to the set inclusion  $PO \subseteq PO^{INT}$ .*

Example (a) illustrates a natural setting in which Social Redistribution holds true, but Strong Redistribution is not satisfied. This is because Strong Redistribution allows for large deviations of internal Pareto improvements. However, in the presence of negative externalities, if  $\sum_{i \in \mathcal{I}} x'_i$  is too large, then it may not be possible to redistribute it to obtain a true Pareto improvement  $y$ .

**Example (a).** There is one commodity and two individuals. The total endowment is  $r = 1$ , and the utility functions are:

$$u_1(x_1, x_2) = x_1 - \lambda \max\{x_2 - \delta, 0\} \text{ and } u_2(x_1, x_2) = x_2 - \lambda \max\{x_1 - \delta, 0\}$$

with  $\lambda > 0$  and  $\frac{1}{2} \leq \delta \leq 1$ . In this example, if the consumption of an individual is above a certain threshold  $\delta$ , then the other suffers because he is jealous. The parameter  $\lambda$  can be interpreted as a measure of jealousy about the excess

<sup>11</sup> We recall that  $u_i$  is strongly monotone in  $x_i$  if for every  $x_{-i} \in \mathbb{R}_+^{L(I-1)}$  and for every  $(x'_i, x_i) \in \mathbb{R}_+^{2L}$  with  $x'_i > x_i$ , it holds that  $u_i(x'_i, x_{-i}) > u_i(x_i, x_{-i})$ .

consumption of the others.<sup>12</sup>

It is easy to check that every feasible allocation  $x = (x_1, x_2) \in \mathbb{R}_+^2$  is internally Pareto optimal. Consequently, Social Redistribution is satisfied. We show that for  $\lambda$  large enough, this economy does not satisfy Strong Redistribution. Consider the feasible allocation  $x = (\delta, 1 - \delta)$ . Strong Redistribution fails at  $x$ , because for internal Pareto improvements  $x'$  that are far enough from the threshold  $\delta$ , it is not possible to redistribute  $x'$  to obtain a Pareto improvement  $y$ . This is because the negative external effect  $\lambda(y_i - \delta)$  becomes too large. Take, for instance, the internal Pareto improvement  $x' = (x'_1, x'_2) = (\delta + \frac{1}{2}, \delta + \frac{1}{2})$  that is not feasible. Let  $y = (y_1, y_2) \in \mathbb{R}_+^2$  satisfy Strong Redistribution, i.e.,  $y_1 + y_2 = x'_1 + x'_2 = 2\delta + 1$  and

$$\begin{aligned} u_1(y) &= y_1 - \lambda \max\{y_2 - \delta, 0\} \geq u_1(x) = \delta, \\ u_2(y) &= y_2 - \lambda \max\{y_1 - \delta, 0\} \geq u_2(x) = 1 - \delta \end{aligned}$$

with at least one strict inequality. Summing up, it must be that

$$y_1 + y_2 > 1 + \lambda \max\{y_2 - \delta, 0\} + \lambda \max\{y_1 - \delta, 0\}.$$

Since  $y_1 + y_2 > 2\delta$ , we have that  $y_i - \delta > 0$  for at least one individual  $i$ . Then, the above inequality fails for  $\lambda > 2\delta$ .

We let the reader check that  $x = (\delta, 1 - \delta)$  is a Pareto allocation. Hence, Proposition 1 and Theorem 1 ensure that  $x$  can be implemented as a competitive equilibrium.  $\square$

### 3.2 Comparison with (Weak) Non-malevolence and Social Monotonicity

In Parks (1991), one finds a notion of Non-malevolence weaker than the one in Winter (1969). We call this assumption Weak Non-malevolence. The difference between Non-malevolence and Weak Non-malevolence is that the latter requires that everybody is strictly better off. In order to remind the notion of Weak Non-malevolence, we need some additional notations.

For every  $i \in \mathcal{I}$  and for every allocation  $x \in \mathbb{R}_+^{LI}$ ,  $S_i(x)$  (resp.,  $S_i^{INT}(x)$ ) denotes the set of all (resp., internal) Pareto improvements of  $x$ , that are not necessarily feasible, for which individual  $i$  is strictly better off. That is,

$$S_i^{INT}(x) = \{y = (y_j)_{j \in \mathcal{I}} \in \mathbb{R}_+^{LI} : \forall j \in \mathcal{I}, y_j \in R_j(x) \text{ and } y_i \in P_i(x)\},$$

$$S_i(x) = \{y \in \mathbb{R}_+^{LI} : \forall j \in \mathcal{I}, u_j(y) \geq u_j(x) \text{ and } u_i(y) > u_i(x)\}.$$

<sup>12</sup> The analysis still holds true with two individual parameters  $\lambda_1 \neq \lambda_2$ .

**Assumption 2 (Weak Non-malevolence)** *The preferences are weakly non-malevolent if for all feasible allocations  $x \in F$ ,  $\bigcap_{h \in \mathcal{I}} S_h^{INT}(x) \subseteq \bigcap_{h \in \mathcal{I}} S_h(x)$ .*

Strong Redistribution is weaker than Weak Non-malevolence.<sup>13</sup> Indeed, consider any internal Pareto improvement for which individual  $h$  is strictly better off. If the preferences are continuous and strongly monotone in own consumption, then it is possible to redistribute some commodity  $\ell$  in such a way that everybody is internally strictly better off. By Weak Non-malevolence, at this new allocation, all the individuals are strictly better off, and consequently Strong Redistribution is satisfied. This result is stated below.

**Proposition 2** *Assume that for all  $i \in \mathcal{I}$ ,  $u_i$  is continuous and strongly monotone in component  $i$ . Then, Weak Non-malevolence entails Strong Redistribution for all individuals.*

In the following example, Strong Redistribution holds true, but Weak Non-malevolence is not satisfied. This is because, starting from an internal Pareto improvement, some individuals get worse without redistribution.

**Example (b)** There is one commodity and two individuals. The utility functions are  $u_1(x_1, x_2) = x_1$  and  $u_2(x_1, x_2) = x_2 - x_1$ . In order to get an internal Pareto improvement  $(x'_1, x'_2)$  of  $(x_1, x_2)$ , both individuals must increase their own consumptions. Then, one easily redistributes  $\varepsilon = (x'_1 + x'_2) - (x_1 + x_2) > 0$  in order to achieve an allocation  $y = (y_1, y_2)$  such that  $y_1 + y_2 = x'_1 + x'_2$  and everybody is strictly better off. Hence, Strong Redistribution is satisfied. However, Assumption 2 does not hold true. It is enough to consider allocations  $(x'_1, x'_2)$  with  $x'_1 > x'_2 > x_2 > x_1$ . All the individuals are strictly internal better off, but individual 2 gets worse, because  $x'_2 - x'_1 < x_2 - x_1$ .  $\square$

We now remind the context of the Social Monotonicity assumption. The individual preferences have a Bergson-Samuelson form, that is, for all  $i \in \mathcal{I}$ :

$$u_i(x) = V_i(m_1(x_1), \dots, m_i(x_i), \dots, m_I(x_I)),$$

where  $V_i$  is strictly increasing in component  $i$ , and that  $m_k$  is continuous, strongly monotone, and strictly quasi-concave for all  $k \in \mathcal{I}$ .

**Assumption 3 (Social Monotonicity)** *For every feasible allocations  $x = (x_i)_{i \in \mathcal{I}} \in F$  and for every  $z \in \mathbb{R}^L$  with  $z > 0$ , there exists  $(z_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{LI}$  such that  $z = \sum_{i=1}^I z_i$ , and  $V_i(m_1(x_1 + z_1), \dots, m_I(x_I + z_I)) \geq V_i(m_1(x_1), \dots, m_I(x_I))$  for all  $i \in \mathcal{I}$  with a strict inequality for some individual  $k$ .*

Remark that the preferences in Assumption 1 are more general than the ones

<sup>13</sup> Hence, Non-malevolence is stronger than Strong Redistribution as well.

in Dufwenberg et al. (2011), because they are not necessarily of the Bergson-Samuelson type. In the context of Dufwenberg et al. (2011), Social Monotonicity is stronger than Strong Redistribution as it is shown in Proposition 3. Another substantial difference is that, in Assumption 1 and Strong Redistribution, it is possible to decrease the consumption of some commodities. This is not allowed in Social Monotonicity since everybody increases his own consumption. This difference is illustrated in Example (c).

**Proposition 3** *Consider the model in Dufwenberg et al. (2011) defined above. Social Monotonicity implies Strong Redistribution for all individuals.*

The next example shows why Strong Redistribution and Social Monotonicity are not equivalent.

**Example (c)** There is one commodity and three individuals. The individual utility functions are:

$$u_1(x_1, x_2, x_3) = x_1 - x_2 - x_3 \text{ and } u_2(x_1, x_2, x_3) = x_2 - x_1 - x_3 = -u_3(x_1, x_2, x_3).$$

For each Pareto improvement, the only individual who can be strictly better off is individual 1. This is because, the utilities of individuals 2 and 3 move in opposite directions. Since individuals 1 and 2 negatively care about the consumption of individual 3, then  $x_3$  should decrease.<sup>14</sup> Hence, Social Monotonicity is not satisfied. However, Strong Redistribution holds true in the interior of the consumption sets. Indeed, since  $x_3 > 0$ , starting from an internal Pareto improvement, it is possible to achieve a Pareto improvement through a redistribution by decreasing  $x_3$ .  $\square$

Finally, we remark that, since Strong Redistribution entails Social Redistribution, either Weak Non-malevolence or Social Monotonicity is stronger than Social Redistribution. Furthermore, Examples (b) and (c) also show that either Weak Non-malevolence or Social Monotonicity is not equivalent to Social Redistribution.

## 4 A differentiable approach

From now on, we focus on a differential setting. For every individual  $i \in \mathcal{I}$ , we assume that the individual consumption set is the open set  $X_i = \mathbb{R}_{++}^L$ ,

<sup>14</sup>Indeed, take any  $(z_1, z_2, z_3)$  such that  $u_i(x_1 + z_1, x_2 + z_2, x_3 + z_3) \geq u_i(x_1, x_2, x_3)$  for all  $i$ , with at least one strict inequality. Then  $z_1 - z_2 - z_3 \geq 0$ ,  $z_2 - z_1 - z_3 = z_3 + z_1 - z_2 = 0$ . This implies that  $z_2 = z_1 + z_3$  and  $z_1 > z_2 + z_3$ . Consequently, it must be that  $z_3 < 0$ .

$e_i \in \mathbb{R}_{++}^L$ , and that the function  $u_i$  is  $\mathcal{C}^1$  on the set of allocations  $X = \prod_{i \in \mathcal{I}} X_i$ . In this context, the set  $F$  of feasible allocations is  $\{x = (x_i)_{i \in \mathcal{I}} \in X : \sum_{i \in \mathcal{I}} x_i = r\}$ .

