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Existence of an Equilibrium in Arrowian Markets for Consumption Externalities

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Abstract

We study the existence of quasi-equilibria and equilibria for pure exchange economies with consumption externalities and Arrowian markets with personalized Lindahl prices. We provide examples showing first that quasi-equilibria and equilibria of the externality economy fail to exist under assumptions guaranteeing existence for economies without externalities. We show that the externality economy has identical equilibrium allocations of an appropriately defined constant returns to scale production economy without externalities. We exploit this equivalence to map sufficient conditions for the existence of quasi equilibria and equilibria of the production economy into sufficient conditions of the pure exchange economy with externalities, thereby unveiling suitable irreducibility conditions and survival conditions.

JEL classification: C62, D50, D62.

Key words: Consumption externalities, markets for externalities, existence of an equilibrium, irreducibility and survival.

1 Introduction

In the presence of consumption externalities restoring the two Fundamental Theorems of Welfare Economics requires setting up markets where individuals

face personalized Lindahl prices and choose allocations, that is, profiles of consumption bundles one for each of the individuals in the economy. This was observed a long time ago by Samuelson (1954), Arrow (1969), Laffont (1976).

Surprisingly, the literature seems to be missing a general argument for the existence of Arrow-Lindahl equilibria¹. This paper fills this void. We study economies with consumption externalities in preferences, and find general conditions for the existence of quasi-equilibria and equilibria. This is not a trivial task. We provide examples of externality economies showing that under standard assumptions of local non-satiation and strict positivity of individual endowments quasi-equilibria and equilibria fail to exist.

Arrow suggested that in such externality economies “individual i ’s consumption is regarded as the production of joint outputs, one for each individual whose utility is affected by individual i ’s consumption” (Arrow, 1969, see pages 9-10). Following this intuition, we show that a pure exchange economy with consumption externalities where trade takes place in Arrowian markets for externalities yields the same equilibrium allocations of an appropriately defined production economy with constant return to scale and no externalities. Once this equivalence is established, existence of equilibrium, quasi-equilibrium, irreducibility and survival for economies with consumption externalities follow from well known and understood results for production economies.

The heavy lifting in our argument is done by the construction of the production economy and in particular of its technology. The technology transforms inputs of physical commodities into a collection of as many identical allocations as individuals in the economy. The total amount of physical commodities in an allocation cannot exceed the total amount of inputs used in the production process. The technology displays constant returns to scale, thereby generating zero equilibrium profits and making irrelevant specifying individual property rights over production activities. Further the chosen form of the technology restricts equilibrium prices to satisfy the classical Lindahl compatibility conditions. The latter establishes equivalence in equilibrium prices, and thus equilibrium allocations of the two economies.

We exploit the equivalence of feasible allocations to map classical irreducibility and survival conditions of the production economy into equivalent such conditions of the economy with externalities. In doing this, we understand

¹ Exceptions are Crès (1996) and Conley and Smith (2005). Crès (1996) focuses on the symmetry breaking properties of equilibria with consumption externalities. Existence is generic in the space of preferences under strong conditions of smoothness and irreducibility. Conley and Smith (2005) proves existence of quasi-equilibria and equilibria treating externality markets directly instead of indirectly through Arrowian commodities. Thus, there is no relation between the prices of the ordinary commodities and the prices of the externality rights.

the reasons for the failure of existence of quasi-equilibria and equilibria of the externality economy.

Without free disposal in the externality economy, the equivalent production economy does not satisfy the standard weak survival assumption, explaining the lack of quasi-equilibria in our first example. In the second example there is free disposal, and thus quasi-equilibria exist, but equilibria do not because the equivalent production economy is not irreducible. Indeed, irreducibility for externality economies is more demanding. In economies without externalities, irreducibility means that at any feasible allocation, any group of individuals can be made better off (in the Pareto sense) by adding to their consumption bundles something of the privately owned resources of the complementary group. For economies with externalities, any group must choose an allocation that specifies not only their consumption bundles, but also those of the individuals in the complementary group. This makes it harder to find an allocation (Pareto) dominating the target feasible allocation for all the members of the chosen group.

The paper is organized as follows. Section 2 describes the fundamentals, the basic assumptions and the structure of the markets. Section 3 provides examples of nonexistence of quasi-equilibrium and equilibrium in economies satisfying the assumptions of Section 2. Section 4 is the key part of the paper describing the equivalent production economy without externalities. Section 5 shows arguments for the existence of a quasi equilibrium and equilibrium and it defines notions of irreducibility and survival for the economies with externalities.

2 The basic model and its assumptions

We consider a pure exchange economy with a finite number of physical commodities labeled by the superscript $\ell \in \mathcal{L} = \{1, \dots, L\}$, a finite set of individuals labeled by the subscript $i \in \mathcal{I} = \{1, \dots, I\}$. Each individual i is characterized by a consumption set $X_i \subset \mathbb{R}^L$, an endowment of commodities $e_i \in \mathbb{R}^L$, and other-regarding preferences represented by a utility function allowing for possible consumption external effects:

$$u_i : x \in X = \prod_{i \in \mathcal{I}} X_i \rightarrow u_i(x) \in \mathbb{R},$$

where $x = (x_i)_{i \in \mathcal{I}} \in X$ is an allocation. An economy with consumption externalities is $\mathcal{E} = (X_i, u_i, e_i)_{i \in \mathcal{I}}$.

Individuals' utility functions, consumption sets and endowments satisfy standard assumptions listed below.

Assumption A

- (1) The set X_i is closed, convex and bounded from below, $0 \in X_i$ and $e_i \geq 0$.
- (2) The function u_i is continuous, quasi-concave and locally non-satiated on X .²

We also assume the existence of a free disposal technology, i.e., an allocation $x = (x_i)_{i \in \mathcal{I}} \in X$ is a feasible allocation of the economy \mathcal{E} if $\sum_{i \in \mathcal{I}} x_i \leq \sum_{i \in \mathcal{I}} e_i$.

