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On the convex hull of k -additive 0-1 capacities and its application to model identification in decision making

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Abstract

The Choquet integral w.r.t. a capacity is a versatile tool commonly used in decision making. Its practical identification requires, however, to solve an optimization problem with exponentially many variables and constraints. The introduction of k -additive capacities, through the use of the Möbius transform, permits to reduce the number of variables to a polynomial size, but leaves the number of constraints exponential. When $k = 2$, the use of vertices of the set of 2-additive capacities permits to solve the problem as the number of vertices is polynomial. When $k > 2$, this solution is no more applicable as the set of vertices of k -additive capacities is not known. We propose in this paper to use instead the set of vertices which are 0-1 valued. We show that the number of such vertices is polynomial, and we observe that the loss of generality is very small for $n = 4$, $k = 3$, and conjecture that this still holds for larger values of n . Also, we study the geometric properties of the convex hull of 0-1 valued k -additive capacities.

Keywords: capacity, k -additive capacity, Choquet integral, vertices, facets

1 Introduction

Capacities, introduced by Choquet [4] and later by Sugeno [13] under the name of fuzzy measures, have been widely applied in the field of decision making, both in decision under risk and uncertainty where capacities play the role of a generalized probability measure (see, e.g., [6, 14]), and in multicriteria decision making, where capacities permit to model weights of importance on groups of criteria (see, e.g., [9]).

While in models based on probability measures, the Lebesgue integral is the adequate tool to compute expectations of random variables, the Choquet integral is the main tool for computing expectations and averages in case of capacity-based models, and it appears to be a very versatile aggregation model able to capture complex decision strategies. However, this richness induces difficulties for the identification of the model in practice, because the Choquet integral contains an exponential number of terms in the number of inputs. Moreover, the identification of the model is done via an optimization problem, where the number of constraints is also exponential. The introduction of k -additive capacities [7] has permitted to make the number of

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variables polynomial in the number of inputs, but the number of constraints is still exponential. By expressing a 2-additive capacity as a convex combination of vertices, whose number is polynomial, some researchers have circumvented the problem and made the number of constraints polynomial [1, 10]. One of the results of this paper is that we can extend this mechanism to 3-additive capacities, by using 0-1 valued 3-additive capacities. In order to achieve this, we have studied the geometrical structure of the convex hull of 0-1 valued k -additive capacities in general.

The paper is organized as follows. Section 2 explains in details the identification problem and the current state of knowledge. Section 3 introduces the necessary notions for the geometrical study of the convex hull of 0-1 valued capacities, while Section 4 is devoted to the study of vertices and facets of this set. The particular case of $n = 4$ and $k = 3$ is given in detail, with a complete list of vertices and facets in Appendices A and B.

2 The identification problem

2.1 Basic definitions

Let $N = [n]$, $n \geq 3$. We consider (*normalized*) *capacities* on N , i.e., mappings $v : 2^N \rightarrow [0, 1]$ satisfying $v(\emptyset) = 0$, $v(N) = 1$ and monotonicity w.r.t inclusion: $v(S) \leq v(T)$ whenever $S \subseteq T$. We denote by $\mathcal{M}(n)$ the set of normalized capacities on N , which is a closed convex polytope of dimension $2^n - 2$.

We sometimes use the notation \overline{ijk} for the set $\{i, j, k\}$, writing, e.g., $\overline{3}$, $\overline{12}$, $\overline{235}$, etc.

Given a capacity $v \in \mathcal{M}(n)$, its *Möbius transform* is the set function $m^v : 2^N \rightarrow \mathbb{R}$ defined by

$$m^v(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T) \quad (S \in 2^N).$$

Equivalently

$$v = \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} m^v(S) u_S, \quad (1)$$

where u_S is a capacity (usually called *unanimity game*) on N defined by $u_S(T) = 1$ if $T \supseteq S$ and 0 otherwise. Note that by definition $m^{u_S}(T) = 1$ if $S = T$ and 0 otherwise.