In this section, we introduce two new notions, dubbed Directional Social Redistribution and Strong Directional Social Redistribution. We provide sufficient conditions for ensuring these assumptions. We then compare (resp., Strong) Directional Social Redistribution with Social (resp., Strong) Redistribution and with Local Non-malevolence. Finally, we remark that our notions and results also encompass the interesting framework where all commodities are not necessarily desirable for all individuals.

We begin with (Strong) Directional Social Redistribution assumptions. The idea is that, starting from directions  $(z'_i)_{i \in \mathcal{I}}$  in which individuals increase their utility levels in own consumption, it is possible to redistribute  $\sum_{i \in \mathcal{I}} z'_i$  in order to find a direction  $z = (z_i)_{i \in \mathcal{I}}$  in which everybody is strictly better-off when changes in the consumptions of the others are taken into account.

**Assumption 4 (Strong/Directional Social Redistribution)** *Directional Social Redistribution holds if for all feasible allocations  $x \in F$ , there is at least one individual  $h$  such that for all  $z' = (z'_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$  with  $\sum_{i \in \mathcal{I}} z'_i = 0$ ,  $D_{x_i} u_i(x) \cdot z'_i \geq 0$  for all  $i \in \mathcal{I}$  and  $D_{x_h} u_h(x) \cdot z'_h > 0$ , there exists  $z = (z_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$  such that*

- (a)  $\sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} z'_i$ , and
- (b)  $D_{x_i} u_i(x) \cdot z > 0$  for all  $i \in \mathcal{I}$ .

*Strong Directional Social Redistribution is defined as above, except that one does not require  $\sum_{i \in \mathcal{I}} z'_i$  to be equal to zero.*

By definition, Directional Social Redistribution is weaker than Strong Directional Social Redistribution. However, these two notions are not equivalent. We come back on this point in Sections 5 and 6.<sup>15</sup>

Observe that, if there are no externalities, i.e.,  $D_{x_j} u_i(x) = 0$  for all  $j \neq i$ , and there is at least one commodity  $\ell$  such that  $D_{x_\ell} u_i(x) > 0$  for all  $i \in \mathcal{I}$ , then both Directional Social Redistribution and its strong version are satisfied. Indeed, if  $D_{x_i} u_i(x) \cdot z'_i \geq 0$  for all  $i$  with a strict inequality for some  $h$ , then one can redistribute commodity  $\ell$  to obtain  $z = (z_i)_{i \in \mathcal{I}}$  such that  $\sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} z'_i$  and  $D_{x_i} u_i(x) \cdot z_i > 0$  for all  $i$ . Hence, conditions (a) and (b) are satisfied, because

<sup>15</sup> See Example (d) in Section 5, and Example (e).(i), (ii) in Subsection 6.1.

$D_x u_i(x) \cdot z = D_{x_i} u_i(x) \cdot z_i$  for all individuals.

Another sufficient condition to make both notions in Assumption 4 hold is that there is one commodity and a pair of individuals such that everybody cares about the consumption of one individual more than the consumption of the other. Formally, for all feasible allocations  $x$ , there exist at least one commodity  $\ell$  and two individuals  $k$  and  $j$  such that

$$D_{x_k^\ell} u_i(x) > D_{x_j^\ell} u_i(x), \forall i \in \mathcal{I}.$$

The idea is that it is possible to transfer commodity  $\ell$  from individual  $j$  to individual  $k$ , so that everybody is strictly better-off.<sup>16</sup> In particular, this condition holds true if  $D_{x_j^\ell} u_i(x) = 0$  and  $D_{x_k^\ell} u_i(x) > 0$  for all  $i$ , meaning that everybody is interested in the consumption of commodity  $\ell$  by individual  $k$  and completely indifferent to the one by individual  $j$ .

Under the usual quasi-concavity of utilities in own consumption, Directional Social Redistribution (resp., its strong version) entails Social Redistribution (resp., Strong Redistribution). The converse is also true, under additional assumptions. First, the utilities are quasi-concave in allocations. Second, it is possible to redistribute some desirable commodity to move from a directional improvement  $(D_x u_i(x) \cdot z)_{i \in \mathcal{I}} > 0$  to a position where everybody is strictly better off. This idea is summarized in conditions (1) and (2) of the next proposition. Further, we show that these conditions are satisfied in many economic environments.

**Proposition 4**

- (i) Assume that for all  $i \in \mathcal{I}$ , the function  $u_i$  is quasi-concave in  $x_i$  and  $D_{x_i} u_i(x) \neq 0$  for all  $x \in X$ , then Directional Social Redistribution (resp., its strong version) implies Social Redistribution (resp., Strong Redistribution).
- (ii) Assume that the function  $u_i$  is quasi-concave and there is one commodity  $\ell$  such that  $D_{x_i^\ell} u_i(x) > 0$  for all  $i \in \mathcal{I}$ . Social Redistribution implies Directional Social Redistribution, provided that for all  $k$  there exists  $v = (v_j)_{j \in \mathcal{I}} \in \mathbb{R}^I$ ,  $v \neq 0$  such that  $\sum_{j \in \mathcal{I}} v_j = 0$  and

$$\sum_{j \in \mathcal{I}} v_j D_{x_j^\ell} u_i(x) > 0, \forall i \neq k. \tag{1}$$

*Strong Redistribution implies Strong Directional Social Redistribution,*

<sup>16</sup> Indeed, one considers  $x_j^\ell - z_j^\ell$  and  $x_k^\ell + z_j^\ell$  with  $z_j^\ell > 0$ , that entail  $D_x u_i(x) \cdot z = (D_{x_k^\ell} u_i(x) - D_{x_j^\ell} u_i(x)) z_j^\ell > 0$  for all  $i \in \mathcal{I}$ .

provided that there exists  $v = (v_j)_{j \in \mathcal{I}} \in \mathbb{R}^I$ ,  $v \neq 0$  such that

$$\sum_{j \in \mathcal{I}} v_j D_{x_j^\ell} u_i(x) > 0, \forall i \in \mathcal{I}. \quad (2)$$

Notice that:

- conditions (1) and (2) are satisfied if commodity  $\ell$  is not an externality good,
- condition (2) holds true if first order external effects in commodity  $\ell$  are not too strong. This means that, as for commodity  $\ell$ , the individual marginal utilities dominate the global first order external effect. That is, for all  $x \in F$  there exists  $0 < \varepsilon_i \leq 1$  such that

$$\sum_{j \neq i} |D_{x_j^\ell} u_i(x)| < \varepsilon_i D_{x_i^\ell} u_i(x), \forall i \in \mathcal{I}.$$

Under this assumption, condition (1) is true if  $\sum_{i \in \mathcal{I}} \varepsilon_i \leq 1$ .

- Condition (2) is satisfied if there exists at least one individual  $i_0$  such that  $D_{x_{i_0}^\ell} u_i(x) > 0$  for all  $i \in \mathcal{I}$ . That is, the consumption of commodity  $\ell$  by one individual  $i_0$  is good for everybody else.

We now remind the notion of Local Non-malevolence.

**Assumption 5 (Local Non-malevolence)** *For all feasible allocations  $x = (x_i)_{i \in \mathcal{I}} \in F$  and for all  $z' = (z'_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$  such that  $D_{x_i} u_i(x) \cdot z'_i > 0$  for all  $i \in \mathcal{I}$ , it holds that  $D_x u_i(x) \cdot z' > 0$ ,  $\forall i \in \mathcal{I}$ .*

If at least one commodity is desirable for everybody, then it is possible to redistribute this commodity to move from a directionally internal improvement (i.e.,  $(D_{x_i} u_i(x) \cdot z'_i)_{i \in \mathcal{I}} > 0$ ) to a situation where everybody is internally strictly better off (i.e.,  $D_{x_i} u_i(x) \cdot z_i > 0$  for all  $i$ ). Therefore, in such a case, Strong Directional Social Redistribution is weaker than Local Non-malevolence, as stated below. Hence, Local Non-malevolence is stronger than Directional Social Redistribution as well.

**Proposition 5** *Assume that there is one commodity  $\ell$  such that  $D_{x_i^\ell} u_i(x) > 0$  for all  $i \in \mathcal{I}$ . Then, Local Non-malevolence implies Strong Directional Social Redistribution for all individuals.*

However, if Strong Directional Social Redistribution holds true, it may be that Local Non-malevolence fails. We come back on this issue in Section 6.<sup>17</sup> Therefore, Local Non-malevolence is not equivalent to Directional Social Redistribution as well.

<sup>17</sup> See Example (f) in Subsection 6.2.

We end this section by emphasizing that we are also interested in describing economic settings where individuals do not necessarily care about the consumption of all commodities. For this purpose, we introduce the following remark. It is worth mentioning that all our previous notions and results are easily adapted to this slightly different setting. Importantly, all the main results stated in the next sections also apply to the following setting.

**Remark 2** Let  $S_i \subseteq \mathcal{L}$  be the non-empty set of all commodities  $\ell$  such that  $u_i$  depends on  $x_i^\ell$ . We assume that  $\bigcup_{i \in \mathcal{I}} S_i = \mathcal{L}$ . For every individual  $i \in \mathcal{I}$ , the utility  $u_i$  is a  $C^1$  function defined on  $(x_{S_h})_{h \in \mathcal{I}}$ , where  $x_{S_h} = (x_h^\ell)_{\ell \in S_h} \in \mathbb{R}_{++}^{|S_h|}$  for all  $h \in \mathcal{I}$ . With innocuous abuse of notation, for every bundle related to individual  $i$ , we add a zero for each commodity  $\ell$  that does not belong to  $S_i$ . That is, for all  $i$ :

- any vector  $v_i = (v_i^\ell)_{\ell \in \mathcal{L}} \in \mathbb{R}^L$  is now defined by  $v_i^\ell = 0$  for all  $\ell \notin S_i$ ,
- the individual consumption set is

$$X_i = \{x_i = (x_i^\ell)_{\ell \in \mathcal{L}} \in \mathbb{R}_+^L : x_{S_i} \in \mathbb{R}_{++}^{|S_i|} \text{ and } x_i^\ell = 0, \forall \ell \notin S_i\},$$

- for all  $x = (x_i)_{i \in \mathcal{I}} \in X$ ,  $u_i(x)$  denotes  $u_i((x_{S_h})_{h \in \mathcal{I}})$ ,
- for all  $x = (x_i)_{i \in \mathcal{I}} \in X$  and for all  $h$ ,  $D_{x_h} u_i(x) \in \mathbb{R}_+^L$  denotes the bundle  $(D_{x_h^\ell} u_i(x))_{\ell \in \mathcal{L}}$  where  $D_{x_h^\ell} u_i(x) = 0$  whenever  $\ell \notin S_h$  and  $D_{x_h^\ell} u_i(x) > 0$  for all  $\ell \in S_i$ ,
- $e_i \in \mathbb{R}_+^L$ ,  $e_{S_i} \in \mathbb{R}_{++}^{|S_i|}$ , and  $r = \sum_{i \in \mathcal{I}} e_i \in \mathbb{R}_{++}^L$ .

