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► **To cite this version:**

Susumu Cato, Stéphane Gonzalez, Eric Rémila, Philippe Solal. Approval voting versus proportional threshold methods: so far and yet so near. 2022. halshs-03858356

HAL Id: halshs-03858356

<https://shs.hal.science/halshs-03858356>

Preprint submitted on 17 Nov 2022

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Approval voting versus proportional threshold methods: so far and yet so near

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Abstract. This paper provides the first axiomatic characterization of a class of certification methods, dubbed proportional threshold methods, that builds on consistency properties across populations and profiles of binary opinions. We then compare proportional threshold methods with the approval voting method. We formally outline the similarities and differences of these two collective decision processes and provide an axiomatic characterization of the approval voting method that, perhaps surprisingly, is closely related to the one of the proportional threshold methods.

Journal of Economic Literature Classification Nos.: C02, D71.

Keywords: approval voting method, certification method, consistency, evaluation, monotonicity, proportional threshold method.

Introduction

When the members of a group express binary (i.e., yes-no, approval-disapproval) opinions on a set of available options, there are two natural methods of deriving a collective decision based on the group's opinions. First, one might ask what options can be considered collectively acceptable. An intuitive manner of determining a set of acceptable options involves setting an approval threshold such that those options that meet (or exceed) it (meaning that they are supported by a sufficiently large share of the electorate) are selected. Second, one might then ask what options can be identified as the best. For this, the most natural method is selecting the one which is supported by the largest portion of the electorate, a procedure known as the approval voting method. Proportional threshold methods and the approval voting method serve substantially different purposes and, as such, suit different situations.

On one hand, proportional threshold methods are appropriate in the case of certification problems, i.e., situations in which the acceptance of one option does not affect the acceptance of another. Common examples include the procedures for accepting new members in a club, religious congregation, or political party, decision to accept or reject a working paper in a peer-reviewed journal, validation of a student's university degree, or acceptance or rejection of a law in a legislative assembly. Notable real-world examples include the practice of impeachment in the United States, which requires two-third of the attending senators to concur with convicting the president, the elections of the president, vice president, and members of the Supreme Court in the Philippines, and those of judges in Japan. In France, most laws are passed if they are supported by a majority of the attending members of each chamber of parliament, while constitutional reforms require a higher rate of support (two-third of the attending members). A related method can be found in the selection process of members of the "Pro Football Hall of Fame." Finalists who are

selected by fans and committee members by a certain process are considered in the “Selection Saturday” the day before the Super Bowl. Approvals of at least four-fifth of the board members are needed to be elected. On the other hand, the approval voting method is a competitive method of selection, insofar as (abstracting from possible ties) the selection of one option excluding the selection of the others. Natural examples include the election of a town mayor, the recruitment of a candidate by a hiring committee, or the selection of a student for a scholarship. The Latvian parliament uses the approval voting method within open-list proportional representation.

Proportional threshold methods and the approval voting method have important differences. First, their outcomes can be substantially different, since the most approved option may fail to meet the approval threshold for certification and, symmetrically, options that meet the certification threshold may not be the most approved. Nevertheless, as long as some option meets the threshold, the approval voting outcome must be included in the set of certified options. Second, the approval voting method always yields a winner, while proportional threshold methods may well be empty¹ (this is the case when no alternative meets the approval threshold). Third, the two methods differ in their treatment of voters who express a negative opinion on all the available candidates (a “none of the above” ballot): while these agents do not affect the outcome of the approval voting method, they do affect the one of proportional threshold methods, as they reduce the share of approvals of all options.

To formally assess the fundamental differences and similarities between the approval voting method and proportional threshold methods, we adopt the axiomatic approach. Our first contribution, Theorem 1, is an axiomatic characterization of proportional threshold methods that relies on seven independent axioms, six of which are also satisfied by the approval voting method. Our axioms are versions (and in some cases, extensions) of well-known principles (e.g., anonymity, neutrality, and unanimity preservation). Three of our axioms are particularly noteworthy. *Bi-consistency* requires that: (i) if an option is selected under two distinct sets of agents, then it is selected under the union of the two sets, and (ii) if an option is not selected under two distinct sets of agents, then it is not selected under the union of the two sets. Part (i) of this axiom is weaker than the well-established principle of consistency, which is used by Smith (1973), Young (1975), and Myerson (1995) to characterize scoring methods. Young (1974) and Fishburn (1978ab) utilize this axiom to characterize the Borda rule and approval voting method, respectively; these are two specific scoring methods. Part (ii) is the equivalent requirement for “unselected” options. Although this axiom is very natural, is not generally met by scoring methods. *Acceptability* demands

¹This is the case for the practice of impeachment; it is quite possible for no president/judge to be convicted. On the contrary, the approval voting method always chooses some outcomes as long as at least one option is approved by some agents.

that if some options are selected under a given opinion profile, and agents expand their set of approved options by adding some previously rejected options, then some options must be selected under the new opinion profile. This axiom is inspired by the monotonicity principle that was originally introduced by Murakami (1966, 1968) in the framework of preference aggregation with two alternatives and used later by Fishburn (1973) to characterize a class of absolute majority rules. Our third key axiom is also a principle of monotonicity, dubbed *Antitonicity*. It demands that if an option is not selected under a given opinion profile, and voters expand their set of accepted options as above but in such a way that the set of supporters of the given option does not change (meaning that voters who did not approve of the given option still do not approve of it), then the latter remains unselected in the new opinion profile. In contrapositive, if an option is selected in the original profile and its supporters are unchanged in a new profile which contains less individually approved options, then this option continues to be selected in the modified profile. In our framework, Maskin monotonicity (1999) implies antitonicity but the converse is not true.

The only axiom in our characterization of proportional threshold methods that is not satisfied by the approval voting method is Biconsistency. Hence, Theorem 1 outlines the formal similarities between these two methods and bears the question of their differences. Our second contribution, Theorem 2, is a novel characterization of the approval voting method obtained by replacing Biconsistency with the standard consistency axiom and adding one additional property, called *Null agent out*, that formally translates the above-mentioned idea that agents who reject all options are irrelevant for determining the outcome of the election. Thus, our two main results offer comparable axiomatic characterizations of the approval voting method and proportional threshold methods. This paper can be seen as a first step, opening a new branch of the literature on collective decision processes, seeking to understand the fundamental properties that separate competitive methods from certification methods.

Related literature

Since the seminal works by Brams and Fishburn (Brams and Fishburn (1978), and Fishburn (1978a,b)), a burgeoning body of research has been dedicated to the analysis of the axioms characterizing the approval voting method. Notably, Fishburn (1978a,b) provides the earliest axiomatic characterization of the approval voting method and it is extended by, for instance, Baigent and Xu (1991), Alós-Ferrer (2006), Massó and Vorsatz (2008), Vorsatz (2007, 2008), Alcalde-Unzu and Vorsatz (2009, 2014), Brandl and Peters (2019), Cato, Rémila, and Solal (2021),

and Ceron and Gonzalez (2021).² Despite the interest this method has generated in the social choice community,³ it remains little used in practice to date. In contrast with the approval voting method, proportional threshold methods have never been characterized axiomatically, although they are commonly used. To the best of our knowledge, Theorem 1 is the first characterization of proportional threshold methods. It is, however, related to a series of characterizations of absolute majority methods. An earlier characterization was classified by Fishburn (1973), and more recently, characterizations by Austen-Smith and Banks (2000), Aşan and Sanver (2006), and Houy (2007) have been found. In these settings, the population is fixed and only one option is considered. Agents must choose between accepting or rejecting the option.⁴ Since agents express their preferences over acceptance and rejection, their opinions are expressed by a single value, which can be “1”, “0”, and “−1”, where “1” means “the option should be accepted”, “0” represents the “indifference between acceptance and rejection”, and “−1” means “the option should be rejected”.

Notably, there is a threshold or quota such that the option is accepted if the number of supporters is strictly greater than the number of opponents and is rejected if this number is strictly less. However, this line of works does not consider a variable population and number of options. In other words, the sets of agents and options considered in the aforementioned existing works are fixed.

A variable population framework is indeed crucial for actual selection processes. As is the case in the absolute majority rules, our method includes quotas, but they must be consistent across different populations. Indeed, in the real world, it is likely to have a change in the number of eligible persons (the number of seats or the number of citizens over a certain age), and thus, the total number of evaluations/ballots can change. It should be emphasized that the class of proportional threshold methods is one of the natural, plausible ways of determining quotas under a variable-population framework, although we do not reject the possibility of other types of quota systems; under a proportional threshold method, each quota is given as the product of the number of agents and a constant, which is between zero and one. As mentioned earlier, this is utilized by impeachment and other actual selection methods, in which participants can vary. On the other

²A comprehensive survey of axiomatic characterizations of the approval voting method (up to 2010) can be found in the research by Xu (2010).