For an integer k such that $1 \leq k \leq n$, a capacity v is (*at most*) k -*additive* if $m^v(S) = 0$ whenever $|S| > k$. We denote by $\mathcal{M}_k(n)$ the set of (at most) k -additive capacities. Clearly, $\mathcal{M}_n(n) = \mathcal{M}(n)$. $\mathcal{M}_k(n)$ is a closed convex polytope of dimension

$$d(n, k) := \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} - 1. \quad (2)$$

2.2 Identification of the Choquet integral

Let v be a capacity on N , and m^v its Möbius transform. The Choquet integral is defined as follows, in terms of the Möbius transform:

$$C_{m^v}(x) = \sum_{S \subseteq N} m^v(S) \min_{i \in S} x_i, \quad (3)$$

where $x \in \mathbb{R}^n$ is the integrand.

The normalization and monotonicity conditions on v expressed in terms of m^v are (see, e.g., [8, Th. 2.33 (ii)]):

$$m^v(\emptyset) = 0 \quad (4)$$

$$\sum_{T \subseteq N} m^v(T) = 1 \quad (5)$$

$$\sum_{T \subseteq S} m^v(T \cup \{i\}) \geq 0, \forall i \in N \forall S \subseteq N \setminus \{i\}. \quad (6)$$

The main interest of k -additive capacities is that they need only $O(k^n)$ coefficients to be defined (see (2)). However, the number of monotonicity constraints is still exponential – more precisely $n 2^{n-1}$ (see (6)). An exponential number of constraints is a limitation for the practical identification of the capacity. The elicitation techniques are indeed based on an optimization problem in which the unknowns are the terms of the Möbius coefficients and the constraints are the normalization and monotonicity conditions. Having an exponential number of monotonicity conditions is a bottleneck for allowing the elicitation of decision models, even for n of relatively small size.

n	5	10	15	20	25
Number of monotonicity conditions	80	5 120	245 760	10 485 760	419 430 400

Therefore, the aim is to have a family of monotone models having a polynomial number of unknowns and monotonicity conditions, so that it might be feasible to learn such a model with a large number of inputs.

Let us consider the case of 2-additive capacities. There exists an alternative representation of 2-additive capacities, which does not use the Möbius transform. As it will be explained in Section 3.3, the *extreme points (vertices)* of \mathcal{M}_2 are the capacities $u_{\{i\}}$ (for $i \in N$), $u_{\{i,j\}}$ (for $\{i, j\} \subseteq N$) and $\overline{u_{\{i,j\}}} := u_{\{i\}} + u_{\{j\}} - u_{\{i,j\}}$ (for $\{i, j\} \subseteq N$). Hence any 2-additive capacity takes the form

$$v = \sum_{i \in N} w_i u_{\{i\}} + \sum_{\{i,j\} \subseteq N} w_{i,j} u_{\{i,j\}} + \sum_{\{i,j\} \subseteq N} \overline{w_{i,j}} \overline{u_{\{i,j\}}} \quad (7)$$

with the constraints

$$\forall i \in N \quad w_i \geq 0 \quad \forall \{i, j\} \subseteq N \quad w_{i,j}, \overline{w_{i,j}} \geq 0 \quad (8)$$

$$\sum_{i \in N} w_i + \sum_{\{i,j\} \subseteq N} w_{i,j} + \sum_{\{i,j\} \subseteq N} \overline{w_{i,j}} = 1. \quad (9)$$

The Choquet integral then takes the following form

$$C_v(x) = \sum_{i \in N} w_i x_i + \sum_{\{i,j\} \subseteq N} w_{i,j} \min(x_i, x_j) + \sum_{\{i,j\} \subseteq N} \overline{w_{i,j}} \max(x_i, x_j). \quad (10)$$

We can summarize the comparison between the Möbius representation and the vertice representation in the following table.

	Möbius representation	vertices representation
Number of unknowns	$\frac{n(n+1)}{2}$	n^2
Number of monotonicity conditions	$n 2^{n-1}$	n^2
Number of normalization conditions	2	1

Some researchers learning a Choquet integral w.r.t. a 2-additive capacity have thus adopted the vertice representation rather than the Möbius one [1, 10].

However, the same technique cannot be applied anymore for $k > 2$. This is because for $k > 2$, the set of vertices of $\mathcal{M}_k(n)$ is not known, and contains k -additive capacities which are not 0-1 valued. Our idea is to restrict precisely to the set of 0-1 valued k -additive capacities, denoted by $\mathcal{M}_k^{0-1}(n)$.