## 5 Implementation of Pareto optimal allocations

In this section, we provide our main results on the decentralized implementation of Pareto optima, namely Theorems 2, 3 and 4. Corollary 1 is an important consequence of the first order conditions associated with Pareto optimality and Theorem 2. Example (d) illustrates how the implementation is done under (Directional) Social Redistribution.

By Social Redistribution and Theorem 1, every Pareto optimal allocation is internally Pareto optimal. Hence, under classical assumptions made in smooth economies, the competitive supporting price is determined through the first order conditions associated with internal Pareto optimality. This is the idea of Theorem 2.

**Theorem 2** Assume that for all  $i \in \mathcal{I}$ , the function  $u_i$  is quasi-concave in  $x_i$  and  $D_{x_i} u_i(x) \gg 0$  for all  $x \in X$ . Social Redistribution implies that for each  $x^* \in PO$ , the supporting price  $p^* = D_{x_1} u_1(x^*)$  implements  $x^*$  as a competitive

equilibrium of the economy  $\mathcal{E}^* = (u_i, x_i^*)_{i \in \mathcal{I}}$ . Furthermore, in the context of Remark 2,  $p^*$  is given by the following algorithm.

**Algorithm.** For all  $\ell \in \mathcal{L}$ ,  $A^\ell = \{i \in \mathcal{I} : \ell \in S_i\}$  is the set of individuals for whom commodity  $\ell$  is desirable. Note that  $A^\ell \neq \emptyset$  for all  $\ell$ .

*Step 1.* Since  $A^1 \neq \emptyset$ , without loss of generality, normalize the price of commodity 1, i.e.,  $p^{*1} = 1$ . Denote  $\mathcal{L}^1 = \{1\}$ . If  $\mathcal{L}^1 = \mathcal{L}$ , then the price  $p^*$  is determined. Otherwise, proceed as follows.

*Step  $k \geq 2$ .* There are only two possible alternatives in Cases 1 and 2 below.

*Case 1.* There exists at least one commodity  $\ell' \in \mathcal{L} \setminus \mathcal{L}^{k-1}$  such that  $A^{\ell'} \cap A^\ell \neq \emptyset$  for some  $\ell \in \mathcal{L}^{k-1}$ . For all such commodities  $\ell'$ , set

$$p^{*\ell'} = p^{*\ell} \frac{D_{x_i^{\ell'}} u_i(x^*)}{D_{x_i^\ell} u_i(x^*)}$$

for some  $i \in A^{\ell'} \cap A^\ell$ .<sup>18</sup> In this case, define  $\mathcal{L}^k := \mathcal{L}^{k-1} \cup \{\ell' \in \mathcal{L} \setminus \mathcal{L}^{k-1} : A^{\ell'} \cap A^\ell \neq \emptyset \text{ for some } \ell \in \mathcal{L}^{k-1}\}$ .

*Case 2.* For all commodities  $\ell' \in \mathcal{L} \setminus \mathcal{L}^{k-1}$  and  $\ell \in \mathcal{L}^{k-1}$ ,  $A^{\ell'} \cap A^\ell = \emptyset$ . Pick one commodity  $\ell' \in \mathcal{L} \setminus \mathcal{L}^{k-1}$  and set  $p^{*\ell'} = 1$ . In this case,  $\mathcal{L}^k := \mathcal{L}^{k-1} \cup \{\ell'\}$ .

At the end of Step  $k$ , if  $\mathcal{L}^k = \mathcal{L}$ , then the competitive price  $p^*$  is completely determined. Otherwise, go for Step  $k + 1$ . Importantly, the algorithm ends after at most  $L$  steps.  $\square$

We now present an interesting consequence of Strong Directional Social Redistribution and Farkas's Lemma, namely Theorem 3.

The marginal utility  $D_{x_j} u_i(x)$  of individual  $i$  with respect to the consumption of individual  $j$  can be interpreted as the personalized price of individual  $i$  for the externality  $x_j$  created by individual  $j$ . Therefore, Theorem 3 is an important result, because it implies that, if the positive weighted sum of the personalized supporting prices  $\sum_{i \in \mathcal{I}} \lambda_i D_{x_j} u_i(x)$  is  $j$ -invariant, then it is positively proportional to the price that supports  $x$  as a competitive equilibrium.

It is worth noting that, at a Pareto optimal allocation  $x$ , the  $j$ -invariance of  $\sum_{i \in \mathcal{I}} \lambda_i D_{x_j} u_i(x)$  is always satisfied, because of the first order conditions associated with Pareto optimality. Thus, the vector  $\beta = \sum_{i \in \mathcal{I}} \lambda_i D_{x_j} u_i(x)$  is interpreted as the *social* price at  $x$ . That is,  $\beta$  is the shadow price that supports  $x$  for

<sup>18</sup> Since  $x^* \in PO^{INT}$ , the price  $p^{*\ell'}$  is well defined, because it does not depend on the specific individual  $i \in A^{\ell'} \cap A^\ell$ .

the utilitarian social planner who achieves Pareto optima by maximizing some weighted averages of the individual utilities under the feasibility constraint. Hence, Theorem 3 ensures that the social price  $\beta$  can be used as a competitive supporting price.

**Theorem 3** *Let  $x = (x_i)_{i \in \mathcal{I}} \in F$  be a feasible allocation with  $D_{x_i} u_i(x) \neq 0$  for all  $i \in \mathcal{I}$ . Assume that there exist  $(\lambda_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^I \setminus \{0\}$  and  $\beta = (\beta^\ell)_{\ell \in \mathcal{L}} \in \mathbb{R}^L$  such that*

$$\sum_{i \in \mathcal{I}} \lambda_i D_{x_j} u_i(x) = \beta, \quad \forall j \in \mathcal{I}. \quad (3)$$

*Strong Directional Social Redistribution implies that  $\beta \neq 0$  and that  $\sum_{i \in \mathcal{I}} \lambda_i D_{x_j} u_i(x)$  is positively proportional to  $D_{x_j} u_j(x)$  for all  $j \in \mathcal{I}$ .*<sup>19</sup>

In the next corollary, we focus on economies where only one commodity is an externality for everybody. Without loss of generality, this is commodity  $L$ , and the utility functions are:

$$u_i(x_i, (x_j^L)_{j \in \mathcal{I}}), \quad \forall i \in \mathcal{I}.$$

This setting is interesting because it covers, for instance, the case of a *public good*, where individual preferences depend on the total amount  $\sum_{j \in \mathcal{I}} x_j^L$  of public good. Using Theorem 3 and first order conditions associated with Pareto optimality, we understand that Strong Social Redistribution fails, because the social price  $\beta$  is not positively proportional to the competitive supporting price. However, under Social Redistribution, we are able to compare these two prices. Importantly, the social price of a public good relative to another good is higher than its relative competitive price. On the other hand, if the externality commodity is a “bad”, its relative social price is lower than its relative competitive price. These results are given in the following corollary.

**Corollary 1** *For all  $i \in \mathcal{I}$ , the individual utility is  $u_i(x_i, (x_j^L)_{j \in \mathcal{I}})$ . Under the assumptions of Theorem 2, assume that there is an individual  $j$  such that  $D_{x_j^L} u_i(x) > 0$  (resp.,  $D_{x_j^L} u_i(x) < 0$ ) for all  $i \neq j$ .<sup>20</sup> Let  $x^*$  be a Pareto optimal allocation and  $\beta$  be the social price at  $x^*$ , Social Redistribution implies that*

$$\frac{\beta^L}{\beta^\ell} > \frac{p^{*L}}{p^{*\ell}} \quad (\text{resp.}, \quad \frac{\beta^L}{\beta^\ell} < \frac{p^{*L}}{p^{*\ell}}),$$

<sup>19</sup> In the context of Remark 2, condition (3) reads  $\sum_{i \in \mathcal{I}} \lambda_i D_{x_j^\ell} u_i(x) = \beta^\ell$ , for all  $j$  such that  $\ell \in S_j$ . In this case, for all  $j \in \mathcal{I}$ , the vector  $(\beta^\ell)_{\ell \in S_j}$  is positively proportional to  $(D_{x_j^\ell} u_j(x))_{\ell \in S_j}$ .

<sup>20</sup> It means that everybody cares positively (resp., negatively) about the consumption of the externality commodity by some individual  $j$ . According to Remark 2, this corollary still holds true if the externality commodity  $L \notin S_i$  for some individuals  $i$ .

where  $p^*$  is the competitive supporting price at  $x^*$ .

Using similar arguments for proving Theorem 3, we clarify Theorem 1 in Parks (1991) by providing a full characterization of Local Non-malevolence. We obtain that Local Non-malevolence is equivalent to having each personalized supporting price  $D_{x_j} u_i(x)$  positively proportional to the marginal utility  $D_{x_j} u_j(x)$  of individual  $j$  with respect to his own consumption. Therefore, if each personalized supporting price is positively proportional to the competitive supporting price, then their weighted average has the same property. Consequently, under Local Non-malevolence, the social price  $\beta$  can be used to implement Pareto optimal allocations as well.

**Theorem 4** *Let  $x = (x_i)_{i \in \mathcal{I}} \in F$  be a feasible allocation with  $D_{x_j} u_i(x) \neq 0$  for all  $(i, j) \in \mathcal{I} \times \mathcal{I}$ . Local Non-malevolence is satisfied if and only if  $D_{x_j} u_i(x)$  is positively proportional to  $D_{x_j} u_j(x)$  for all  $(i, j) \in \mathcal{I} \times \mathcal{I}$ .*

Finally, we provide an example of an economy where individual preferences are negatively impacted by the global emission of gas. Using our results, we illustrate how the decentralized implementation is done under Directional Social Redistribution, while this implementation could not be done if the assumptions were instead other assumptions proposed in the literature.

**Example (d).** There are three individuals and two commodities, e.g., two drivers and one non-driver. Commodity 1 is food and it is desirable for all the individuals. Every driver consumes gas (commodity 2) and pollutes. The non-driver does not care about consuming gas. All of them suffer from the global gas emission in the economy.

According to Remark 2, an allocation is  $x = ((x_1^1, x_1^2), (x_2^1, x_2^2), (x_3^1, 0)) \in X = \mathbb{R}_{++}^5 \times \{0\}$ , and the total endowment is  $r = (r^1, r^2) \in \mathbb{R}_{++}^2$ . The utility functions are of the CARA form. Remark that the following preferences are non-separable.<sup>21</sup>

$$\forall i = 1, 2, u_i(x) = -e^{-x_i^1} - e^{-x_i^2 + \varepsilon(x_1^2 + x_2^2)} \text{ and } u_3(x) = -e^{-x_3^1} - e^{\delta(x_1^2 + x_2^2)}.$$

The parameters  $\varepsilon$  and  $\delta$  measure the dissatisfaction about the global gas emission of drivers and non-driver, respectively. We assume that  $0 < \varepsilon < \frac{1}{2}$  and  $0 < \delta < 1$  is small enough.<sup>22</sup>

We prove that Directional Social Redistribution is satisfied. Indeed, take an

<sup>21</sup> The utility function  $u_i$  is separable if for all allocations  $x = (x_i, x_{-i}) \in \mathbb{R}_+^{LI}$  and  $x' = (x'_i, x'_{-i}) \in \mathbb{R}_+^{LI}$ , we have that  $u_i(x_i, x_{-i}) \geq u_i(x'_i, x_{-i})$  if and only if  $u_i(x_i, x'_{-i}) \geq u_i(x'_i, x'_{-i})$ .