³More anecdotally, at a 2010 workshop in Normandy (France), several voting theorists voted on the question “Which is the best voting rule to use in your city to elect the mayor?” The approval voting method won the contest, which compared a total of 18 different voting methods. Details can be found in the work of Laslier (2012).

⁴Equivalently, acceptance and rejection can be viewed as two distinct options x and y . This way of looking at the problem allows Ozkes and Sanver (2017) to consider the case where there are more than two options to define a “competitive” social ranking.

hand, it is also crucial to consider an arbitrary number of options when comparing proportional threshold methods to the approval voting method, which is our intention.

One may wonder if the fundamental role of Biconsistency is to identify the proportional structure over quotas instead of allowing an arbitrary combination of quotas. Indeed, it is easy to see that Biconsistency necessarily restricts quotas over different population sizes. However, our parallel characterizations show that the approval voting method, which is not a quota method, satisfies all the axioms for characterizing the proportional threshold methods, other than Biconsistency. This implies that Biconsistency has another fundamental role. That is, it is crucial for deriving the quota structure. This double-layered function of Biconsistency is novel and has not been established in the existing works on absolute majority methods, which employ quotas.

Moreover, our setting is also closely related to the problem of group identification, which is examined by Kasher and Rubinstein (1997), Samet and Schmeidler (2003), Ju (2005, 2010, 2013), and Cho and Ju (2017), and Cho and Park (2018). In their setting, each agent writes the names of the agents who are personally considered to be qualified for a certain membership. It is a particular type of social approval. A key difference is that a set of options is arbitrary in our framework, while it coincides with the set of agents in the group-identification setting. More importantly, most works in group identification utilizes what is called *Independence*, which requires that each option (or individual's membership in their context) is dependent only on opinions over this option. Notably, this axiom is strong. Although it makes sense when defining a certification rule,⁵ the Independence axiom is “untenable”⁶ when it comes to choosing the number of options in a competitive manner. A novel contribution this paper makes is that the Independence axiom is derived from others. That is, we do not impose Independence. While distinguishing the approval voting method from the proportional threshold method using the Independence axiom is completely transparent, our result establishes a more surprising difference between the two rules. This distinction involves null agents, and a form of consistency in the way rejected options are treated. This also suggests that our technique of deriving Independence may apply to the framework of group identification with a variable population domain.⁷

⁵In this paper, we propose to define acceptance rules precisely as selection rules verifying the Independence property.

⁶Samet and Schmeidler (2003: 226) wrote: “In certain social situations the Independence axiom is untenable [...] for example the election of a committee of a certain fixed size from a list of candidates. [...] A well-known rule for such a problem is approval voting”.

⁷Similar observations can be made about the strong monotonicity axiom, used by Samet and Schmeidler (2003), Kasher and Rubinstein (1997), and Miller (2008) in the context of group identification. Lemmas 4 and 5 on this paper establish how to derive these axioms from weaker axioms.

1 Definitions and main concepts

1.1 Evaluating profiles and certification methods

Let \mathcal{X} be an arbitrary infinite **set of options**, and \mathbb{N} be the set of potential agents, where $\mathbb{N} = \{0, 1, 2, \dots\}$. An electorate N is a nonempty finite subset of \mathbb{N} . Each agent $i \in \mathbb{N}$ can either **approve** of an option $x \in \mathcal{X}$ or **disapprove** of this option. Each option is evaluated by the agents. For each electorate N of \mathbb{N} and each finite subset X of \mathcal{X} , an **evaluation profile** on (N, X) collects the evaluations of the electorate and is represented by a matrix $B_N^X = [b_i^x]_{i \in N, x \in X}$, where for each $i \in N$ and each $x \in X$, $b_i^x \in \{0, 1\}$. The value $b_i^x = 1$ means that i approves of x whereas the value $b_i^x = 0$ stands for the situation where i disapproves of x . For $M \subseteq N$, wherein with a slight abuse of notation, we write B_M^X for $[b_i^x]_{i \in M, x \in X}$. Let O_N^X be the evaluation profile such that each agent disapproves of each option in X . Given a set X of options, two sets N_1, N_2 of agents such that $N_1 \cap N_2 = \emptyset$, and two profiles $B_{N_1}^X = [b_i^x]_{i \in N_1, x \in X}$, $B_{N_2}^X = [b_i^x]_{i \in N_2, x \in X}$, let $B_{N_1 \cup N_2}^X$ be such that

$$B_{N_1 \cup N_2}^X = [b_i^x]_{i \in N_1 \cup N_2, x \in X}.$$

That is, $B_{N_1 \cup N_2}^X$ is the evaluation profile on $N_1 \cup N_2$ that is the union of the individual evaluations contained in $B_{N_1}^X$ and in $B_{N_2}^X$.

Let \mathcal{B}_N^X be the set of all evaluation profiles on (N, X) , and let \mathcal{B} be the set of all possible evaluation profiles. For each (N, X) , the set \mathcal{B}_N^X is endowed with the partial ordering $\geq_{\mathcal{B}_N^X}$, where

$$\forall B_N^X, B'_N{}^X \in \mathcal{B}_N^X, \quad [B_N^X \geq_{\mathcal{B}_N^X} B'_N{}^X] \iff [\forall i \in N, \forall x \in X, \quad b_i^x \geq b_i'^x].$$

Given an electorate N , a finite set of options $X \subseteq \mathcal{X}$, an evaluation profile $B_N^X \in \mathcal{B}_N^X$, and an option $x \in X$, let $s(B_N^X, x)$ be the **total evaluation** received by x under B_N^X :

$$s(B_N^X, x) = \sum_{i \in N} b_i^x.$$

Thus, $s(B_N^X, x)$ is the total number of agents who approves of the option x in B_N^X .

A **selection method** is a mapping $f : \mathcal{B} \rightarrow 2^{\mathcal{X}}$ that maps each evaluation profile $B_N^X \in \mathcal{B}$ to a possibly empty set $f(B_N^X)$ of options in X . Among the entire class of all possible selection methods, it is worthwhile to distinguish a subclass of selection methods that are called **certification methods**. Roughly speaking, a certification method is a selection method, under which the options are not in competition. More formally, such a method satisfies the Independence principle requiring that the certification of an option is independent of the evaluation of the remaining options.

Independence: For each pair of evaluation profiles $B_N^X, B_N'^X \in \mathcal{B}_N^X$ and each option $x \in X$, if, for each $i \in N$, $b_i^x = b_i'^x$ and $x \in f(B_N^X)$, then $x \in f(B_N'^X)$.

This contrasts with other selection methods, such as the scoring methods where an option is selected if it maximizes a certain score based on the evaluations of the electorate. We refer to Myerson (1995) for the definition of scoring methods in a more general context where the form of the evaluation is not specified. Among the scoring methods, the **approval voting method** f^A , developed by Brams and Fishburn (1978), is one of the most popular scoring methods. The approval voting method is defined as follows:

$$\forall B_N^X \in \mathcal{B}, B_N^X \neq O_N^X, \forall x \in X, \quad x \in f^A(B_N^X) \iff x \in \arg \max_{y \in X} s(B_N^X, y), \text{ and } f^A(O_N^X) = \emptyset.$$

Notably, options are in competition in the sense that no matter how many endorsements an option gets, if this option does not get the greatest number of approval evaluations, then it not be selected.

1.2 Proportional threshold methods

This section focuses on certification methods that aggregate the evaluation of any (finite) set of options by any (finite) set of agents. We introduce the family of proportional threshold methods and the larger family of quota methods for this domain.

A **weak r -proportional threshold method** on \mathcal{B} is defined as follows: There is a fixed proportion $r \in]0, 1]$ such that

$$\forall B_N^X \in \mathcal{B}, \forall x \in X, \quad x \in f^r(B_N^X) \iff \frac{s(B_N^X, x)}{n} \geq r. \quad (1)$$

And a **strict r -proportional threshold method** on \mathcal{B} is defined as follows: There is a fixed proportion $r \in [0, 1[$ such that

$$\forall B_N^X \in \mathcal{B}, \forall x \in X, \quad x \in f^r(B_N^X) \iff \frac{s(B_N^X, x)}{n} > r. \quad (2)$$

We note that weak and strict r -proportional threshold methods require that an option obtains at least one approval to be selected and if each agent approves of some option, then this option is selected. That explains why $r > 0$ for the weak r -proportional threshold method and $r < 1$ for the strict r -proportional threshold method. We say that f is a **r -proportional threshold method** if it is a weak r -proportional threshold method for some $r \in]0, 1]$ or a strict r -proportional threshold method for some $r \in [0, 1[$. The set of r -proportional threshold methods together is the class of **proportional threshold methods**.

