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**Unknottedness of the Graph of Pairwise Stable
Networks & Network Dynamics**

Julien FIXARY

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Unknottedness of the Graph of Pairwise Stable Networks & Network Dynamics

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Abstract

We extend Bich-Fixary's theorem ([2]) about the topological structure of the *graph of pairwise stable networks*. Namely, we show that the graph of pairwise stable networks is not only homeomorphic to the space of societies, but that it is ambient isotopic to a trivial copy of this space (a result in the line of Demichelis-Germano's *unknottedness theorem* ([7])). Furthermore, we introduce the notion of (*extended*) *network dynamics* which refers to families of vector fields on the set of weighted networks whose zeros correspond to pairwise stable networks. We use our version of the unknottedness theorem to show that most of network dynamics can be continuously connected to each other, without adding additional zeros. Finally, we prove that this result has an important consequence on the indices of these network dynamics at any pairwise stable network, a concept that we link to genericity using Bich-Fixary's oddness theorem ([2]).

Keywords– Pairwise Stability, Unknottedness Theorem, Network Dynamics, Genericity

JEL classification– C61, C62, D85

I INTRODUCTION

For several years, network formation theory is gaining importance in economic theory. Informally, a network is composed by a set of nodes (or agents) and a set of links, where a link between two nodes indicates a certain relationship between these nodes (e.g. a network defined by a group of people and friendship relationships, or a network defined by a group of researchers and co-author relationships). In 1996, Jackson-Wolinsky ([19]) introduced *pairwise stability* concept which is often applied in network formation theory in order to predict which networks are likely to arise in a strategic setting (every agent is given a payoff function, which represents its linking preferences). Briefly, a network is said to be pairwise stable if no two agents could gain from linking and no single agent could gain by severing one of his or her link. In 2020, Bich-Morhaim ([3]) extended pairwise stability concept to *weighted networks*, i.e. networks where the strength of any link is measured by a real number between 0 and 1, and proved the existence of a pairwise stable weighted network for large classes of payoff functions.

Recently, Bich-Fixary ([2]) made a contribution to the study of the topological structure of the *graph of pairwise stable networks*, which is the set of all couples $((v_1, \dots, v_n), g)$, where $v = (v_1, \dots, v_n) \in \mathcal{F}$ is a profile of payoff functions of the agents involved in the network formation (\mathcal{F} being a particular space of profiles of payoff functions that satisfy some concavity assumption and some differentiability assumption; details are given in the rest of this paper), and g is a pairwise stable weighted network associated to v . One consequence of this contribution (that is stated in the same paper) is that for large classes of polynomial payoff functions, there exists *generically* an odd number of pairwise stable networks.¹ These results are in line with many game theoretic results (e.g. [4, 7, 9, 11, 14, 16, 20, 23, 28]). The general idea of this paper is to emphasize even more the similarities between game theory (with Nash equilibrium concept) and network formation theory (with pairwise stability concept).² To do so, we will focus on two specific topics:

1. The first goal is to strengthen Bich-Fixary's result ([2]) to show that the graph \mathcal{P} of pairwise stable networks is not only homeomorphic to \mathcal{F} , but that it can be continuously deformed to a copy $\mathcal{F}^0 \subset \mathcal{F} \times [0, 1]^L$ of the space \mathcal{F} within the *ambient space* $\mathcal{F} \times [-\varepsilon, 1+\varepsilon]^L$ (for $\varepsilon > 0$), which can be seen as a kind of enlargement of the base space $\mathcal{F} \times [0, 1]^L \supset \mathcal{P}$ (here, we consider a fixed set of agents: L is the set of links, i.e. the set of all pairs of agents, and $[0, 1]^L$ is the set of weighted networks, i.e. the set of all maps from L to $[0, 1]$). We will refer to this result (Theorem 3.1) by calling it the *unknottedness theorem* (more explanations are provided in Section 3). We will show that this result remains valid if we replace the space \mathcal{F} by some of its subspaces which behave "nicely"

¹In the field of game theory, Bich-Fixary ([1]) proved a similar result for large classes of polynomial payoff functions and semi-algebraic sets of strategies.

²Nevertheless, understand that by nature, pairwise stability concept is not a priori comparable to Nash equilibrium concept since the notion of pairwise stable network involves unordered pairs of agents and is characterized both by a cooperative aspect and by a non-cooperative aspect (to use the words of Jackson-Wolinsky ([19]): "*The formation of a link requires the consent of both parties involved, but severance can be done unilaterally*").

(Corollary 3.1), and which we will call *regular sets*.³

These results are parallel to those of game theory, and in particular to the work of Predtetchinski ([25]) and of Demichelis-Germano ([7]).

2. The second goal is to introduce what we will call (*extended*) *network dynamics*. Briefly, a network dynamics is a family of vector fields $(\tilde{D}_v)_{v \in \mathcal{F}}$ on the (enlarged) set $[-\varepsilon, 1 + \varepsilon]^L$ of weighted networks (for $\varepsilon > 0$) whose zeros coincide with the set of pairwise stable networks of v ($v \in \mathcal{F}$). Furthermore, we use the unknottedness theorem to prove the second main result of this paper (Theorem 4.1), which states that any two *strongly inward-pointing* network dynamics (more explanations are provided in Section 4) are homotopic within the set of all network dynamics on \mathcal{F} (i.e. these network dynamics can be continuously connected to each other, without adding additional zeros). From this theorem follows two important corollaries: the first one (Corollary 4.1) asserts that Theorem 4.1 is still valid in the case where we replace the space \mathcal{F} by any regular set, and the second one (Corollary 4.2) states the index at any pairwise stable network does not depend on the choice of the strongly-inward pointing network dynamics. Finally, some links are made with network dynamics on the ("non-enlarged") set $[0, 1]^L$ of weighted networks (Proposition 4.1), and with genericity (Corollary 4.3).

Remark that the notion of network dynamics is very similar to the one of *Nash field* (or *Nash dynamics*) in game theory⁴, which has been studied extensively (e.g. [6, 7, 10, 13, 26]).⁵ Again, the results about network dynamics stated here in the present paper are parallel to the work that has been made in game theory by Demichelis-Germano ([7]), with the difference that the classes of payoff functions considered in our case are larger.

This paper is organized as follows. In Section 2, we recall some elementary notions of network formation theory, and also the main result of Bich-Fixary ([2]), which is a key element for the rest of this paper. In Section 3, we present the unknottedness theorem, which is the first main result of this paper. In Section 4, we introduce the notion of network dynamics, and we present the second main result of this paper (Theorem 4.1), along with the several previously mentioned corollaries concerning indices of network dynamics and genericity. Finally, in Section 5, we provide some mathematical reminders regarding general topology and differential geometry (Subsection 5.1), together with the proofs of every theorems of this paper (Subsection 5.2 and Subsection 5.3).

³These particular sets have already been introduced in Bich-Fixary ([1, 2]).

⁴For a reminder about Nash fields: see Laraki-Renault-Sorin, pp.84-85 ([21]), or Demichelis-Germano ([7]).

⁵Past years, evolution processes in unweighted networks have also been studied a lot (e.g. [17, 18, 27]). In this paper, the aim is to provide some theoretical basics about evolution processes in weighted networks starting from a more differential-geometric approach.

2 PAIRWISE STABILITY AND THE GRAPH OF PAIRWISE STABLE NETWORKS

Throughout this paper, consider a set $N = \{1, \dots, n\}$, called the *set of agents* (for some fixed integer $n > 1$). The *set of links* is defined as $L = \{\{i, j\} : (i, j) \in N^2, i \neq j\}$, and the *set of (weighted) networks*⁶ is defined as $\mathbb{G} = [0, 1]^L$.

Definition 2.1. (Society)

A (*weighted*) *society* is defined as a n -tuple $v = (v_i)_{i \in N} \in \mathcal{F}(\mathbb{G}, \mathbb{R})^n$, where for every agent $i \in N$, v_i is called the *payoff function of agent i*.

Notations. Every link $\{i, j\} \in L$ is denoted ij . For every link $ij \in L$, $L_{-ij} := L - ij$ and $\mathbb{G}_{-ij} := [0, 1]^{L_{-ij}}$. For every link $ij \in L$, every $g_{-ij} \in \mathbb{G}_{-ij}$, and every $w \in [0, 1]$, $g' = (w, g_{-ij}) \in \mathbb{G}$ is the network defined in the following way: $g'_{kl} = g_{kl}$, for every $kl \neq ij$, and $g'_{ij} = w$. For every link $ij \in L$, and every network $g \in \mathbb{G}$, $g_{-ij} \in \mathbb{G}_{-ij}$ is defined as $g_{-ij} = (g_{kl})_{kl \neq ij}$.

Recall the seminal concept of pairwise stability introduced by Jackson-Wolinsky ([19]):

Definition 2.2. (Pairwise stable network)

Let v be a society. A network $g = (g_{ij})_{ij \in L} \in \mathbb{G}$ is said to be *pairwise stable (with respect to v)* if for every link $ij \in L$, the following conditions hold:

1. For every $w \in [0, g_{ij})$, $v_i(w, g_{-ij}) \leq v_i(g)$ and $v_j(w, g_{-ij}) \leq v_j(g)$.
2. For every $w \in (g_{ij}, 1]$, $v_i(w, g_{-ij}) \leq v_i(g)$ or $v_j(w, g_{-ij}) \leq v_j(g)$.

Notations. For every society v , the set of all pairwise stable networks of v is denoted $\mathbf{P}(v)$.

The payoff functions considered in this paper satisfy some concavity assumption and some differentiability assumption. More precisely, for every agent $i \in N$, define the vector space

$$\mathcal{V}_i = \{v_i \in \mathcal{C}^0(\mathbb{G}, \mathbb{R}) : \forall j \neq i, \forall g_{-ij} \in \mathbb{G}_{-ij}, v_i(\cdot, g_{-ij}) \in \mathcal{C}^1([0, 1], \mathbb{R})\},$$

endowed with the following norm:

$$\left\{ \begin{array}{l} \|\cdot\|_i : \mathcal{V}_i \rightarrow \mathbb{R} \\ v_i \mapsto \max\{\max\{\|v_i\|_\infty, \|\partial_{ij} v_i\|_\infty\} : j \neq i\} \end{array} \right\},$$

where for every $v_i \in \mathcal{V}_i$, and every $j \neq i$,

$$\left\{ \begin{array}{l} \partial_{ij} v_i : \mathbb{G} \rightarrow \mathbb{R} \\ g = (g_{ij})_{ij \in L} \mapsto \frac{\partial v_i}{\partial g_{ij}}(g_{ij}, g_{-ij}) \end{array} \right\},$$

⁶Throughout this paper: (i) for every sets X and Y , the set of all maps from X to Y is denoted $\mathcal{F}(X, Y)$ or X^Y , (ii) for every finite set X , the set \mathbb{R}^X is endowed with its usual (Euclidean) topology and its usual smooth structure, and for every compact interval I of \mathbb{R} , the set I^X is seen as an embedded submanifold (with corners) of \mathbb{R}^X .

Also, for every agent $i \in N$, define the topological subspaces

$$\mathcal{F}_i = \{v_i \in \mathcal{V}_i : \forall j \neq i, \forall g_{-ij} \in \mathbb{G}_{-ij}, v_i(\cdot, g_{-ij}) \text{ is concave}\} \text{ and}$$

$$\mathcal{A}_i = \{g \in \mathbb{G} \mapsto \sum_{j \neq i} \alpha_{ij} g_{ij} + c_i \in \mathbb{R} : \forall j \neq i, \alpha_{ij} \in \mathbb{R}, c_i \in \mathbb{R}\}$$

of \mathcal{V}_i . Moreover⁷, let $\mathcal{V} = \prod_{i \in N} \mathcal{V}_i$, $\mathcal{F} = \prod_{i \in N} \mathcal{F}_i$ and $\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$.

The following regularity concept plays a central role in the main theorems of this paper:

Definition 2.3. (Regular set)

A subset \mathcal{R} of \mathcal{V} is called *regular* if the following conditions hold:

1. **(Affine stability assumption).** $\mathcal{R} = \mathcal{R} + \mathcal{A}$.
2. **(Concavity assumption).** $\mathcal{R} \subset \mathcal{F}$.

The sets \mathcal{A} and \mathcal{F} are examples of regular sets. Less trivial examples of regular sets are the following ones: if we consider a given profile $\bar{v} = (\bar{v}_1, \dots, \bar{v}_n)$ of payoff functions in \mathcal{F} , then the sets

$$\{(g \mapsto \bar{v}_i(g) + \sum_{j \neq i} \alpha_{i,j} g_{ij} + c_i)_{i \in N} : \forall i \in N, \forall j \neq i, \alpha_{i,j} \in \mathbb{R}, c_i \in \mathbb{R}\}$$

and

$$\{(g \mapsto \bar{v}_i(g) + \sum_{j \neq i} \beta_{i,j} g_{ij}^2 + \sum_{j \neq i} \alpha_{i,j} g_{ij} + c_i)_{i \in N} : \forall i \in N, \forall j \neq i, \alpha_{i,j} \in \mathbb{R}, \beta_{i,j} \in (-\infty, 0], c_i \in \mathbb{R}\}$$

are both regular.

Definition 2.4. (PSN correspondence, graph of pairwise stable networks)

The *pairwise stable networks correspondence*, or *PSN correspondence*, is defined as the correspondence

$$\begin{cases} \mathbf{P} : \mathcal{F} \rightarrow \mathbb{G} \\ v \mapsto \mathbf{P}(v) \end{cases},$$

and the *graph of pairwise stable networks* is defined as the graph of the PSN correspondence:

$$\mathcal{P} := \text{Gr}(\mathbf{P}) = \{(v, g) \in \mathcal{F} \times \mathbb{G} : g \in \mathbf{P}(v)\}.$$

Moreover, for every subset \mathcal{U} of \mathcal{F} ,

$$\mathcal{P}_{\mathcal{U}} := \mathcal{P} \cap (\mathcal{U} \times \mathbb{G}).$$

Notations. The projection $\mathcal{P} \rightarrow \mathcal{F}$, $(v, g) \mapsto v$ is denoted $\pi_{\mathcal{F}}$. Moreover, for every subset \mathcal{U} of \mathcal{F} , the map $\pi_{\mathcal{F}}|_{\mathcal{P}_{\mathcal{U}}}$ is denoted $\pi_{\mathcal{U}}$.

The following theorem and corollary of Bich-Fixary ([2]) characterize the topological structure of the graph of pairwise stable networks in the case of regular sets:

⁷Throughout this paper, the cartesian product of topological spaces is always endowed with the product topology.