<sup>22</sup> The upper bound  $\delta \leq \min \left\{ \varepsilon, \frac{1-2\varepsilon}{e^{(r^1+r^2)}} \right\}$  ensures the existence of Pareto optimal allocations in  $X$ .

allocation  $x$  and  $(z'_1, z'_2, z'_3) = ((z_1^1, z_1^2), (z_2^1, z_2^2), (z_3^1, 0))$  as in Assumption 4.

We first show that  $D_{x_1}u_1(x)$  and  $D_{x_2}u_2(x)$  are not positively collinear. Otherwise,  $D_{x_1}u_1(x) = \lambda D_{x_2}u_2(x)$  for some  $\lambda > 0$ . Furthermore,  $D_{x_1}u_1(x) = \gamma D_{x_3}u_3(x)$  for some  $\gamma > 0$ . Then,  $D_{x_1}u_1(x) \cdot z'_i \geq 0$  for all  $i$  with at least one strict inequality. Summing over  $i$ , we get  $D_{x_1}u_1(x) \cdot (z'_1 + z'_2 + z'_3) > 0$ , which is impossible because  $\sum_{i=1}^3 z'_i = 0$ .

Second, we need to verify that there exists  $z = ((z_1^1, z_1^2), (z_2^1, z_2^2), (z_3^1, 0))$  such that  $D_x u_i(x) \cdot z > 0$  for all  $i$ , and  $\sum_{i=1}^3 z_i = 0$ . It suffices to prove that the system below has a solution  $(z_1^1, z_1^2, z_3^1)$ , because  $z_1^1 + z_2^1 + z_3^1 = 0$  and  $z_1^2 + z_2^2 = 0$ .

$$\begin{pmatrix} D_{x_1}u_1(x) & \frac{D_{x_1^2}u_1(x)}{1-\varepsilon} & 0 \\ -D_{x_2}u_2(x) & -\frac{D_{x_2^2}u_2(x)}{1-\varepsilon} & -D_{x_2}u_2(x) \\ 0 & 0 & D_{x_3}u_3(x) \end{pmatrix} \begin{pmatrix} z_1^1 \\ z_1^2 \\ z_3^1 \end{pmatrix} \gg 0$$

Since  $D_{x_1}u_1(x)$  and  $D_{x_2}u_2(x)$  are not positively collinear, and  $D_{x_3}u_3(x) > 0$ , the above matrix has full rank. Hence, this system has a solution.

Then, Social Redistribution holds true by (i) of Proposition 4. Let  $x^*$  be a Pareto optimal allocation.<sup>23</sup> Applying the Algorithm in Theorem 2, we determine the competitive supporting price at  $x^*$ . Normalize the price of food, i.e.,  $p^1 = 1$ . Since  $A^1 \cap A^2 = \{1, 2\}$ , the price  $p^{*2}$  of gas is given by  $\frac{D_{x_1^2}u_1(x^*)}{D_{x_1}u_1(x^*)} = (1 - \varepsilon)e^{(x_1^{*1} - x_1^{*2}) + \varepsilon r^2}$ .

We now apply Corollary 1, in the case where gas is not desirable for the non-diver. Since gas is a negative externality for everybody, then the social price  $\frac{\beta^2}{\beta^1}$  of gas relative to food must be strictly lower than its relative competitive price  $p^{*2}$ . Consequently, by Theorem 3, Strong Directional Social Redistribution does not hold.

We end this example by noting that the implementation above cannot be obtained using all other assumptions. Indeed, Proposition 5 implies that Local Non-malevolence is not satisfied. From (ii) of Proposition 4, Strong Redistribution is not satisfied as well, and consequently, by Propositions 2 and 3, (Weak) Non-malevolence and Social Monotonicity do not hold.  $\square$

<sup>23</sup> Since  $u_i$  is  $\mathcal{C}^1$  and concave, using necessary and sufficient conditions for Pareto optimality, one easily verifies that  $x_1^{*1} - x_1^{*2} = x_2^{*1} - x_2^{*2}$ ,  $x_3^{*1} = r^1 - (x_1^{*1} + x_2^{*1})$ , and  $x_1^{*2} + x_2^{*2} = r^2$ .

## 6 Bergson-Samuelson utility functions and beyond

In this section, we study an economy with Bergson-Samuelson utility functions. That is, for all  $i \in \mathcal{I}$ ,  $u_i(x) = V_i(m_1(x_1), \dots, m_i(x_i), \dots, m_I(x_I))$  where  $V_i : D \subseteq \mathbb{R}^I \rightarrow \mathbb{R}$  is strictly increasing in component  $i$ , and  $m_j : X_i \rightarrow \mathbb{R}$  is monotone for all individuals  $j$ .<sup>24</sup>

In this setting, the set inclusion  $PO \subseteq PO^{INT}$  is ensured by positive externalities, but this is not the case for more general preferences, as it is shown in the first part of Example (e), see Subsection 6.1. The second part of Example (e) illustrates the difference between the social price of a public good and the competitive price in the decentralization of Pareto optimal allocations.

In Subsection 6.2, we characterize Strong Directional Social Redistribution in terms of properties of the Jacobian matrix of the utilities, which can be also extended to a larger class of preferences. Furthermore, Example (f) shows that this characterization can be used for the decentralized implementation, while this implementation could not be done using Local Non-malevolence.

### 6.1 Positive externalities

As pointed out in Winter (1969), in the context of the Bergson-Samuelson utility functions, positive externalities imply Non-malevolence, because  $V_i$  is increasing in every component  $j \neq i$ . Consequently, one obtains the inclusion  $PO \subseteq PO^{INT}$  by Theorem 1 and Proposition 2. However, the following example shows that positive externalities are not enough for getting this inclusion in the case of preferences that are not of the Bergson-Samuelson type, and consequently, the decentralized implementation is not sustainable via a competitive price. This example also shows that, if the positive external effects are the same for all the individuals, then the relative difference between the social price and the competitive price grows up with the number of individuals.

**Example (e).** There are two commodities and two individuals. Commodity 1 is a private commodity. Commodity 2 creates positive externalities, for instance, it is a public good. The total endowment is  $r = (r^1, r^2) \in \mathbb{R}_{++}^2$ . The log-linear utility functions are:

$$\begin{aligned} u_1(x_1, x_2) &= \ln x_1 + \ln [x_1^2 + \varepsilon_1(x_1^2 + x_2^2)], \\ u_2(x_1, x_2) &= \ln x_2 + \ln [x_2^2 + \varepsilon_2(x_1^2 + x_2^2)], \end{aligned}$$

<sup>24</sup> The domain  $D$  of  $V_i$  includes  $\prod_{i \in \mathcal{I}} m_i(X_i)$  where  $m_i(X_i)$  is the image of  $m_i$ .

where  $0 < \varepsilon_1, \varepsilon_2 < 1$ . The parameters  $\varepsilon_1$  and  $\varepsilon_2$  represent how much each individual cares about the global consumption of the public good.

We show that Directional Social Redistribution is not satisfied if  $\varepsilon_1 \neq \varepsilon_2$ . Consequently, from (ii) of Proposition 4 and Theorem 1, one concludes that  $PO \not\subseteq PO^{INT}$ .

We first provide the characterization of Pareto optimal allocations. Since  $u_i$  is  $\mathcal{C}^1$  and concave on  $\mathbb{R}_{++}^4$ , following Theorems 3.1.1 and 3.1.8 in Part I of Miettinen (1998), the necessary and sufficient conditions for Pareto optimality are satisfied. That is, there exist  $\lambda = (\lambda_1, \lambda_2) \gg 0$  and  $\beta = (\beta^1, \beta^2) \geq 0$  such that for all commodities  $\ell = 1, 2$ :

$$\begin{cases} \lambda_1 D_{x_j^\ell} u_1(x_1, x_2) + \lambda_2 D_{x_j^\ell} u_2(x_1, x_2) = \beta^\ell, \quad \forall j = 1, 2 \\ \beta^\ell (x_1^\ell + x_2^\ell - r^\ell) = 0 \end{cases}$$

Since  $D_x u_i(x_1, x_2) \gg 0$  and  $\lambda_i > 0$  for all  $i$ , then  $\beta = (\beta^1, \beta^2) \gg 0$ . Therefore, a Pareto optimal allocation  $(x_1, x_2) \in \mathbb{R}_{++}^4$  is completely characterized by the following system.

$$\begin{cases} \frac{x_1^1}{x_1^2 + \varepsilon_1(x_1^2 + x_2^2)} = \frac{x_2^1}{x_2^2 + \varepsilon_2(x_1^2 + x_2^2)} \\ x_1^\ell + x_2^\ell = r^\ell, \quad \forall \ell = 1, 2 \end{cases} \quad (4)$$

Since  $0 < \varepsilon_1, \varepsilon_2 < 1$ , one deduces that the set of Pareto optimal allocations is non-empty.<sup>25</sup>

Consider now a Pareto optimal allocation  $x$ . From the first equation in system (4), the individuals' marginal utilities with respect to the own consumption  $D_{x_i} u_i(x) = \left( \frac{1}{x_i^1}, \frac{1+\varepsilon_i}{x_i^2 + \varepsilon_i r^2} \right)$  are not collinear. Then, one deduces that there exists  $(z'_1, z'_2) \in \mathbb{R}^4$  with  $z'_1 + z'_2 = 0$  such that  $D_{x_i} u_i \cdot z'_i > 0$  for all  $i = 1, 2$ . However, there is no  $z = (z_1, z_2) = ((z_1^1, z_1^2), (z_2^1, z_2^2)) \in \mathbb{R}^4$  with  $z_1 + z_2 = 0$  such that  $D_x u_i \cdot z > 0$  for all  $i = 1, 2$ . Otherwise, one gets

$$\frac{1}{x_1^1} z_1^1 + \frac{1}{x_1^2 + \varepsilon_1 r^2} z_1^2 > 0 > \frac{1}{x_2^1} z_1^1 + \frac{1}{x_2^2 + \varepsilon_2 r^2} z_1^2,$$

which is impossible by the first equation in system (4). Therefore, Directional Social Redistribution is not satisfied.

We end this subsection by noting that, if the individuals have the same measure of appreciation of the global consumption of commodity 2, i.e.,  $\varepsilon_1 = \varepsilon_2 = \varepsilon$ , then:

<sup>25</sup> From system (4), every Pareto optimal allocation is determined by  $x_1^1 = r^1 \left[ \frac{x_1^2 + \varepsilon_1 r^2}{(1 + \varepsilon_1 + \varepsilon_2) r^2} \right]$  and  $x_1^\ell + x_2^\ell = r^\ell$  for all  $\ell = 1, 2$ .