2.1 Arrowian markets for consumption externalities

Market structure and equilibrium follow the classical definitions in Arrow (1969) and Laffont (1976).

Thus, an extended consumption vector of individual i is

$$\tilde{x}_i = (x_{ih})_{h \in \mathcal{I}} \in X,$$

where x_{ii} is the effective consumption of individual i , while x_{ih} is the external effect of individual h 's consumption as perceived by individual i . Equivalently, x_{ih} is the demand of i for the consumption of h . A price system is

$$\tilde{p} = (p, ((p_{ih})_{h \neq i})_{i \in \mathcal{I}}) \in \mathbb{R}^L \times \mathbb{R}^{L(I-1)I},$$

where $p \in \mathbb{R}^L$ is the price of the L physical commodities, while $p_{ih} \in \mathbb{R}^L$ is the price paid by individual i to h for the consumption externality created by individual h on i . Individual i faces a *personalized* price for her own consumption, i.e., the price $p_{ii} = p - \sum_{h \neq i} p_{hi}$. By definition of p_{ii} , \tilde{p} satisfies the classical compatibility condition on prices à la Lindahl, that is:

$$p = \sum_{h \in \mathcal{I}} p_{hi}, \forall i \in \mathcal{I}.$$

With this notation, individual i 's budget constraint is³:

$$(p - \sum_{h \neq i} p_{hi}) \cdot x_{ii} + \sum_{h \neq i} p_{ih} \cdot x_{ih} \leq p \cdot e_i.$$

At equilibrium, the extended consumption vectors are coherent, that is, $x_{ih} = x_{hh}$ for all pairs (i, h) . Hence, $p_{ih} \cdot x_{ih} (= p_{ih} \cdot x_{hh})$ is the value transfer (positive or negative) from individual i to h .

² That is, for every $x \in X$ the set $\{x' \in X : u_i(x') \geq u_i(x)\}$ is convex, and for every open neighborhood N of x there exists $x' \in N \cap X$ such that $u_i(x') > u_i(x)$.

³ The (unique) budget constraints reflects the completeness of the market structure. In principle one could generalize the analysis of this paper by explicitly introducing time, uncertainty and an incomplete set of assets.

Definition 1 (Equilibrium with markets for externalities) *An equilibrium of \mathcal{E} is a pair of an allocation $x^* = (x_i^*)_{i \in \mathcal{I}} \in X$ and a price system $\tilde{p}^* \neq 0$ such that:*

(1) for all i ,

$$x^* \in \operatorname{argmax}_{x \in X} \{u_i(x) \text{ s.to } (p^* - \sum_{h \neq i} p_{hi}^*) \cdot x_i + \sum_{h \neq i} p_{ih}^* \cdot x_h \leq p^* \cdot e_i\},$$

(2) x^* is feasible, i.e., $\sum_{i \in \mathcal{I}} x_i^* \leq \sum_{i \in \mathcal{I}} e_i$, and $p^* \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) = 0$.

As known, for a quasi-equilibrium, (1) in the definition above is replaced by:

$$(p^* - \sum_{h \neq i} p_{hi}^*) \cdot x_i^* + \sum_{h \neq i} p_{ih}^* \cdot x_h^* = p^* \cdot e_i,$$

and all $x \in X$, $u_i(x) \geq u_i(x^*)$ implies that

$$p^* \cdot e_i \leq (p^* - \sum_{h \neq i} p_{hi}^*) \cdot x_i + \sum_{h \neq i} p_{ih}^* \cdot x_h.$$

3 Examples of non-existence of (quasi-)equilibrium

This section consists of two examples. The first shows that quasi-equilibria may fail to exist in absence of a free disposal technology. The second example suggests that the notion of irreducibility is for economies with externalities far more demanding than for standard economies. Indeed, strictly positive endowments and local non-satiation are not sufficient for irreducibility.

3.1 Non-existence of a quasi-equilibrium without free disposal

For standard pure exchange economies (without external effects) satisfying Assumption A and $e_i \gg 0$, all $i \in \mathcal{I}$, quasi-equilibria exist even when the presence of free disposal is dispensed with, that is, when feasibility reads $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i$.⁴ The example shows that this is not the case for economies with consumption externalities. We come back on this issue in Subsection 5.1.

Consider an economy with one physical commodity, three individuals, $X_i = \mathbb{R}_+$, $e_i \in \mathbb{R}_{++}$, for $i \in \mathcal{I} = \{1, 2, 3\}$, and utility functions:

$$u_1(x_1, x_2, x_3) = x_1 \text{ and } u_i(x_1, x_2, x_3) = x_i - 2 \sum_{h \neq i} x_h, \quad i = 2, 3.$$

⁴ See Hart and Kuhn (1975).

Consider a quasi-equilibrium (x^*, \tilde{p}^*) . The (first order) optimality conditions of individual 1's cost minimization problem imply that $(p_{11}^*, p_{12}^*, p_{13}^*) \geq 0$ and the definition of quasi-equilibrium that $p^* e_1 \geq 0$ and then $p^* \geq 0$.