Theorem 2.1. *There exists a homeomorphism η from \mathcal{P} to \mathcal{F} (with inverse ρ), and a proper homotopy between $\pi_{\mathcal{F}}$ and η .*

Corollary 2.1. *For every regular set \mathcal{R} , there exists a homeomorphism $\eta_{\mathcal{R}}$ from $\mathcal{P}_{\mathcal{R}}$ to \mathcal{R} (with inverse $\rho_{\mathcal{R}}$), and a proper homotopy between $\pi_{\mathcal{R}}$ and $\eta_{\mathcal{R}}$.*

Recall that the homeomorphism η of Theorem 2.1 is built in the following way: consider a network $g^0 \in \mathbb{G}$ and define

$$\begin{cases} \eta: \mathcal{P} & \rightarrow \mathcal{F} \\ (v, g) & \mapsto v^g \end{cases},$$

where for every agent $i \in \mathbb{N}$, and every network $\gamma \in \mathbb{G}$,

$$v_i^g(\gamma) = v_i(\gamma) + \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0)) (\gamma_{ij} - g_{ij}) + \sum_{j \neq i} \gamma_{ij} g_{ij}.$$

The inverse of η is defined as

$$\begin{cases} \rho: \mathcal{F} & \rightarrow \mathcal{P} \\ v & \mapsto (\hat{v}, g^v) \end{cases},$$

where for every link $ij \in \mathbb{L}$, $g_{ij}^v = \min\{w_{ij}^v, w_{ji}^v\}$, with $w_{ij}^v \in [0, 1]$ (resp. $w_{ji}^v \in [0, 1]$) the unique solution of the strictly concave maximization problem

$$\max_{w \in [0, 1]} v_i(w, g_{-ij}^0) - \frac{w^2}{2} \quad (\text{resp.} \quad \max_{w \in [0, 1]} v_j(w, g_{-ij}^0) - \frac{w^2}{2}),$$

and where for every agent $i \in \mathbb{N}$, and every network $\gamma \in \mathbb{G}$,

$$\hat{v}_i(\gamma) = v_i(\gamma) - \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij} v_i(g_{ij}^v, g_{-ij}^0)) (\gamma_{ij} - g_{ij}^v) - \sum_{j \neq i} \gamma_{ij} g_{ij}^v.$$

To simplify what comes next in the rest of this paper, for every agent $i \in \mathbb{N}$, every network $g^0 \in \mathbb{G}$, and every $(v, g) \in \mathcal{F} \times \mathbb{G}$, define the maps

$$\begin{cases} h_i^g[v]: \mathbb{G} & \rightarrow \mathbb{R} \\ \gamma & \mapsto \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}, g_{-ij}) - \partial_{ij} v_i(g_{ij}, g_{-ij}^0)) (\gamma_{ij} - g_{ij}) \end{cases}$$

and

$$\begin{cases} l_i^g: \mathbb{G} & \rightarrow \mathbb{R} \\ \gamma & \mapsto \sum_{j \neq i} \gamma_{ij} g_{ij} \end{cases},$$

so that $v_i^g = v_i + h_i^g[v] + l_i^g$ and $\hat{v}_i = v_i - h_i^{g^v}[v] - l_i^{g^v}$.

3 UNKNOTTEDNESS THEOREM

The aim of this section is to present the unknottedness theorem, which states that the graph \mathcal{P} of pairwise stable networks can be continuously deformed to a trivial copy \mathcal{F}^0 of the space \mathcal{F} of societies (satisfying some concavity assumption and some differentiability assumption) within some ambient space, larger than $\mathcal{F} \times \mathbb{G}$, and which will be described in the following. Before that, let us introduce some notations.

Notations. Denote $\mathcal{E} = \mathcal{F} \times \mathbb{G}$, and for every network $g^0 \in \mathbb{G}$, $\mathcal{F}^0 = \mathcal{F} \times \{g^0\}$. For every $\varepsilon > 0$, denote $\mathbb{G}^\varepsilon = [-\varepsilon, 1 + \varepsilon]^L$, and $\mathcal{E}^\varepsilon = \mathcal{F} \times \mathbb{G}^\varepsilon$. The projection $\mathcal{P} \rightarrow \mathcal{F}$, $(v, g) \mapsto g$ is denoted $\pi_{\mathbb{G}}$. Moreover, for every subset \mathcal{U} of \mathcal{F} , and every network $g^0 \in \mathbb{G}$, $\mathcal{U}^0 := \mathcal{U} \times \{g^0\}$, $\mathcal{E}_{\mathcal{U}} := \mathcal{U} \times \mathbb{G}$, $\mathcal{E}_{\mathcal{U}}^\varepsilon := \mathcal{U} \times \mathbb{G}^\varepsilon$, and the map $\pi_{\mathbb{G}}|_{\mathcal{P}_{\mathcal{U}}}$ is also denoted $\pi_{\mathbb{G}}$ (by abuse of notation).

From now on and until the end of this paper, consider a fixed network $g^0 \in \mathbb{G}$, and a fixed $\varepsilon > 0$. Moreover, consider the homeomorphism η of Theorem 2.1 and its inverse ρ , and for every regular set \mathcal{R} , consider the homeomorphism $\eta_{\mathcal{R}}$ of Corollary 2.1 and its inverse $\rho_{\mathcal{R}}$.

Recall that for every topological spaces X and Y , an ambient isotopy between two topological embeddings $f^1, f^2 : X \rightarrow Y$ is a continuous map $\varphi : [0, 1] \times Y \rightarrow Y$ such that $\varphi_0 = \text{id}_Y$, $\varphi_1 \circ f^1 = f^2$ and for every $t \in [0, 1]$, φ_t is a homeomorphism (see Section 5.1 for more details). In our case, we will consider two particular topological embeddings from \mathcal{F} to \mathcal{E}^ε , and two particular topological embeddings from \mathcal{R} to $\mathcal{E}_{\mathcal{R}}^\varepsilon$, where \mathcal{R} is an arbitrary regular set.

Theorem 3.1. (*Unknottedness theorem*)

There exists an ambient isotopy φ between the topological embeddings $\kappa^1 : \mathcal{F} \rightarrow \mathcal{E}^\varepsilon, v \mapsto \rho(v) = (\hat{v}, g^v)$ and $\kappa^2 : \mathcal{F} \rightarrow \mathcal{E}^\varepsilon, v \mapsto (v, g^0)$ within \mathcal{E}^ε .

The proof of Theorem 3.1 is given in Section 5.2.

As stated at the beginning of this section, and since $\kappa^1(\mathcal{F}) = \mathcal{P}$ and $\kappa^2(\mathcal{F}) = \mathcal{F}^0$, Theorem 3.1 states that the graph \mathcal{P} of pairwise stable networks is continuously deformed to a copy of the space \mathcal{F} within the ambient space \mathcal{E}^ε . By analogy to knot theory (a branch of topology which study topological embeddings of the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ in \mathbb{R}^3), the term "unknottedness" comes from the fact that κ^2 can be seen as *the unknot*, and that κ^1 is ambient isotopic to κ^2 within \mathcal{E}^ε .

Notice that Theorem 3.1 is stronger than Theorem 2.1 of Bich-Fixary ([2]): since κ^1 and κ^2 are ambient isotopic within \mathcal{E}^ε , we get that $\kappa^1(\mathcal{F}) = \mathcal{P}$ and $\kappa^2(\mathcal{F}) = \mathcal{F}^0$ are homeomorphic, where \mathcal{F}^0 is itself homeomorphic to \mathcal{F} .

The following corollary of Theorem 3.1 states that if we consider a regular set \mathcal{R} , then we still obtain that the graph $\mathcal{P}_{\mathcal{R}}$ of pairwise stable networks (restrained to payoff functions in \mathcal{R}) can be continuously deformed to a copy of the space \mathcal{R} within the ambient space $\mathcal{E}_{\mathcal{R}}^\varepsilon$:

Corollary 3.1. (*Unknottedness theorem - regular sets*)

Let \mathcal{R} be a regular set. There exists an ambient isotopy $\varphi_{\mathcal{R}}$ between the topological embeddings $\kappa_{\mathcal{R}}^1 : \mathcal{R} \rightarrow \mathcal{E}_{\mathcal{R}}^\varepsilon, v \mapsto \rho_{\mathcal{R}}(v)$ and $\kappa_{\mathcal{R}}^2 : \mathcal{R} \rightarrow \mathcal{E}_{\mathcal{R}}^\varepsilon, v \mapsto (v, g^0)$ within $\mathcal{E}_{\mathcal{R}}^\varepsilon$.

The proof of Corollary 3.1 is given in Section 5.2. Similarly as before, remark that Corollary 3.1 strengthens Corollary 2.1 of Bich-Fixary ([2]).

4 DYNAMICS IN NETWORKS

In this section, we introduce the notions of network dynamics and of extended network dynamics on an arbitrary subset \mathcal{U} of \mathcal{F} . Briefly, the former notion refers to families of vector

fields⁸ $(D_v)_{v \in \mathcal{U}}$ on \mathbb{G} whose zeros coincide with the set $\mathbf{P}(v)$ of pairwise stable networks of v (for every $v \in \mathcal{U}$), while the latter notion refers to families of vector fields on the enlarged space \mathbb{G}^ε , with the same property as before concerning their zeros.

The main theorem of this section (Theorem 4.1) states that any two extended network dynamics \tilde{D}^1 and \tilde{D}^2 on \mathcal{F} satisfying some additional properties (these extended network dynamics will be called *strongly inward-pointing*) are homotopic within the set of all extended network dynamics on \mathcal{F} , i.e. there exists a homotopy \tilde{H} between \tilde{D}^1 and \tilde{D}^2 such that for every $t \in [0, 1]$, \tilde{H}_t is an extended network dynamics on \mathcal{F} . An important corollary (Corollary 4.2) is that for every $v \in \mathcal{F}$ and every isolated zero g of \tilde{D}_v^1 and \tilde{D}_v^2 , the index of \tilde{D}_v^1 at g has to be equal to the index of \tilde{D}_v^2 at g .

Definition 4.1. (Network dynamics, extended network dynamics)

Let \mathcal{U} be a subset of \mathcal{F} :

- A *network dynamics on \mathcal{U}* is defined as a continuous map $D : \mathcal{E}_{\mathcal{U}} \rightarrow T\mathbb{G}$ such that the following conditions hold:
 1. For every $v \in \mathcal{U}$, $D_v := D(v, \cdot)$ is a vector field on \mathbb{G} .
 2. $\mathcal{Z}(D) = \mathcal{P}_{\mathcal{U}}$, where $\mathcal{Z}(D) := \{(v, g) \in \mathcal{E}_{\mathcal{U}} : g \in \mathcal{Z}(D_v)\}$, and where $\mathcal{Z}(D_v)$ is the set of zeros of the vector field D_v .
- An *extended network dynamics on \mathcal{U}* is defined as a continuous map $\tilde{D} : \mathcal{E}_{\mathcal{U}}^\varepsilon \rightarrow T\mathbb{G}^\varepsilon$ such that the following conditions hold:
 1. For every $v \in \mathcal{U}$, $\tilde{D}_v := \tilde{D}(v, \cdot)$ is a vector field on \mathbb{G}^ε .
 2. $\mathcal{Z}(\tilde{D}) = \mathcal{P}_{\mathcal{U}}$, where $\mathcal{Z}(\tilde{D}) := \{(v, g) \in \mathcal{E}_{\mathcal{U}}^\varepsilon : g \in \mathcal{Z}(\tilde{D}_v)\}$, and where $\mathcal{Z}(\tilde{D}_v)$ is the set of zeros of the vector field \tilde{D}_v .

Remark that \mathbb{G}^ε is a convex subset of \mathbb{R}^L , which implies that it is contractible, thus that $T\mathbb{G}^\varepsilon$ is trivial (see Proposition 5.1 in Section 5.1). Consequently, Proposition 5.3 (see Section 5.1) tells us that any network dynamics D on \mathcal{U} can be treated as a continuous map from $\mathcal{E}_{\mathcal{U}}$ to \mathbb{R}^L such that $D^{-1}(\{0\}) = \mathcal{P}$, and that any extended network dynamics \tilde{D} on \mathcal{U} can be treated as a continuous map from $\mathcal{E}_{\mathcal{U}}^\varepsilon$ to \mathbb{R}^L such that $\tilde{D}^{-1}(\{0\}) = \mathcal{P}$, where \mathcal{U} is any subset of \mathcal{F} . Sometimes, this identification will be used in order to avoid complex details, and in particular in the following definitions:

Definition 4.2. (Inward-pointing network dynamics)

Let \mathcal{U} be a subset of \mathcal{F} . A network dynamics D on \mathcal{U} is said to be *inward-pointing* if for every $v \in \mathcal{U}$, every $g \in \partial\mathbb{G}$, and every $x \in N_{\partial\mathbb{G}}(g)$,

$$\langle D_v(g), x \rangle \leq 0,$$

where $N_{\partial\mathbb{G}}(g)$ is the normal cone to $\partial\mathbb{G}$ at g .

⁸Recall that the tangent bundle $T\mathbb{G}$ of \mathbb{G} is the disjoint union of all tangent spaces $T_g\mathbb{G}$ to \mathbb{G} at g ($g \in \mathbb{G}$), and that a vector field on \mathbb{G} is an assignment of a tangent vector in $T_g\mathbb{G}$ to each network $g \in \mathbb{G}$ (see Section 5.1 for more details). Moreover, each tangent space $T_g\mathbb{G}$ ($g \in \mathbb{G}$) can be identified to \mathbb{R}^L , and we will see that each vector field on \mathbb{G} can be seen as a continuous map from \mathbb{G} to \mathbb{R}^L .

Definition 4.3. (Strongly inward-pointing extended network dynamics)

Let \mathcal{U} be a subset of \mathcal{F} . An extended network dynamics \tilde{D} on \mathcal{U} is said to be *strongly inward-pointing* if for every $v \in \mathcal{U}$, every $\varepsilon' \in (0, \varepsilon]$, every $g \in \partial\mathbb{G}^{\varepsilon'}$, and every $x \in N_{\partial\mathbb{G}^{\varepsilon'}}(g)$,

$$\langle \tilde{D}_v(g), x \rangle < 0,$$

where $N_{\partial\mathbb{G}^{\varepsilon'}}(g)$ is the normal cone to $\partial\mathbb{G}^{\varepsilon'}$ at g .

The two notions of inward-pointing network dynamics and of strongly inward-pointing extended network dynamics are in fact quite natural. If we want to deal with flows associated to vector fields on \mathbb{G} , then the first notion is worth to consider (at the boundary of \mathbb{G} , we do not want to move "outside"). The second notion reflects the artificial feature of the enlargement \mathcal{E}^ε of the space \mathcal{E} : even if we want to be able to describe what happens a little bit outside \mathbb{G} (in particular, around isolated zeros on $\partial\mathbb{G}$ of extended network dynamics), starting on $\mathbb{G}^\varepsilon - \mathbb{G}$, we want to be sure to move back to \mathbb{G} . Moreover, we will see that this extra assumption on extended network dynamics is useful in the proof of the following theorem:

Theorem 4.1. *Let \tilde{D}^1 and \tilde{D}^2 be strongly inward-pointing extended network dynamics on \mathcal{F} . There exists a homotopy \tilde{H} between \tilde{D}^1 and \tilde{D}^2 such that for every $t \in [0, 1]$, \tilde{H}_t is an extended network dynamics on \mathcal{F} .*

The following result states that Theorem 4.1 still holds if we consider a regular set \mathcal{R} :

Corollary 4.1. *Let \mathcal{R} be a regular set, and \tilde{D}^1 and \tilde{D}^2 be strongly inward-pointing extended network dynamics on \mathcal{R} . There exists a homotopy $\tilde{H}_{\mathcal{R}}$ between \tilde{D}^1 and \tilde{D}^2 such that for every $t \in [0, 1]$, $(\tilde{H}_{\mathcal{R}})_t$ is an extended network dynamics on \mathcal{R} .*

Recall that for a given vector field V on an arbitrary smooth manifold, and a given isolated zero z of V , the index of V at z is an indicator which helps to quantify the behavior of V around z (i.e. V may circulate around z , it may have a source, a sink, a saddle, etc.). This definition is recalled in Section 5.1. The following corollary of Theorem 4.1 establishes a link between the indices of strongly inward-pointing extended network dynamics on \mathcal{R} at any isolated zero (i.e. at any isolated pairwise stable network), where \mathcal{R} is a regular set.