- (i)  $PO \subseteq PO^{INT}$ , because, at a PO allocation, the first equation in (4) implies that the individuals' marginal rates of substitutions coincide:

$$MRS_{2,1}u_1(x) = \frac{x_1^1(1 + \varepsilon)}{x_1^2 + \varepsilon(x_1^2 + x_2^2)} = \frac{x_2^1(1 + \varepsilon)}{x_2^2 + \varepsilon(x_1^2 + x_2^2)} = MRS_{2,1}u_2(x).$$

Hence, by Theorem 1, Social Redistribution holds true. Then, the relative competitive price  $\frac{p^2}{p^1} = MRS_{2,1}u_1(x) = MRS_{2,1}u_2(x)$  implements the Pareto optimal allocation  $x$ , according to Theorem 2. Further, using similar arguments as in Example (d), one easily checks that this implementation could not be done using all the other assumptions.

- (ii) According to Corollary 1, the relative social price  $\frac{\beta^2}{\beta^1}$  is strictly greater than the relative price  $\frac{p^2}{p^1}$  that works as competitive price. Then, Strong Directional Social Redistribution is not satisfied, because of Theorem 3. However, Directional Social Redistribution is satisfied by (ii) of Proposition 4. This shows that Directional Social Redistribution is weaker than its strong version. Furthermore, the relative difference between these two prices increases when the number of individuals grows up. More precisely, if  $I \geq 2$  and  $u_i(x) = \ln x_i^1 + \ln [x_i^2 + \varepsilon \sum_{i=1}^I x_i^2]$  for all  $i$ , then

$$\frac{\beta^2/\beta^1 - p^2/p^1}{p^2/p^1} = \frac{(I - 1)\varepsilon}{1 + \varepsilon},$$

which goes to  $+\infty$  when  $I$  tends to  $+\infty$ .  $\square$

## 6.2 Bergson-Samuelson utility functions: A differential viewpoint

In this subsection, the domain  $D$  of  $V_i$  is an open subset of  $\mathbb{R}^I$ ,  $V_i$  and  $m_i$  are  $\mathcal{C}^1$  functions for all  $i \in \mathcal{I}$ , with  $D_{v_i}V_i(v) > 0$  for all  $v \in D$ , and  $D_{x_i}m_i(x_i) > 0$  for all  $x_i \in X_i$ . For a given  $x \in X$ ,  $V_{ij}(x)$  denotes the derivative of  $V_i$  with respect to component  $j$  at the point  $(m_i(x))_{i \in \mathcal{I}} \in D$ , and  $J_V(x) = [V_{ij}(x)]_{I \times I}$  is the Jacobian matrix of the mapping  $V = (V_i)_{i \in \mathcal{I}}$  at the point  $(m_i(x))_{i \in \mathcal{I}} \in D$ .

Note that the marginal utility  $D_{x_j}u_i(x)$  of individual  $i$  with respect to the consumption  $x_j$  of individual  $j$  is equal to  $\frac{V_{ij}(x)}{V_{jj}(x)}D_{x_j}u_j(x)$ . Then,  $V_{ij}(x)$  can be interpreted as a measure of how the change in the marginal utility  $D_{x_j}u_j(x)$  of individual  $j$  with respect to his own consumption affects  $D_{x_j}u_i(x)$ .

We now provide below three sufficient conditions on the entries of the Jacobian matrix  $J_V(x)$  that can be easily checked to ensure Strong Directional Social Redistribution in Assumption 4. We deduce these conditions from a

more general result stated at the end of this subsection, namely Theorem 5. These conditions are interesting because the entries of the matrix  $J_V(x)$  have a natural interpretation in terms of positive or negative externalities.

- (i)  $J_V(x)$  has all positive entries, i.e.,  $V_{ij}(x) \geq 0$  for all  $i \neq j$ . That is, we are in the presence of positive externalities. This is the case, for instance, of the altruistic utilities considered in Bourlès et al. (2017), where for all  $i$ :

$$u_i(x) = m_i(x_i) + \sum_{j \neq i} \alpha_{ij} m_j(x_j), \text{ with } 0 \leq \alpha_{ij} < 1.$$

It also covers the case without externalities, i.e.,  $J_V(x)$  is diagonal.

- (ii)  $J_V(x)$  has at least one column  $j$  with strictly positive entries, i.e.,  $V_{ij}(x) > 0$  for all  $i \neq j$ . This means that everyone positively cares about one specific individual  $j$ .
- (iii)  $J_V(x)$  has negative and positive entries, and there exists  $d = (d_i)_{i \in \mathcal{I}} \in \mathbb{R}_{++}^I$  such that

$$V_{ii}(x) > \sum_{j \neq i} \frac{d_j}{d_i} |V_{ij}(x)|, \forall i \in \mathcal{I}.$$

That is, we allow for a combination of negative and positive external effects, but they are not too strong, because they are dominated by  $V_{ii}(x)$ .

The next example is an application of condition (iii) and Theorem 3.

**Example (f).** We consider an economy of the Bergson-Samuelson type with two individuals and negative external effects, where

$$u_1(x) = m_1(x_1) - \varepsilon_1 m_2(x_2) \text{ and } u_2(x) = m_2(x_2) - \varepsilon_2 m_1(x_1),$$

with  $0 < \varepsilon_i < 1$  for all  $i = 1, 2$ . It is worth noting that this economy does not satisfy Local Non-malevolence, because of Theorem 4. However, this economy satisfies condition (iii) above, and consequently, Strong Directional Social Redistribution holds true. Importantly, as a consequence of Theorem 3, the social price  $\beta$  and the competitive price can be both used for the decentralized implementation, because they are positively proportional.  $\square$

The following theorem is the main result of this subsection.

In order to introduce this theorem, let  $\mathcal{H}$  be any non-empty subset of  $\mathcal{I}$  with cardinality  $H$ , and denote  $J_{V,\mathcal{H}}(x)$  the  $I \times H$  submatrix obtained from  $J_V(x)$  by deleting the column  $j$  for all  $j \in \mathcal{I} \setminus \mathcal{H}$ . Starting from a directionally internal Pareto improvement  $(D_{x_i} m_i(x_i) \cdot z'_i)_{i \in \mathcal{I}} > 0$ , we are interested in focusing on the set  $\mathcal{H}$  of individuals who are strictly better off, i.e.,  $D_{x_h} m_h(x_h) \cdot z'_h > 0$  for all  $h \in \mathcal{H}$ . Property (5) below ensures that it is possible to redistribute  $\sum_{h \in \mathcal{H}} z'_h$  among the individuals in  $\mathcal{H}$  to strictly improve all the individuals in  $\mathcal{I}$ .

Consequently, Strong Directional Social Redistribution holds true by simply checking property (5). Furthermore, if at least one commodity is desirable for everybody, the set  $\mathcal{H}$  can be easily extended to whole set  $\mathcal{I}$ .

**Theorem 5** *Let  $x \in F$  be a feasible allocation and let  $z' = (z'_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$  such that  $D_{x_i} m_i(x_i) \cdot z'_i \geq 0$  for all  $i \in \mathcal{I}$  with  $D_{x_h} m_h(x_h) \cdot z'_h > 0$  for some  $h$ . Denote  $\mathcal{H} = \{h \in \mathcal{I} \mid D_{x_h} m_h(x_h) \cdot z'_h > 0\}$ . Assume that*

$$\exists \theta \in \mathbb{R}_+^H \setminus \{0\} : J_{V, \mathcal{H}}(x) \theta^T \gg 0, \quad (5)$$

*then Strong Directional Social Redistribution is satisfied for all  $h \in \mathcal{H}$ .*<sup>26</sup>

One gets conditions (i), (ii), and (iii), by applying this theorem with  $\mathcal{H} = \mathcal{I}$ .

This analysis is easily extended to a large class of preferences that we call the class of *proportional marginal utilities*. More precisely, for all  $i, j$ , and for all  $x \in X$ , there exists  $a_{ij}(x) \in \mathbb{R}$  such that

$$D_{x_j} u_i(x) = a_{ij}(x) D_{x_j} u_j(x).$$

Remark that the class of Bergson-Samuelson utility functions is a proper subset of this class, where  $a_{ij}(x) = \frac{V_{ij}(x)}{V_{jj}(x)}$ .<sup>27</sup> Denote  $A_u(x) = [a_{ij}(x)]_{I \times I}$ . If there is  $\theta \in \mathbb{R}_+^I \setminus \{0\}$  such that  $A_u(x) \theta^T \gg 0$ , then using similar arguments as in Theorem 5, Strong Directional Social Redistribution holds true.

We end this subsection by observing that the preferences satisfying Local Non-malevolence also belong to the class of proportional marginal utilities. Indeed, Theorem 4 shows that Local Non-malevolence is equivalent to have that  $a_{ij}(x) \geq 0$  for all  $i, j$ .

## 7 Appendix

**Proof of Proposition 1.** Since  $PO \subseteq PO^{INT}$ , we have that  $x^* \in PO^{INT}$ . This means, by Definition 2, that  $x^*$  is a Pareto optimal allocation of the economy  $(u_i(\cdot, x_{-i}^*))_{i \in \mathcal{I}}$ , where the externalities have been fixed at  $x^*$ . Hence, the total endowment  $r$  does not belong to the set  $P_{\bar{h}}(x^*) + R_{-\bar{h}}(x^*)$ , where  $\bar{h}$  is

<sup>26</sup> This is because one redistributes  $\sum_{h \in \mathcal{H}} z'_i$  to obtain  $(z_h)_{h \in \mathcal{H}}$  such that  $(D_{x_h} m_h(x_h) \cdot z_h)_{h \in \mathcal{H}}$  is positively proportional to some  $\theta = (\theta_h)_{h \in \mathcal{H}}$ . Hence, everybody in  $\mathcal{I}$  is strictly better off, because  $D_{x_h} u_i(x) = V_{ih}(x) D_{x_h} m_h(x_h)$  implies that  $(D_{x_h} u_i(x) \cdot z)_{i \in \mathcal{I}}$  is positively proportional to  $J_{V, \mathcal{H}}(x) \theta^T$ .

<sup>27</sup> Take, for instance, any non-separable utility function  $u(x)$  and  $u_i(x) = F_i(u(x))$  for all  $i$ , where  $F_i$  is some  $C^1$  function. These functions are in the proportional marginal utilities class, but they are not of the Bergson-Samuelson type.

the individual given by (a). Furthermore, from (a) one gets  $P_h(x^*) + R_{-h}(x^*) \neq \emptyset$ . The proof then follows as in Section 6.4 of Debreu (1959), so that there exists  $p^* \in \mathbb{R}^L \setminus \{0\}$  such that  $(x^*, p^*)$  satisfies (ii) of Definition 3, if the initial endowment is  $x_i^*$  for all  $i$ . ■

**Proof of Theorem 1.** We first prove that Social Redistribution entails  $PO \subseteq PO^{INT}$ . Let  $x^* \in PO$  and assume by contradiction that  $x^* \notin PO^{INT}$ . Hence, by Definition 2, there exists  $x = (x_i)_{i \in \mathcal{I}} \in F$  such that  $u_i(x_i, x_{-i}^*) \geq u_i(x^*)$  for all  $i \in \mathcal{I}$  and  $u_j(x_j, x_{-j}^*) > u_j(x^*)$  for some  $j$ . Define the set  $\mathcal{J}(x, x^*) = \{i \in \mathcal{I} : u_i(x_i, x_{-i}^*) > u_i(x^*)\}$  and observe that it is non-empty since  $j \in \mathcal{J}(x, x^*)$ . Consider the individual  $h$  given by Assumption 1 for the feasible allocation  $x^*$ . Then, there are two possible cases.