The market clearing (or feasibility) condition $\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i$ implies that aggregate consumption is strictly positive, $\sum_{i \in \mathcal{I}} x_i^* > 0$. Therefore, since $u_2(x^*) + u_3(x^*) = -(x_2^* + x_3^*) - 4x_1^* < 0$ either $u_2(x^*) < 0$ or $u_3(x^*) < 0$. Assume without loss of generality that $u_2(x^*) < 0$. Then it must be that $(p_{21}^*, p_{22}^*, p_{23}^*) \in \mathbb{R}_+^3$ and $p^* = 0$. Indeed, if $(p_{21}^*, p_{22}^*, p_{23}^*) \notin \mathbb{R}_+^3$, it is $\sum_{h=1}^3 p_{2h}^* x_h < p^* e_2$ for some $x \in \mathbb{R}_{++}^3$, and similarly if $p^* > 0$ it is $\sum_{h=1}^3 p_{2h}^* x'_h < p^* e_2$ for some $x' \in \mathbb{R}_{++}^3$. In both cases, the interior of the budget set is nonempty implying that x^* (solution to the cost minimization problem) is a solution to individual 2's utility maximization problem. Further, in both cases 0 is in the budget set. But, then $u_2(x^*)$ must be greater than $u_2(0) = 0$ contradicting $u_2(x^*) < 0$.

We have established so far that $(p_{11}^*, p_{12}^*, p_{13}^*) \geq 0$, $(p_{21}^*, p_{22}^*, p_{23}^*) \geq 0$ and $p^* = 0 = \sum_{h=1}^3 p_{hi}^*$ thereby implying that $p_{3h}^* \leq 0$ for all h . But the optimality conditions of individual 3 imply that $p_{33}^* \geq 0$ so that $0 = p_{13}^* = p_{23}^* = p_{33}^*$. Then, since $p_{33}^* = 0$, once again the optimality conditions imply that $(p_{31}^*, p_{32}^*) \geq 0$. Therefore, $0 = p^* = p_{11}^* = p_{21}^* = p_{31}^* = p_{12}^* = p_{22}^* = p_{32}^*$, which leads to a contradiction with the fact that $\tilde{p}^* \neq 0$. Therefore, without free disposal, there is no quasi-equilibrium.

If we allow for free disposal, the allocation $x^* = 0$ and the price \tilde{p}^* defined by $p^* = p_{12}^* = p_{13}^* = p_{21}^* = p_{31}^* = 0$, and $p_{23}^* = p_{32}^* = -1$ is a quasi-equilibrium. Indeed, $\sum x_i^* < \sum e_i$, $(p^* - \sum_{h \neq i} p_{hi}^*) \cdot x_i^* + \sum_{h \neq i} p_{ih}^* \cdot x_h^* = p^* \cdot e_i$ and it is easily verified that at \tilde{p}^* the allocation x^* solves the cost minimization problems of every individual.

3.2 Non-existence of an equilibrium

The economy satisfies Assumption A with strictly positive endowments and individuals' utility functions strictly increasing in their own consumption. Under such conditions, all quasi-equilibria of economies without externalities are equilibria. However, the example shows that for economies with externalities equilibria fail to exist.

There is one physical commodity and three individuals, $X_i = \mathbb{R}_+$, $e_i \in \mathbb{R}_{++}$ for $i \in \mathcal{I} = \{1, 2, 3\}$. Utility functions are:

$$u_1(x_1, x_2, x_3) = x_1, \quad u_2(x_1, x_2, x_3) = x_2 - x_3, \quad \text{and} \quad u_3(x_1, x_2, x_3) = x_3 - x_2.$$

Let (x^*, \tilde{p}^*) be an equilibrium. Since u_i is strictly increasing in x_i , we get $p_{ii}^* > 0$ for every i . Furthermore, if u_i is independent of x_h , then $p_{ih}^* \geq 0$ and

hence $(p_{12}^*, p_{13}^*, p_{21}^*, p_{31}^*) \geq 0$. Thus $p^* = p_{11}^* + p_{21}^* + p_{31}^* > 0$. Consequently, as $e_i > 0$, $p^* e_i$ is strictly positive for every i . Then, each individual can reach a strictly positive utility level in her budget constraint. Therefore, at equilibrium, $u_2(x_1^*, x_2^*, x_3^*) = x_2^* - x_3^* > 0$ as well as $u_3(x_1^*, x_2^*, x_3^*) = x_3^* - x_2^* > 0$, which is clearly impossible.

This economy has a continuum of quasi-equilibria. They are described by $p^* = p_{12}^* = p_{13}^* = p_{21}^* = p_{31}^* = 0$, $p_{23}^* = p_{32}^* = -1$, and $x_2^* = x_3^* \leq \frac{1}{2}(r - x_1^*)$ with $x_1^* \in [0, r]$ and $r = e_1 + e_2 + e_3$.

In the economy with individuals 2 and 3 only, equilibria exist with $p^* = 0$, $p_{23}^* = p_{32}^* = -1$, and $x_2^* = x_3^* \leq \frac{1}{2}(e_2 + e_3)$. Hence, the presence of individual 1, who is not affected by any external effects and who is not affecting any other individual prevents the existence of an equilibrium. We further discuss this issue in Subsection 5.2.

4 The production economy behind the consumption externalities

In this section, we construct a standard production economy without externalities, \mathcal{E}_Y . We show that there is a simple one-to-one mapping between the equilibria of the two economies \mathcal{E}_Y and \mathcal{E} . Thus existence of (quasi-)equilibria of \mathcal{E} follows from the existence of equilibria of \mathcal{E}_Y . Similarly, survival and irreducibility conditions for \mathcal{E} are derived from survival and irreducibility conditions for \mathcal{E}_Y . The main idea behind the construction of \mathcal{E}_Y is in the definition of the production set Y that transforms inputs of L physical commodities into I identical allocations, one for each individual.

The production economy \mathcal{E}_Y has $L + (LI)I$ commodities. A generic bundle of $\mathbb{R}^L \times \mathbb{R}^{(LI)I}$ is denoted as

$$(a, \tilde{a}_1, \dots, \tilde{a}_i, \dots, \tilde{a}_I)$$

where $a \in \mathbb{R}^L$ and $\tilde{a}_i = (a_{ih})_{h \in \mathcal{I}} \in \mathbb{R}^{LI}$ for every $i \in \mathcal{I}$.