It is worthwhile to compare an r -proportional threshold method to the approval voting method f^A defined above. Note that if $f^r(B_N^X) \neq \emptyset$, then

$$f^r(B_N^X) \supseteq f^A(B_N^X).$$

That is, the approval voting method is more selective than any r -proportional threshold method whenever the latter chooses some option.

In the course of this paper, we also consider a larger family of certification methods. Let $q : \mathbb{N} \rightarrow \mathbb{N}$ be an integer function. A **q -quota method** on \mathcal{B} is defined as follows:

$$\forall B_N^X \in \mathcal{B}, \forall x \in X, \quad x \in f^q(B_N^X) \iff s(B_N^X, x) \geq q(n). \quad (3)$$

According to this method, an option that gets approvals from at least $q(n)$ agents among the n agents of the electorate N is socially approved under the q -quota method. By definition, if $q(n) = 0$, then $f^q(B_N^X) = X$ for each $B_N^X \in \mathcal{B}_N^X$. And, if $q(n) > n$, then it holds that $f^q(B_N^X) = \emptyset$ for each $B_N^X \in \mathcal{B}_N^X$. The family of q -quota methods f^q , where q is an integer function, constitutes the class of the **quota methods**.

Remark 1. We note that the family of quota methods contains the family of proportional threshold methods. To see this, let us take the following two cases:

- Consider the weak proportional threshold method f^r for some $r \in]0, 1]$. For each $n \in \mathbb{N}$, define $q(n)$ as the smallest integer such that $q(n) \geq nr$. In this case, f^r coincides with the corresponding q -quota method f^q .
- Consider the strict proportional threshold method f^r for some $r \in [0, 1[$. For each $n \in \mathbb{N}$, define $q(n)$ as the largest integer such that $q(n) \leq nr$. In this case, f^r coincides with the corresponding q -quota method f^q .

2 Main results

2.1 Axiomatic characterization of the proportional threshold methods

We now introduce a set of axioms for a selection method f on \mathcal{B} . The first axiom is a weak condition of nonemptiness which indicates that, for each nonempty set of options and agents, at least one evaluation profile selects some options.

Nontriviality: For each electorate N and each finite subset X of \mathcal{X} , there is an evaluation profile $B_N^X \in \mathcal{B}_N^X$ such that $f(B_N^X) \neq \emptyset$.

The following axiom essentially requires that the option is invariant to exchanges of the agents' names. It relies on a principle of impartiality of collective-decision making with respect to the agents, and it is uncontroversial especially in the case of a finite population. Given two finite sets $N = \{i_1, i_2, \dots, i_n\}$ and $N' = \{i'_1, i'_2, \dots, i'_n\}$ with same cardinality n of agents, and two evaluation profiles B_N^X and $B_{N'}^X$ on the option set $X \subseteq \mathcal{X}$, we say that $B_{N'}^X$ is an **agent copy** of B_N^X if, for each $k \in \{1, 2, \dots, n\}$ and each $x \in X$, we have $b_{i_k}^x = b_{i'_k}^x$.

Anonymity: If $B_{N'}^X \in \mathcal{B}_{N'}^X$ is an agent copy of $B_N^X \in \mathcal{B}_N^X$, then $f(B_N^X) = f(B_{N'}^X)$.

Anonymity states that exchanging the evaluations of any two agents does not modify the set of selected options. There exists a stronger form of Anonymity, dubbed Anonymity for a given option (see Gonzalez, Laruelle, and Solal [2019]), which says that if the number of approvals and disapprovals is the same for each option in two different evaluation profiles, then the selected options are the same.

Anonymity for a given option: For each pair of electorates N and N' of the same size n , and each pair of evaluation profiles $B_N^X \in \mathcal{B}_N^X$ and $B_{N'}^X \in \mathcal{B}_{N'}^X$, if, for each $y \in X$, $s(B_N^X, y) = s(B_{N'}^X, y)$, then $f(B_N^X) = f(B_{N'}^X)$.

The next axiom relies on a principle of impartiality of collective-decision making with respect to the options. It states that the labeling of options can be changed without affecting the outcome. Given two option sets $X = \{x_1, x_2, \dots, x_k\}$ and $X' = \{x'_1, x'_2, \dots, x'_k\}$ in \mathcal{X} with same number k of options, and two evaluation profiles B_N^X and $B_N^{X'}$ on the electorate N , we say that $B_N^{X'}$ is an **option copy** of B_N^X if, for each $\ell \in \{1, 2, \dots, k\}$ and each $i \in N$, we have $b_i^{x_\ell} = b_i^{x'_\ell}$.

Neutrality: If $B_N^{X'} \in \mathcal{B}_N^{X'}$ is an option copy of $B_N^X \in \mathcal{B}_N^X$, then

$$\forall \ell \in \{1, 2, \dots, k\}, \quad [x_\ell \in f(B_N^X)] \iff [x'_\ell \in f(B_N^{X'})].$$

We now introduce axioms on monotonic changes of individual evaluations. If an option is chosen in a profile, it continues to be chosen if the number of approvals of each other option decreases. Specifically, if an option is chosen in the original profile and its supporters are not changed in a new profile obtained by an antitonic change yielding less personally approved of options, then it continues to be selected in the new profile. This axiom is formally stated as follows.

Antitonicity: For each pair of evaluation profiles $B_N^X, B_N^{X'} \in \mathcal{B}_N^X$ and each $x \in X$, if $B_N^{X'} \leq_{\mathcal{B}_N^X} B_N^X$

and $b_i^x = b_i'^x$ for each $i \in N$, then $x \in f(B_N^X)$ implies $x \in f(B_N'^X)$.

The essence of this axiom is closely related to Maskin monotonicity (Maskin, 1999). However, Antitonicity is substantially weaker than it. To understand this point, let us define the lower counter set, $L_i(B_N^X, x)$, of an option $x \in X$ at B_N^X as follows:

$$\forall i \in N, \quad L_i(B_N^X, x) = \{y \in X : b_i^x \geq b_i^y\}.$$

The exact translation of Maskin monotonicity requires that if $x \in f(B_N^X)$, then $x \in f(B_N'^X)$ for each evaluation profile such that $L_i(B_N'^X, x) \supseteq L_i(B_N^X, x)$ for each $i \in N$. In particular, if $B_N'^X \leq_{\mathcal{B}_N^X} B_N^X$ and $b_i^x = b_i'^x$ for each $i \in N$, then $L_i(B_N'^X, x) \supseteq L_i(B_N^X, x)$ for each $i \in N$. Thus, Maskin monotonicity implies Antitonicity. The converse is not always true. Consider the selection method that selects an option x if x is the unique option unanimously approved. Otherwise, the selection method does not select any option. This selection method satisfies Antitonicity but violates Maskin monotonicity.

Notably, the aforementioned axioms focus on the case where the number of agents does not change. An axiom concerning changes in the number of agents has been effectively employed in the literature of voting. The most well-known axiom is Consistency, which requires that if two disjoint sets of agents separately select two non-disjoint subsets of options, then the union of these sets of agents should exactly select the intersection of the two subsets.⁸

Consistency: For each pair of electorates N_1, N_2 such that $N_1 \cap N_2 = \emptyset$ and each pair of evaluation profiles $B_{N_1}^X \in \mathcal{B}_{N_1}^X, B_{N_2}^X \in \mathcal{B}_{N_2}^X$ such that $f(B_{N_1}^X) \cap f(B_{N_2}^X) \neq \emptyset$, the following equality holds:

$$f(B_{N_1 \cup N_2}^X) = f(B_{N_1}^X) \cap f(B_{N_2}^X).$$

We note that, for $r \notin \{0, 1\}$, no r -proportional threshold method satisfies Consistency. The following example illustrates this point.