**Case 1:**  $h \in \mathcal{J}(x, x^*)$ . By Assumption 1, there exists a feasible allocation  $y = (y_i)_{i \in \mathcal{I}} \in F$  such that  $u_i(y) \geq u_i(x^*)$  for all  $i \in \mathcal{I}$  with  $u_k(y) > u_k(x^*)$  for some individual  $k$ , which contradicts the Pareto optimality of  $x^*$ . This concludes the proof.

**Case 2:**  $h \notin \mathcal{J}(x, x^*)$ . Take  $j \in \mathcal{J}(x, x^*)$ . Assume that  $x_j^* \geq x_j$ . Since  $u_j$  is strongly monotone in component  $j$ , we get  $u_j(x^*) \geq u_j(x_j, x_{-j}^*)$ . But, this is not possible because  $u_j(x_j, x_{-j}^*) > u_j(x^*)$ . Then, there exists  $\ell \in \mathcal{L}$  such that  $x_j^\ell > x_j^{*\ell}$ . Since  $u_j$  is continuous in component  $j$ , there is  $\varepsilon > 0$  such that  $x_j - \varepsilon \mathbf{1}_\ell \in \mathbb{R}_+^L$  and  $u_j(x_j - \varepsilon \mathbf{1}_\ell, x_{-j}^*) > u_j(x^*)$ , where  $\mathbf{1}_\ell$  is the bundle whose component  $\ell$  is 1 and 0 otherwise. Furthermore,  $u_h(x_h + \varepsilon \mathbf{1}_\ell, x_{-h}^*) > u_h(x^*)$ , because  $u_h$  is strongly monotone in component  $h$ . Consider now the allocation  $\bar{x} = (\bar{x}_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{LI}$  with  $\bar{x}_j = x_j - \varepsilon \mathbf{1}_\ell$ ,  $\bar{x}_h = x_h + \varepsilon \mathbf{1}_\ell$ , and  $\bar{x}_i = x_i$  for all  $i \neq j, h$ . The feasibility of  $x$  clearly implies the feasibility of  $\bar{x}$ . Hence,  $\bar{x}$  is an internal feasible Pareto improvement of  $x^*$  with  $h \in \mathcal{J}(\bar{x}, x^*)$ , and one comes back to Case 1.

We now assume that  $PO \subseteq PO^{INT}$ , and we show that Social Redistribution holds true. Let  $x$  and  $x'$  be two feasible allocations as in Assumption 1. Then,  $x$  is not a Pareto optimal allocation of the economy  $(u_i(\cdot, x_{-i}))_{i \in \mathcal{I}}$ , where the externalities have been fixed at  $x$ . Hence, by Definition 2,  $x \notin PO^{INT}$ . Consequently,  $x \notin PO$ , and then there exists a feasible Pareto improvement  $y$  of  $x$ . Then Assumption 1 is satisfied. ■

**Proof of Proposition 3.** Take a feasible allocation  $x = (x_i)_{i \in \mathcal{I}} \in F$  and an allocation  $x' = (x'_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{LI}$  such that

$$V_i(m_1(x_1), \dots, m_i(x'_i), \dots, m_I(x_I)) \geq V_i(m_1(x_1), \dots, m_i(x_i), \dots, m_I(x_I))$$

for all  $i \in \mathcal{I}$ , with a strict inequality for some  $h$ . Since  $V_i$  is strictly increasing in component  $i$ , we must have  $m_i(x'_i) \geq m_i(x_i)$  for all  $i$ , with  $m_h(x'_h) > m_h(x_h)$ . Denote  $\mathcal{I}_1$  the set of individuals  $j$  where  $m_j(x'_j) > m_j(x_j)$ ,  $\mathcal{I}_1 \neq \emptyset$  since  $h \in \mathcal{I}_1$ . It follows from the strong monotonicity and the continuity of  $m_j$

that for all  $j \in \mathcal{I}_1$  there exists  $\tilde{x}_j \in \mathbb{R}_+^L$  such that  $\tilde{x}_j < x'_j$  and  $m_j(\tilde{x}_j) = m_j(x_j)$ . Then, consider the allocation  $\tilde{x} = (\tilde{x}_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{LI}$ , where  $\tilde{x}_i = x'_i$  for all  $i \in \mathcal{I} \setminus \mathcal{I}_1$ . Set  $\tilde{z} = \sum_{i=1}^I (x'_i - \tilde{x}_i)$  and notice that  $\tilde{z} > 0$  by construction.

By Assumption 3, there is  $(\tilde{z}_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{LI}$  such that  $\sum_{i=1}^I \tilde{z}_i = \sum_{i=1}^I (x'_i - \tilde{x}_i)$  and  $V_i(m_1(\tilde{x}_1 + \tilde{z}_1), \dots, m_I(\tilde{x}_I + \tilde{z}_I)) \geq V_i(m_1(\tilde{x}_1), \dots, m_I(\tilde{x}_I))$  for all  $i \in \mathcal{I}$ , with a strict inequality for some  $k$ . In order to ensure Strong Redistribution, it is enough to set  $y_i = \tilde{x}_i + \tilde{z}_i$  for all  $i \in \mathcal{I}$ . ■

**Proof of Proposition 4.** (i) Let  $x = (x_i)_{i \in \mathcal{I}} \in X$  and  $x' = (x'_i)_{i \in \mathcal{I}} \in X$  be two feasible allocations as in Assumption 1. Since  $D_{x_i} u_i(x) \neq 0$  and  $u_i$  is quasi-concave with respect to  $x_i$ , we have  $D_{x_i} u_i(x) \cdot (x'_i - x_i) \geq 0$  for all  $i \in \mathcal{I}$  with  $D_{x_h} u_h(x) \cdot (x'_h - x_h) > 0$ . Define  $z'_i = x'_i - x_i$  for all  $i \in \mathcal{I}$ . Since both  $x$  and  $x'$  are feasible allocations,  $\sum_{i \in \mathcal{I}} z'_i = 0$ . Then, Directional Social Redistribution in

Assumption 4 implies that there exists  $z = (z_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$  such that  $\sum_{i \in \mathcal{I}} z_i = 0$  and  $D_x u_i(x) \cdot z > 0$  for all  $i \in \mathcal{I}$ . Then, there exists  $0 < t < 1$  such that  $x + tz \in X$  and  $u_i(x + tz) - u_i(x) > 0$  for all  $i \in \mathcal{I}$ . Hence, the allocation  $y$  defined by  $y_i = x_i + tz_i$  for all  $i$ , satisfies conditions (1) and (2) in Assumption 1.

The arguments above also apply to show that Strong Directional Social Redistribution implies Strong Redistribution.

(ii) We first prove that Assumption 1 implies Directional Social Redistribution in Assumption 4. Let  $x = (x_i)_{i \in \mathcal{I}} \in F$  and  $z' = (z'_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$  as in Assumption 4, with  $\sum_{i \in \mathcal{I}} z'_i = 0$ . Since  $D_{x'_i} u_i(x) > 0$  for all  $i \in \mathcal{I}$ , one finds  $\bar{z} = (\bar{z}_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$

such that  $\sum_{i \in \mathcal{I}} \bar{z}_i = \sum_{i \in \mathcal{I}} z'_i = 0$  and  $D_{x_i} u_i(x) \cdot \bar{z}_i > 0$  for all  $i \in \mathcal{I}$ . Then, there exists  $t > 0$  such that  $x'_i := x_i + t\bar{z}_i \in X_i$ ,  $u_i(x'_i, x_{-i}) - u_i(x) > 0$  for all  $i \in \mathcal{I}$ , and  $x'$  is feasible since  $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$ . Then, Assumption 1 implies that there is

$y = (y_i)_{i \in \mathcal{I}} \in F$  such that  $u_i(y) \geq u_i(x)$  for all  $i \in \mathcal{I}$ , with a strict inequality for some  $k$ . Then, by quasi-concavity of  $u_i$ ,  $D_x u_i(x) \cdot (y - x) \geq 0$  for all  $i \in \mathcal{I}$ , and  $D_x u_k(x) \cdot (y - x) > 0$  since  $D_x u_k(x) \neq 0$ . By (1),  $\sum_{j \in \mathcal{I}} v_j D_{x'_j} u_j(x) > 0$  for

all  $i \neq k$ , for some  $v \neq 0$  such that  $\sum_{j \in \mathcal{I}} v_j = 0$ . Consider  $z = (z_i)_{i \in \mathcal{I}}$  defined by  $z_i = (y_i - x_i) + \varepsilon v_i \mathbf{1}_\ell$ , where  $\mathbf{1}_\ell \in \mathbb{R}_+^L$  is the bundle whose component  $\ell$  is 1

and 0 otherwise. We get  $\sum_{i \in \mathcal{I}} z_i = 0$ , and

$$\begin{aligned} D_x u_k(x) \cdot z &= D_x u_k(x) \cdot (y - x) + \varepsilon \sum_{j \in \mathcal{I}} v_j D_{x_j^\ell} u_k(x), \\ D_x u_i(x) \cdot z &= D_x u_i(x) \cdot (y - x) + \varepsilon \sum_{j \in \mathcal{I}} v_j D_{x_j^\ell} u_i(x), \forall i \neq k. \end{aligned}$$

Since  $D_x u_k(x) \cdot z > 0$  for some  $\varepsilon > 0$ , then  $z$  satisfies conditions (a) and (b) in Assumption 4.