The production technology is described by the production set is

$$Y = \left\{ y = (-z, \tilde{y}_1, \dots, \tilde{y}_i, \dots, \tilde{y}_I) \in \mathbb{R}_-^L \times \mathbb{R}^{(LI)I} : \tilde{y}_i = \tilde{y}_h, \forall i, h \in \mathcal{I} \text{ and } \sum_{i \in \mathcal{I}} y_{ii} \leq z \right\}$$

It is verified that Y is a closed convex cone that satisfies the possibility of inaction and the impossibility of free production, i.e., $Y \cap (\mathbb{R}_+^L \times \mathbb{R}_+^{(LI)I}) = \{0\}$. The condition $\sum_{i \in \mathcal{I}} y_{ii} \leq z$ allows for free disposal of physical commodities.

The consumption set of individual i is now defined as:

$$\Xi_i = \{\xi_i = (0_L, 0_{LI}, \dots, 0_{LI}, \tilde{x}_i, 0_{LI}, \dots, 0_{LI}) \in \mathbb{R}^L \times \mathbb{R}^{(LI)I} : \tilde{x}_i \in X\} \quad (1)$$

and the set $\Xi = \prod_{i \in \mathcal{I}} \Xi_i$.

The utility function of individual i is $\tilde{u}_i : \Xi_i \rightarrow \mathbb{R}$ defined by $\tilde{u}_i(\xi_i) = u_i(\tilde{x}_i)$. The endowment of individual i is $\eta_i = (e_i, 0_{LI}, \dots, 0_{LI}, \dots, 0_{LI}) \in \mathbb{R}_+^L \times \mathbb{R}^{(LI)I}$.

An allocation of the production economy is a pair $(\xi, y) \in \Xi \times Y$, and (ξ, y) is feasible if $\sum_{i \in \mathcal{I}} \xi_i = \sum_{i \in \mathcal{I}} \eta_i + y$.

The price system of \mathcal{E}_Y is

$$\pi = (p, \tilde{p}_1, \dots, \tilde{p}_i, \dots, \tilde{p}_I) \in \mathbb{R}^L \times \mathbb{R}^{(LI)I}.$$

The next lemma describes the relation between the feasible allocations of the economies \mathcal{E} and \mathcal{E}_Y .

Lemma 2

- a) Let $(\xi, y) \in \Xi \times Y$ be a feasible allocation of \mathcal{E}_Y . Let $(\tilde{x}_i)_{i \in \mathcal{I}}$, $\tilde{x}_i \in X$, be such that $\xi_i = (0_L, 0_{LI}, \dots, 0_{LI}, \tilde{x}_i, 0_{LI}, \dots, 0_{LI})$ for all $i \in \mathcal{I}$. Then, $x = \tilde{x}_1$ is a feasible allocation of \mathcal{E} .
- b) Conversely, let $x \in X$ be a feasible allocation of \mathcal{E} . Let

$$\xi_i = (0_L, 0_{LI}, \dots, 0_{LI}, x, 0_{LI}, \dots, 0_{LI})$$

for all $i \in \mathcal{I}$ and $y = (-\sum_{i \in \mathcal{I}} e_i, x, \dots, x, \dots, x)$, then $(\xi, y) \in \Xi \times Y$ is a feasible allocation of \mathcal{E}_Y .

Proof. a) By definition of Y , $\tilde{x}_1 = \tilde{x}_i = \tilde{y}_i$ for all $i \in \mathcal{I}$ and $\sum_{i \in \mathcal{I}} x_{1i} = \sum_{i \in \mathcal{I}} y_{ii} \leq \sum_{i \in \mathcal{I}} e_i$. Then $x = \tilde{x}_1$ is a feasible allocation of \mathcal{E} .

b) Since $x \in X$ is a feasible allocation of \mathcal{E} , $\sum_{i \in \mathcal{I}} x_i \leq \sum_{i \in \mathcal{I}} e_i$ and by definition of Y , $y = (-\sum_{i \in \mathcal{I}} e_i, x, \dots, x, \dots, x)$ belongs to Y . Further, $\sum_{i \in \mathcal{I}} \xi_i = (0_L, x, \dots, x, \dots, x)$ and

$$\sum_{i \in \mathcal{I}} \eta_i + y = \left(\sum_{i \in \mathcal{I}} e_i - \sum_{i \in \mathcal{I}} e_i, x, \dots, x, \dots, x \right) = (0_L, x, \dots, x, \dots, x)$$

implying that $(\xi, y) \in \Xi \times Y$ is a feasible allocation of \mathcal{E}_Y . \square

Since the production set Y exhibits constant returns to scale, at equilibrium optimal profit must be zero. Thus profit maximization restricts the search of an equilibrium price of \mathcal{E}_Y to the price domain

$$\Pi^* = \{\pi : \pi \cdot y \leq 0, \forall y \in Y\}.$$

A price π is in Π^* if and only if it satisfies restrictions identical to those generated by the Lindahl conjecture in the price domain of \mathcal{E} , the economy with externalities. This is spelled out in the next key lemma.