Corollary 4.2. *Let \mathcal{R} be a regular set, \tilde{D}^1 and \tilde{D}^2 be strongly inward-pointing extended network dynamics on \mathcal{R} , and $v \in \mathcal{R}$. For every isolated point g of $\mathcal{Z}(\tilde{D}_v^1) = \mathcal{Z}(\tilde{D}_v^2)$, the index of \tilde{D}_v^1 at g is equal to the index of \tilde{D}_v^2 at g .*

The proofs of Theorem 4.1, Corollary 4.1 and Corollary 4.2 are given in Section 5.3.

4.1 INDEX OF A ZERO OF A NETWORK DYNAMICS

The next proposition establishes a link between inward-pointing network dynamics and strongly inward-pointing extended network dynamics:

Proposition 4.1. *Let \mathcal{U} be a subset of \mathcal{F} , and D be an inward-pointing network dynamics on \mathcal{U} . Moreover, consider:*

- *the projection r from \mathbb{G}^ε to the closest point in \mathbb{G} ,*
- *a continuous map b from \mathbb{G}^ε to $[0, 1]$ such that $b(\mathbb{G}) = 1$ and $b(\partial\mathbb{G}^\varepsilon) = 0$,*
- *a vector field $V : \mathbb{G}^\varepsilon \rightarrow \mathbb{R}^L$ which is identically equal to zero on \mathbb{G} , and such that for every $\varepsilon' \in (0, \varepsilon]$, every $g \in \partial\mathbb{G}^{\varepsilon'}$, and every $x \in N_{\partial\mathbb{G}^{\varepsilon'}}(g)$, $\langle V(g), x \rangle < 0$.*

The map

$$\begin{cases} D^\varepsilon : & \mathcal{E}_\mathcal{U}^\varepsilon & \rightarrow & \mathbb{R}^L \\ & (v, g) & \mapsto & b(g)D(v, r(g)) + V(g) \end{cases}$$

(called extension of D) is a strongly inward-pointing extended network dynamics on \mathcal{U} such that $D^\varepsilon|_{\mathcal{E}_\mathcal{U}} = D$.

The proof of Proposition 4.1 is given in Section 5.3.

A consequence of Proposition 4.1 and of Corollary 4.2 is the following one: if we consider a regular set \mathcal{R} , an inward-pointing network dynamics D on \mathcal{R} and an extension D^ε of D , then for every $v \in \mathcal{R}$, and every isolated zero g of D^ε , the index of D^ε at g does not depend on the choice of this extension. Thus, for every $v \in \mathcal{R}$, even if we technically cannot talk about the index of D_v at an isolated zero g on $\partial\mathbb{G}$, we can still describe how D_v behaves around g using any extension D^ε (because such an extension is strongly inward-pointing, no zero is added outside \mathbb{G} , which makes it a "good representation" of D on $\mathcal{E}_\mathcal{R}^\varepsilon$). In particular, we can see why the enlarged space $\mathcal{E}_\mathcal{R}^\varepsilon$ is useful to consider.

4.2 ZEROS OF AN EXTENDED NETWORK DYNAMICS AND GENERICITY

For every agent $i \in N$, consider some integer $\delta_i \in \mathbb{N}$, and the vector space $\mathbb{R}_{\delta_i}[g]$ of polynomial payoff functions whose degree is less or equal to δ_i . In particular, for every network $g \in \mathbb{G}$, any element $v_i \in \mathbb{R}_{\delta_i}[g]$ can be expressed as

$$v_i(g) = \sum_{k \in \mathbb{N}^L} d^{i,k} g^k,$$

where⁹ for every multi-index $k = (k_{ij})_{ij \in L} \in \mathbb{N}^L$, $d^{i,k} = 0$ if $\sum_{ij \in L} k_{ij} > \delta_i$, and $g^k := \prod_{ij \in L} g_{ij}^{k_{ij}}$.

For every $i \in N$, denote ϕ_i the vector space isomorphism which associates to every polynomial payoff function in $\mathbb{R}_{\delta_i}[g]$ its coefficients in \mathbb{R}^{m_i} (with respect to some predefined order on \mathbb{N}^L), for some integer m_i . Moreover, define $m = \sum_{i \in N} m_i$, and $\phi := \times_{i \in N} \phi_i : \prod_{i \in N} \mathbb{R}_{\delta_i}[g] \rightarrow \mathbb{R}^m$, the product map which associates to every polynomial society its coefficients in \mathbb{R}^m .

⁹For example, if we consider three agents, and if $v_1(g_{12}, g_{13}, g_{23}) = -g_{12}^2 g_{23} + 3g_{12} g_{13}^5 g_{23}^4 - g_{13}^2 g_{23}$, then we can write $v_1(g) = d^{1,k_1} g^{k_1} + d^{1,k_2} g^{k_2} + d^{1,k_3} g^{k_3}$ with $k_1 = (2, 0, 3)$, $k_2 = (1, 5, 4)$ and $k_3 = (0, 2, 1)$, and with $d^{1,k_1} = -1$, $d^{1,k_2} = 3$ and $d^{1,k_3} = -1$.

In this setting, the next definition of regularity (stronger than the one in Section 2) is worth to consider in order to refine Corollary 4.2. A subset \mathcal{R} of $\prod_{i \in \mathbb{N}} \mathbb{R}_{\delta_i}[g]$ is called *semi-algebraically regular* if it is regular and if $\phi(\mathcal{R})$ is a semi-algebraic set.

The sets \mathcal{A} and $\mathcal{F} \cap \prod_{i \in \mathbb{N}} \mathbb{R}_{\delta_i}[g]$ are examples of semi-algebraically regular sets. In a similar way to regular sets, less trivial examples of semi-algebraically regular sets are the following ones: if we consider a given profile $\bar{v} = (\bar{v}_1, \dots, \bar{v}_n)$ of payoff functions in $\mathcal{F} \cap \prod_{i \in \mathbb{N}} \mathbb{R}_{\delta_i}[g]$, then the sets

$$\{(g \mapsto \bar{v}_i(g) + \sum_{j \neq i} \alpha_{i,j} g_{ij} + c_i)_{i \in \mathbb{N}} : \forall i \in \mathbb{N}, \forall j \neq i, \alpha_{i,j} \in \mathbb{R}, c_i \in \mathbb{R}\}$$

and

$$\{(g \mapsto \bar{v}_i(g) + \sum_{j \neq i} \beta_{i,j} g_{ij}^2 + \sum_{j \neq i} \alpha_{i,j} g_{ij} + c_i)_{i \in \mathbb{N}} : \forall i \in \mathbb{N}, \forall j \neq i, \alpha_{i,j} \in \mathbb{R}, \beta_{i,j} \in (-\infty, 0], c_i \in \mathbb{R}\}$$

are both semi-algebraically regular (see Bich-Fixary ([2]) for some practical applications).

The following theorem of Bich-Fixary ([2]) states that *almost every*¹⁰ society in a given semi-algebraically regular set has an odd number of pairwise stable networks.

Theorem 4.2. *For every semi-algebraically regular set \mathcal{R} , there exists a generic semi-algebraic subset G of $\phi(\mathcal{R})$ such that for every $x \in G$, the society $v^x := \phi^{-1}(x)$ has an odd number of pairwise stable networks.*

Thanks to this result, we are now able to provide a slightly more precise version of Corollary 4.2:

Corollary 4.3. *Let \mathcal{R} be a semi-algebraically regular set, and \tilde{D}^1 and \tilde{D}^2 be strongly inward-pointing extended network dynamics on \mathcal{R} . There exists a generic semi-algebraic subset G of $\phi(\mathcal{R})$ such that for every $x \in G$, and every $g \in \mathcal{Z}(\tilde{D}_{v^x}^1) = \mathcal{Z}(\tilde{D}_{v^x}^2)$, the index of $\tilde{D}_{v^x}^1$ at g is equal to the index of $\tilde{D}_{v^x}^2$ at g .*

The proof of Corollary 4.3 follows directly from Corollary 4.2 and Theorem 4.2.

The difference between Corollary 4.2 and Corollary 4.3 lies in the fact that if we consider a semi-algebraically regular set \mathcal{R} of societies, and strongly inward-pointing extended network dynamics \tilde{D}^1 and \tilde{D}^2 on \mathcal{R} , then for *almost every* society $v \in \mathcal{R}$, we are now able to talk about the equality of the indices of \tilde{D}_v^1 and \tilde{D}_v^2 at any zero, since in that case, each zero is isolated.

¹⁰For more details about genericity and Theorem 4.2, see Bich-Fixary ([2]).

5 APPENDIX

5.1 REMINDERS ABOUT GENERAL TOPOLOGY AND DIFFERENTIAL GEOMETRY

5.1.1 General topology and vector bundles

Recall first the general notions of homotopy, ambient isotopy and vector bundle, which will be useful throughout this paper.

Definition 5.1. (Homotopy, ambient isotopy)

Let X and Y be topological spaces, and $f, g : X \rightarrow Y$ be continuous maps. A *homotopy between f and g* is a continuous map $H : [0, 1] \times X \rightarrow Y$ such that $H_0 = f$ and $H_1 = g$, where for every $t \in [0, 1]$, $H_t := H(t, \cdot)$. If there exists a homotopy between f and g , then f and g are said to be *homotopic*. Suppose now that f and g are topological embeddings. An *ambient isotopy between f and g within Y* is a continuous map $\varphi : [0, 1] \times Y \rightarrow Y$ such that $\varphi_0 = \text{id}_Y$, $\varphi_1 \circ f = g$ and for every $t \in [0, 1]$, φ_t is a homeomorphism. If there exists an ambient isotopy between f and g within Y , then f and g are said to be *ambient isotopic within Y* .

Definition 5.2. (Vector bundle, section)

Let E and B be topological spaces, $\pi : E \rightarrow B$ be a continuous surjective map, and $k \in \mathbb{N}$. The tuple (E, B, π) is called a *(real) vector bundle of rank k* if the two following conditions are fulfilled:

1. For every $x \in B$, the fiber $\pi^{-1}(\{x\})$ is endowed with the structure of a k -dimensional real vector space.
2. For every $x \in B$, there exists an open neighborhood U of x in B and a homeomorphism $\theta : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ (where $U \times \mathbb{R}^k$ is endowed with the product topology), called *local trivialization*, such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\theta} & U \times \mathbb{R}^k \\ \downarrow \pi & \swarrow p_U & \\ U & & \end{array}$$

(where $p_U : U \times \mathbb{R}^k \rightarrow U$ is the natural projection) and such that for every $y \in U$, the restricted map $\pi^{-1}(\{y\}) \rightarrow \{y\} \times \mathbb{R}^k, z \mapsto \theta(z)$ is a vector space isomorphism.

A *section* of E is a continuous map $s : B \rightarrow E$ such that $\pi \circ s = \text{id}_B$.

Definition 5.3. (Vector bundle homomorphism, trivial vector bundle)

Let (E, B, π) and (E', B, π') be vector bundles over the same base space B , and $F : E \rightarrow E'$ be a continuous map. The map F is called a *vector bundle homomorphism* if the following diagram commutes:

$$\begin{array}{ccc}
 E & \xrightarrow{F} & E' \\
 \downarrow \pi & \swarrow \pi' & \\
 B & &
 \end{array}$$

and if for every $x \in B$, the restricted map $\pi^{-1}(\{x\}) \rightarrow (\pi')^{-1}(\{x\}), y \mapsto F(y)$ is a linear map. The map F is called a *vector bundle isomorphism* if it is a vector bundle homomorphism, F is a bijective map and F^{-1} is a vector bundle homomorphism. A vector bundle (E, B, π) of rank k is called *trivial* if it is isomorphic to the trivial bundle $(B \times \mathbb{R}^k, B, p_B)$, where $p_B : B \times \mathbb{R}^k \rightarrow B$ is the natural projection.

As trivial vector bundles have nice properties (some of them will be stated hereafter in this section), the next proposition is worth to consider as it provides a useful sufficient condition for a vector bundle to be trivial.

Proposition 5.1. *Let (E, B, π) be a vector bundle of rank k . If B is a contractible space, then (E, B, π) is trivial.*

Proof. See Bott-Tu, Corollary 6.9, p.59 ([5]). □

A particular case of vector bundle which is considered in this paper is the tangent bundle of a smooth k -manifold M (with or without boundary, with or without corners):

Definition 5.4. (Tangent space)

Let M be a smooth k -manifold, and $m \in M$. A linear map $d : \mathcal{C}^\infty(M, \mathbb{R}) \rightarrow \mathbb{R}$ is called a derivation at m if it satisfies the Leibniz identity: for every $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$, $d(fg) = d(f)g(m) + f(m)d(g)$. The vector space of all derivations at m is denoted $T_m M$ and is called the *tangent space to M at m* .

Proposition 5.2. (Tangent bundle)

Let M be a smooth k -manifold, and

$$TM := \coprod_{m \in M} T_m M.$$

The tuple (TM, M, π) is a smooth vector bundle of rank k and is called the tangent bundle of M , where $\pi : TM \rightarrow M, (m, v) \mapsto m$.

Proof. See Lee, Proposition 3.18, p.66, and Proposition 10.4, p.252 ([22]). □

Definition 5.5. (Vector field, set of zeros of a vector field)

Let M be a smooth k -manifold. A section $V : M \rightarrow TM$ of the tangent bundle of M is called a *vector field on M* . The vector space of all vector fields on M is denoted $\mathfrak{X}(M)$. For every vector field $V \in \mathfrak{X}(M)$, the *set of zeros of V* is defined as $\mathcal{Z}(V) = \{m \in M : V(m) = (m, d_0)\}$, where d_0 is the derivation constantly equal to $0 \in \mathbb{R}$.

The next proposition highlight the interests of trivial tangent bundles in this paper. Briefly, it states that dynamics can actually be treated as continuous maps.

Proposition 5.3. *Let M be a smooth k -manifold with trivial tangent bundle, and X be a topological space. There exists a bijection Ψ from $\mathcal{C}^0(X \times M, \mathbb{R}^k)$ to*

$$\mathcal{D} = \{D \in \mathcal{C}^0(X \times M, TM) : \forall x \in X, D(x, \cdot) \in \mathfrak{X}(M)\}$$

with the following properties:

1. *For every $f_1, f_2 \in \mathcal{C}^0(X \times M, \mathbb{R}^k)$, if there exists a homotopy H between f_1 and f_2 , then there exists a homotopy H' between $\Psi(f_1)$ and $\Psi(f_2)$ such that for every $t \in [0, 1]$, $H'_t \in \mathcal{D}$. Conversely, for every $D_1, D_2 \in \mathcal{D}$, if there exists a homotopy H between D_1 and D_2 such that for every $t \in [0, 1]$, $H_t \in \mathcal{D}$, then there exists a homotopy H' between $\Psi^{-1}(D_1)$ and $\Psi^{-1}(D_2)$.*
2. *For every $f \in \mathcal{C}^0(X \times M, \mathbb{R}^k)$, $f^{-1}(\{0\}) = \mathcal{Z}(\Psi(f))$, where $\mathcal{Z}(\Psi(f)) = \{(x, m) \in X \times M : m \in \mathcal{Z}(\Psi(f)(x, \cdot))\}$. Conversely, for every $D \in \mathcal{D}$, $\mathcal{Z}(D) = [\Psi^{-1}(D)]^{-1}(\{0\})$.*

Proof. Since TM is trivial, let F be a vector bundle isomorphism from (TM, M, π) to $(M \times \mathbb{R}^k, M, p_M)$. Define the map

$$\left\{ \begin{array}{l} \Psi : \mathcal{C}^0(X \times M, \mathbb{R}^k) \rightarrow \mathcal{D} \\ f \mapsto \left\{ \begin{array}{l} \Psi(f) : X \times M \rightarrow TM \\ (x, m) \mapsto F^{-1}(m, f(x, m)) \end{array} \right. \end{array} \right.$$

It is a bijection with inverse

$$\left\{ \begin{array}{l} \Psi^{-1} : \mathcal{D} \rightarrow \mathcal{C}^0(X \times M, \mathbb{R}^k) \\ D \mapsto \left\{ \begin{array}{l} \Psi^{-1}(D) : X \times M \rightarrow \mathbb{R}^k \\ (x, m) \mapsto (p_{\mathbb{R}^k} \circ F \circ D(x, \cdot))(m) \end{array} \right. \end{array} \right.,$$

where $p_{\mathbb{R}^k} : M \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ is the natural projection.