Next, we show that Strong Redistribution implies Strong Directional Social Redistribution in Assumption 4. Let  $x = (x_i)_{i \in \mathcal{I}} \in F$  and  $z' = (z'_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$  as in Assumption 4. The proof is slightly different from the above one, because now  $\sum_{i \in \mathcal{I}} z'_i$  is not necessarily equal to zero. By (2), for some  $v \neq 0$ ,  $\sum_{j \in \mathcal{I}} v_j D_{x_j^\ell} u_i(x) > 0$  for all  $i \in \mathcal{I}$ . Denote  $\delta = \sum_{j \in \mathcal{I}} v_j$ , then  $D_{x_h} u_h(x) \cdot (z'_h - \varepsilon \delta \mathbf{1}_\ell) > 0$  for some  $\varepsilon > 0$ . Since  $D_{x_i^\ell} u_i(x) > 0$  for all  $i \in \mathcal{I}$ , one can find  $\bar{z} = (\bar{z}_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$  such that  $\sum_i \bar{z}_i = \sum_i z'_i - \varepsilon \delta \mathbf{1}_\ell$  and  $D_{x_i} u_i(x) \cdot \bar{z}_i > 0$  for all  $i \in \mathcal{I}$ . Similarly, as explained above, take  $z_i = \frac{y_i - x_i}{t} + \varepsilon v_i \mathbf{1}_\ell$  for all  $i \in \mathcal{I}$ . It is easy to check that  $\sum_i z_i = \sum_i z'_i$ , and  $D_x u_i(x) \cdot z = \frac{1}{t} D_x u_i(x) \cdot (y - x) + \varepsilon \sum_{j \in \mathcal{I}} v_j D_{x_j^\ell} u_i(x) > 0$  for all  $i \in \mathcal{I}$ . Hence,  $z$  satisfies Strong Directional Social Redistribution. ■

**Proof of Theorem 2.** Let  $x^* = (x_i^*)_{i \in \mathcal{I}} \in PO$  be a Pareto optimal allocation, from Theorem 1 one deduces that  $x^* \in PO^{INT}$ . That is,  $x^*$  is a Pareto optimal allocation of the economy  $(u_i(\cdot, x_{-i}^*))_{i \in \mathcal{I}}$  where the externalities have been fixed at  $x^*$ . Then, the vectors  $(D_{x_i} u_i(x_i^*, x_{-i}^*))_{i \in \mathcal{I}}$  are positively collinear, because of the standard first order sufficient conditions for Pareto optimality without externalities, and consequently, one sets  $p^* = D_{x_1} u_1(x_1^*, x_{-1}^*)$ .

In the context of Remark 2, following similar arguments as above, one easily verifies that the Algorithm works for determining the competitive supporting price. This concludes the proof of the theorem. ■

**Proof of Theorem 3.** Take any  $z' = (z'_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$  that satisfies Strong Directional Social Redistribution in Assumption 4, with  $\sum_{i \in \mathcal{I}} z'_i \neq 0$ . We first show that

$$\beta \cdot \sum_{i \in \mathcal{I}} z'_i > 0. \quad (6)$$

Indeed, let  $z = (z_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$  be the bundle satisfying conditions (a) and (b) of Assumption 4. Since  $(\lambda_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^I$  and  $\lambda_i > 0$  for some  $i$ , from condition

(b) of Assumption 4, one gets

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \lambda_i D_{x_j} u_i(x) \cdot z_j > 0$$

which is equivalent to  $\sum_{j \in \mathcal{I}} \sum_{i \in \mathcal{I}} \lambda_i D_{x_j} u_i(x) \cdot z_j > 0$ , so that by (3) we obtain  $\beta \cdot \sum_{j \in \mathcal{I}} z_j > 0$ . Then, using condition (a) of Assumption 4, we get (6). Notice that, condition (6) implies that  $\beta \neq 0$ .<sup>28</sup>

We now claim that for every  $j \in \mathcal{I}$ , we have that

$$\tilde{z}_j \in \mathbb{R}^L \text{ and } D_{x_j} u_j(x) \cdot \tilde{z}_j \geq 0 \implies \beta \cdot \tilde{z}_j \geq 0. \quad (7)$$

Consider individual  $h$  given in Assumption 4. Let  $j \neq h$ , and consider the sequence  $(z'_n)_{n \in \mathbb{N}}$ , where for every  $n \in \mathbb{N}$ ,  $z'_n = (z'_{ni})_{i \in \mathcal{I}} \in \mathbb{R}^{L\mathcal{I}}$  is defined by  $z'_{nj} = \tilde{z}_j$ ,  $z'_{nh} = \frac{1}{n} D_{x_h} u_h(x)$ , and  $z'_{ni} = 0$  for all  $i \neq j, h$ . By construction,  $D_{x_i} u_i(x) \cdot z'_{ni} \geq 0$  for all  $n$  and for all  $i$ , with  $D_{x_h} u_h(x) \cdot z'_{nh} > 0$  since  $D_{x_h} u_h(x) \neq 0$ . Then, by (6) we get  $\beta \cdot \sum_{i \in \mathcal{I}} z'_{ni} > 0$  for all  $n$  and, consequently, we obtain

$$\beta \cdot \left( \tilde{z}_j + \frac{1}{n} D_{x_h} u_h(x) \right) > 0, \quad \forall n \in \mathbb{N}.$$

Taking the limit over  $n$ , one gets  $\beta \cdot \tilde{z}_j \geq 0$ . If  $j = h$ , take  $z'_{nh} = \tilde{z}_h + \frac{1}{n} D_{x_h} u_h(x)$ , and  $z'_{ni} = 0$  for all  $i \neq h$ . Using the same arguments, (7) holds true for  $j = h$ , which completes the proof of the claim.

From (7), the system  $D_{x_j} u_j(x) \cdot \tilde{z}_j \geq 0$  and  $\beta \cdot \tilde{z}_j < 0$  does not have any solution. Then, applying Farkas's lemma, there exists  $\gamma_j \geq 0$  such that

$$\beta = \gamma_j D_{x_j} u_j(x).$$

Since  $\beta \neq 0$ , it must be that  $\gamma_j > 0$ . ■

**Proof of Corollary 1.** We remind that  $u_i(x) := u_i(x_i, (x_j^L)_{j \in \mathcal{I}})$  for all  $i \in \mathcal{I}$ . We provide the proof for the more general case where  $A^L \subseteq \mathcal{I}$ . According to Definition 1, following Theorem 3.1.1 in Part I of Miettinen (1998), the Fritz John necessary conditions for Pareto optimality hold true at  $x^*$ . That is, there exist  $\lambda = (\lambda_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{\mathcal{I}}$  and  $\beta \in \mathbb{R}_+^L$  with  $(\lambda, \beta) \neq 0$  such that

$$\begin{cases} (i) \sum_{i \in \mathcal{I}} \lambda_i D_{x_j^\ell} u_i(x^*) = \beta^\ell, \quad \forall j \in \mathcal{I} \text{ and } \forall \ell \neq L \\ (ii) \sum_{i \in \mathcal{I}} \lambda_i D_{x_j^L} u_i(x^*) = \beta^L, \quad \forall j \in A^L \end{cases}$$

<sup>28</sup> Using exactly the same arguments, one easily checks that  $\sum_{i \in \mathcal{I}} z'_i = 0$  is not possible.

Since  $D_{x_j^\ell} u_i(x^*) = 0$  for all  $j \neq i$  and for all  $\ell \neq L$ , the equations in (i) become:

$$\lambda_i D_{x_i^\ell} u_i(x^*) = \beta^\ell, \quad \forall i \in \mathcal{I} \text{ and } \forall \ell \neq L.$$

If  $\lambda_i = 0$  for some  $i$ , then  $\beta^\ell = 0$  for all  $\ell \neq L$ , which implies that  $\lambda_i = 0$  for all  $i$ , because  $D_{x_i^\ell} u_i(x^*) > 0$  by assumption. Hence, from (ii) one gets  $\beta^L = 0$ , and consequently  $(\lambda, \beta) = 0$ , which is a contradiction. Therefore, we must have  $\lambda_i > 0$  for all  $i$ , and consequently  $\beta^\ell > 0$  for all  $\ell \neq L$ . Then, for all  $\ell \neq L$  and for all  $j \in A^L$  we get

$$\frac{\beta^L}{\beta^\ell} = \sum_{i \in \mathcal{I}} \frac{\lambda_i D_{x_j^L} u_i(x^*)}{\lambda_i D_{x_i^\ell} u_i(x^*)} = \frac{D_{x_j^L} u_j(x^*)}{D_{x_j^\ell} u_j(x^*)} + \sum_{i \neq j} \frac{D_{x_j^L} u_i(x^*)}{D_{x_i^\ell} u_i(x^*)} \quad (8)$$

Now apply the Algorithm in Theorem 2 to determine the supporting price  $p^*$ . Notice that  $p^{*1} = 1$  and the Algorithm stops at Case 1 of Step 2 since  $\mathcal{L}^2 = \mathcal{L}$ . This is because  $A^1 \cap A^\ell = \mathcal{I} \neq \emptyset$  for all  $\ell \notin \{1, L\}$ , and  $A^1 \cap A^L \neq \emptyset$  since  $\emptyset \neq A^L \subseteq \mathcal{I} = A^1$ . The prices are then defined by

$$\begin{cases} p^{*\ell} = \frac{D_{x_i^\ell} u_i(x^*)}{D_{x_i^1} u_i(x^*)}, \quad \forall i \in \mathcal{I} \text{ and } \forall \ell \neq L \\ p^{*L} = \frac{D_{x_j^L} u_j(x^*)}{D_{x_j^1} u_j(x^*)}, \quad \forall j \in A^L \end{cases}$$

Let  $j$  such that  $D_{x_j^L} u_i(x) > 0$  (resp.,  $D_{x_j^L} u_i(x) < 0$ ) for all  $i \neq j$ . According to the notation given in Remark 2, we must have  $j \in A^L$ . From the system above, we get  $\frac{p^{*L}}{p^{*\ell}} = \frac{D_{x_j^L} u_j(x^*)}{D_{x_j^\ell} u_j(x^*)}$ . Hence, by (8) we obtain

$$\frac{\beta^L}{\beta^\ell} = \frac{p^{*L}}{p^{*\ell}} + \sum_{i \neq j} \frac{D_{x_j^L} u_i(x^*)}{D_{x_i^\ell} u_i(x^*)}$$

that concludes the proof, since  $D_{x_i^\ell} u_i(x^*) > 0$  for all  $i$  and for all  $\ell \neq L$ . ■

**Proof of Theorem 4.** Assume Assumption 5. We first claim that for all  $(i, j) \in \mathcal{I} \times \mathcal{I}$ ,

$$\tilde{z}_j \in \mathbb{R}^L : D_{x_j} u_j(x) \cdot \tilde{z}_j \geq 0 \implies D_{x_j} u_i(x) \cdot \tilde{z}_j \geq 0.$$

Take  $\tilde{z}_j$  such that  $D_{x_j} u_j(x) \cdot \tilde{z}_j \geq 0$ . Consider the sequence  $(z'_n)_{n \in \mathbb{N}}$ , where for every  $n > 0$ ,  $z'_n = (z'_{nh})_{h \in \mathcal{I}}$  is defined by  $z'_{nj} = \tilde{z}_j + \frac{D_{x_j} u_j(x)}{n}$  and  $z'_{nh} = \frac{D_{x_h} u_h(x)}{n}$  for all  $h \neq j$ . By construction,  $D_{x_h} u_h(x) \cdot z'_{nh} > 0$  for all  $n > 0$  and for all  $h$ , because  $D_{x_h} u_h(x) \neq 0$ . By Assumption 5, we have that  $D_x u_i(x) \cdot z'_n =$

$\sum_{h \neq j} D_{x_h} u_i(x) \cdot z'_{nh} + D_{x_j} u_i(x) \cdot z'_{nj} > 0$  for all  $n > 0$ , that is

$$\left[ \frac{1}{n} \left( \sum_{h \in \mathcal{I}} D_{x_h} u_i(x) \cdot D_{x_h} u_h(x) \right) + D_{x_j} u_i(x) \cdot \tilde{z}_j \right] > 0, \forall n > 0.$$

Taking the limit over  $n$ , we obtain  $D_{x_j} u_i(x) \cdot \tilde{z}_j \geq 0$ , and the claim is then proved. Hence, using Farkas's Lemma as in the proof of Theorem 3, we have that  $D_{x_j} u_i(x)$  is positively proportional to  $D_{x_j} u_j(x)$ .