⁸Smith (1973), Young (1975), and Myerson (1995) use the axiom of Consistency to characterize the scoring methods. Young (1974) and Fishburn (1978a,b) uses it to characterize the Borda method and the approval voting method, respectively. To be precise, Smith (1973) and Young (1974, 1975) consider the context where agents vote by expressing a rank ordering of the options while Fishburn (1978a,b) assumes that each agent casts a ballot containing a subset of options. Myerson (1995) shows that the characterization of the scoring methods by Smith (1973) and Young (1975) can be extended by dropping the ordering assumption. Smith (1973) assumes also that the outcome of the selection method determines a complete rank ordering of the options. Young (1974, 1975), Fishburn (1978a,b), and Myerson (1995) drop this assumption and specify that a selection method only needs to choose a nonempty subset of options. We follow Young (1974, 1975), Fishburn (1978a,b), and Myerson (1985) in this regard except that the empty set is allowed as an outcome.

Example 1. Let $X = \{a, b\}$ be the set of options. Take $N_1 = \{1\}$ and $N_2 = \{2\}$. Now, we consider the following two evaluation profiles: $B_{N_1}^X$ is such that agent 1 approves a and b ; $B_{N_2}^X$ is such that agent 2 approves a and disapproves b . Let f^r be the weak r -proportional threshold method where $0 < r < 1/2$. Then,

$$f^r(B_{N_1}^X) = \{a, b\} \text{ and } f^r(B_{N_2}^X) = \{a\}.$$

Assume now that the two electorates merge into one larger electorate $N_1 \cup N_2$ so that the evaluation profile $B_{N_1 \cup N_2}^X$ is obtained by putting together the two previously separated evaluation profiles. Because each option is approved at least by one agent over two in $B_{N_1 \cup N_2}^X$,

$$f^r(B_{N_1 \cup N_2}^X) = \{a, b\}.$$

On the other hand, Consistency requires that the outcome $f^r(B_{N_1 \cup N_2}^X)$ is equal to

$$f^r(B_{N_1}^X) \cap f^r(B_{N_2}^X) = \{a\}.$$

Thus, f^r violates Consistency. Nevertheless, remark that

$$f^r(B_{N_1}^X) \cap f^r(B_{N_2}^X) \subseteq f^r(B_{N_1 \cup N_2}^X).$$

Example 1 motivates us to consider a weaker alternative of consistency, which requires that if an option is selected separately by two disjoint electorates, then this option is also selected on the union of these electorates. This version is satisfied by the r -proportional threshold methods but does not discriminate them from the scoring methods. Therefore, it is natural to keep this weak version of consistency and add another requirement, saying that if an option is not selected on any of these two disjoint sets of agents, then it is also not selected on the union of these sets of agents. Interestingly, the scoring rules do not satisfy in general this requirement. Indeed, it is easy to construct an instance where the approval voting method violates this principle. Combining the aforementioned two requirements yields the following axiom called Biconsistency.⁹

Biconsistency: For each pair of electorates N_1, N_2 such that $N_1 \cap N_2 = \emptyset$ and each pair of evaluation profiles $B_{N_1}^X \in \mathcal{B}_{N_1}^X$, $B_{N_2}^X \in \mathcal{B}_{N_2}^X$, the following two conditions hold:

- (i) if $x \in f(B_{N_1}^X)$ and $x \in f(B_{N_2}^X)$, then $x \in f(B_{N_1 \cup N_2}^X)$;

⁹Saari (1990) offers a notable attempt with regard to consistency axioms. He refers the Smith-Young consistency as to Strong consistency. In addition to it, Saari (1990) introduces Weak consistency that requires that if the outcomes are the same between evaluations under two distinct set of agents, then the union of the two sets of evaluations yields the same outcome. Notably, this axiom is weaker than biconsistency.

(ii) if $x \notin f(B_{N_1}^X)$ and $x \notin f(B_{N_2}^X)$, then $x \notin f(B_{N_1 \cup N_2}^X)$.

The next axiom states that if an evaluation profile contains enough supporters for some options to be deemed acceptable, i.e., if the outcome is non-empty, then adding supporters to some options should not make all options simultaneously unacceptable.

Acceptability: For any two evaluation profiles $B_N^X, B'_N^X \in \mathcal{B}_N^X$, if $f(B'_N^X) \neq \emptyset$ and $B_N^X \geq_{\mathcal{B}_N^X} B'_N^X$, then $f(B_N^X) \neq \emptyset$.

The last axiom states that adding or deleting an option that only receives disapprovals does not change the outcome.

Null option out: For each evaluation profile $B_N^X \in \mathcal{B}_N^X$ and each option $x \in \mathcal{X} \setminus X$ such that, for each $i \in N$, $b_i^x = 0$, it holds that $f(B_N^X) = f(B_N^{X \cup \{x\}})$.

Now, we are ready to state our characterization of the proportional threshold methods.

Theorem 1. *A selection method f on \mathcal{B} satisfies Nontriviality, Anonymity for a given option, Neutrality, Antitonicity, Biconsistency, Acceptability, and Null option out if and only if it belongs to the class of proportional threshold methods.*

2.2 Comparison with the approval voting method

The approval voting method satisfies all the axioms used in Theorem 1 except Biconsistency; see the logical independence established in the Appendix. Thus, competition among options, which is the fundamental nature of the approval voting method, is not necessarily excluded unless Biconsistency is imposed. Note that Biconsistency requires that the selection methods make consistent choices in different subgroups. More specifically, this axiom requires that the selection method selects/rejects exactly those candidates that are selected/rejected in each group individually. As alluded earlier, the approval voting method is compatible with this consistency requirement only when each group of agents accepts an option; it violates the requirement when each group of agents rejects this option. This illustrates the difference between the treatment of options that are accepted and rejected. On the contrary, the proportional threshold methods handle them in the same manner.

Another important difference between the approval voting method and proportional threshold methods is the way in which agents who disapprove of all options are considered. One may call them **null agents**. Let us assume that such an agent withdraws from the vote. In proportional

threshold methods, this withdrawal can influence the outcome because it modifies the proportion of supporters. Thus, the presence or absence of null agents can influence outcomes. On the contrary, such withdrawal does not influence the outcome obtained by the approval voting method. Deciding whether a null agent can influence the outcome is an important normative issue. If null agents can influence the outcome, then this provides them with an incentive to participate in the evaluation process even if they do not approve of any option. In such a case, the participation of these agents conveys a more accurate idea of the popularity of the options selected by the selection method. This argument is less obvious when the goal is to design a selection method that selects the best options and not certify certain options. Indeed, this would be tantamount to providing agents who are indifferent to all the options the possibility or opportunity of changing the outcome.

The following axiom expresses the idea that the outcome should not be affected by the withdrawal or participation of null agents.

Null agent out: For each evaluation profile $B_N^X \in \mathcal{B}_N^X$, and for each agent $i \in N$ such that $b_i^x = 0$ for each $x \in X$, it holds that:

$$f(B_N^X) = f(B_{N \setminus i}^X).$$

As mentioned above, the approval voting method satisfies Null agent out. Dropping Biconistency in the statement of Theorem 1 and adding Null agent out and Consistency yield an axiomatic characterization of the approval voting method. Antitonicity is satisfied by the approval voting method but is redundant in the following result.

Theorem 2. *A selection method f on \mathcal{B} satisfies Nontriviality, Anonymity for a given option, Neutrality, Consistency, Acceptability, Null option out, and Null agent out if and only if it is the approval voting method.*

Theorem 1 and Theorem 2 together provide the parallel structure between the proportional threshold methods and the approval voting method. Unlike the original axiomatic characterization of the approval voting method provided by Fishburn (1978a,b), Theorem 2 utilizes neither Faithfulness nor Cancellation; see Section 3.1 for the definition of Faithfulness. In the present context, Cancellation states that if all options are approved by the same number of agents, then all options are selected. Anonymity for a given option and Neutrality imply that in such a case either all options are selected or no option is selected. Considering that Fishburn (1978a,b) does not allow the set of chosen options to be empty, the axioms of Theorem 2 imply Cancellation.

Although the proof of Theorem 2 is similar to that of Fishburn (1978a,b), its statement is interesting for at least two reasons. First, it states an alternative characterization of the approval voting method.¹⁰ Second, and more significantly, this characterization is comparable with the axiomatic characterization of proportional methods contained in Theorem 1.

3 Intermediary results and proof of Theorems 1 and 2

Before turning to the proof of Theorem 1, we conduct an axiomatic analysis that aims at showing that the combination of some of the aforementioned axioms implies some other axioms used in the literature. These implications are contained in several lemmas that will be used to prove Theorem 1.