3 Geometrical properties of $\mathcal{M}(n)$ and $\mathcal{M}_k(n)$

3.1 Volume of $\mathcal{M}(n)$

$\mathcal{M}(n)$ is a convex closed polyhedron, whose properties are well known. It is an order polytope as defined by Stanley [12]. Given a poset (P, \preceq) , its corresponding *order polytope* is defined by

$$\mathcal{O}(P) := \{f : P \rightarrow [0, 1] : f(x) \leq f(y) \text{ if } x \preceq y\}.$$

The volume of $\mathcal{O}(P)$ is given by

$$V(\mathcal{O}(P)) = \frac{e(P)}{|P|!} \quad (11)$$

where $e(P)$ is the number of linear extensions of P . A *linear extension* of P is a total order \leq on P which is compatible with \preceq in the following sense: $x \preceq y$ implies $x \leq y$. For $\mathcal{M}(n)$, the underlying poset is $(2^N \setminus \{\emptyset, N\}, \subseteq)$, as $v(\emptyset)$ and $v(N)$ have fixed values 0 and 1. No closed-form formula is known for $e(2^N)$ (equal to $e(2^N \setminus \{\emptyset, N\})$), but its value grows extremely fast and is known till $n = 7$ (this is sequence A046873 in the *Online Encyclopedia of Integer Sequences*; see Table below for $n \leq 6$)(see also [2] for bounds on $e(2^N)$). Using Table 1 and

n	$e(2^N)$
1	1
2	2
3	48
4	1680384
5	14807804035657359360
6	141377911697227887117195970316200795630205476957716480

Table 1: Number of linear extensions of 2^N

(11), we can compute the volume of $\mathcal{M}(n) = \frac{e(2^N)}{(2^n - 2)!}$ for $n \leq 6$ (see Table 2), which can be seen to decrease very fast with n .

n	$V(\mathcal{M}(n))$
1	1
2	1
3	0.0666667
4	0.0000192753
5	0.0000000000000558252
6	0.0000000000000000000000000000449247

Table 2: Volume of $\mathcal{M}(n)$

3.2 Vertices of $\mathcal{M}(n)$

The vertices of an order polytope $\mathcal{O}(P)$ are the characteristic functions of the filters of (P, \preceq) , where a *filter* I is a subset of P such that $x \in I$ and $x \preceq y$ with $y \in P$ imply $y \in I$. Therefore, the vertices of $\mathcal{O}(P)$ are in bijection with the antichains of P .

For normalized capacities, this amounts to saying that vertices of $\mathcal{M}(n)$ are all the 0-1 capacities, i.e., capacities taking only the values 0 and 1 (in game theory, these are called *simple games*, up to the difference that the 0 function is also a simple game). We denote by $\mathcal{M}^{0-1}(n)$ the set of 0-1 capacities. Clearly, 0-1 capacities are in bijection with the antichains of $(2^N \setminus \{\emptyset, N\}, \subseteq)$, as explained above for order polytopes. Specifically, given a 0-1 capacity v , its corresponding antichain W is the set of minimal elements (w.r.t. inclusion) of the collection $\{S \in 2^N \setminus \{\emptyset, N\} : v(S) = 1\}$.

Now, the number of antichains of $(2^N \setminus \{\emptyset, N\}, \subseteq)$ is the number of antichains of $(2^N, \subseteq)$ minus 2, the latter being given by the Dedekind numbers $M(n)$ (sequence A000372 in OEIS), known till $n = 8$. Table 3 gives the number of vertices of $\mathcal{M}(n)$.

n	$M(n) - 2$
1	1
2	4
3	18
4	166
5	7579
6	7828352
7	2414682040996
8	56130437228687557907786

Table 3: The number of vertices of $\mathcal{M}(n)$

3.3 Vertices of $\mathcal{M}_k(n)$

For $k = 1, 2$, it is known that the vertices of $\mathcal{M}_k(n)$ are the 0-1 k -additive capacities, i.e., the set $\mathcal{M}_k^{0-1}(n)$ [11]. This is no more true for $k \geq 3$: every 0-1 k -additive capacity is still a vertex of $\mathcal{M}_k(n)$, but (many) other vertices exist, whose expression is still unknown [5] (see below).