It is easy to verify that the converse holds true. ■

Let  $\mathcal{H}$  be any non-empty subset of  $\mathcal{I}$  with cardinality  $H$ . The following lemma is used for proving Theorem 5.

**Lemma 1** *Let  $x \in F$  be a feasible allocation, and  $\mathcal{H}$  be a non-empty subset of  $\mathcal{I}$ . Take  $z' = (z'_h)_{h \in \mathcal{H}} \in \mathbb{R}^{LH}$  such that  $D_{x_h} m_h(x_h) \cdot z'_h > 0$  for all  $h \in \mathcal{H}$ , then for every  $\theta = (\theta_h)_{h \in \mathcal{H}} \in \mathbb{R}_{++}^H$ , there exist  $\lambda > 0$  and  $z = (z_h)_{h \in \mathcal{H}} \in \mathbb{R}^{LH}$  such that*

- (1)  $\sum_{h \in \mathcal{H}} z_h = \sum_{h \in \mathcal{H}} z'_h$ , and
- (2)  $D_{x_h} m_h(x_h) \cdot z_h = \lambda \theta_h$  for all  $h \in \mathcal{H}$ .<sup>29</sup>

**Proof of Lemma 1.** Without loss of generality,  $\mathcal{H} = \{1, \dots, H\}$ . Denote  $Z = \sum_{h \in \mathcal{H}} z'_h$  and fix  $\theta \gg 0$ . The proof is developed through two cases.

**Case 1:** all vectors  $(D_{x_h} m_h(x_h))_{h \in \mathcal{H}}$  are positively collinear. In this case, for all  $h \in \mathcal{H}$ , there is  $\alpha_h > 0$  such that  $D_{x_h} m_h(x_h) = \alpha_h D_{x_1} m_1(x_1)$ . Hence,  $D_{x_1} m_1(x_1) \cdot Z > 0$  since  $D_{x_h} m_h(x_h) \cdot z'_h > 0$  for all  $h$ , and consequently,  $Z \neq 0$ . In order to get the two conditions above, set  $\lambda = \frac{D_{x_1} m_1(x_1) \cdot Z}{\sum_{j \in \mathcal{H}} \frac{\theta_j}{\alpha_j}}$  and

$$z_h = \left( \frac{Z}{\sum_{j \in \mathcal{H}} \frac{\theta_j}{\alpha_j}} \right) \frac{\theta_h}{\alpha_h} \text{ for all } h \in \mathcal{H}.$$

**Case 2:**  $(D_{x_h} m_h(x_h))_{h \in \mathcal{H}}$  are not positively collinear. Hence, we must have  $H, L \geq 2$ . Conditions (1) and (2) hold true if there is  $z = (z_h)_{h \in \mathcal{H}}$  such that  $\sum_{h \in \mathcal{H}} z_h = Z$ ,  $\frac{D_{x_1} m_1(x_1)}{\theta_1} \cdot z_1 > 0$ , and  $\frac{D_{x_1} m_1(x_1)}{\theta_1} \cdot z_1 = \frac{D_{x_h} m_h(x_h)}{\theta_h} \cdot z_h$ ,  $\forall h \neq 1$ ,

<sup>29</sup> The idea is that, if it is possible to internally strictly improve individuals along some directions  $(z'_h)_{h \in \mathcal{H}}$ , then for any level of improvements  $\theta = (\theta_h)_{h \in \mathcal{H}} \in \mathbb{R}_{++}^H$ , it is possible to redistribute  $\sum_{h \in \mathcal{H}} z'_h$  in such a way that one achieves a proportion of these levels by using a same ratio  $\lambda$ .

that we rewrite as follows.

$$\begin{cases} \frac{D_{x_1} m_1(x_1)}{\theta_1} \cdot z_1 > 0 \\ \frac{D_{x_1} m_1(x_1)}{\theta_1} \cdot z_1 - \frac{D_{x_h} m_h(x_h)}{\theta_h} \cdot z_h = 0, \forall h = 2, \dots, H-1 \\ \left( \frac{D_{x_1} m_1(x_1)}{\theta_1} + \frac{D_{x_H} m_H(x_H)}{\theta_H} \right) \cdot z_1 + \sum_{h=2}^{H-1} \frac{D_{x_h} m_h(x_h)}{\theta_h} \cdot z_h = \frac{D_{x_H} m_H(x_H)}{\theta_H} \cdot Z \end{cases} \quad (9)$$

Define now the  $H \times L(H-1)$  matrix  $M$  associated with the system above. Notice that  $H \leq L(H-1)$  since  $H, L \geq 2$ .

$$M = \begin{pmatrix} \frac{D_{x_1} m_1(x_1)}{\theta_1} & 0 & 0 & \dots & 0 & 0 \\ \frac{D_{x_1} m_1(x_1)}{\theta_1} & -\frac{D_{x_2} m_2(x_2)}{\theta_2} & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \\ \frac{D_{x_1} m_1(x_1)}{\theta_1} & 0 & 0 & \dots & 0 & -\frac{D_{x_{H-1}} m_{H-1}(x_{H-1})}{\theta_{H-1}} \\ \frac{D_{x_1} m_1(x_1)}{\theta_1} + \frac{D_{x_H} m_H(x_H)}{\theta_H} & \frac{D_{x_H} m_H(x_H)}{\theta_H} & \frac{D_{x_H} m_H(x_H)}{\theta_H} & \dots & \frac{D_{x_H} m_H(x_H)}{\theta_H} & \frac{D_{x_H} m_H(x_H)}{\theta_H} \end{pmatrix}$$

We first prove that the matrix  $M$  has full row rank. Otherwise, there exists  $\gamma = (\gamma_h)_{h \in \mathcal{H}} \in \mathbb{R}^H$  with  $\gamma \neq 0$  such that  $\gamma M = 0$ . Then, we get  $\gamma_H \frac{D_{x_H} m_H(x_H)}{\theta_H} = \gamma_h \frac{D_{x_h} m_h(x_h)}{\theta_h}$  for all  $h = 2, \dots, H-1$ , and  $\gamma_H \frac{D_{x_H} m_H(x_H)}{\theta_H} = \left( \sum_{h=1}^{H-1} \gamma_h - \gamma_H \right) \frac{D_{x_1} m_1(x_1)}{\theta_1}$ . If  $\gamma_H \neq 0$ , then all gradients  $(D_{x_h} m_h(x_h))_{h \in \mathcal{I}}$  are positively collinear since  $D_{x_h} m_h(x_h) > 0$  for all  $h$ , a contradiction. Therefore, we must have  $\gamma_H = 0$ , and consequently, one obtains  $\gamma = 0$ , another contradiction. Therefore, system (9) has a solution  $(z_h)_{h \neq H}$  in  $\mathbb{R}^{L(H-1)}$ . Taking  $z_H = Z - \sum_{h \neq H} z_h$ , conditions (1) and (2) in the lemma are satisfied, that completes the proof. ■

**Proof of Theorem 5** Assume that  $\theta \in \mathbb{R}_{++}^H$ . Then there exist  $\lambda > 0$  and  $z = (z_h)_{h \in \mathcal{H}} \in \mathbb{R}^{LH}$  such that conditions (1) and (2) of the above lemma hold true. By definition of  $\mathcal{H}$ , for all  $j \in \mathcal{I} \setminus \mathcal{H}$ ,  $D_{x_j} u_j(x) \cdot z'_j = 0$  since  $D_{x_j} u_j(x) = V_{jj}(x) D_{x_j} m_j(x_j)$ . Taking  $z_j = z'_j$  for all  $j \in \mathcal{I} \setminus \mathcal{H}$ , we get  $\sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} z'_i$ , and for all  $i \in \mathcal{I}$ :

$$D_x u_i(x) \cdot z = \sum_{j \in \mathcal{I}} V_{ij}(x) D_{x_j} m_j(x_j) \cdot z_j = \sum_{h \in \mathcal{H}} V_{ih}(x) D_{x_h} m_h(x_h) \cdot z_h = \lambda \sum_{h \in \mathcal{H}} V_{ih}(x) \theta_h.$$

Therefore, conditions (a) and (b) of Assumption 4 are satisfied, because  $J_{V, \mathcal{H}}(x) \theta^T \gg 0$ .

Assume now that some components of  $\theta$  are equal to zero, then there is  $\bar{\theta} \in \mathbb{R}_{++}^H$  such that  $J_{V, \mathcal{H}}(x) \bar{\theta}^T \gg 0$ , because  $\theta \in \mathbb{R}_+^H \setminus \{0\}$ . Hence, applying the above arguments to  $\bar{\theta} \in \mathbb{R}_{++}^H$  completes the proof. ■

## References

- Arrow, K.J., 1969. The Organization of Economic Activity: Issues Pertinent to the Choice of Market versus Non-market Allocation. In: *The Analysis and Evaluation of Public Expenditures: The P.P.B. System* (Joint Economic Committee of the Congress of the United States, Washington, D.C.), 47–64.
- Balasko, Y., 1988. *Foundations of the theory of general equilibrium*. Academic Press, Boston.
- Bergson, A., 1938. A reformulation of certain aspects of welfare economics. *Q. J. Econ.* 52, 310–334.
- Bourlès, R., Bramoullé, Y., Perez-Richet, E., 2017. Altruism in networks. *Econometrica* 85, 675–689.
- Dufwenberg, M., Heidhues, P., Kirchsteiger, G., Riedel, F., Sobel, J., 2011. Other-regarding preferences in general equilibrium. *Rev. Econ. Stud.* 78, 613–639.
- Debreu, G., 1959. *Theory of Value*. Yale University Press.
- Geanakoplos, J.D., Polemarchakis, H.M., 2008. Pareto improving taxes. *J. Math. Econ.* 44, 682–696.
- Laffont, J.-J., 1976. Decentralization with externalities. *Eur. Econ. Rev.* 7, 359–375.
- Mas-Colell, A., 1985. *The theory of general economic equilibrium. A differentiable approach*. Cambridge University Press.
- Miettinen, K., 1998. *Nonlinear Multiobjective Optimization*. Kluwer Academic Publishers.
- Osana, H., 1972. Externalities and the basic theorems of welfare economics. *J. Econ. Theory* 4, 401–414.
- Parks, R.P., 1991. Pareto irrelevant externalities. *J. Econ. Theory* 54, 165–179.
- Rader, T., 1972. *Theory of General Economic Equilibrium*. Academic Press, New York and London.
- Samuelson, P.A., 1947. *Foundations of Economic Analysis*. Harvard University Press.
- Sobel, J., 2005. Interdependent preferences and reciprocity. *J. Econ. Lit.* 43, 392–436.
- Winter, S.G., 1969. A simple remark on the second optimality theorem of welfare economics. *J. Econ. Theory* 1, 99–103.