3.1 Axiomatic analysis

To formulate our first lemma, we introduce an axiom widely used in existing axiomatic characterizations of the approval voting method. It says that when the electorate is reduced to one agent, the chosen options are the ones approved by this agent.

Faithfulness: For each $i \in \mathbb{N}$ and each $B_{\{i\}}^X \in \mathcal{B}_{\{i\}}^X$, $f(B_{\{i\}}^X) = \{x \in X : b_i^x = 1\}$.

The following lemma shows that this axiom is implied by Nontriviality, Neutrality, Acceptability, and Null option out.

Lemma 1. *If a selection method f satisfies Nontriviality, Neutrality, Acceptability, and Null option out, then it satisfies Faithfulness.*

Proof. Pick any $i \in \mathbb{N}$ and any $B_{\{i\}}^X \in \mathcal{B}_{\{i\}}^X$. From Null option out, it follows that $f(B_{\{i\}}^X) = f(B_{\{i\}}^{\bar{X}})$, where $\bar{X} = \{x \in X : b_i^x = 1\}$. By Nontriviality, there is at least one $\tilde{B}_{\{i\}}^{\bar{X}} \in \mathcal{B}_{\{i\}}^{\bar{X}}$ such that $f(\tilde{B}_{\{i\}}^{\bar{X}}) \neq \emptyset$. Because $\tilde{B}_{\{i\}}^{\bar{X}} \leq_{\mathcal{B}_{\{i\}}^{\bar{X}}} B_{\{i\}}^{\bar{X}}$, Acceptability implies that $f(B_{\{i\}}^{\bar{X}}) \neq \emptyset$. Finally, Neutrality implies that $f(B_{\{i\}}^{\bar{X}}) = \bar{X}$. Therefore, $f(B_{\{i\}}^X) = f(B_{\{i\}}^{\bar{X}}) = \bar{X}$, which shows that f satisfies Faithfulness.

The second lemma establishes that if a selection method satisfies Faithfulness and Biconsistency, then it satisfies two variations of the unanimity principle, which are commonly imposed in a wide range of collective decision methods. The first variation requires that if x is approved by

¹⁰See Brandl and Peters (2022) for a catalogue of axiomatic characterizations of the approval voting method.

all agents, it is socially approved.

Full approval: For each $B_N^X \in \mathcal{B}_N^X$ and each $x \in X$, if, for each $i \in N$, $b_i^x = 1$, then $x \in f(B)$.

The second variation of the unanimity principle is a counterpart of Full approval. It requires that x is socially disapproved whenever no agent approves of it.

Full disapproval: For each $B_N^X \in \mathcal{B}_N^X$ and each $x \in X$, if, for each $i \in N$, $b_i^x = 0$, then $x \notin f(B)$.

Now, the lemma is stated as follows.

Lemma 2. *If a selection method f satisfies Faithfulness and Biconsistency, then it satisfies Full approval and Full disapproval.*

The proof of this lemma is straightforward and so it is left to the reader.

As mentioned earlier, Antitonicity is related to Maskin monotonicity. To formulate our next result, we introduce another weak version of Maskin monotonicity. Specifically, the following axiom states that if an option is selected from an evaluation profile, then it remains selected in evaluation profiles where this option receives more supporters, ceteris paribus.

Monotonicity: Consider two evaluation profiles $B_N^X, B'_N^X \in \mathcal{B}_N^X$, and an option $x \in X$ such that, for each $i \in N$, and each $y \neq x$, $b_i^y = b_i'^y$. If $x \in f(B_N^X)$ and $\{i \in N : b_i^x = 1\} \subseteq \{i \in N : b_i'^x = 1\}$ then $x \in f(B'_N^X)$.

The following proposition shows that this axiom is implied by Faithfulness and Biconsistency. Although this result is not necessary for proving our main result, it is interesting in itself since Maskin monotonicity has attracted attention.

Proposition 1. *If a selection method f on \mathcal{B} satisfies Faithfulness and Biconsistency, then it satisfies Monotonicity.*

Proof. Assume that the profiles B_N^X, B'_N^X and the option $x \in X$ are as required for Monotonicity to apply. The proof is done by induction on the number p of agents $i \in N$ such that $b_i^x = 0$ and $b_i'^x = 1$.

INITIALIZATION: For $p = 0$, $B_N^X = B'_N^X$ and so the result trivially follows.

For $p = 1$, let $i \in N$ be the unique agent such that $b_i^x = 0$ and $b_i'^x = 1$. Because it is assumed that $x \in f(B_N^X)$, Biconsistency implies that either $x \in f(B_{N \setminus \{i\}}^X)$ or $x \in f(B_{\{i\}}^X)$. By Faithfulness,

$x \notin f(B_{\{i\}}^X)$. Thus, $x \in f(B_{N \setminus \{i\}}^X)$. Note that, by definition, $B_{N \setminus \{i\}}^X = B'_{N \setminus \{i\}}^X$, thus $x \in f(B'_{N \setminus \{i\}}^X)$. By Faithfulness, $x \in f(B''_{\{i\}}^X)$. Therefore, by Biconsistency, $x \in f(B'_N{}^X)$, which is the result.

INDUCTION HYPOTHESIS: Assume that f satisfies Monotonicity for any given $p \geq 1$.

INDUCTION STEP: Consider that the number of agents $i \in N$ such that $b_i^x = 0$ and $b_i^{x'} = 1$ is equal to $p + 1$. Let i_0 be such an agent, and consider the profile $B''_N{}^X$ such that $b_i^{y'} = b_i^{y''}$ for any pair $(i, y) \in N \times X$ such that $(i, y) \neq (i_0, x)$, and $b_{i_0}^{x''} = 0$. Hence, the number p of agents $i \in N$ such that $b_i^x = 0$ and $b_i^{x''} = 1$ is p . Thus, we can apply the induction hypothesis to the pair $(B_N^X, B''_N{}^X)$ to obtain $x \in f(B''_N{}^X)$. Afterwards, by another application of the induction hypothesis to the pair $(B''_N{}^X, B_N^X)$, we get the desired result $x \in f(B'_N{}^X)$. ■

Let x and y be two options. We say that y weakly dominates x via B_N^X if, for each $i \in N$, $b_i^{y'} \geq b_i^{x'}$. The next property is in the same spirit as the Pareto principle. It indicates that if x is selected and y weakly dominates x , then y is selected as well.

Pareto consistency: Consider any evaluation profile $B_N^X \in \mathcal{B}_N^X$ and two distinct options $x, y \in X$ such that y weakly dominates x via B_N^X . If $x \in f(B_N^X)$, then $y \in f(B_N^X)$.

The next lemma establishes that Pareto consistency is implied by Faithfulness, Anonymity for a given option, Neutrality, and Biconsistency.

Lemma 3. *If a selection method f on \mathcal{B} satisfies Faithfulness, Anonymity for a given option, Neutrality, and Biconsistency then it satisfies Pareto Consistency.*

Proof. Let $B_N^X \in \mathcal{B}_N^X$ be an evaluation profile, x and y be two options of X such that y weakly dominates x via B_N^X . Because y weakly dominates x via B_N^X , we have $s(B_N^X, x) \leq s(B_N^X, y)$. Hence, it suffices to prove that if $s(B_N^X, x) \leq s(B_N^X, y)$ and $x \in f(B_N^X)$ then $y \in f(B_N^X)$. We proceed with contradiction by assuming that

$$\exists x, y \in X : s(B_N^X, x) \leq s(B_N^X, y), y \notin f(B_N^X) \text{ and } x \in f(B_N^X). \quad (4)$$

We distinguish two cases. If $s(B_N^X, x) = s(B_N^X, y)$, then Anonymity for a given option and Neutrality imply that either $x, y \in f(B_N^X)$ or $x, y \notin f(B_N^X)$ holds, which contradicts (4). Next, assume that $s(B_N^X, x) < s(B_N^X, y)$. Let $k = s(B_N^X, y) - s(B_N^X, x)$ and K be an electorate of k elements such that $N \cap K = \emptyset$. Let B_K^X be the evaluation profile such that

1. for each $z \neq x$ and each $j \in K$, $b_j^z = 0$;
2. for each $j \in K$, $b_j^x = 1$.