Lemma 3 *The price system $\pi \in \Pi^*$ if and only if $0 \leq \sum_{h \in \mathcal{I}} p_{hi} \leq p$ and $\sum_{h \in \mathcal{I}} p_{hi}$ is i -invariant.*

Proof: By definition, $y \in Y$ implies that $z = \sum_{i \in \mathcal{I}} y_{ii} + b$ for some $b \in \mathbb{R}_+^L$. Thus, profits at (π, y) are

$$\pi \cdot y = \sum_{i \in \mathcal{I}} \left(\sum_{h \in \mathcal{I}} p_{hi} \right) \cdot y_{ii} - p \cdot z = \sum_{i \in \mathcal{I}} \left(\sum_{h \in \mathcal{I}} p_{hi} - p \right) \cdot y_{ii} - p \cdot b$$

Therefore, $\pi \in \Pi^*$ iff the system of linear inequalities

$$\begin{aligned} \sum_{i \in \mathcal{I}} \left(\sum_{h \in \mathcal{I}} p_{hi} - p \right) \cdot y_{ii} - p \cdot b &> 0, \\ \sum_{i \in \mathcal{I}} y_{ii} + b &\geq 0, \\ b &\geq 0 \end{aligned}$$

does not have a solution $((y_{ii})_{i \in \mathcal{I}}, -b)$. Let v be a vector of dimension $1 \times (LI + L)$ defined as $v = (\sum_h p_{h1} - p, \dots, \sum_h p_{hI} - p, -p)$. Then

$$v \cdot ((y_{ii})_{i \in \mathcal{I}}, -b) = \sum_i \left(\sum_h p_{hi} - p \right) \cdot y_{ii} - p \cdot b$$

Let A be a matrix of dimension $2L \times (LI + L)$ defined as

$$A = \begin{bmatrix} I_L & \cdots & I_L & -I_L \\ 0 & \cdots & 0 & -I_L \end{bmatrix}$$

Then,

$$A \cdot ((y_{ii})_{i \in \mathcal{I}}, -b) = \left(\sum_i y_{ii} + b, b \right).$$

Thus $\pi \in \Pi^*$ iff the system of linear inequalities $v \cdot c > 0, A \cdot c \geq 0$ does not have a solution $c \in \mathbb{R}^{LI+L}$. By the Theorem of the Alternative, this is true iff there exists $\theta = (\theta_1, \theta_2) \in \mathbb{R}_+^{2L} \setminus \{0\}$ such that $-v = A^T \cdot \theta$. That is, $(p - \sum_h p_{hi}) = \theta_1 \geq 0$, for all $i \in \mathcal{I}$, and $p = \theta_1 + \theta_2 \geq 0$. \square

Since \mathcal{E}_Y is a standard production economy (without externalities), we dispense with the definitions of equilibrium and quasi-equilibrium.

The next proposition exploits Lemmas 2 and 3 in order to show the existence of a simple one-to-one mapping between the equilibria of the economies \mathcal{E} and \mathcal{E}_Y . Importantly such a mapping yields equivalent allocation-price pairs in the sense specified by the statement of the next proposition.

Proposition 4

- a) Let $(x^*, \tilde{p}^*) \in X \times \mathbb{R}_+^L \times \mathbb{R}^{(L-I)I}$ be an equilibrium (resp. a quasi-equilibrium) of the economy \mathcal{E} . Consider the pair (ξ, y) where for all $i \in \mathcal{I}$, $\xi_i = (0_L, 0_{LI}, \dots, 0_{LI}, x_i^*, 0_{LI}, \dots, 0_{LI})$, and $y = (-\sum_{i \in \mathcal{I}} e_i, x^*, \dots, x^*, \dots, x^*)$. Further define the price system $\pi = (p, \tilde{p}_1, \dots, \tilde{p}_i, \dots, \tilde{p}_I)$ as $p = p^*$ and $p_{ih} = p_{ih}^*$ for $h \neq i$ and $p_{ii} = p^* - \sum_{h \neq i} p_{hi}^*$, for every $i \in \mathcal{I}$. Then, $(\xi, y, \pi) \in \Xi \times Y \times \mathbb{R}_+^L \times \mathbb{R}^{(LI)I}$ is an equilibrium (resp. a quasi-equilibrium) of the production economy \mathcal{E}_Y .
- b) Conversely, let $(\xi, y, \pi) \in \Xi \times Y \times \mathbb{R}_+^L \times \mathbb{R}^{(LI)I}$ be an equilibrium (resp. a quasi-equilibrium) of the production economy \mathcal{E}_Y . Let $x^* \in X$ such that $\xi_i = (0_L, 0_{LI}, \dots, x_i^*, \dots, 0_{LI})$ and $\tilde{p}^* = (\sum_{h \in \mathcal{I}} p_{h1}, ((p_{ih})_{h \neq i})_{i \in \mathcal{I}})$. Then, (x^*, \tilde{p}^*) is an equilibrium (resp. a quasi-equilibrium) of the economy \mathcal{E} .

Proof. First, by Lemma 2, the mappings between $(\xi, y) \in \Xi \times Y$ and $x^* \in X$ preserve feasibility. Next we pick π, y as defined in a) and we show that $\pi \in \Pi^*$ and that y maximizes profits at π . Since by construction $\sum_{h \in \mathcal{I}} p_{hi} = p^* \geq 0$, for all $i \in \mathcal{I}$, Lemma 3 implies $\pi \in \Pi^*$. Then,

$$\begin{aligned}
 \pi \cdot y &= \sum_{i \in \mathcal{I}} \left(\sum_{h \in \mathcal{I}} p_{hi} \right) \cdot y_{ii} - p \cdot z \\
 &= \sum_{i \in \mathcal{I}} \left(\sum_{h \in \mathcal{I}} p_{hi} \right) \cdot x_i^* - p \cdot \sum_{i \in \mathcal{I}} e_i \\
 &= \sum_{i \in \mathcal{I}} (p_{ii} + \sum_{h \neq i} p_{hi}) \cdot x_i^* - p \cdot \sum_{i \in \mathcal{I}} e_i \\
 &= p^* \cdot \sum_{i \in \mathcal{I}} (x_i^* - e_i) = 0
 \end{aligned}$$

implies that y maximizes profits at π .