First, let $f_1, f_2 \in \mathcal{C}^0(X \times M, \mathbb{R}^k)$, and H be a homotopy between f_1 and f_2 . Define the map

$$\left\{ \begin{array}{l} H' : [0, 1] \times X \times M \rightarrow TM \\ (t, x, m) \mapsto \Psi(H_t)(x, m) = F^{-1}(m, H_t(x, m)) \end{array} \right.$$

This map is continuous, $H'_0 = \Psi(f_1)$, and $H'_1 = \Psi(f_2)$. Moreover, for every $t \in [0, 1]$, and every $(x, m) \in X \times M$, notice that by definition of F :

$$(\pi \circ H'_t(x, \cdot))(m) = (\pi \circ F^{-1})(m, H_t(x, m)) = p_M(m, H_t(m, x)) = m,$$

so that $H'_t(x, \cdot) \in \mathfrak{X}(M)$, and $H'_t \in \mathcal{D}$.

Conversely, if $D_1, D_2 \in \mathcal{D}$, and if H is a homotopy between D_1 and D_2 such that for every $t \in [0, 1]$, $H_t \in \mathcal{D}$, then the map

$$\left\{ \begin{array}{l} H' : [0, 1] \times X \times M \rightarrow \mathbb{R}^k \\ (t, x, m) \mapsto \Psi^{-1}(H_t)(x, m) = (p_{\mathbb{R}^k} \circ F \circ H_t(x, \cdot))(m) \end{array} \right.$$

is the required homotopy between $\Psi^{-1}(D_1)$ and $\Psi^{-1}(D_2)$.

Finally, let $f \in \mathcal{C}^0(X \times M, \mathbb{R}^k)$. By definition,

$$\begin{aligned} \{(x, m) \in X \times M : m \in \mathcal{Z}(\Psi(f)(x, \cdot))\} &= \{(x, m) \in X \times M : \Psi(f)(x, m) = (m, d_0)\} \\ &= \{(x, m) \in X \times M : F^{-1}(m, f(x, m)) = (m, d_0)\}. \end{aligned}$$

First, let $(x, m) \in f^{-1}(\{0\})$. As F^{-1} is a vector bundle homomorphism, the restricted map $\{m\} \times \mathbb{R}^k \rightarrow \{m\} \times T_m M, (m, y) \mapsto F^{-1}(m, y)$ is a linear map, so that $F^{-1}(m, f(x, m)) = F^{-1}(m, 0) = (m, d_0)$. Now, let $(x, m) \in X \times M$ such that $F^{-1}(m, f(x, m)) = (m, d_0)$. Thus, $F(F^{-1}(m, f(x, m))) = (m, f(x, m)) = F(m, d_0)$. As F is a smooth vector bundle isomorphism, the restricted map $\{m\} \times T_m M \rightarrow \{m\} \times \mathbb{R}^k, (m, v) \mapsto F(m, v)$ is a linear map, so that $(m, f(x, m)) = F(m, d_0) = (m, 0)$.

Conversely, if $D \in \mathcal{D}$, then since Ψ is a bijection, there exists a unique element $f_D \in \mathcal{C}^0(X \times M, \mathbb{R}^k)$ such that $\Psi(f_D) = D$. The result follows from the previous proof: $\mathcal{Z}(D) = \mathcal{Z}(\Psi(f_D)) = f_D^{-1}(\{0\}) = [\Psi^{-1}(D)]^{-1}(\{0\})$. \square

Corollary 5.1. *Let M be a smooth k -manifold with trivial tangent bundle. There exists a bijection Φ from $\mathcal{C}^0(M, \mathbb{R}^k)$ to $\mathfrak{X}(M)$ with the following properties:*

1. *For every $f_1, f_2 \in \mathcal{C}^0(M, \mathbb{R}^k)$, if there exists a homotopy H between f_1 and f_2 , then there exists a homotopy H' between $\Phi(f_1)$ and $\Phi(f_2)$ such that for every $t \in [0, 1]$, $H'_t \in \mathfrak{X}(M)$. Conversely, for every $V_1, V_2 \in \mathfrak{X}(M)$, if there exists a homotopy H between V_1 and V_2 such that for every $t \in [0, 1]$, $H_t \in \mathfrak{X}(M)$, then there exists a homotopy H' between $\Phi^{-1}(V_1)$ and $\Phi^{-1}(V_2)$.*
2. *For every $f \in \mathcal{C}^0(M, \mathbb{R}^k)$, $f^{-1}(\{0\}) = \mathcal{Z}(\Phi(f))$. Conversely, for every $V \in \mathfrak{X}(M)$, $\mathcal{Z}(V) = [\Phi^{-1}(V)]^{-1}(\{0\})$.*

Proof. The bijection Φ is constructed in a similar way as the one of Proposition 5.3:

$$\left\{ \begin{array}{l} \Phi : \mathcal{C}^0(M, \mathbb{R}^k) \rightarrow \mathfrak{X}(M) \\ f \mapsto \left\{ \begin{array}{l} \Phi(f) : M \rightarrow TM \\ m \mapsto F^{-1}(m, f(m)) \end{array} \right. \end{array} \right.,$$

with its inverse

$$\left\{ \begin{array}{l} \Phi^{-1} : \mathfrak{X}(M) \rightarrow \mathcal{C}^0(M, \mathbb{R}^k) \\ V \mapsto \left\{ \begin{array}{l} \Phi^{-1}(V) : M \rightarrow \mathbb{R}^k \\ m \mapsto (p_{\mathbb{R}^k} \circ F \circ V)(m) \end{array} \right. \end{array} \right.,$$

where F is a vector bundle isomorphism from (TM, M, π) to $(M \times \mathbb{R}^k, M, p_M)$. The others constructions are analogous to the ones of Proposition 5.3. \square

5.1.2 Index of a zero of a vector field

In this section, we recall the definition of the index of a vector field (on an arbitrary smooth manifold) at one of its isolated zeros.

Definition 5.6. (Index of a zero of a continuous map on an open subset of \mathbb{R}^k)

Let U be an open subset of \mathbb{R}^k , $f : U \rightarrow \mathbb{R}^k$ be a continuous map, and $z \in f^{-1}(\{0\})$. Suppose that there exists $\epsilon > 0$ such that $x \notin B(z, \epsilon)$, for every $x \in f^{-1}(\{0\}) - \{z\}$. The *index of f at z* is defined as the topological degree¹¹ of the map

$$\begin{cases} f/\|f\| : \partial B(z, \epsilon) \rightarrow \mathbb{S}^{k-1} \\ x \mapsto f(x)/\|f(x)\|_2 \end{cases},$$

and is denoted $\text{ind}_z(f)$.¹²

For the next definition, consider the bijection Φ of Corollary 5.1.

Definition 5.7. (Index of a zero of a vector field on an open subset of \mathbb{R}^k)

Let U be an open subset¹³ of \mathbb{R}^k , $V : U \rightarrow TU$ be a vector field on U , and $z \in \mathcal{Z}(V)$. Suppose that there exists $\epsilon > 0$ such that $x \notin B(z, \epsilon)$, for every $x \in \mathcal{Z}(V) - \{z\}$. The *index of V at z* is defined as the index of $\Phi^{-1}(V) : U \rightarrow \mathbb{R}^k$ at z , and is denoted $\text{ind}_z(V)$.¹⁴

Definition 5.8. (Related vector fields)

Let M and N be smooth manifolds, $V \in \mathfrak{X}(M)$, $W \in \mathfrak{X}(N)$, and $f : M \rightarrow N$ be a smooth map. The vector fields V and W are said to be *f-related* if for every $m \in M$, $(T_m f \circ V)(m) = (W \circ f)(m)$, where $T_m f : T_m M \rightarrow T_{f(m)} N$ is the tangent map of f at m .

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ \uparrow V & & \uparrow W \\ M & \xrightarrow{f} & N \end{array}$$

Proposition 5.4. *Let M and N be smooth manifolds, $V \in \mathfrak{X}(M)$, and $f : M \rightarrow N$ be a diffeomorphism. There exists a unique vector field $f_*V \in \mathfrak{X}(N)$ such that V and f_*V are f-related.*

Proof. This vector field is defined as follows: for every $n \in N$,

$$f_*V(n) = (T_{f^{-1}(n)}f \circ V \circ f^{-1})(n),$$

and is called the *pushforward of V by f* (see Lee, Proposition 8.19, p.183 ([22])). □

¹¹For a reminder about the main properties of topological degree that are used in this paper: see Hatcher ([15]), or Dold ([8]).

¹² $\text{ind}_z(f)$ does not depend on ϵ (see Guillemin-Pollack, p.133 ([12])).

¹³ U is seen here as an embedded submanifold of \mathbb{R}^k . The tangent bundle of U is trivial (see Lee, Propositions 3.18 and 3.20, pp.66-67 ([22])), so that the bijection Φ of Corollary 5.1 can be used.

¹⁴ $\text{ind}_z(V)$ is well-defined since it does not depend on the choice of the vector bundle isomorphism F in the proof of Corollary 5.1. Indeed, suppose that F' is another vector bundle isomorphism from (TU, U, π) to $(U \times \mathbb{R}^k, U, p)$, and that Φ' is defined in the same way as Φ , but using F' instead of F . In that case, since $z \in [\Phi^{-1}(V)]^{-1}(\{0\}) = [(\Phi')^{-1}(V)]^{-1}(\{0\})$ (see Corollary 5.1), the maps $\Phi^{-1}(V)/\|\Phi^{-1}(V)\|$ and $(\Phi')^{-1}(V)/\|(\Phi')^{-1}(V)\|$ are well-defined and homotopic: the map $[0, 1] \times \partial B(z, \epsilon) \rightarrow \mathbb{S}^{k-1}$, $(t, x) \mapsto [(1-t)\Phi^{-1}(V)(x) + t(\Phi')^{-1}(V)(x)]/\|(1-t)\Phi^{-1}(V)(x) + t(\Phi')^{-1}(V)(x)\|_2$ is such a homotopy. Thus, $\text{deg}(\Phi^{-1}(V)/\|\Phi^{-1}(V)\|) = \text{ind}_z(\Phi^{-1}(V)) = \text{ind}_z((\Phi')^{-1}(V)) = \text{deg}((\Phi')^{-1}(V)/\|(\Phi')^{-1}(V)\|)$.

Definition 5.9. (Index of a zero of a vector field on a smooth manifold)

Let M be a smooth manifold, $V \in \mathfrak{X}(M)$, $z \in \mathcal{Z}(V)$, and (U, ϕ) be a chart of M such that $z \in U$. Suppose that $z \in \text{int}(M)$ and that z is an isolated point of $\mathcal{Z}(V)$, and consider an open neighborhood U' of z in M such that $x \notin U'$, for every $x \in \mathcal{Z}(V) - \{z\}$. Moreover, consider the chart $(U \cap U', \psi)$ of M , where $\psi = \phi|_{U \cap U'}$. The *index of V at z* is defined as the index of the pushforward ψ_*V of V by ψ at $\psi(z)$:

$$\begin{cases} \psi_*V : \phi(U \cap U') & \rightarrow & T\phi(U \cap U') \\ x & \mapsto & (T_{\psi^{-1}(x)}\psi \circ V \circ \psi^{-1})(x) \end{cases} ,$$

and is denoted $\text{ind}_z(V)$.¹⁵

5.2 PROOFS OF SECTION 3

Proof of Theorem 3.1. Recall that we consider a fixed network $g^0 \in \mathbb{G}$, and a fixed $\varepsilon > 0$. Also, recall that η is the homeomorphism of Theorem 2.1, and that

$$\begin{cases} \rho : \mathcal{F} & \rightarrow & \mathcal{P} \\ v & \mapsto & (\hat{v}, g^v) = ((\hat{v}_i)_{i \in \mathbb{N}}, g^v) \end{cases} ,$$

is its inverse, where for every agent $i \in \mathbb{N}$, and every network $\gamma \in \mathbb{G}$,

$$\begin{aligned} \hat{v}_i(\gamma) &= v_i(\gamma) - \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij} v_i(g_{ij}^0, g_{-ij}^0)) (\gamma_{ij} - g_{ij}^v) - \sum_{j \neq i} \gamma_{ij} g_{ij}^v \\ &= v_i(\gamma) - h_i^{g^v}[v](\gamma) - l_i^{g^v}(\gamma), \end{aligned}$$

with $h_i^{g^v}[v](\gamma) = \sum_{j \neq i} (\partial_{ij} v_i(g_{ij}^v, g_{-ij}^v) - \partial_{ij} v_i(g_{ij}^0, g_{-ij}^0)) (\gamma_{ij} - g_{ij}^v)$ and $l_i^{g^v}(\gamma) = \sum_{j \neq i} \gamma_{ij} g_{ij}^v$ (see Section 2).

The proof of Theorem 3.1 goes as follows:

1. In Step I, we show that $\kappa^1 : \mathcal{F} \rightarrow \mathcal{E}^\varepsilon, v \mapsto \rho(v) = (\hat{v}, g^v)$ and the topological embedding $e : \mathcal{F} \rightarrow \mathcal{E}^\varepsilon, v \mapsto (v, (\pi_{\mathbb{G}} \circ \rho)(v)) = (v, g^v)$ are ambient isotopic within \mathcal{E}^ε (through an ambient isotopy φ^1). The idea is to continuously deform the graph \mathcal{P} of pairwise stable networks to the graph of the continuous map $\pi_{\mathbb{G}} \circ \rho$.
2. In Step II, for every continuous map $f : \mathcal{F} \rightarrow \mathbb{G}$, we show that the topological embedding $e_f : \mathcal{F} \rightarrow \mathcal{E}^\varepsilon, v \mapsto (v, f(v))$ and $\kappa^2 : \mathcal{F} \rightarrow \mathcal{E}^\varepsilon, v \mapsto (v, g^0)$ are ambient isotopic within \mathcal{E}^ε (through an ambient isotopy φ_f^2). The idea is to continuously deform the graph of any continuous map $\mathcal{F} \rightarrow \mathbb{G}$ to the trivial copy \mathcal{F}^0 of \mathcal{F} .
3. In Step III, using ambient isotopies φ^1 of Step I and φ_f^2 of Step II (for $f = \pi_{\mathbb{G}} \circ \rho$), we construct an ambient isotopy φ between κ^1 and κ^2 within \mathcal{E}^ε , which ends the proof of this theorem.