By Lemma 2, f satisfies Full approval and Full disapproval, from which we conclude that $y \notin f(B_K^X)$ and $x \in f(B_K^X)$. On the other hand, $s(B_{N \cup K}^X, x) = s(B_{N \cup K}^X, y)$, and so, by the previous case, either $x, y \in f(B_{N \cup K}^X)$ or $x, y \notin f(B_{N \cup K}^X)$, which contradicts (4). The statement of Lemma 3 follows from these two cases. ■

We now introduce a strong axiom associated with monotonic changes in individual evaluations. It requires that if agents approve more options, then more of them are socially approved. This axiom is first proposed by Murakami (1968, 1968) in a different model. A closely related version is examined by Samet and Schmeidler (2003), Kasher and Rubinstein (1997), and Miller (2008) within a model that is similar to ours. Also, Bossert and Cato (2021) formulate the corresponding axiom in the Arrovian framework.

Strong monotonicity: For each pair of evaluation profiles $B_N^X, B_N'^X \in \mathcal{B}_N^X$, if $B_N^X \leq_{\mathcal{B}_N^X} B_N'^X$, then $f(B_N^X) \subseteq f(B_N'^X)$.

Note that Strong monotonicity implies Monotonicity but not the way around. We prove that Strong monotonicity is implied by the conjunction of Faithfulness, Anonymity for a given option, Neutrality, Biconsistency, Acceptability, and Null option out.

Lemma 4. *If a selection method f on \mathcal{B} satisfies Faithfulness, Anonymity for a given option, Neutrality, Biconsistency, Acceptability, and Null option out, then it satisfies Strong monotonicity.*

Proof. For each $N = \{i_1, \dots, i_n\}$ and each evaluation profile B_N^X , define its normalization as the evaluation profile $\hat{B}_N^X = (\hat{b}_i^x)_{i \in N, x \in X}$ given by:

1. for each $x \in X$ and each $k \in \{1, \dots, s(x, B_N^X)\}$, $\hat{b}_{i_k}^x = 1$;
2. for each $x \in X$ and each $k \in \{s(x, B_N^X) + 1, \dots, n\}$, $\hat{b}_{i_k}^x = 0$.

The following points hold:

- (a) by Anonymity for a given option, $f(B_N^X) = f(\hat{B}_N^X)$;
- (b) if $B_N^X \leq_{\mathcal{B}_N^X} B_N'^X$ then $\hat{B}_N^X \leq_{\mathcal{B}_N^X} \hat{B}_N'^X$;
- (c) y weakly dominates x in \hat{B}_N^X if and only if $s(y, \hat{B}_N^X) \geq s(x, \hat{B}_N^X)$.

By points (a) and (b), we can restrict ourselves to the case where $B_N^X = \hat{B}_N^X$, $B_N'^X = \hat{B}_N'^X$, and $B_N^X \leq_{\mathcal{B}_N^X} B_N'^X$. Now, assume, for the sake of contradiction, that there exists $x \in X$ such that $x \in f(B_N^X)$ and $x \notin f(B_N'^X)$. Two exclusive cases arise.

Case 1: For each $y \in X$, $s(B_N^X, x) \geq s(B_N^X, y)$.

First, by point (c), x weakly dominates any other option of X in B_N^X . By Lemma 3, f satisfies Pareto Consistency, and, by hypothesis, $x \notin f(B_N^X)$. Hence, $f(B_N^X) = \emptyset$. Second, $B_N^X \leq_{\mathcal{B}_N^X} B_N^X$, $f(B_N^X) \neq \emptyset$ since it contains x by hypothesis, and f satisfies Acceptability. Hence, $f(B_N^X) \neq \emptyset$. Therefore, we simultaneously have $f(B_N^X) = \emptyset$ and $f(B_N^X) \neq \emptyset$, a contradiction.

Case 2: There exists $y \in X$ such that $s(B_N^X, x) < s(B_N^X, y)$. Let

$$Y = \{y \in X : s(B_N^X, y) > s(B_N^X, x)\}.$$

Let k be a fixed positive integer (k is assumed to be sufficiently large as explained below). For each $y \in Y$, consider a new option set $X^y = \{x_1^y, \dots, x_k^y\}$ such that, $X \cap X^y = \emptyset$ and, for each pair of distinct elements (y, y') of $Y \times Y$, $X^y \cap X^{y'} = \emptyset$. For any integer $\ell \in \{1, \dots, k\}$, let $Y^\ell = \{x_\ell^y : y \in Y\}$.

Denote by X' the set of options $(X \setminus Y) \cup (\cup_{y \in Y} X^y)$. Note that $(X \setminus Y, Y^1, \dots, Y^k)$ is a partition of X' . For each integer $\ell \in \{1, \dots, k\}$, let $\bar{B}_{\ell, N}^{X'}$ and $\tilde{B}_{\ell, N}^{X'}$ be two evaluation profiles in $\mathcal{B}_N^{X'}$ such that, for each $i \in N$,

1. for each $x \in X \setminus Y$, $\bar{b}_{\ell, i}^x = b_i^x$ and $\tilde{b}_{\ell, i}^x = b_i^x$;
2. for each $x_\ell^y \in Y^\ell$, $\bar{b}_i^{x_\ell^y} = b_i^y$ and $\tilde{b}_i^{x_\ell^y} = b_i^y$,
3. $\bar{b}_i^x = \tilde{b}_i^x = 0$, otherwise.

Informally, the evaluations for the options of Y in the original evaluation profiles with option set X are given to the corresponding options of Y^ℓ in the new profiles $\bar{B}_{\ell, N}^{X'}$ and $\tilde{B}_{\ell, N}^{X'}$.

By Null option out, $f(\bar{B}_{\ell, N}^{X'}) = f(\bar{B}_{\ell, N}^{(X \setminus Y) \cup Y^\ell})$. Note that Y^ℓ and Y have the same size. Thus, by Neutrality, we obtain

$$x \in f(\bar{B}_{\ell, N}^{(X \setminus Y) \cup Y^\ell}) \iff x \in f(B_N^{(X \setminus Y) \cup Y}) = f(B_N^X),$$

and, since, by hypothesis, $x \in f(B_N^X)$, we necessarily have $x \in f(\bar{B}_{\ell, N}^{X'})$. By a similar argument, we obtain $f(\tilde{B}_{\ell, N}^{X'}) = f(B_N^X)$. Because $x \notin f(B_N^X)$ by hypothesis, we necessarily have $x \notin f(\tilde{B}_{\ell, N}^{X'})$.

Next, for each $\ell \in \{1, \dots, k\}$, take a copy $\bar{B}_{N_\ell}^{X'}$ of $\bar{B}_{\ell, N}^{X'}$ in such a way that the sets N_ℓ are pairwise disjoint. Let $N'' = \cup_{1 \leq \ell \leq k} N_\ell$ be a new electorate, and consider the evaluation profile $\bar{B}_{N''}^{X'} \in \mathcal{B}_{N''}^{X'}$. By Anonymity (for a given option), $f(\bar{B}_{N_\ell}^{X'}) = f(\bar{B}_{\ell, N}^{X'})$, which proves that $x \in f(\bar{B}_{N_\ell}^{X'})$. Thus, by Biconsistency, we obtain that $x \in f(\bar{B}_{N''}^{X'})$. Proceeding in a similar way, we easily conclude that $x \notin f(\tilde{B}_{N''}^{X'})$. Note that, by construction, $\bar{B}_{N''}^{X'} \leq_{\mathcal{B}_{N''}^{X'}} \tilde{B}_{N''}^{X'}$. Furthermore, for each $y \in Y$ and each $\ell \in \{1, \dots, k\}$, it holds that $s(\bar{B}_{N''}^{X'}, x_\ell^y) = s(\bar{B}_{\ell, N}^{X'}, y) \leq n$; and, for each $z \in X \setminus Y$, it holds that

$$s(\tilde{B}_{N''}^{X'}, z) = k \cdot s(B_N^X, z) \tag{5}$$

Because $x \in f(B_N^X)$, by Full disapproval, $s(B_N^X, x) > 0$. In addition, from the hypothesis $B_N^X \leq_{\mathcal{B}_N^X} B_N'^X$, we have $s(B_N'^X, z) \geq s(B_N^X, z)$ for each option $z \in X$. Therefore, by (5), $s(\tilde{B}_{N''}^{X'}, x) \geq k$. Thus, for k chosen sufficiently large, that is, $k > n$, we have, for each $z \in X$, $s(\tilde{B}_{N''}^X, z) \leq s(\tilde{B}_{N''}^{X'}, x)$. This amounts to saying that all the hypotheses of **Case 1** apply to evaluation profiles $\tilde{B}_{N''}^{X'}$ and $\tilde{B}_{N''}^X$, that is, $\tilde{B}_{N''}^{X'} \leq_{\mathcal{B}_{N''}^{X'}} \tilde{B}_{N''}^X$, $x \in f(\tilde{B}_{N''}^{X'})$ and $x \notin f(\tilde{B}_{N''}^X)$. By the result of **Case 1**, we have the desired contradiction. ■

In the literature of group identification, it is usually required that the decision on each option should only depend on agents' evaluations of this option (Rubinstein and Fishburn, 1986; Kasher and Rubinstein 1997; Samet and Schmeidler 2003; Ju 2005, 2010, 2013). This corresponds to Independence in our framework; see Section 1 for its definition. As shown in the next lemma, the conjunction of Strong monotonicity and Antitonicity yields Independence. This means that a selection method satisfying Strong monotonicity and Antitonicity is a certification method in our sense.