Therefore, in order to conclude the argument it suffices to verify the optimality of individuals' choices in a) and b) at respectively π and \tilde{p}^* . This is implied by the equalities $\pi \cdot \eta_i = p^* \cdot e_i$ and $\pi \cdot \xi'_i = \tilde{p}_i^* \cdot \tilde{x}'_i$, for $\xi'_i = (0, \dots, 0, \tilde{x}'_i, 0, \dots, 0)$, all $\tilde{x}'_i \in X$, and $i \in \mathcal{I}$. It is verified that the later holds true since in b), by Lemma 3, $\sum_{h \in \mathcal{I}} p_{h1} = \sum_{h \in \mathcal{I}} p_{hi}$, for all $i \in \mathcal{I}$. \square

5 Existence of an equilibrium with markets for externalities

We prove that quasi-equilibria exist and we provide sufficient conditions under which quasi-equilibria are equilibria, namely an irreducibility condition combined with a survival condition. Contrary to what happens in exchange economies without externalities, the argument requires an irreducibility con-

dition even with strictly positive endowments and locally non-satiated preferences.

5.1 Existence of a quasi-equilibrium with markets for externalities

The following theorem is a consequence of Lemma 3, Proposition 4, and a general result, Proposition 2.2.2 in Florenzano (2003), for the existence of a quasi-equilibrium in production economies without externalities.

Theorem 5 (Existence of a quasi-equilibrium of the economy \mathcal{E}) *Under Assumption A, there exists a quasi-equilibrium (x^*, \tilde{p}^*) of the economy \mathcal{E} with $p^* \geq 0$.*

Proof. Consider the production economy \mathcal{E}_Y defined in Section 4. By Assumption A, consumption sets are closed, convex and bounded from below, $P_i(\xi_i) = \{\xi'_i \in \Xi_i : \tilde{u}_i(\xi'_i) > \tilde{u}_i(\xi_i)\}$ are nonempty and convex, while $P_i^{-1}(\xi'_i) = \{\xi_i \in \Xi_i : \xi'_i \in P_i(\xi_i)\}$ are open in Ξ_i . Further, by construction, the production set Y is closed, convex, and satisfies the possibility of inaction and the impossibility of free production. Therefore, the attainable consumption sets are nonempty and compact. Furthermore, the weak survival assumption $\eta_i \in \Xi_i - Y$ for all $i \in \mathcal{I}$ is satisfied since $0 \in \Xi_i$ and $-\eta_i \in Y$. Thus the assumptions of Proposition 2.2.2 in Florenzano (2003) are satisfied: \mathcal{E}_Y has a quasi-equilibrium with $\pi \neq 0$ that by Lemma 3 and Proposition 4 implies the existence of a quasi-equilibrium of \mathcal{E} with $p^* \geq 0$. \square

Example in Subsection 3.1 shows that without free disposal the economy with externalities \mathcal{E} may fail to have a quasi-equilibrium. This is because the production economy \mathcal{E}_Y does not satisfy the weak survival condition $\eta_i \in \Xi_i - Y$ of Florenzano (2003) in absence of the free disposal condition $\sum_{i \in \mathcal{I}} y_{ii} \leq z$.

5.2 Irreducibility conditions

We derive equivalent irreducibility conditions of the economy with externalities from known conditions of irreducibility for production economies without externalities. We adapt the classical irreducibility conditions of McKenzie (1959, 1961, 1981) and Bergstrom (1976) to the production economy \mathcal{E}_Y .

Assumption I *The economy \mathcal{E}_Y is McKenzie-Bergstrom irreducible if for every feasible allocation $((\xi_i)_{i \in \mathcal{I}}, y) \in \Xi \times Y$ and for every pair of non-empty sets I_1 and I_2 with $I_1 \cup I_2 = \mathcal{I}$ and $I_1 \cap I_2 = \emptyset$, there exist $\theta_h > 0$, $h \in I_2$, $(\xi'_i)_{i \in \mathcal{I}} \in \Xi$ and $y' \in Y$ satisfying the two following conditions:*

- (1) $\sum_{k \in I_1} (\xi'_k - \eta_k) + \sum_{h \in I_2} \theta_h (\xi'_h - \eta_h) = y'$,
- (2) $\tilde{u}_k(\xi'_k) \geq \tilde{u}_k(\xi_k)$ for all $k \in I_1$ and $\tilde{u}_{k_0}(\xi'_{k_0}) > \tilde{u}_{k_0}(\xi_{k_0})$ for some $k_0 \in I_1$.

It is verified that:

- (i) if $Y = \{0\}$ and I_2 is a singleton, Assumption I coincides with the irreducibility condition of Bergstrom (1976) for a classical exchange economy without free disposal, and
- (ii) if the economy satisfies the irreducibility conditions of McKenzie (1959, 1961, 1981), then it satisfies Assumption I.

We now translate Assumption I into an equivalent irreducibility condition for the economy with consumption externalities \mathcal{E} . An externality economy is irreducible when any subset of individuals can pick a Pareto improving allocation whose total resources are less or equal than the sum of the endowments of the first group plus a weighted sum of endowments of the residual.

Assumption E (Irreducibility in \mathcal{E}) *The economy \mathcal{E} is irreducible if for every feasible allocation $x \in X$ and for every pair of non-empty sets I_1 and I_2 with $I_1 \cup I_2 = \mathcal{I}$ and $I_1 \cap I_2 = \emptyset$, there exist $\theta_h > 0$, $h \in I_2$, and an allocation $\bar{x} = (\bar{x}_i)_{i \in \mathcal{I}} \in X$ satisfying the two following conditions:*

- (1) $\sum_{i \in \mathcal{I}} \bar{x}_i \leq \sum_{k \in I_1} e_k + \sum_{h \in I_2} \theta_h e_h$,
- (2) $u_k(\bar{x}) \geq u_k(x)$ for all $k \in I_1$ and $u_{k_0}(\bar{x}) > u_{k_0}(x)$ for some $k_0 \in I_1$.