¹⁵ $\text{ind}_z(V)$ does not depend on the choice of the chart containing z (see Guillemin-Pollack, p.134 ([12]), or Milnor, pp.33-34 ([24])).

Step I. Let $e : \mathcal{F} \rightarrow \mathcal{E}^\varepsilon, v \mapsto (v, (\pi_{\mathbb{G}} \circ \rho)(v)) = (v, g^v)$. There exists an ambient isotopy φ^1 between κ^1 and e within \mathcal{E}^ε .

Let r be the projection from \mathbb{G}^ε to the closest point in \mathbb{G} (this map is a retraction of \mathbb{G}^ε on \mathbb{G} , i.e. a continuous map such that its composition with the inclusion $\mathbb{G} \hookrightarrow \mathbb{G}^\varepsilon$ is the identity on \mathbb{G}), and b be a continuous map from \mathbb{G}^ε to $[0, 1]$ such that $b(\mathbb{G}) = 1$ and $b(\partial\mathbb{G}^\varepsilon) = 0$. Moreover, let

$$\left\{ \begin{array}{l} \varphi^1 : [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathcal{E}^\varepsilon \\ (t, (v, g)) \mapsto (v^+, g) \end{array} \right\},$$

where for every $i \in \mathbb{N}$, $v_i^+ = v_i + tb(g)(h_i^{r(g)}[v] + l_i^{r(g)})$, and let

$$\left\{ \begin{array}{l} \xi^1 : [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathcal{E}^\varepsilon \\ (t, (v, g)) \mapsto (v^-, g) \end{array} \right\},$$

where for every $i \in \mathbb{N}$, $v_i^- = v_i - tb(g)(h_i^{r(g)}[v] + l_i^{r(g)})$.

First, φ^1 is a continuous map, as both r and b are continuous maps (for more details, see the proof of continuity of the homeomorphism η in Bich-Fixary ([2]), which is quite similar).

Second, for every $i \in \mathbb{N}$, every $(v, g) \in \mathcal{E}^\varepsilon$, and for $t = 0$,

$$v_i^+ = v_i + 0b(g)(h_i^{r(g)}[v] + l_i^{r(g)}) = v_i,$$

so that $\varphi_0^1 = \text{id}_{\mathcal{E}^\varepsilon}$.

Third, for every $v \in \mathcal{F}$, and for $t = 1$, $(\varphi_1^1 \circ \kappa^1)(v) = \varphi_1^1(\rho(v)) = \varphi_1^1(\hat{v}, g^v) = (\hat{v}^+, g^v)$, where for every $i \in \mathbb{N}$,

$$\hat{v}_i^+ = \hat{v}_i + b(g^v)(h_i^{r(g^v)}[v] + l_i^{r(g^v)}) = (v_i - h_i^{g^v}[v] - l_i^{g^v}) + b(g^v)(h_i^{r(g^v)}[v] + l_i^{r(g^v)}) = v_i$$

(since $g^v \in \mathbb{G}$), so that $\varphi_1^1 \circ \kappa^1 = e$.

Last, it remains to show that for every $t \in [0, 1]$, φ_t^1 is a homeomorphism with inverse ξ_t^1 . First, both φ_t^1 and ξ_t^1 are continuous maps (as both φ^1 and ξ^1 are continuous maps). Second, for every $(v, g) \in \mathcal{E}^\varepsilon$, $(\varphi_t^1 \circ \xi_t^1)(v, g) = \varphi_t^1(v^-, g) = ((v^-)^+, g)$, where for every $i \in \mathbb{N}$,

$$\begin{aligned} (v^-)_i^+ &= v_i^- + tb(g)(h_i^{r(g)}[v^-] + l_i^{r(g)}) \\ &= (v_i - tb(g)(h_i^{r(g)}[v] + l_i^{r(g)})) + tb(g)(h_i^{r(g)}[v^-] + l_i^{r(g)}) \\ &= v_i + tb(g)(h_i^{r(g)}[v^-] - h_i^{r(g)}[v]). \end{aligned}$$

For every $\gamma \in \mathbb{G}$,

$$\begin{aligned} & (h_i^{r(g)}[v^-] - h_i^{r(g)}[v])(\gamma) \\ &= \sum_{j \neq i} (\partial_{ij} v_i^-(r(g)_{ij}, r(g)_{-ij}) - \partial_{ij} v_i^-(r(g)_{ij}, g_{-ij}^0))(\gamma_{ij} - r(g)_{ij}) \\ & \quad - \sum_{j \neq i} (\partial_{ij} v_i(r(g)_{ij}, r(g)_{-ij}) - \partial_{ij} v_i(r(g)_{ij}, g_{-ij}^0))(\gamma_{ij} - r(g)_{ij}) \\ &= \sum_{j \neq i} (\partial_{ij} (v_i^- - v_i)(r(g)_{ij}, r(g)_{-ij}) - \partial_{ij} (v_i^- - v_i)(r(g)_{ij}, g_{-ij}^0))(\gamma_{ij} - r(g)_{ij}), \end{aligned}$$

where $v_i^- - v_i = (v_i - \text{tb}(g)(h_i^{r(g)}[v] + l_i^{r(g)})) - v_i = -\text{tb}(g)(h_i^{r(g)}[v] + l_i^{r(g)})$. Since $v_i^- - v_i$ is an element of \mathcal{A}_i (that is, an affine map in $(g_{ij})_{j \neq i}$), notice that $\partial_{ij}(v_i^- - v_i)(r(g)_{ij}, r(g)_{-ij}) - \partial_{ij}(v_i^- - v_i)(r(g)_{ij}, g_{-ij}^0) = 0$, so that $h_i^{r(g)}[v^-] = h_i^{r(g)}[v]$ and $(v_i^-)^+ = v_i$. In a similar way, notice that $(\xi_t^1 \circ \varphi_t^1)(v, g) = (v, g)$.

Finally, φ^1 is an ambient isotopy between κ^1 and e within \mathcal{E}^ε . Remark that for every $t \in [0, 1]$, $\varphi_t^1|_{\mathcal{F} \times \partial \mathbb{G}^\varepsilon} = \text{id}_{\mathcal{F} \times \partial \mathbb{G}^\varepsilon}$, i.e. the ambient isotopy φ^1 does not deform the boundary of the ambient space.

Step II. Let $f : \mathcal{F} \rightarrow \mathbb{G}$ be a continuous map, and $e_f : \mathcal{F} \rightarrow \mathcal{E}^\varepsilon, v \mapsto (v, f(v))$. There exists an ambient isotopy φ^2 between e_f and κ^2 within \mathcal{E}^ε .

For every link $ij \in L$, and every $w \in [0, 1]$, consider the piecewise linear maps $\chi_{ij}(w, \cdot)$ and $\zeta_{ij}(w, \cdot)$ defined by

$$\left\{ \begin{array}{l} \chi_{ij}(w, \cdot) : [-\varepsilon, 1 + \varepsilon] \rightarrow [-\varepsilon, 1 + \varepsilon] \\ \quad -\varepsilon \mapsto -\varepsilon \\ \quad g_{ij}^0 \mapsto w \\ \quad 1 + \varepsilon \mapsto 1 + \varepsilon \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \zeta_{ij}(w, \cdot) : [-\varepsilon, 1 + \varepsilon] \rightarrow [-\varepsilon, 1 + \varepsilon] \\ \quad -\varepsilon \mapsto -\varepsilon \\ \quad w \mapsto g_{ij}^0 \\ \quad 1 + \varepsilon \mapsto 1 + \varepsilon \end{array} \right. ,$$

and consider the maps $\chi_{ij} : [0, 1] \times [-\varepsilon, 1 + \varepsilon] \rightarrow [-\varepsilon, 1 + \varepsilon], (w, \omega) \mapsto \chi_{ij}(w, \omega)$ and $\zeta_{ij} : [0, 1] \times [-\varepsilon, 1 + \varepsilon] \rightarrow [-\varepsilon, 1 + \varepsilon], (w, \omega) \mapsto \zeta_{ij}(w, \omega)$. For every $ij \in L$, the maps χ_{ij} and ζ_{ij} have the following properties:

1. For every $w \in [0, 1]$, $\chi_{ij}(w, \cdot)$ is a homeomorphism¹⁶ with inverse $\zeta_{ij}(w, \cdot)$.
2. For every $w \in [0, 1]$, $\chi_{ij}(w, \cdot)|_{[-\varepsilon, 1 + \varepsilon]} = \text{id}_{[-\varepsilon, 1 + \varepsilon]}$.
3. For every $w \in [0, 1]$, $\chi_{ij}(w, g_{ij}^0) = w$.
4. $\chi_{ij}(g_{ij}^0, \cdot) = \text{id}_{[-\varepsilon, 1 + \varepsilon]}$.

Let

$$\left\{ \begin{array}{l} \varphi_f^2 : [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathcal{E}^\varepsilon \\ \quad (t, (v, g)) \mapsto (v, g^t) \end{array} \right. ,$$

where for every $ij \in L$, $g_{ij}^t = \zeta_{ij}((1 - t)g_{ij}^0 + tf(v)_{ij}, g_{ij})$, and let

$$\left\{ \begin{array}{l} \xi_f^2 : [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathcal{E}^\varepsilon \\ \quad (t, (v, g)) \mapsto (v, g^x) \end{array} \right. ,$$

where for every $ij \in L$, $g_{ij}^x = \chi_{ij}((1 - t)g_{ij}^0 + tf(v)_{ij}, g_{ij})$.

First, φ_f^2 is a continuous map, as for every $ij \in L$, ζ_{ij} is a continuous map.

Second, for every $ij \in L$, every $(v, g) \in \mathcal{E}^\varepsilon$, and for $t = 0$,

$$g_{ij}^t = \zeta_{ij}((1 - 0)g_{ij}^0 + 0f(v)_{ij}, g_{ij}) = \zeta_{ij}(g_{ij}^0, g_{ij}).$$

¹⁶These two maps are continuous, and for every $\omega \in [-\varepsilon, 1 + \varepsilon]$, $(\zeta_{ij}(w, \cdot) \circ \chi_{ij}(w, \cdot))(\omega) = \omega$ (resp. $(\chi_{ij}(w, \cdot) \circ \zeta_{ij}(w, \cdot))(\omega) = \omega$), since $(\zeta_{ij}(w, \cdot) \circ \chi_{ij}(w, \cdot))(g_{ij}^0) = g_{ij}^0$ (resp. $(\chi_{ij}(w, \cdot) \circ \zeta_{ij}(w, \cdot))(g_{ij}^0) = g_{ij}^0$).

Now, by properties 1 and 4, notice that $\zeta_{ij}(g_{ij}^0, \cdot)$ is the identity on $[-\varepsilon, 1 + \varepsilon]$, so that $(\varphi_f^2)_0 = \text{id}_{\mathcal{E}^\varepsilon}$.

Third, for every $v \in \mathcal{F}$, and for $t = 1$, $((\varphi_f^2)_1 \circ e_f)(v) = (\varphi_f^2)_1(v, f(v)) = (v, f(v)^\zeta)$, where for every $ij \in L$,

$$f(v)_{ij}^\zeta = \zeta_{ij}(f(v)_{ij}, f(v)_{ij}) = g_{ij}^0$$

(from properties 1 and 3), so that $(\varphi_f^2)_1 \circ e_f = \kappa^2$.

Last, it remains to show that for every $t \in [0, 1]$, $(\varphi_f^2)_t$ is a homeomorphism with inverse $(\xi_f^2)_t$. First, both $(\varphi_f^2)_t$ and $(\xi_f^2)_t$ are continuous maps (as both φ_f^2 and ξ_f^2 are continuous maps). Second, for every $(v, g) \in \mathcal{E}^\varepsilon$, $((\varphi_f^2)_t \circ (\xi_f^2)_t)(v, g) = (\varphi_f^2)_t(v, g^x) = (v, (g^x)^\zeta)$, where for every $ij \in L$,

$$\begin{aligned} (g^x)_{ij}^\zeta &= \zeta_{ij}((1-t)g_{ij}^0 + tf(v)_{ij}, g_{ij}^x) \\ &= \zeta_{ij}((1-t)g_{ij}^0 + tf(v)_{ij}, \chi_{ij}((1-t)g_{ij}^0 + tf(v)_{ij}, g_{ij})) \\ &= (\zeta_{ij}((1-t)g_{ij}^0 + tf(v)_{ij}, \cdot) \circ \chi_{ij}((1-t)g_{ij}^0 + tf(v)_{ij}, \cdot))(g_{ij}) \\ &= g_{ij} \text{ (from property 1)}. \end{aligned}$$

In a similar way, notice that $((\xi_f^2)_t \circ (\varphi_f^2)_t)(v, g) = (v, g)$.

Finally, φ_f^2 is an ambient isotopy between e_f and κ^2 within \mathcal{E}^ε . Remark that for every $t \in [0, 1]$, $(\varphi_f^2)_t|_{\mathcal{F} \times \partial \mathbb{G}^\varepsilon} = \text{id}_{\mathcal{F} \times \partial \mathbb{G}^\varepsilon}$, i.e. the ambient isotopy φ_f^2 does not deform the boundary of the ambient space.

Step III. *There exists an ambient isotopy φ between κ^1 and κ^2 within \mathcal{E}^ε .*

To finish, from Step I and Step II, the map

$$\begin{cases} \varphi : [0, 1] \times \mathcal{E}^\varepsilon & \rightarrow & \mathcal{E}^\varepsilon \\ (t, (v, g)) & \mapsto & ((\varphi_{(\pi_{\mathbb{G} \circ \rho})}^2)_t \circ \varphi_t^1)(v, g) \end{cases}$$

is an ambient isotopy¹⁷ between κ^1 and κ^2 within \mathcal{E}^ε . Remark that for every $t \in [0, 1]$, $\varphi_t|_{\mathcal{F} \times \partial \mathbb{G}^\varepsilon} = \text{id}_{\mathcal{F} \times \partial \mathbb{G}^\varepsilon}$, i.e. the ambient isotopy φ does not deform the boundary of the ambient space.

Proof of Corollary 3.1. Recall that $\eta_{\mathcal{R}}$ is the homeomorphism of Corollary 2.1, and that $\rho_{\mathcal{R}}$ is its inverse.

The proof is similar to the one of Theorem 3.1. Since \mathcal{R} is a regular set, one can show that for every continuous map $f : \mathcal{R} \rightarrow \mathbb{G}$, the maps

$$\begin{cases} (\varphi^1)_{\mathcal{R}} : [0, 1] \times \mathcal{E}_{\mathcal{R}}^\varepsilon & \rightarrow & \mathcal{E}_{\mathcal{R}}^\varepsilon \\ (t, (v, g)) & \mapsto & (v^+, g) \end{cases} \quad \text{and} \quad \begin{cases} (\varphi_f^2)_{\mathcal{R}} : [0, 1] \times \mathcal{E}_{\mathcal{R}}^\varepsilon & \rightarrow & \mathcal{E}_{\mathcal{R}}^\varepsilon \\ (t, (v, g)) & \mapsto & (v, g^\zeta) \end{cases}$$

are well-defined¹⁸ (from affine stability assumption in Definition 2.3).