For further use, we explicit the set of vertices of $\mathcal{M}_2(n)$, i.e., $\mathcal{M}_2^{0-1}(n)$ (see, e.g., [8, Th. 2.65]). There are three types of vertices:

- (i) The unanimity games $u_{\{i\}}$, corresponding to the antichain $\{\{i\}\}$, for $i \in N$;
- (ii) The unanimity games $u_{\{i,j\}}$, corresponding to the antichain $\{\{i,j\}\}$, for $\{i,j\} \subseteq N$;
- (iii) The conjugate unanimity game $\overline{u_{\{i,j\}}} = u_{\{i\}} + u_{\{j\}} - u_{\{i,j\}}$, corresponding to the antichain $\{\{i\}, \{j\}\}$, for $\{i,j\} \subseteq N$.

4 The set $\text{conv}(\mathcal{M}_k^{0-1}(n))$

Let us consider for the rest of the paper a fixed integer $k > 2$. As the set of 0-1 vertices of $\mathcal{M}_k(n)$ is strictly included into the set of vertices of $\mathcal{M}_k(n)$, it follows that $\text{conv}(\mathcal{M}_k^{0-1}(n)) \subsetneq \mathcal{M}_k(n)$, where $\text{conv}(\cdot)$ indicates the convex hull of a set.

k -additive capacities are known to be useful in practice, especially in decision making. The question is whether it could be advantageous to use the smaller set $\text{conv}(\mathcal{M}_k^{0-1}(n))$ instead.

The sets $\mathcal{M}_k(n)$ and $\text{conv}(\mathcal{M}_k^{0-1}(n))$ have contrasted properties. As explained above, the vertices of $\mathcal{M}_k(n)$ are not known, but its facets are known. Indeed, the monotonicity constraints in terms of the Möbius transform are, as we have seen in Section 2.2:

$$\sum_{\substack{T \in \{\{i\}, S\} \\ |T| \leq k}} m^v(T) \geq 0 \quad (S \subseteq [n], i \in S), \quad (12)$$

with the notation $[A, B] := \{C \subseteq N : A \subseteq C \subseteq B\}$. Each of these inequalities determines a facet, as they come from the inequalities $v(A \cup i) - v(A) \geq 0$, which are facets. Note that the number of facets is $\sum_{i=1}^n i \binom{n}{i}$.

On the other hand, the vertices of $\text{conv}(\mathcal{M}_k^{0-1}(n))$ are known (these are precisely all 0-1 k -additive capacities), but the facets seem to be extremely difficult to obtain, and they are quite numerous.

4.1 Vertices of $\text{conv}(\mathcal{M}_k^{0-1}(n))$

Let us elaborate on the vertices of $\text{conv}(\mathcal{M}_k^{0-1}(n))$. Even if they are known to be the 0-1 k -additive capacities, it is nevertheless difficult to get their analytical expression. From [8, Th. 2.35], we know that the Möbius transform of a 0-1 capacity v generated by an antichain $\mathcal{A} = \{A_1, \dots, A_\ell\}$ is given by

$$m^v(S) = \sum_{\substack{I \subseteq \{1, \dots, \ell\} \\ I \neq \emptyset \\ \bigcup_{i \in I} A_i = S}} (-1)^{|I|+1} \quad (S \subseteq N, S \neq \emptyset) \quad (13)$$

with the usual convention $\sum_{\emptyset} = 0$. It remains to identify which antichains yield $m^v(S) = 0$ for all S of cardinality greater than k . A first obvious fact is given below.

Lemma 1. Consider an antichain \mathcal{A} in $(2^N \setminus \{\emptyset, N\}, \subseteq)$ such that $|\bigcup \mathcal{A}| \leq k$. Then the capacity generated by \mathcal{A} belongs to $\mathcal{M}_k^{0-1}(n)$.

Proof. Take \mathcal{A} such that $|\bigcup \mathcal{A}| \leq k$ and the corresponding capacity v . Then by (13), $m^v(S) = 0$ as soon as $|S| > k$. Therefore, v is k -additive, and is 0-1-valued because it is generated by an antichain. □

Trivial examples of such antichains are $\mathcal{A} = \{S\}$, with $1 \leq |S| \leq k$, which yield the vertices u_S , $1 \leq |S| \leq k$. Another simple example is the vertices generated by a collection of singletons, say $\{\{i_1\}, \dots, \{i_s\}\}$, with $s \leq k$. Letting $S = \{i_1, \dots, i_s\}$, according to (13), this yields the vertex

$$\begin{aligned} \bar{u}_S &:= u_{\{i_1\}} + \dots + u_{\{i_s\}} - u_{\{i_1, i_2\}} - \dots + u_{\{i_1, i_2, i_3\}} + \dots + (-1)^{s+1} u_S \\ &= \sum_{T \subseteq S, T \neq \emptyset} (-1)^{|T|+1} u_T. \end{aligned} \quad (14)$$