Assumption E⁵ is satisfied in an economy without externalities if $e_i \gg 0$, all $i \in \mathcal{I}$, and the utility functions are locally non-satiated. It is also satisfied with externalities if there are only two consumers, $e_i > 0$, all $i = 1, 2$, and preferences are strongly monotonic in own consumption. Indeed, without loss of generality, set $I_1 = \{1\}$ and $I_2 = \{2\}$ and choose $\bar{x}_2 = x_2$ and $\bar{x}_1 \leq x_1 + e_2$ with $u_1(\bar{x}_1, \bar{x}_2) > u_1(x_1, \bar{x}_2)$. However, in the economy of the example in Subsection 3.2 with three individuals, although $e_i > 0$ and preferences are strongly monotonic in own consumption, an equilibrium does not exist. Hence such an economy does not satisfy Assumption E with $I_1 = \{2, 3\}$, $I_2 = \{1\}$ at the feasible allocation $(0, r/2, r/2)$ with $r = e_1 + e_2 + e_3$.

Assumption E is also satisfied in the presence of positive externalities (as for instance public goods). This is stated in the next proposition whose simple proof is left to the reader. Further, it is verified that Assumption N in Crès (1996) implies Assumption E.

⁵ Definition 4 in Geistdorfer-Florenzano (1982) is more general than Assumption I as it allows $\theta_k > 0$, $k \in I_1$. However, by the definition of our technology Y , the allocation \bar{x} in (2) of Assumption E must be invariant in I_1 implying that the weights θ_k must be invariant in I_1 .

Proposition 6 *Under Assumption A, suppose that there exists at least one commodity $\ell \in \mathcal{L}$ such that for all $x \in X$, $i \in \mathcal{I}$ and $h \neq i$, $u_i(\bar{x}) > u_i(x)$ and $u_h(\bar{x}) \geq u_h(x)$, for $\bar{x}_h^{\ell'} = x_h^{\ell'}$, for all $(h, \ell') \neq (i, \ell)$, and $\bar{x}_i^\ell > x_i^\ell$. Then, if $e_h^\ell > 0$ for all $h \in \mathcal{I}$, Assumption E is satisfied.*

Finally, we exploit the definition of the production technology Y to show that Assumptions I and E are equivalent.

Proposition 7 *The economy \mathcal{E}_Y is McKenzie-Bergstrom irreducible if and only if the economy \mathcal{E} satisfies Assumption E.*

Proof. Let us assume that Assumption I is satisfied. Let x be a feasible allocation of \mathcal{E} and I_1 and I_2 as in Assumption E. Then, by definition of Y and feasibility, $y = (-\sum_{i \in \mathcal{I}} e_i, x, \dots, x, \dots, x)$ and $\xi_i = (0_L, 0_{LI}, \dots, 0_{LI}, x, 0_{LI}, \dots, 0_{LI})$ for all $i \in \mathcal{I}$ is a feasible allocation of \mathcal{E}_Y . Let $y' \in Y$ and $(\xi'_i)_{i \in \mathcal{I}}$ as given by Assumption I. Then $y' = (-z', (\tilde{y}'_i)_{i \in \mathcal{I}})$ with $z' \in \mathbb{R}_+^L$ and $\tilde{y}'_i = \bar{x}$ for all $i \in \mathcal{I}$ for some $\bar{x} = (\bar{x}_i)_{i \in \mathcal{I}} \in \mathbb{R}^{LI}$ such that $\sum_{i \in \mathcal{I}} \bar{x}_i \leq z'$.

For all i , $\xi'_i = (0_L, 0_{LI}, \dots, 0_{LI}, \tilde{x}'_i, 0_{LI}, \dots, 0_{LI})$ for some $\tilde{x}'_i \in X$. Since $\eta_i = (e_i, 0_{LI}, \dots, 0_{LI}, \dots, 0_{LI})$ for all $i \in \mathcal{I}$, Condition 1 in Assumption I reads:

$$\begin{cases} \forall k \in I_1, \tilde{x}'_k = \tilde{y}'_k = \bar{x} \\ \forall h \in I_2, \theta_h \tilde{x}'_h = \tilde{y}'_h = \bar{x} \\ z' = \sum_{k \in I_1} e_k + \sum_{h \in I_2} \theta_h e_h \end{cases}$$

Then $\bar{x} \in X$ because $\bar{x} = \tilde{x}'_k \in X$ for all $k \in I_1$. Therefore, the economy \mathcal{E} satisfies Assumption E since for all $k \in I_1$, $u_k(\bar{x}) = \tilde{u}_k(\xi'_k)$ and $u_k(x) = \tilde{u}_k(\xi_k)$.

Conversely, assume now that the economy \mathcal{E} satisfies Assumption E. Let $((\xi'_i)_{i \in \mathcal{I}}, y)$ be a feasible allocation of \mathcal{E}_Y and I_1 and I_2 as in Assumption I. Then, x such that $\xi_1 = (0, x, 0, \dots, 0)$ is a feasible allocation of \mathcal{E} . Let \bar{x} as given by Assumption E.