¹⁷In particular, by definition of φ^1 and $\varphi_{(\pi_{\mathbb{G} \circ \rho})}^2$: $((\varphi_{(\pi_{\mathbb{G} \circ \rho})}^2)_1 \circ \varphi_1^1) \circ \kappa^1 = ((\varphi_{(\pi_{\mathbb{G} \circ \rho})}^2)_1 \circ e) = ((\varphi_{(\pi_{\mathbb{G} \circ \rho})}^2)_1 \circ e_{(\pi_{\mathbb{G} \circ \rho})}) = \kappa^2$.

¹⁸ v^+ and g^ζ are defined as in the proof of Theorem 3.1.

The map $(\varphi^1)_{\mathcal{R}}$ is an ambient isotopy between $\kappa_{\mathcal{R}}^1 : \mathcal{R} \rightarrow \mathcal{E}_{\mathcal{R}}^{\varepsilon}, v \mapsto \rho_{\mathcal{R}}(v)$ and the topological embedding $\mathcal{R} \rightarrow \mathcal{E}_{\mathcal{R}}^{\varepsilon}, v \mapsto (v, (\pi_{\mathbb{G}} \circ \rho_{\mathcal{R}})(v)) = (v, g^v)$ within $\mathcal{E}_{\mathcal{R}}^{\varepsilon}$, and the map $(\varphi_f^2)_{\mathcal{R}}$ is an ambient isotopy between the topological embedding $\mathcal{R} \rightarrow \mathcal{E}_{\mathcal{R}}^{\varepsilon}, v \mapsto (v, f(v))$ and $\kappa_{\mathcal{R}}^2 : \mathcal{R} \rightarrow \mathcal{E}_{\mathcal{R}}^{\varepsilon}, v \mapsto (v, g^0)$ within $\mathcal{E}_{\mathcal{R}}^{\varepsilon}$, for every continuous map $f : \mathcal{R} \rightarrow \mathbb{G}$. Thus, the map

$$\left\{ \begin{array}{l} \varphi_{\mathcal{R}} : [0, 1] \times \mathcal{E}_{\mathcal{R}}^{\varepsilon} \rightarrow \mathcal{E}_{\mathcal{R}}^{\varepsilon} \\ (t, (v, g)) \mapsto [(\varphi_{(\pi_{\mathbb{G}} \circ \rho_{\mathcal{R}})}^2)_{\mathcal{R}}]_t \circ [(\varphi^1)_{\mathcal{R}}]_t(v, g) \end{array} \right.$$

is an ambient isotopy between $\kappa_{\mathcal{R}}^1$ and $\kappa_{\mathcal{R}}^2$ within $\mathcal{E}_{\mathcal{R}}^{\varepsilon}$.

5.3 PROOFS OF SECTION 4

Proof of Theorem 4.1. First of all, remark that \mathbb{G}^{ε} is a convex subset of \mathbb{R}^L , which implies that it is contractible. Thus, from Proposition 5.1 (see Section 5.1), $T\mathbb{G}^{\varepsilon}$ is trivial. Consequently, using Proposition 5.3 (see Section 5.1), \tilde{D}^1 and \tilde{D}^2 are treated throughout all this proof as continuous maps from $\mathcal{E}^{\varepsilon}$ to \mathbb{R}^L such that $(\tilde{D}^1)^{-1}(\{0\}) = (\tilde{D}^2)^{-1}(\{0\}) = \mathcal{P}$, for the sake of simplicity.

The proof of Theorem 4.1 goes as follows: the required homotopy \tilde{H} between \tilde{D}^1 and \tilde{D}^2 is built in a two-step process:

$$\left\{ \begin{array}{l} \tilde{H} : [0, 1] \times \mathcal{E}^{\varepsilon} \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto \left\{ \begin{array}{l} \tilde{H}^1(2t, (v, g)) \quad \text{if } t \in [0, \frac{1}{2}] \\ \tilde{H}^2(2t - 1, (v, g)) \quad \text{if } t \in [\frac{1}{2}, 1] \end{array} \right. \end{array} \right. ,$$

where:

1. The map \tilde{H}^1 is built in Step I. The idea is to deform the map \tilde{D}^1 to a map $\tilde{D}^{1,1}$ which is equal to \tilde{D}^1 on $\mathbb{G}^{\varepsilon} - \mathbb{G}$, and which is equal to \tilde{D}^2 on $\partial\mathbb{G}^{\varepsilon}$. This property will be crucial for the construction of the map \tilde{H}^2 .
2. The map \tilde{H}^2 is built from Step II to Step VIII. The idea is to deform the map $\tilde{D}^{1,1}$ to the map \tilde{D}^2 . Briefly:
 - In Step II, we use the ambient isotopy φ of Theorem 3.1 to simplify this construction. Our aim is to deform the map $\tilde{\Delta}^{1,1} = \tilde{D}^{1,1} \circ \varphi_1^{-1}$ to the map $\tilde{\Delta}^2 = \tilde{D}^2 \circ \varphi_1^{-1}$.
 - From Step III to Step VI, we proceed to a kind of "backand-forth motion". First, the map $\tilde{\Delta}^{1,1}$ is deformed to a certain map $\tilde{\Delta}^{1,2}$, itself deformed to a certain map $\tilde{\Delta}^{1,3}$ (Step III and Step IV). Second, we notice that $\tilde{\Delta}^{1,3}$ is equal to a certain map $\tilde{\Delta}^{2,4}$ (Step V), which is a consequence of the crucial property explained previously. Last, in a very similar way, the map $\tilde{\Delta}^{2,4}$ is deformed to a certain map $\tilde{\Delta}^{2,5}$, itself deformed to the map $\tilde{\Delta}^2$ (Step V and Step VI).
 - We conclude this construction in Step VII and Step VIII.

Step I. *First step in the construction of the homotopy \tilde{H} between \tilde{D}^1 and \tilde{D}^2 .*

First, define

$$\left\{ \begin{array}{l} \tilde{D}^{1,1} : \mathcal{E}^\varepsilon \rightarrow \mathbb{R}^L \\ (v, g) \mapsto \begin{cases} \lambda_g \tilde{D}^1(v, g) + (1 - \lambda_g) \tilde{D}^2(v, g) & \text{if } g \in \mathbb{G}^\varepsilon - \mathbb{G} \\ \tilde{D}^1(v, g) & \text{otherwise} \end{cases} \end{array} \right. ,$$

where for every $g \in \mathbb{G}^\varepsilon - \mathbb{G}$,

$$\lambda_g = \frac{\|g - g^{1/2}\|_\infty - \frac{1}{2} - \varepsilon}{-\varepsilon} \in [0, 1],$$

and $g^{1/2}$ is the network such that for every $ij \in L$, $g_{ij}^{1/2} = \frac{1}{2}$. The map $\tilde{D}^{1,1}$ is continuous since (i) the map \tilde{D}^1 is continuous, (ii) the map $(v, g) \in \mathcal{E}^\varepsilon \mapsto \lambda_g \tilde{D}^1(v, g) + (1 - \lambda_g) \tilde{D}^2(v, g) \in \mathbb{R}^L$ is continuous (because the map $g \in \mathbb{G}^\varepsilon - \mathbb{G} \mapsto \lambda_g \in [0, 1]$ is continuous), and (iii) for every $(v, g) \in \mathcal{F} \times \partial\mathbb{G}$, $\lambda_g \tilde{D}^1(v, g) + (1 - \lambda_g) \tilde{D}^2(v, g) = \tilde{D}^1(v, g)$.

Moreover, notice that $\tilde{D}^{1,1}|_{\mathcal{E}} = \tilde{D}^1|_{\mathcal{E}}$ and $\tilde{D}^{1,1}|_{\mathcal{F} \times \partial\mathbb{G}^\varepsilon} = \tilde{D}^2|_{\mathcal{F} \times \partial\mathbb{G}^\varepsilon}$.

Second, define

$$\left\{ \begin{array}{l} \tilde{H}^1 : [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto (1 - t) \tilde{D}^1(v, g) + t \tilde{D}^{1,1}(v, g) \end{array} \right.$$

The map \tilde{H}^1 is a homotopy between \tilde{D}^1 and $\tilde{D}^{1,1}$: it is continuous (since both \tilde{D}^1 and $\tilde{D}^{1,1}$ are continuous maps), and for every $(v, g) \in \mathcal{E}^\varepsilon$, $\tilde{H}_0^1(v, g) = \tilde{D}^1(v, g)$ and $\tilde{H}_1^1(v, g) = \tilde{D}^{1,1}(v, g)$. Furthermore, for every $t \in [0, 1]$, the map \tilde{H}_t^1 is an extended network dynamics on \mathcal{F} :

- For every $t \in [0, 1]$, \tilde{H}_t^1 is continuous since \tilde{H}^1 is a continuous map.
- $(\tilde{H}_t^1)^{-1}(\{0\}) = \mathcal{P}$. Indeed, remark first that, since \tilde{D}^1 and \tilde{D}^2 are strongly inward-pointing, and since for every $g \in \mathbb{G}^\varepsilon - \mathbb{G}$, $\lambda_g \in [0, 1]$, we get that for every $v \in \mathcal{F}$, every $\varepsilon' \in (0, \varepsilon]$, every $g \in \partial\mathbb{G}^{\varepsilon'}$, and every $x \in N_{\partial\mathbb{G}^{\varepsilon'}}(g)$,

$$\langle \tilde{D}^{1,1}(v, g), x \rangle = \langle \lambda_g \tilde{D}^1(v, g) + (1 - \lambda_g) \tilde{D}^2(v, g), x \rangle = \lambda_g \langle \tilde{D}^1(v, g), x \rangle + (1 - \lambda_g) \langle \tilde{D}^2(v, g), x \rangle < 0,$$

and we obtain that for every $v \in \mathcal{F}$, $\tilde{D}_v^{1,1}$ has no zeros on $\mathbb{G}^\varepsilon - \mathbb{G}$. Thus, $(\tilde{D}^{1,1})^{-1} = \mathcal{P}$, since $\tilde{D}^{1,1}|_{\mathcal{E}} = \tilde{D}^1|_{\mathcal{E}}$. Finally, for every $v \in \mathcal{F}$, every $\varepsilon' \in (0, \varepsilon]$, every $g \in \partial\mathbb{G}^{\varepsilon'}$, every $x \in N_{\partial\mathbb{G}^{\varepsilon'}}(g)$, and every $t \in [0, 1]$,

$$\langle \tilde{H}^1(v, g), x \rangle = \langle (1 - t) \tilde{D}^1(v, g) + t \tilde{D}^{1,1}(v, g), x \rangle = (1 - t) \langle \tilde{D}^1(v, g), x \rangle + t \langle \tilde{D}^{1,1}(v, g), x \rangle < 0,$$

and we obtain that for every $v \in \mathcal{F}$, $\tilde{H}_t^1(v, \cdot)$ has no zeros on $\mathbb{G}^\varepsilon - \mathbb{G}$. Thus, for every $t \in [0, 1]$, $(\tilde{H}_t^1)^{-1} = \mathcal{P}$, since for every $(v, g) \in \mathcal{E}$,

$$\tilde{H}_t^1(v, g) = (1 - t) \tilde{D}^1(v, g) + t \tilde{D}^{1,1}(v, g) = (1 - t) \tilde{D}^1(v, g) + t \tilde{D}^1(v, g) = \tilde{D}^1(v, g).$$

Step II. *Use of the ambient isotopy φ .*

Now, consider the ambient isotopy

$$\begin{cases} \varphi : [0, 1] \times \mathcal{E}^\varepsilon & \rightarrow & \mathcal{E}^\varepsilon \\ (t, (v, g)) & \mapsto & ((\varphi_{(\pi_{\mathbb{G}} \circ \rho)}^2)_t \circ \varphi_t^1)(v, g) \end{cases}$$

between $\kappa^1 : \mathcal{F} \rightarrow \mathcal{E}^\varepsilon, v \mapsto \rho(v) = (\hat{v}, g^v)$ and $\kappa^2 : \mathcal{F} \rightarrow \mathcal{E}^\varepsilon, v \mapsto (v, g^0)$ within \mathcal{E}^ε of Theorem 3.1, and define $\tilde{\Delta}^{1,1} = \tilde{D}^{1,1} \circ \varphi_1^{-1}$ and $\tilde{\Delta}^2 = \tilde{D}^2 \circ \varphi_1^{-1}$. The map $\tilde{\Delta}^{1,1}$ is continuous (as the composition of two continuous maps) and is such that

$$(\tilde{\Delta}^{1,1})^{-1}(\{0\}) = (\tilde{D}^{1,1} \circ \varphi_1^{-1})^{-1}(\{0\}) = \varphi_1((\tilde{D}^{1,1})^{-1}(\{0\})) = \varphi_1(\mathcal{P}) = \mathcal{F}^0 = \mathcal{F} \times \{g^0\},$$

and similarly for the map $\tilde{\Delta}^2$. Thus, if $\tilde{\Phi}$ is a homotopy between $\tilde{\Delta}^{1,1}$ and $\tilde{\Delta}^2$ such that for every $t \in [0, 1]$, $(\tilde{\Phi}_t)^{-1}(\{0\}) = \mathcal{F}^0$, then

$$\begin{cases} \tilde{H}^2 : [0, 1] \times \mathcal{E}^\varepsilon & \rightarrow & \mathbb{R}^L \\ (t, (v, g)) & \mapsto & (\tilde{\Phi}_t \circ \varphi_1)(v, g) \end{cases}$$

is a homotopy¹⁹ between $\tilde{D}^{1,1}$ and \tilde{D}^2 such that for every $t \in [0, 1]$, the map \tilde{H}_t^2 is an extended network dynamics on \mathcal{F} .

Also, notice that by definition of φ , we get that $\tilde{\Delta}^{1,1}|_{\mathcal{F} \times \partial \mathbb{G}^\varepsilon} = \tilde{\Delta}^2|_{\mathcal{F} \times \partial \mathbb{G}^\varepsilon}$ (the ambient isotopy φ does not deform the boundary of the ambient space).

Step III. *First step in the construction of the homotopy $\tilde{\Phi}$ between $\tilde{\Delta}^{1,1}$ and $\tilde{\Delta}^2$.*

First, define

$$\begin{cases} \tilde{\Delta}^{1,2} : \mathcal{E}^\varepsilon & \rightarrow & \mathbb{R}^L \\ (v, g) & \mapsto & \begin{cases} (\tilde{\Delta}^{1,1}(v, g) / \|\tilde{\Delta}^{1,1}(v, g)\|_2) \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0 & \text{otherwise} \end{cases} \end{cases} .$$

Continuity of this map is a consequence of the (more general) following lemma:

Lemma 5.1. *Let X and Y be metrizable topological spaces, A be a subspace of X , $f_1 : A \rightarrow Y$, $f_2 : X - A \rightarrow Y$, and*

$$\begin{cases} f : X & \rightarrow & Y \\ x & \mapsto & \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \in X - A \end{cases} \end{cases} .$$

Suppose that both f_1 and f_2 are continuous, and that for every $\bar{x} \in \partial_X A$, every sequence (u^k) in X such that $\{u^k : k \in \mathbb{N}\} \subset A$, and every sequence (v^k) in X such that $\{v^k : k \in \mathbb{N}\} \subset X - A$, $u^k \rightarrow \bar{x}$ and $v^k \rightarrow \bar{x}$ implies that $f_1(u^k) \rightarrow f(\bar{x})$ and $f_2(v^k) \rightarrow f(\bar{x})$. Then, the map f is continuous.