More generally, we give below all possible forms of antichains of support at most 3, with distinct $i, j, \ell \in N$:

- (i) Support with size 1: $\mathcal{A} = \{\{i\}\}$;
- (ii) Support with size 2: $\mathcal{A} = \{\{i\}, \{j\}\}$, $\mathcal{A} = \{\{i, j\}\}$;

- (iii) Support with size 3: $\mathcal{A} = \{\{i\}, \{j\}, \{\ell\}\}$, $\mathcal{A} = \{\{i\}, \{j, \ell\}\}$, $\mathcal{A} = \{\{i, j\}, \{i, \ell\}\}$, $\mathcal{A} = \{\{i, j\}, \{i, \ell\}, \{j, \ell\}\}$, $\mathcal{A} = \{\{i, j, \ell\}\}$.

However, there are capacities in $\mathcal{M}_k^{0-1}(n)$ generated by antichains with a support larger than k . Consider for example $n = 4$, $k = 3$ and the antichain $\mathcal{A} = \{\overline{14}, \overline{23}, \overline{24}\}$. Then $m^v(\overline{1234}) = -1 + 1 = 0$. It seems very difficult in the general case to find all antichains of support larger than k yielding a 0-1 k -additive capacity. However, they can be obtained when $k = 3$.

Let us call *support* of an antichain \mathcal{A} the set $\bigcup \mathcal{A}$.

Theorem 1. Let \mathcal{A} be an antichain on $(2^N \setminus \{\emptyset, N\}, \subseteq)$, with $|\bigcup \mathcal{A}| > 3$, and denote by $v_{\mathcal{A}}$ the corresponding 0-1 capacity. Then $v_{\mathcal{A}} \in \mathcal{M}_3^{0-1}(n)$ if and only if

- (i) No element of \mathcal{A} has a cardinality larger than 3 and smaller than 2.
- (ii) If all elements of \mathcal{A} have cardinality 2, then \mathcal{A} has the form

$$\mathcal{A} = \{\overline{12}, \overline{34}, \overline{13}\},$$

i.e., a partition of the support in two blocks and a set formed by an element of each block (support of size 4).

- (iii) If \mathcal{A} has an element of cardinality 3, say, $\overline{123}$, then \mathcal{A} has the form

$$\mathcal{A} = \{\overline{123}, \overline{14}, \overline{24}, \overline{34}\}$$

(support of size 4).

- (iv) The size of the support of \mathcal{A} is at most 4.

Proof. We prove the necessity part.

(i) Suppose $A \in \mathcal{A}$ with $|A| > 3$. As \mathcal{A} is an antichain, by (13), $m^{v_{\mathcal{A}}}(A) = 1$, hence $v_{\mathcal{A}}$ is not 3-additive, a contradiction.

Now, suppose \mathcal{A} contains a singleton, say, $\{1\}$. As \mathcal{A} is an antichain, no other set in \mathcal{A} contains 1. Let us put $\mathcal{A}' = \mathcal{A} \setminus \{\{1\}\}$. We have by (13) for any $S \subseteq \bigcup \mathcal{A}'$

$$m^{v_{\mathcal{A}}}(S \cup \{1\}) = -m^{v_{\mathcal{A}'}}(S), \quad m^{v_{\mathcal{A}}}(S) = m^{v_{\mathcal{A}'}}(S).$$

Then $m^{v_{\mathcal{A}}}$ is at most 3-additive if $m^{v_{\mathcal{A}'}}$ is at most 2-additive. Indeed, suppose there exists S with $|S| > 3$ such that $m^{v_{\mathcal{A}}}(S) \neq 0$. If $S \not\ni 1$, $0 \neq m^{v_{\mathcal{A}}}(S) = m^{v_{\mathcal{A}'}}(S)$, which contradicts 2-additivity of $m^{v_{\mathcal{A}'}}$. If $S \ni 1$, $m^{v_{\mathcal{A}'}}(S \setminus \{1\}) = -m^{v_{\mathcal{A}}}(S) \neq 0$, which also contradicts 2-additivity.