Lemma 5. *If a selection method f on \mathcal{B} satisfies Strong monotonicity and Antitonicity, then it satisfies Independence.*

Proof. Consider the evaluation profiles $B_N^X, B_N'^X \in \mathcal{B}_N^X$ and an option $x \in X$ such that, for each $i \in N$, $b_i^x = b_i'^x$ and $b_i^y = 0$ for each $y \in X \setminus \{x\}$. Assume that $x \in f(B_N^X)$. On the one hand, by Antitonicity, $x \in f(B_N'^X)$. On the other hand, by Strong monotonicity, $x \in f(B_N''^X)$ for all $B_N''^X \in \mathcal{B}_N^X$ such that $b_i^x = b_i''^x$ for each $i \in N$, the desired result. ■

The following presents a characterization of the class of quota methods.

Proposition 2. *A selection method f on \mathcal{B} satisfies Anonymity, Neutrality, Strong monotonicity, and Antitonicity if and only if it belongs to the class of quota methods.*

Proof. Pick any electorate N , any finite set of options X , any evaluation profile $B_N'^X \in \mathcal{B}_N^X$, and any option $x \in X$. Assume that $x \in f(B_N'^X)$. By Lemma 5, for each evaluation profile $B_N^X \in \mathcal{B}_N^X$ such that

$$\{i \in N : b_i^x = 1\} = \{i \in N : b_i'^x = 1\},$$

it holds that $x \in f(B_N^X)$. By Anonymity, for each evaluation profile $B_N^X \in \mathcal{B}_N^X$ such that

$$|\{i \in N : b_i^x = 1\}| = |\{i \in N : b_i'^x = 1\}|,$$

it holds that $x \in f(B_N^X)$. Then, by Strong monotonicity, for each evaluation profile $B_N^X \in \mathcal{B}_N^X$ such that

$$|\{i \in N : b_i^x = 1\}| \geq |\{i \in N : b_i'^x = 1\}|,$$

it holds that $x \in f(B_N^X)$. This ensures that there exists an integer $q_x(N)$ such that, for each $B_N^X \in \mathcal{B}_N^X$, $x \in f(B_N^X)$ if and only if $|\{i \in N : b_i^x = 1\}| \geq q_x(N)$. By Anonymity, for each pair of finite set of agents N, N' of size n , we get $q_x(N) = q_x(N')$. This allows to define $q_x(n) = q_x(N)$. Finally, because f satisfies Neutrality, the threshold $q_x(n)$ does not depend on x . Therefore, $f = f^q$ for some integer function $q : \mathbb{N} \rightarrow \mathbb{N}$, as desired. ■

Note also that the statement of Proposition 2 allows $q(n)$ to be zero or a number larger than the number of agents. The aforementioned unanimity principles yield appropriate restrictions on the integer function q .

Corollary 1. *Assume that f satisfies all the axioms listed in Proposition 2. Assume further that f satisfies Full disapproval, then $f = f^q$ for some integer function q such that $q(n) \geq 1$ for each integer $n \geq 1$. Assume instead that f satisfies Full approval, then $f = f^q$ for some integer function q such that $q(n) \leq n$ for each integer $n \geq 1$.*

In practice, a relevant q -quota method f^q on \mathcal{B} is such that $1 \leq q(n) \leq n$ so that such f^q satisfies Full disapproval and Full approval.

3.2 Proof of Theorem 1

First, it is straightforward to show that the proportional threshold methods satisfy Nontriviality, Anonymity for a given option, Neutrality, Antitonicity, Biconsistency, Acceptability, and Null option out. Second, let f be a selection method satisfying the axioms of Theorem 1. To show: f is a proportional threshold method.

By Lemma 1, f satisfies Faithfulness and, by Lemma 4, f satisfies Strong monotonicity. Consequently, it follows from Proposition 2 that f is a q -quota method (i.e., there exists $q : \mathbb{N} \rightarrow \mathbb{N}$ such that f is a q -quota method). Moreover, Lemma 2 implies that f satisfies Full approval and Full disapproval. Therefore, by Corollary 1, for each $n \in \mathbb{N}$, we have $1 \leq q(n) \leq n$. Let

$$r = \inf_{n \geq 1} \frac{q(n)}{n}.$$

It is immediate that, for each finite set of agents N such that $|N| = n$ and each $B_N^X \in \mathcal{B}_N^X$,

$$x \in f(B_N^X) \implies \frac{s(B_N^X, x)}{n} \geq r,$$

that is,

$$x \in f(B_N^X) \implies x \in f^r(B_N^X), \tag{6}$$

where f^r stands for a weak r -proportional threshold method. The rest of the proof is completed in two steps.

Step 1: Let n and m be two positive integers. For each electorate M of size m and each evaluation profile $B_M^X \in \mathcal{B}_M^X$, the following implication holds:

$$\frac{s(B_M^X, x)}{m} \geq \frac{q(n)}{n} \implies x \in f(B_M^X). \quad (7)$$

To show this, pick any electorate N of size n and any evaluation profile $B_N^X \in \mathcal{B}_N^X$ such that $s(B_N^X, x) = q(n)$. Because $s(B_N^X, x) = q(n)$ and f is a q -quota method, we have $x \in f(B_N^X)$.

Let $\tilde{B}_{\{1, \dots, nm\}}^X$ be the evaluation profile on the electorate $\{1, \dots, nm\}$ such that, for each $k \in \{0, \dots, m-1\}$, each $\ell \in \{1, \dots, n\}$ and each $x \in X$, $\tilde{b}_{kn+\ell}^x = b_\ell^x$. In other words, $\tilde{B}_{\{1, \dots, nm\}}^X$ is obtained by “concatenating” m agent copies of the evaluation profile B_N^X . Because $x \in f(B_N^X)$, Biconsistency implies that $x \in f(\tilde{B}_{\{1, \dots, nm\}}^X)$. From the left-hand side of (7), observe that $s(\tilde{B}_{\{1, \dots, nm\}}^X, x) = mq(n) \leq ns(B_M^X, x)$. Consider any evaluation profile $\tilde{B}_{\{1, \dots, nm\}}''^X$ such that $s(\tilde{B}_{\{1, \dots, nm\}}''^X, x) = ns(B_M^X, x)$ and $\tilde{B}_{\{1, \dots, nm\}}^X \leq_{\mathcal{B}_{\{1, \dots, nm\}}^X} \tilde{B}_{\{1, \dots, nm\}}''^X$. By Strong monotonicity, we have

$$x \in f(\tilde{B}_{\{1, \dots, nm\}}''^X). \quad (8)$$

Next, pick any other evaluation profile $\bar{B}_{\{1, \dots, nm\}}^X$ on $\{1, \dots, nm\}$ such that, for each $k \in \{0, \dots, n-1\}$ each $\ell \in \{1, \dots, m\}$, and each $x \in X$, $\bar{b}_{kn+\ell}^x = b_\ell^x$. In other words $\bar{B}_{\{1, \dots, nm\}}^X$ is obtained by “concatenating” n agent copies of the evaluation profile B_M^X .

It follows that $\bar{B}_{\{1, \dots, nm\}}^X$ and $\tilde{B}_{\{1, \dots, nm\}}^X$ are two evaluation profiles with the same electorate and such that $s(\bar{B}_{\{1, \dots, nm\}}^X, x) = s(\tilde{B}_{\{1, \dots, nm\}}^X, x) = ns(B_M^X, x)$. By (8), $x \in f(\tilde{B}_{\{1, \dots, nm\}}''^X)$ so that, by Anonymity for a given option and Lemma 5, we get

$$x \in f(\bar{B}_{\{1, \dots, nm\}}^X). \quad (9)$$

Assume, for the sake of contradiction, that $x \notin f(B_M^X)$. Then, Biconsistency implies that

$$x \notin f(\bar{B}_{\{1, \dots, nm\}}^X),$$

which contradicts (9). Therefore, (7) holds, as claimed.