¹⁹For every $(v, g) \in \mathcal{E}^\varepsilon$, $\tilde{H}_0^2(v, g) = (\tilde{\Phi}_0 \circ \varphi_1)(v, g) = (\tilde{\Delta}^{1,1} \circ \varphi_1)(v, g) = \tilde{D}^1(v, g)$ and $\tilde{H}_1^2(v, g) = (\tilde{\Phi}_1 \circ \varphi_1)(v, g) = (\tilde{\Delta}^2 \circ \varphi_1)(v, g) = \tilde{D}^2(v, g)$. Moreover, for every $t \in [0, 1]$, $(\tilde{H}_t^2)^{-1}(\{0\}) = (\tilde{\Phi}_t \circ \varphi_1)^{-1}(\{0\}) = \varphi_1^{-1}(\tilde{\Phi}_t^{-1}(\{0\})) = \varphi_1^{-1}(\mathcal{F}^0) = \mathcal{P}$.

Proof. Let $x \in X$, and (x^k) be a sequence in X such that $(x^k) \longrightarrow x$:

1. First, suppose that $x \in \text{int}_X A$. Since $(x^k) \longrightarrow x$, there exists $K \in \mathbb{N}$ such that for every $k \geq K$, $x^k \in \text{int}_X A$. Now, consider an open set U in Y containing $f(x)$ (since $x \in \text{int}_X A$, $f(x) = f_1(x)$). Since $(x^k)_{k \geq K}$ is a sequence in A which converges to x , we get that $f_1(x^k) \xrightarrow{k \geq K} f_1(x)$ (by continuity of f_1). In particular, there exists $K' \in \mathbb{N}$ such that for every $k \geq K' \geq K$, $f_1(x^k) \in U$. Since $K' \geq K$, we get that for every $k \geq K'$, $x^k \in \text{int}_X A$, thus that $f_1(x^k) = f(x)$ (i.e. that f is continuous at x).
2. The case where $x \in \text{int}_X X - A$ is similar to the previous one.
3. Finally, suppose that $x \in \partial_X A$. Let $I_A = \{k \in \mathbb{N} : x^k \in A\}$ and $I_{X-A} = \{k \in \mathbb{N} : x^k \in X - A\}$, and consider the two subsequences $(x^k_A) = (x^k)_{k \in I_A}$ and $(x^k_{X-A}) = (x^k)_{k \in I_{X-A}}$ of (x^k) . Since $x^k_A \longrightarrow x$ and $x^k_{X-A} \longrightarrow x$, we get that $f_1(x^k_A) \longrightarrow f(x)$ and $f_2(x^k_{X-A}) \longrightarrow \bar{f}(x)$ (by assumption). Now, consider an open set U in Y containing $f(x)$. Since $f_1(x^k_A)$ is a sequence in Y which converges to $f(x)$, we get that there exists $K_A \in I_A$ such that for every $k \geq K_A$ (in I_A), $f_1(x^k_A) \in U$. Similarly, we get that there exists $K_{X-A} \in I_{X-A}$ such that for every $k \geq K_{X-A}$ (in I_{X-A}), $f_2(x^k_{X-A}) \in U$. Let $K = \max\{K_A, K_{X-A}\}$. Since $\mathbb{N} = I_A \cup I_{X-A}$, we get that either $f(x^k) = f_1(x^k_A)$, or $f(x^k) = f_2(x^k_{X-A})$, for every $k \geq K$ (by definition of f). Thus, we get that for every $k \geq K$, $f(x^k) \in U$ (i.e. that f is continuous at x).

□

In our case, we obtain the result by setting $X = \mathcal{E}^\varepsilon$, $Y = \mathbb{R}^L$, $A = \mathcal{F} \times (\mathbb{G}^\varepsilon - \{g^0\})$, $f_1 : (v, g) \mapsto (\tilde{\Delta}^{1,1}(v, g) / \|\tilde{\Delta}^{1,1}(v, g)\|_2) \|g - g^0\|_2$, $f_2 : (v, g) \mapsto 0$ and $f = \tilde{\Delta}^{1,2}$ in Lemma 5.1. Indeed, for every $v \in \mathcal{F}$, and every sequence (v^k, g^k) in \mathcal{E}^ε such that $\{(v^k, g^k) : k \in \mathbb{N}\} \subset \mathcal{F} \times (\mathbb{G}^\varepsilon - \{g^0\})$, if $(v^k, g^k) \longrightarrow (v, g^0)$, then we get that for every $k \in \mathbb{N}$,

$$\begin{aligned} & \|(\tilde{\Delta}^{1,1}(v^k, g^k) / \|\tilde{\Delta}^{1,1}(v^k, g^k)\|_2) \|g^k - g^0\|_2 - \tilde{\Delta}^{1,2}(v, g^0)\|_2 \\ &= \|(\tilde{\Delta}^{1,1}(v^k, g^k) / \|\tilde{\Delta}^{1,1}(v^k, g^k)\|_2) \|g^k - g^0\|_2\|_2 \\ &= \|g^k - g^0\|_2. \end{aligned}$$

Thus, $\|(\tilde{\Delta}^{1,1}(v^k, g^k) / \|\tilde{\Delta}^{1,1}(v^k, g^k)\|_2) \|g^k - g^0\|_2 - \tilde{\Delta}^{1,2}(v, g^0)\|_2$ converges to 0.

Second, define

$$\begin{cases} \tilde{\Phi}^1 : [0, 1] \times \mathcal{E}^\varepsilon & \rightarrow & \mathbb{R}^L \\ (t, (v, g)) & \mapsto & (1-t)\tilde{\Delta}^{1,1}(v, g) + t\tilde{\Delta}^{1,2}(v, g) \end{cases}$$

The map $\tilde{\Phi}^1$ is a homotopy between $\tilde{\Delta}^{1,1}$ and $\tilde{\Delta}^{1,2}$: it is continuous (since both $\tilde{\Delta}^{1,1}$ and $\tilde{\Delta}^{1,2}$ are continuous maps), and for every $(v, g) \in \mathcal{E}^\varepsilon$, $\tilde{\Phi}_0^1(v, g) = \tilde{\Delta}^{1,1}(v, g)$ and $\tilde{\Phi}_1^1(v, g) = \tilde{\Delta}^{1,2}(v, g)$. Furthermore, for every $t \in [0, 1]$, the map $\tilde{\Phi}_t^1$ has the following properties:

- $\tilde{\Phi}_t^1$ is continuous (since $\tilde{\Phi}^1$ is a continuous map).

- $(\tilde{\Phi}_t^1)^{-1}(\{0\}) = \mathcal{F}^0$. Indeed, for every $t \in [0, 1]$, an element $(v, g) \in \mathcal{E}^\varepsilon$ is in $(\tilde{\Phi}_t^1)^{-1}(\{0\})$ if and only if either $g \neq g^0$ and

$$(\tilde{\Delta}^{1,1}(v, g) / \|\tilde{\Delta}^{1,1}(v, g)\|_2) ((1-t)\|\tilde{\Delta}^{1,1}(v, g)\|_2 + t\|g - g^0\|_2) = 0, \quad (1)$$

or $g = g^0$. Thus, since Equation (1) has no solution, we get the result.

Step IV. *Second step in the construction of the homotopy $\tilde{\Phi}$ between $\tilde{\Delta}^{1,1}$ and $\tilde{\Delta}^2$.*

For every $g \in \mathbb{G}^\varepsilon$ such that $g \neq g^0$, consider the unique element $\beta_g \in \partial\mathbb{G}^\varepsilon$ such that $\beta_g \in \{g' \in \mathbb{G}^\varepsilon : \exists t \in \mathbb{R}^+, g' = g^0 + t(g - g^0)\}$ (i.e. such that β_g belongs to the half-line starting at g^0 and passing through g).

First, define

$$\begin{cases} f : [0, 1] \times (\mathbb{G}^\varepsilon - \{g^0\}) & \rightarrow & \mathbb{G}^\varepsilon \\ (t, g) & \mapsto & g + t(\beta_g - g) \end{cases}$$

This map is continuous: it maps $(t, g) \in [0, 1] \times (\mathbb{G}^\varepsilon - \{g^0\})$ to the point $g + t(\beta_g - g)$, which belongs to the line segment between g and β_g .

Second, define

$$\begin{cases} \tilde{\Delta}^{1,3} : \mathcal{E}^\varepsilon & \rightarrow & \mathbb{R}^L \\ (v, g) & \mapsto & \begin{cases} (\tilde{\Delta}^{1,1}(v, \beta_g) / \|\beta_g - g^0\|_2) \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0 & \text{otherwise} \end{cases} \end{cases} .$$

Continuity of this map is a consequence of Lemma 5.1 by setting $X = \mathcal{E}^\varepsilon$, $Y = \mathbb{R}^L$, $A = \mathcal{F} \times (\mathbb{G}^\varepsilon - \{g^0\})$, $f_1 : (v, g) \mapsto (\tilde{\Delta}^{1,1}(v, \beta_g) / \|\beta_g - g^0\|_2) \|g - g^0\|_2$, $f_2 : (v, g) \mapsto 0$ and $f = \tilde{\Delta}^{1,3}$.²⁰

Third, define

$$\begin{cases} \tilde{\Phi}^2 : [0, 1] \times \mathcal{E}^\varepsilon & \rightarrow & \mathbb{R}^L \\ (t, (v, g)) & \mapsto & \begin{cases} (\tilde{\Delta}^{1,1}(v, f(t, g)) / \|f(t, g) - g^0\|_2) \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0 & \text{otherwise} \end{cases} \end{cases} .$$

The map $\tilde{\Phi}^2$ is a homotopy between $\tilde{\Delta}^{1,2}$ and $\tilde{\Delta}^{1,3}$: it is continuous (again, it is a consequence of Lemma 5.1 by setting $X = [0, 1] \times \mathcal{E}^\varepsilon$, $Y = \mathbb{R}^L$, $A = [0, 1] \times (\mathcal{F} \times (\mathbb{G}^\varepsilon - \{g^0\}))$, $f_1 : (t, (v, g)) \mapsto (\tilde{\Delta}^{1,1}(v, f(t, g)) / \|f(t, g) - g^0\|_2) \|g - g^0\|_2$, $f_2 : (v, g) \mapsto 0$ and $f = \tilde{\Phi}^2$), and for every $(v, g) \in \mathcal{E}^\varepsilon$, $\tilde{\Phi}_0^2(v, g) = \tilde{\Delta}^{1,2}(v, g)$ and $\tilde{\Phi}_1^2(v, g) = \tilde{\Delta}^{1,3}(v, g)$. Furthermore, for every $t \in [0, 1]$, the map $\tilde{\Phi}_t^2$ has the following properties:

- $\tilde{\Phi}_t^2$ is continuous (since $\tilde{\Phi}^2$ is a continuous map).

²⁰For every $v \in \mathcal{F}$, and every sequence (v^k, g^k) in \mathcal{E}^ε such that $\{(v^k, g^k) : k \in \mathbb{N}\} \subset \mathcal{F} \times (\mathbb{G}^\varepsilon - \{g^0\})$, if $(v^k, g^k) \rightarrow (v, g^0)$, then notice that 0 is a limit point of the sequence $((\tilde{\Delta}^{1,1}(v^k, \beta_{g^k}) / \|\beta_{g^k} - g^0\|_2) \|g^k - g^0\|_2)$ (because $\partial\mathbb{G}^\varepsilon$ is compact, and because $\tilde{\Delta}^{1,1}$ is continuous). Moreover, remark that this is the only limit point of this sequence, thus that it converges to 0: if ℓ is a limit point of $((\tilde{\Delta}^{1,1}(v^k, \beta_{g^k}) / \|\beta_{g^k} - g^0\|_2) \|g^k - g^0\|_2)$, then there exists a subsequence of this sequence which converges to ℓ , but we can still find a subsequence of this subsequence which converges to 0 (again, because $\partial\mathbb{G}^\varepsilon$ is compact, and because $\tilde{\Delta}^{1,1}$ is continuous), which implies that $\ell = 0$.

- $(\tilde{\Phi}_t^2)^{-1}(\{0\}) = \mathcal{F}^0$. Indeed, for every $t \in [0, 1]$, an element $(v, g) \in \mathcal{E}^\varepsilon$ is in $(\tilde{\Phi}_t^2)^{-1}(\{0\})$ if and only if either $g \neq g^0$ and

$$\tilde{\Delta}^{1,1}(v, f(t, g)) = 0, \quad (2)$$

or $g = g^0$. Thus, since Equation (2) has no solution, we get the result.

Step V. *Third step in the construction of the homotopy $\tilde{\Phi}$ between $\tilde{\Delta}^{1,1}$ and $\tilde{\Delta}^2$.*

First, remark that the maps $\tilde{\Delta}^{1,3}$ (defined in Step IV) and

$$\left\{ \begin{array}{l} \tilde{\Delta}^{2,4}: \mathcal{E}^\varepsilon \rightarrow \mathbb{R}^L \\ (v, g) \mapsto \begin{cases} (\tilde{\Delta}^2(v, \beta_g) / \|\beta_g - g^0\|_2) \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0 & \text{otherwise} \end{cases} \end{array} \right. .$$

are equal. Indeed, from Step II, recall that $\tilde{\Delta}^{1,1}|_{\mathcal{F} \times \partial G^\varepsilon} = \tilde{\Delta}^2|_{\mathcal{F} \times \partial G^\varepsilon}$.

Second, define

$$\left\{ \begin{array}{l} \tilde{\Delta}^{2,5}: \mathcal{E}^\varepsilon \rightarrow \mathbb{R}^L \\ (v, g) \mapsto \begin{cases} (\tilde{\Delta}^2(v, g) / \|\tilde{\Delta}^2(v, g)\|_2) \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0 & \text{otherwise} \end{cases} \end{array} \right. .$$

This map is continuous (see the proof for $\tilde{\Delta}^{1,2}$ in Step III, which is similar).

Third, define

$$\left\{ \begin{array}{l} F: [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto \begin{cases} (\tilde{\Delta}^{1,1}(v, f(t, g)) / \|f(t, g) - g^0\|_2) \|g - g^0\|_2 & \text{if } g \neq g^0 \\ 0 & \text{otherwise} \end{cases} \end{array} \right. .$$

(where the map f is defined in Step IV), and define

$$\left\{ \begin{array}{l} \tilde{\Phi}^3: [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto F(1 - t, (v, g)) \end{array} \right. .$$

The map $\tilde{\Phi}^3$ is a homotopy between $\tilde{\Delta}^{2,4}$ and $\tilde{\Delta}^{2,5}$: it is continuous, and for every $(v, g) \in \mathcal{E}^\varepsilon$, $\tilde{\Phi}_0^3(v, g) = \tilde{\Delta}^{2,4}(v, g)$ and $\tilde{\Phi}_1^3(v, g) = \tilde{\Delta}^{2,5}(v, g)$ (see the proof for $\tilde{\Phi}^2$ in Step IV, which is similar). Furthermore, for every $t \in [0, 1]$, the map $\tilde{\Phi}_t^3$ is continuous and is such that $(\tilde{\Phi}_t^3)^{-1}(\{0\}) = \mathcal{F}^0$ (see the proof for $\tilde{\Phi}^2$ in Step IV, which is similar).