According to §3.3, there are three types of 2-additive 0-1 capacities, whose antichains have the form $\mathcal{A}' = \{\{i\}\}$, $\mathcal{A}' = \{\{i, j\}\}$ and $\mathcal{A}' = \{\{i\}, \{j\}\}$, respectively. Clearly, in any case, the support of \mathcal{A} cannot have a cardinality greater than 3. Therefore, no singleton can exist in \mathcal{A} .

(ii) Assume that no element of size 3 exists in \mathcal{A} and w.l.o.g. that $\overline{12} \in \mathcal{A}$. a) Suppose that the size of the support of \mathcal{A} is 4, say, $\bigcup \mathcal{A} = \overline{1234}$. Then, it must exist elements in \mathcal{A} containing 3, 4. First, suppose $\overline{34} \in \mathcal{A}$. In order to avoid $m^{v_{\mathcal{A}}}(\overline{1234}) \neq 0$, \mathcal{A} must contain exactly one other set $A \subset \overline{1234}$ forming an antichain with $\overline{12}$ and $\overline{34}$, so that $m^{v_{\mathcal{A}}}(\overline{1234}) = -1 + 1 = 0$. Second, suppose that $\overline{34} \notin \mathcal{A}$. Then \mathcal{A} contains at least two sets of the form $\overline{i3}, \overline{j4}$, with $i, j \in \{1, 2\}$. In order to avoid $m^{v_{\mathcal{A}}}(\overline{1234}) = 1 \neq 0$ (unique decomposition of $\overline{1234}$), \mathcal{A} must contain at least another set A . If $A \ni 3$, then $\overline{13}, \overline{23} \in \mathcal{A}$, therefore $\overline{1234}$ can be decomposed into $\overline{12} \cup \overline{13} \cup \overline{j4}$, $\overline{12} \cup \overline{23} \cup \overline{j4}$ and $\overline{12} \cup \overline{13} \cup \overline{23} \cup \overline{j4}$, which yields $m^{v_{\mathcal{A}}}(\overline{1234}) = 1 \neq 0$ (similarly if $A \ni 4$). Consequently, the only possibility left is $\mathcal{A} = \{\overline{12}, \overline{13}, \overline{23}, \overline{14}, \overline{24}\}$, which yields $m^{v_{\mathcal{A}}}(\overline{1234}) = -1 - 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 - 1 - 1 - 1 - 1 - 1 = 1 \neq 0$.