Step 2: Pick any $m \geq 1$, any electorate M of size m and any evaluation profile $B_M^X \in \mathcal{B}_M^X$. Two cases arise, depending on whether r is reached or not.

- There exists a positive integer n_0 such that $q(n_0)/n_0 = r$.

By (6), if $x \in f(B_M^X)$, then $x \in f^r(B_M^X)$, where f^r stands for a weak r -proportional threshold method.

Reciprocally, assume that $x \in f^r(B_M^X)$. By the definition of f^r and hypothesis,

$$\frac{s(B_M^X, x)}{m} \geq r = \frac{q(n_0)}{n_0}.$$

Using **Step 1**, we conclude that $x \in f(B_M^X)$. Therefore, $f = f^r$.

- For each positive integer n , $q(n)/n > r$.

Because f is a q -quota method and $q(n)/n > r$ for each n ,

$$x \in f(B_M^X) \implies \frac{s(B_M^X, x)}{m} \geq \frac{q(m)}{m} > r.$$

Consequently, $x \in f^r(B_M^X)$, where f^r stands for a strict r -proportional threshold method. Reciprocally, assume that $x \in f^r(B_M^X)$, that is, $s(B_M^X, x)/m > r$. By the definition of r as an infimum, there exists n_0 such that

$$\frac{s(B_M^X, x)}{m} > \frac{q(n_0)}{n_0} \geq r.$$

By **Step 1**, we obtain that $x \in f(B_M^X)$. This allows to conclude that $f = f^r$.

This completes the proof of Theorem 1. ■

3.3 Proof of Theorem 2

Because the necessity of the axioms is easy to check, it suffices to establish the sufficiency part of the theorem. By Lemma 1, f satisfies Faithfulness. Let B_N^X be an evaluation profile, and set

$$K = \max_{x \in X} s(B_N^X, x).$$

Observe that $K \leq |N|$. For each $k \in \{1, \dots, K\}$, consider the subset of options $X_k = \{x \in X : s(B_N^X, x) \geq k\}$. By the definition of the approval voting method, we obtain $f^A(B_N^X) = X_K$. Next, we consider a new electorate $N' = \{i_1, i_2, \dots, i_{|N|}\}$ of the same size as N . Define the evaluation profile $B_{N'}^X \in \mathcal{B}_{N'}^X$ as follows:

1. for each $k \in \{1, \dots, K\}$, $b'_{i_k}{}^x = 1$ if $x \in X_k$, and $b'_{i_k}{}^x = 0$ otherwise;
2. for each $k \in \{K + 1, \dots, |N|\}$, $b'_{i_k}{}^x = 0$ for each $x \in X$.

Note that if $x \in X$ receives k approvals in B_N^X , then it also receives exactly k approvals in $B_{N'}^X$ from agents i_1, \dots, i_k . Thus, from Anonymity for a given option, it follows that

$$f(B_N^X) = f(B_{N'}^X), \quad (10)$$

and, by Null agent out, we obtain

$$f(B_{N'}^X) = f(B_{\{i_1, \dots, i_K\}}^X). \quad (11)$$

Faithfulness implies that

$$\forall k \in \{1, \dots, K\}, \quad f(B_{\{i_k\}}^X) = X_k, \quad (12)$$

and, because $X_1 \supseteq X_2 \supseteq \dots \supseteq X_K$, Consistency and (12) lead to

$$f(B_{\{i_1, \dots, i_K\}}^X) = X_K. \quad (13)$$

Combining (10), (11), and (13) with the fact that $X_K = f^A(B_N^X)$, we can conclude that $f(B_N^X) = f^A(B_N^X)$. This is the desired result. ■

4 Concluding remarks

This paper axiomatically examined the differences and similarities between the proportional threshold methods and the approval voting method. While the two types of decision-making methods appear to be very different, they share many important normative properties. Our examination shows that their fundamental difference lies in how these methods respond to changes in electorates. Under the proportional threshold methods, an option cannot be selected if the electorate can be divided into two subgroups such that neither would choose this option. However, this is not correct under the approval voting method. To be precise, our results indicate that both methods are consistent in terms of changes in electorates, but in a different manner. We identified Bconsistency as the fundamental consistency axiom for the proportional threshold methods and examined its contrast with the classical consistency axiom, which is satisfied by the approval voting method. We believe that the two consistency axioms employed in this paper are helpful to understand an important distinction between rules offering acceptable outcomes and those offering the best outcomes.

Appendix

A.1 Logical independence of the axioms in Theorem 1

This part aims to establish the logical independence of the listed in Theorem 1.

- The empty selection method that always selects the empty set satisfies all of our axioms used in the theorem except Nontriviality.
- The approval voting method satisfies all our axioms except Biconsistency.
- Assign a positive weight $w(i) > 0$ to each potential agent $i \in \mathbb{N}$. Define f^w by letting:

$$x \in f^w(B_N^X) \iff \sum_{i \in N} b_i^x w(i) \geq \frac{\sum_{i \in N} w(i)}{2}.$$

This selection method f^w satisfies all the axioms except Anonymity for a given option. Note that Anonymity, which is weaker than Anonymity for a given option, is also violated.

- Define f by letting:

$$x \in f(B_N^X) \iff \frac{s(B_N^X, x)}{n} \geq \frac{1}{|X|}.$$

This satisfies all of our axioms except Null option out.

- Let f be the selection method such that $x \in f(B_N^X)$ if $s(B_N^X, x) > \sum_{y \in X \setminus \{x\}} s(B, y)$. This satisfies each axiom except Acceptability.
- Take any $x \in \mathcal{X}$ and define f^x as the selection method such that, for each N and X , the following holds: for each $z \in X \setminus \{x\}$,

$$z \in f^x(B_N^X) \text{ if } \frac{s(B_N^X, z)}{|N|} > \frac{1}{3},$$

and

$$x \in f^x(B_N^X) \text{ if } \frac{s(B_N^X, x)}{|N|} > \frac{1}{2}.$$

The selection method $f^x(B_N^X)$ satisfies all axioms but Neutrality.

A.2 Logical independence of the axioms in Theorem 2

We now establish that the axioms of Theorem 2 are logically independent.

- The empty selection method, which always selects the empty set, satisfies all of our axioms in Theorem 2 except Nontriviality.
- Assign a positive weight $w(i) > 0$ to each potential agent $i \in \mathbb{N}$. Define the weighted approval voting method $f^{A,w}$ by letting:

$$x \in f^{A,w}(B_N^X) \iff x \in \arg \max_{x \in X} \sum_{i \in N} b_i^x w(i).$$

This selection method $f^{A,w}$ satisfies all axioms except Anonymity for a given option. Anonymity is also violated.

- Define f by letting:

$$x \in f(B_N^X) \iff \left[x \in \arg \max_{x \in X} s(B_N^X, x) \text{ and } \frac{s(B_N^X, x)}{|N|} \geq \frac{1}{2} \right].$$

This selection method satisfies all the axioms except Null agent out.

- Define f by letting:

$$x \in f(B_N^X) \iff x \in \arg \max_{x \in X} \sum_{i \in N} (1 - b_i^x).$$

This selection method satisfies all the axioms except Null option out.

- The strict proportional threshold method f^0 satisfies all the axioms except Consistency.
- Endow \mathcal{X} with a strict total order $>$. Consider the selection method f that selects among $f^A(B_N^X)$ the (unique) option which is minimal with respect to $>$. It satisfies all the axioms except Neutrality.
- Let f be the selection method such that $x \in f(B_N^X)$ if $s(B_N^X, x) > \sum_{y \in X \setminus \{x\}} s(B, y)$. It satisfies each axiom except Acceptability.

Acknowledgments

Financial support from JSPS KAKENHI (JP20H01446, JP 22K01387), the Mitsubishi Foundation (No. ID201920011), research programs “Mathématiques de la décision pour l’ingénierie physique et sociale” (MODMAD) and “Design de Budgets Participatifs” are gratefully acknowledged. We are indebted to Federica Ceron, Antonin Macé and Adriana Navarro Ramos for their benevolent remarks. We would also like to thank the audience of the “interaction seminar” at the Aix-Marseille School of Economics (AMSE) and the participants of the SING 17 conference for their valuable comments. All remaining errors are our own.

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