Set $\gamma_h = \max\{1, \theta_h\} \geq 1$. Then, since by Assumption A, $e_h \geq 0$:

$$\sum_{i \in \mathcal{I}} \bar{x}_i \leq \sum_{k \in I_1} e_k + \sum_{h \in I_2} \theta_h e_h \leq \sum_{k \in I_1} e_k + \sum_{h \in I_2} \gamma_h e_h,$$

For $k \in I_1$ take $\tilde{x}'_k = \bar{x}$, and for $h \in I_2$ set $\tilde{x}'_h = (1/\gamma_h)\bar{x}$. Since $0 \in X$, X is convex and $\gamma_h \geq 1$, it must be that $\tilde{x}'_h \in X$ for all $h \in I_2$. Then the economy \mathcal{E}_Y is McKenzie-Bergstrom irreducible with $\xi'_k = (0, \dots, 0, \bar{x}, 0, \dots, 0)$ for $k \in I_1$ and $\xi'_h = (0, \dots, 0, (1/\gamma_h)\bar{x}, 0, \dots, 0)$ for $h \in I_2$, and $y = (-\sum_{k \in I_1} e_k + \sum_{h \in I_2} \gamma_h e_h, \bar{x}, \dots, \bar{x})$ with the coefficients γ_h for $h \in I_2$. \square

5.3 Survival conditions

The survival assumption is a sufficient condition guaranteeing that at a quasi-equilibrium at least one individual's consumption is not a minimal expenditure bundle. When combined with the irreducibility conditions, it implies that at a quasi-equilibrium no individual's consumption is a minimal expenditure bundle, that is, that individuals consumptions are utility maximizers. Hence a quasi-equilibrium is an equilibrium. We posit the following assumption.

Assumption S (Survival in \mathcal{E}) For all $i \in \mathcal{I}$, $\text{Int } X_i \neq \emptyset$ and $\sum_{i \in \mathcal{I}} e_i \in \mathbb{R}_{++}^L \cap (\text{Int } \sum_{i \in \mathcal{I}} X_i)$.

Assumption S is clearly satisfied under standard survival assumptions for exchange economies without externalities, i.e., $X_i = \mathbb{R}_+^L$ for all $i \in \mathcal{I}$ and $\sum_{i \in \mathcal{I}} e_i \gg 0$.

The following proposition shows that if the economy \mathcal{E} satisfies Assumption S, then the production economy \mathcal{E}_Y satisfies the survival assumption of Florenzano (2003), under which at a quasi-equilibrium at least one individual's consumption is a utility maximizer.

Proposition 8 *If the economy \mathcal{E} satisfies Assumptions A and S, then*

$$\sum_{i \in \mathcal{I}} \eta_i \in \text{Int} \left(\sum_{i \in \mathcal{I}} \Xi_i - Y \right).$$

Proof. By Assumption A, consumption sets X_i are convex. By Assumption S, $\text{Int } X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Thus, $\text{Int } \sum_{i \in \mathcal{I}} X_i = \sum_{i \in \mathcal{I}} \text{Int } X_i$. Then, by Assumption S, there exists $t > 0$ small enough and $\underline{x} = (\underline{x}_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} \text{Int } X_i$ such that $\sum_{i \in \mathcal{I}} e_i - t\mathbf{1}_L \in \mathbb{R}_{++}^L$ and $\sum_{i \in \mathcal{I}} e_i - t\mathbf{1}_L = \sum_{i \in \mathcal{I}} \underline{x}_i$, where $\mathbf{1}_L$ is the vector of \mathbb{R}^L with all its components equal to 1. Consider now the following open set:

$$A = \left\{ r \in \mathbb{R}_{++}^L \mid r \gg \sum_{i \in \mathcal{I}} e_i - t\mathbf{1}_L \right\} \times \left(\prod_{j \in \mathcal{I}} \text{Int } X_j \right)^I.$$

For all $i \in \mathcal{I}$, pick $\tilde{x}_i \in \prod_{j \in \mathcal{I}} \text{Int } X_j$, then $(r, (\tilde{x}_i)_{i \in \mathcal{I}}) \in A$. For all $i \in \mathcal{I}$, set $\xi_i = (0_L, 0_{LI}, \dots, 0_{LI}, \tilde{x}_i, 0_{LI}, \dots, 0_{LI})$, and $y = (-r, \underline{x}, \dots, \underline{x}, \dots, \underline{x})$. Thus $\sum_{i \in \mathcal{I}} e_i - t\mathbf{1}_L = \sum_{i \in \mathcal{I}} \underline{x}_i \ll r$ implies that $y \in Y$, and then

$$(r, \tilde{x}_1 - \underline{x}, \dots, \tilde{x}_i - \underline{x}, \dots, \tilde{x}_I - \underline{x}) = \sum_{i \in \mathcal{I}} \xi_i - y \in \sum_{i \in \mathcal{I}} \Xi_i - Y.$$

Let Φ be the affine mapping from $\mathbb{R}^L \times \mathbb{R}^{(LI)I}$ to itself defined by

$$\Phi(r, \tilde{x}_1, \dots, \tilde{x}_i, \dots, \tilde{x}_I) = (r, \tilde{x}_1 - \underline{x}, \dots, \tilde{x}_i - \underline{x}, \dots, \tilde{x}_I - \underline{x}).$$

The mapping Φ is clearly one-to-one and onto and Φ^{-1} is continuous. Thus, Φ is an open mapping, that is, the image of an open set is an open set. Therefore $\Phi(A)$ is contained in $\sum_{i \in \mathcal{I}} \Xi_i - Y$, so in its interior. Furthermore,

$$\sum_{i \in \mathcal{I}} \eta_i = \left(\sum_{i \in \mathcal{I}} e_i, 0_{LI}, \dots, 0_{LI}, \dots, 0_{LI} \right) = \Phi \left(\sum_{i \in \mathcal{I}} e_i, \underline{x}, \dots, \underline{x}, \dots, \underline{x} \right) \in \Phi(A).$$

Hence $\sum_{i \in \mathcal{I}} \eta_i$ belongs to the interior of $\sum_{i \in \mathcal{I}} \Xi_i - Y$. \square

5.4 Statement of the Existence Theorem

The existence of an equilibrium with markets for externalities of \mathcal{E} is the consequence of Proposition 4, Theorem 5, Propositions 7 and 8.

Theorem 9 (Existence of an equilibrium with markets for externalities). *Under Assumptions A, E and S, the economy \mathcal{E} has an equilibrium with $p^* \geq 0$.*

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