b) Suppose now that the support is $\overline{12345}$. A first remark is that by combining elements of \mathcal{A} , at least 2 sets of size 4 can be formed. Indeed, as at least two elements of \mathcal{A} must be disjoint sets, a set of size 4 can be formed. Now, if only one can be formed, say, $\overline{1234}$, it implies that \mathcal{A} contains only subsets of $\overline{1234}$ and the singleton $\overline{5}$, but this is contrary to the assumption (ii). Suppose then w.l.o.g. that at least $\overline{1234}$ and $\overline{1235}$ can be formed by unions of elements of \mathcal{A} . Then, by (ii)(a), we must have w.l.o.g. $\overline{12}$, $\overline{34}$ and $\overline{ij} \in \mathcal{A}$ with $i \in \{1, 2\}$ and $j \in \{3, 4\}$. As for $\overline{1235}$, the set $\overline{12}$ can play the role of one block of the partition (in which case we have in addition $\overline{35}, \overline{st} \in \mathcal{A}$ with $s \in \{1, 2\}$ and $t \in \{3, 5\}$) or the partition is w.l.o.g. $\overline{13}, \overline{25}$. In the latter case, as $\overline{13}$ is a candidate for \overline{ij} , the only possibility is then $\mathcal{A} = \{\overline{12}, \overline{13}, \overline{25}, \overline{34}\}$, which yields $m^{v\mathcal{A}}(\overline{12345}) = 1 + 1 - 1 \neq 0$, contradicting 3-additivity. We examine then the former case, i.e., $\overline{12}, \overline{34}, \overline{ij}, \overline{35}, \overline{st} \in \mathcal{A}$ with $i, s \in \{1, 2\}$, $j \in \{3, 4\}$, and $t \in \{3, 5\}$, with no other subset of $\overline{1234}, \overline{1235}$, which means that only $\overline{45}$ is possibly in addition in \mathcal{A} . Note in addition that $\overline{ij} = \overline{st}$ is forced if one of them is subset of both $\overline{1234}$ and $\overline{1235}$, i.e., $\overline{13}$ or $\overline{23}$. Case 1: suppose $\overline{45} \in \mathcal{A}$, $\overline{ij} = \overline{st} = \overline{13}$. This is ruled out by (ii)(a) considering the set $\overline{1345}$ (4 subsets of it). The case where $\overline{ij} = \overline{st} = \overline{23}$ is the same, considering $\overline{2345}$. Case 2: $\overline{45} \in \mathcal{A}$, $\overline{ij} \neq \overline{st}$. If $\overline{ij} = \overline{14}$, then this case is ruled out by considering $\overline{1345}$ (4 subsets). Otherwise, $\overline{ij} = \overline{24}$, which is ruled out by $\overline{2345}$. Case 3: $\overline{45} \notin \mathcal{A}$, $\overline{ij} = \overline{st} = \overline{13}$. This case is ruled out by considering $\overline{1345}$ (no partition). The case where $\overline{ij} = \overline{st} = \overline{23}$ is the same, considering $\overline{2345}$. Case 4: $\overline{45} \notin \mathcal{A}$, $\overline{ij} \neq \overline{st}$. Take $\overline{ij} = \overline{14}$. Considering the subset $\overline{1345}$, this forces to take $\overline{st} = \overline{25}$ as $\overline{st} = \overline{15}$ is not possible (4 subsets). Then $m^{v\mathcal{A}}(\overline{12345}) = 1 + 1 + 1 + 1 + 1 - 1 - 1 - 1 - 1 - 1 + 1 \neq 0$, which contradicts 3-additivity. The case $\overline{ij} = \overline{24}$ yields $\overline{st} = \overline{15}$ is much the same.

As a conclusion, there is no possibility for a support of size 5.

c) Suppose that the support is $\overline{123456}$. If elements of \mathcal{A} can form a subset of support 5, the above argument applies. Apart this, the only possibility would be to consider that $\overline{12}, \overline{34}, \overline{56}$ are in \mathcal{A} . But considering $\overline{1234}$, the above result shows that a set \overline{ij} with $i \in \{1, 2\}$, $j \in \{3, 4\}$ must belong to \mathcal{A} . Then $\overline{34} \cup \overline{56} \cup \overline{ij}$ has support of size 5, and no configuration is possible. Therefore, there is no possibility for a support of size 6, and using a similar argument, for any support of greater size.

(iii) Assume $\overline{123} \in \mathcal{A}$. a) We suppose first that the support of \mathcal{A} has size 4, say, $\bigcup \mathcal{A} = \overline{1234}$. Then \mathcal{A} must contain another set containing 4 as element, of the form $A' = A \cup \{4\}$ with $\emptyset \neq A \subset \overline{123}$ (strict inclusion). To make $m^{v\mathcal{A}}(\overline{1234}) = 0$, \mathcal{A} must contain another set $B' = B \cup \{4\}$ so that 1) B is nonempty and strictly included in $\overline{123}$ (because support is $\overline{1234}$) 2) B and A form an antichain in $\overline{123}$. Then $\overline{1234}$ can be written as $\overline{123} \cup A'$, $\overline{123} \cup B'$ and $\overline{123} \cup A' \cup B'$, and possibly also $A' \cup B'$. Observe that in the first case $m^{v\mathcal{A}}(\overline{1234}) = -1 - 1 + 1 \neq 0$, while in the latter case $m^{v\mathcal{A}}(\overline{1234}) = -1 - 1 + 1 - 1 \neq 0$. Therefore, \mathcal{A} must contain another set $C' \subseteq \overline{1234}$ containing 4 (otherwise $C' \subseteq \overline{123}$ which is not possible). Hence $C' = C \cup \{4\}$ with $\emptyset \neq C \subset \overline{123}$ and A, B, C must form an antichain of $\overline{123}$. It can be checked that in order to get $m^{v\mathcal{A}}(\overline{1234}) = 0$, the only possibility is to take $A = \{1\}, B = \{2\}, C = \{3\}$, i.e.,

$$\mathcal{A} = \{\overline{123}, \overline{14}, \overline{24}, \overline{34}\}.$$