Step VI. *Fourth step in the construction of the homotopy $\tilde{\Phi}$ between $\tilde{\Delta}^{1,1}$ and $\tilde{\Delta}^2$.*

Define

$$\left\{ \begin{array}{l} \tilde{\Phi}^4: [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto (1 - t)\tilde{\Delta}^{2,5}(v, g) + t\tilde{\Delta}^2(v, g) \end{array} \right.$$

The map $\tilde{\Phi}^4$ is a homotopy between $\tilde{\Delta}^{2,5}$ and $\tilde{\Delta}^2$: it is continuous, and for every $(v, g) \in \mathcal{E}^\varepsilon$, $\tilde{\Phi}_0^4(v, g) = \tilde{\Delta}^{2,5}(v, g)$ and $\tilde{\Phi}_1^4(v, g) = \tilde{\Delta}^2(v, g)$ (see the proof for $\tilde{\Phi}^1$ in Step III, which is similar). Furthermore, for every $t \in [0, 1]$, the map $\tilde{\Phi}_t^4$ is continuous and is such that $(\tilde{\Phi}_t^4)^{-1}(\{0\}) = \mathcal{F}^0$ (see the proof for $\tilde{\Phi}^1$ in Step III, which is similar).

Step VII. *Last step in the construction of the homotopy $\tilde{\Phi}$ between $\tilde{\Delta}^{1,1}$ and $\tilde{\Delta}^2$.*

From Step III, Step IV and Step V, the map

$$\left\{ \begin{array}{l} \tilde{\Phi} : [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto \begin{cases} \tilde{\Phi}^1(4t, (v, g)) & \text{if } t \in [0, \frac{1}{4}] \\ \tilde{\Phi}^2(4t - 1, (v, g)) & \text{if } t \in [\frac{1}{4}, \frac{1}{2}] \\ \tilde{\Phi}^3(4t - 2, (v, g)) & \text{if } t \in [\frac{1}{2}, \frac{3}{4}] \\ \tilde{\Phi}^4(4t - 3, (v, g)) & \text{if } t \in [\frac{3}{4}, 1] \end{cases} \end{array} \right.$$

is a homotopy between $\tilde{\Delta}^{1,1}$ and $\tilde{\Delta}^2$ such that for every $t \in [0, 1]$, $(\tilde{\Phi}_t)^{-1}(\{0\}) = \mathcal{F}^0$.

Step VIII. *Last step in the construction of the homotopy \tilde{H} between \tilde{D}^1 and \tilde{D}^2 .*

Following our remark in Step II, the map

$$\left\{ \begin{array}{l} \tilde{H}^2 : [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto (\tilde{\Phi}_t \circ \varphi_1)(v, g) \end{array} \right.$$

is a homotopy between $\tilde{D}^{1,1}$ and \tilde{D}^2 such that for every $t \in [0, 1]$, the map \tilde{H}_t^2 is an extended network dynamics on \mathcal{F} .

Finally, from Step I, the map

$$\left\{ \begin{array}{l} \tilde{H} : [0, 1] \times \mathcal{E}^\varepsilon \rightarrow \mathbb{R}^L \\ (t, (v, g)) \mapsto \begin{cases} \tilde{H}^1(2t, (v, g)) & \text{if } t \in [0, \frac{1}{2}] \\ \tilde{H}^2(2t - 1, (v, g)) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{array} \right.$$

is the required homotopy between \tilde{D}^1 and \tilde{D}^2 such that for every $t \in [0, 1]$, the map \tilde{H}_t is an extended network dynamics on \mathcal{F} .

Proof of Corollary 4.1. The proof is similar to the one of Theorem 4.1. Since \mathcal{R} is a regular set, we can just consider the ambient isotopy $\varphi_{\mathcal{R}}$ of Corollary 3.1 instead of the ambient isotopy φ , and we can construct a homotopy $\tilde{H}_{\mathcal{R}}$ between \tilde{D}^1 and \tilde{D}^2 in a similar way.

Proof of Corollary 4.2. Before to start, we precise that we do not use here the identification of Proposition 5.3 (see Section 5.1), unlike in the proof of Theorem 4.1.

Consider the homotopy $\tilde{H}_{\mathcal{R}}$ of Corollary 4.2 between \tilde{D}^1 and \tilde{D}^2 (in the following, we simply denote it by \tilde{H}). Moreover, let g be an isolated zero of \tilde{D}_v^1 and \tilde{D}_v^2 , and (U, ϕ) be a chart on \mathbb{G}^ε such that $g \in U$ (remark that, since $g \in \mathbb{G}$ and $\mathbb{G} \subset \text{int}(\mathbb{G}^\varepsilon)$, we get $g \in \text{int}(\mathbb{G}^\varepsilon)$). Since g is an isolated zero of \tilde{D}_v^1 and \tilde{D}_v^2 , consider an open neighborhood U' of g in \mathbb{G}^ε such that $g' \notin U'$, for every $g' \in \mathcal{Z}(\tilde{D}_v^1) - \{g\} = \mathcal{Z}(\tilde{D}_v^2) - \{g\}$, and consider the chart $(U \cap U', \psi)$ of \mathbb{G}^ε , where $\psi = \phi|_{U \cap U'}$. For $k = 1, 2$, $\text{ind}_g(\tilde{D}_v^k)$ corresponds to the index of the map

$$\left\{ \begin{array}{l} \psi_* D_v^k : \phi(U \cap U') \rightarrow T\phi(U \cap U') \\ x \mapsto (T_{\psi^{-1}(x)} \psi \circ D_v^k \circ \psi^{-1})(x) \end{array} \right.$$

at g (see Definition 5.9 in Section 5.1), which corresponds itself to the topological degree of the map

$$\left\{ \begin{array}{l} \Phi^{-1}(\psi_* D_v^k) / \|\Phi^{-1}(\psi_* D_v^k)\| : \partial B(\psi(g), \epsilon) \rightarrow \mathbb{S}^{|\mathcal{U}|-1} \\ x \mapsto [\Phi^{-1}(\psi_* D_v^k)](x) / \|\Phi^{-1}(\psi_* D_v^k)](x)\|_2 \end{array} \right\},$$

where $B(\psi(g), \epsilon)$ is any ϵ -ball around $\psi(g)$ in $\phi(\mathcal{U} \cap \mathcal{U}')$ (see Corollary 5.1 for the definition of Φ , Definition 5.6 and Definition 5.7 in Section 5.1).²¹

Now, notice that it is sufficient to show that the maps $\Phi^{-1}(\psi_* D_v^1) / \|\Phi^{-1}(\psi_* D_v^1)\|$ and $\Phi^{-1}(\psi_* D_v^2) / \|\Phi^{-1}(\psi_* D_v^2)\|$ are homotopic to obtain that $\text{ind}_g(\tilde{D}_v^1) = \text{ind}_g(\tilde{D}_v^2)$.

First, remark that the map $\tilde{H}_v : [0, 1] \times \mathbb{G}^\epsilon \rightarrow \mathbb{T}\mathbb{G}^\epsilon$, $(t, g') \mapsto \tilde{H}(t, (v, g'))$ is a homotopy between \tilde{D}_v^1 and \tilde{D}_v^2 such that for every $t \in [0, 1]$, $\mathcal{Z}((\tilde{H}_v)_t) = \mathcal{Z}(\tilde{D}_v^1) = \mathcal{Z}(\tilde{D}_v^2)$ (since \tilde{H}_t is an extended network dynamics on \mathcal{R}).

Second, the following map is a homotopy between $\psi_* D_v^1$ and $\psi_* D_v^2$:

$$\left\{ \begin{array}{l} \tilde{\mathcal{H}}_v : [0, 1] \times \phi(\mathcal{U} \cap \mathcal{U}') \rightarrow \mathbb{T}\phi(\mathcal{U} \cap \mathcal{U}') \\ (t, x) \mapsto (\mathbb{T}_{\psi^{-1}(x)} \psi \circ (\tilde{H}_v)_t \circ \psi^{-1})(x) \end{array} \right\}$$

Indeed, it is continuous (as the composition of continuous maps), and for every $x \in \phi(\mathcal{U} \cap \mathcal{U}')$, $(\tilde{\mathcal{H}}_v)_0(x) = \psi_* \tilde{D}_v^1$ and $(\tilde{\mathcal{H}}_v)_1(x) = \psi_* \tilde{D}_v^2$. Moreover, for every $t \in [0, 1]$, $\mathcal{Z}((\tilde{\mathcal{H}}_v)_t) = \mathcal{Z}(\psi_* \tilde{D}_v^1) = \mathcal{Z}(\psi_* \tilde{D}_v^2) = \{\psi(g)\}$ (since for every $t \in [0, 1]$, $\mathcal{Z}((\tilde{H}_v)_t) = \mathcal{Z}(\tilde{D}_v^1) = \mathcal{Z}(\tilde{D}_v^2)$, and by definition of the chart $(\mathcal{U} \cap \mathcal{U}', \psi)$).

Finally, from Corollary 5.1 (see Section 5.1), the map

$$\left\{ \begin{array}{l} [0, 1] \times \partial B(\psi(g), \epsilon) \rightarrow \mathbb{S}^{|\mathcal{U}|-1} \\ (t, x) \mapsto \Phi^{-1}((\tilde{\mathcal{H}}_v)_t)(x) / \|\Phi^{-1}((\tilde{\mathcal{H}}_v)_t)(x)\|_2 \end{array} \right\}$$

is a well-defined homotopy between the maps $\Phi^{-1}(\psi_* D_v^1) / \|\Phi^{-1}(\psi_* D_v^1)\|$ and $\Phi^{-1}(\psi_* D_v^2) / \|\Phi^{-1}(\psi_* D_v^2)\|$. Thus, we get that $\text{ind}_g(\tilde{D}_v^1) = \text{ind}_g(\tilde{D}_v^2)$.

Proof of Proposition 4.1. First, we obtain directly that for every $(v, g) \in \mathcal{E}_\mathcal{U}$, $D^\epsilon(v, g) = b(g)D(v, r(g)) + V(g) = D(v, g)$. Indeed, since $g \in \mathbb{G}$, we get that $b(g) = 1$, $r(g) = g$ and $V(g) = 0$.

Second, $D^\epsilon = b(D \circ (\text{id}_\mathcal{U} \times r)) + V$ is a continuous map since D , b , r and V are continuous maps. Moreover, notice that for every $v \in \mathcal{U}$, every $\epsilon' \in (0, \epsilon]$, every $g \in \partial \mathbb{G}^{\epsilon'}$, and every $x \in \mathbb{N}_{\partial \mathbb{G}^{\epsilon'}}(g)$,

$$\langle D_v^\epsilon(g), x \rangle = \langle b(g)D(v, r(g)) + V(g), x \rangle = b(g)\langle D(v, r(g)), x \rangle + \langle V(g), x \rangle < 0,$$

since (i) by definition, $b(g) \in [0, 1]$, (ii) $\langle D(v, r(g)), x \rangle \leq 0$ because $r(g) \in \partial \mathbb{G}$ and because D is inward-pointing, and (iii) by definition, $\langle V(g), x \rangle < 0$. Thus, for every $v \in \mathcal{U}$, D_v^ϵ has no zeros on $\mathbb{G}^\epsilon - \mathbb{G}$. Since we proved that $D^\epsilon|_{\mathcal{E}_\mathcal{U}} = D$, we get that $(D^\epsilon)^{-1}(\{0\}) = D^{-1}(\{0\}) = \mathcal{P}$. Finally, we get that D^ϵ is a strongly inward-pointing extended network dynamics on \mathcal{U} .

²¹By definition of the chart $(\mathcal{U} \cap \mathcal{U}', \psi)$, any ϵ -ball $B(\psi(g), \epsilon)$ around $\psi(g)$ is such that $x \notin B(\psi(g), \epsilon)$, for every $x \in [\Phi^{-1}(\psi_* D_v^k)]^{-1}(\{0\}) - \{\psi(g)\}$.

REFERENCES

- [1] BICH, P., AND FIXARY, J. Oddness of the number of nash equilibria: the case of polynomial payoff functions. *Preprint (<https://halshs.archives-ouvertes.fr/halshs-03354269>)* (2021).
- [2] BICH, P., AND FIXARY, J. Structure and oddness theorems for pairwise stable networks. *Preprint (<https://halshs.archives-ouvertes.fr/halshs-03287524>)* (2021).
- [3] BICH, P., AND MORHAIM, L. On the existence of pairwise stable weighted networks. *Mathematics of Operations Research* (2020).
- [4] BLUME, L. E., AND ZAME, W. R. The algebraic geometry of perfect and sequential equilibrium. *Econometrica* (1994).
- [5] BOTT, R., AND TU, L. W. Differential forms in algebraic topology. *Springer* (1982).
- [6] DEMICHELIS, S., AND GERMANO, F. On the indices of zeros of nash fields. *Journal of Economic Theory* (2000).
- [7] DEMICHELIS, S., AND GERMANO, F. On (un)knots and dynamics in games. *Games and Economic Behavior* (2002).
- [8] DOLD, A. Lectures on algebraic topology. *Springer* (1995).
- [9] GOVINDAN, S., AND MCLENNAN, A. On the generic finiteness of equilibrium outcome distributions in game forms. *Econometrica* (2001).
- [10] GOVINDAN, S., AND WILSON, R. Equivalence and invariance of the index and degree of nash equilibria. *Games and Economic Behavior* (1997).
- [11] GOVINDAN, S., AND WILSON, R. Direct proofs of generic finiteness of nash equilibrium outcomes. *Econometrica* (2001).
- [12] GUILLEMIN, V., AND POLLACK, A. Differential topology. *Pearson* (1975).
- [13] GUL, F., PEARCE, D., AND STACCHETTI, E. A bound of the proportion of pure strategy equilibria in generic games. *Mathematics of Operations Research* (1993).
- [14] HARSANYI, J. Oddness of the number of equilibrium points: a new proof. *International Journal of Game Theory* (1973).
- [15] HATCHER, A. Algebraic topology. *Cambridge University Press* (2002).
- [16] HERINGS, P. J.-J., AND PEETERS, R. J. A. P. A differentiable homotopy to compute nash equilibria of n-person games. *Economic Theory* (2001).
- [17] JACKSON, M. O., AND WATTS, A. The existence of pairwise stable networks. *Seoul Journal of Economics* (2001).
- [18] JACKSON, M. O., AND WATTS, A. The evolution of social and economic networks. *Journal of Economic Theory* (2002).

- [19] JACKSON, M. O., AND WOLINSKY, A. A strategic model of social and economic networks. *Journal of Economic Theory* (1996).
- [20] KOHLBERG, E., AND MERTENS, J.-F. On the strategic stability of equilibria. *Econometrica* (1986).
- [21] LARAKI, R., RENAULT, J., AND SORIN, S. Mathematical foundations of game theory. *Springer* (2019).
- [22] LEE, J. M. Introduction to smooth manifolds. *Springer* (2012).
- [23] MAS-COLELL, A. Generic finiteness of equilibrium payoffs for bimatrix games. *Journal of Mathematical Economics* (2010).
- [24] MILNOR, J. W. Topology from the differentiable viewpoint. *Princeton University Press* (1998).
- [25] PREDTETCHINSKI, A. A general structure theorem for the nash equilibrium correspondence. *Games and Economic Behavior* (2009).
- [26] RITZBERGER, K. The theory of normal form games from the differentiable viewpoint. *International Journal of Game Theory* (1994).
- [27] WATTS, A. A dynamic model of network formation. *Games and Economic Behavior* (2001).
- [28] WILSON, R. Computing equilibria of n-person games. *SIAM Journal on Applied Mathematics* (1971).