b) Suppose now the support is $\overline{12345}$. We consider two cases: a) none of $\overline{1234}$ nor $\overline{1235}$ can be formed by union of the sets in \mathcal{A} b) only one among $\overline{1234}$ and $\overline{1235}$ can be formed c) $\overline{1234}$ and $\overline{1235}$ can be both formed. In case a) it means that $\overline{45}$ always appear together in the sets in \mathcal{A} , hence considering $\overline{45}$ as a single element, the above reasoning (a) applies and the only possible configuration is $\mathcal{A} = \{\overline{123}, \overline{145}, \overline{245}, \overline{345}\}$. However, we obtain then $m^{v\mathcal{A}}(\overline{2345}) = -1 \neq 0$, a contradiction with 3-additivity. Consider now case b) with $\overline{1234}$ and not $\overline{1235}$. Then, \mathcal{A} contains $\{\overline{123}, \overline{14}, \overline{24}, \overline{34}\}$, plus other subsets (at least one) which must contain $\overline{45}$. If $\overline{45} \in \mathcal{A}$, then $\mathcal{A} = \{\overline{123}, \overline{14}, \overline{24}, \overline{34}, \overline{45}\}$, and considering the subset $\overline{2345}$, \mathcal{A} has not the desired configuration required by (ii). If $\overline{145} \in \mathcal{A}$, then considering $\overline{1245}$, \mathcal{A} should contain $\overline{12}, \overline{24}$ and $\overline{25}$, which is

not the case. The reasoning is similar with $\overline{245}$ and $\overline{345}$. For case c) by the above result, \mathcal{A} must contain $\overline{123}, \overline{14}, \overline{24}, \overline{34}, \overline{15}, \overline{25}, \overline{35}$ but not other subset of $\overline{1234}$ nor $\overline{1235}$, i.e., only $\overline{45}$ would be a possibility. However, in both cases, considering the subset $\overline{2345}$, by (ii), \mathcal{A} has not the desired configuration. As a conclusion, no configuration is valid with a support of size 5.

c) Finally, if the support is $\overline{123456}$, no combination of sets should yield a set of cardinality 5, hence an antichain containing $\{\overline{123}, \overline{456}\}$ would be the only possibility. However, this would yield a 6-additive capacity, so that other sets should be added to \mathcal{A} , yielding sets of cardinality 5 or with an invalid configuration.

Consequently, supports of size greater than 4 are not possible with this configuration.

(iv) This is a consequence of (i), (ii) and (iii).

As for the sufficiency part, by Lemma 1, we just have to check the case where the support of \mathcal{A} is equal to 4, i.e., cases (ii) and (iii). Using (13), it is easy to check that $v_{\mathcal{A}}$ is 3-additive in both cases. □

Based on the above results and on (13), we are now in position to give a complete list of the vertices of $\mathcal{M}_3^{0-1}(n)$ (Table 4).

antichain	vertex	number
$\{\{i\}\}$	$u_{\overline{i}}$	n
$\{\{i, j\}\}$	$u_{\overline{ij}}$	$\binom{n}{2}$
$\{\{i\}, \{j\}\}$	$u_{\overline{i}} + u_{\overline{j}} - u_{\overline{ij}} =: \overline{u}_{ij}$	$\binom{n}{2}$
$\{\{i\}, \{j\}, \{\ell\}\}$	$u_{\overline{i}} + u_{\overline{j}} + u_{\overline{\ell}} - u_{\overline{ij}} - u_{\overline{i\ell}} - u_{\overline{j\ell}} + u_{\overline{ij\ell}} =: \overline{u}_{ij\ell}$	$\binom{n}{3}$
$\{\{i\}, \{j, \ell\}\}$	$u_{\overline{i}} + u_{\overline{j\ell}} - u_{\overline{ij\ell}} =: \overline{u}_{i,j\ell}$	$n \binom{n-1}{2}$
$\{\{i, j\}, \{j, \ell\}\}$	$u_{\overline{ij}} + u_{\overline{j\ell}} - u_{\overline{ij\ell}} =: \overline{u}_{ij,j\ell}$	$(n-2) \binom{n}{2}$
$\{\{i, j\}, \{j, \ell\}, \{i, \ell\}\}$	$u_{\overline{ij}} + u_{\overline{j\ell}} + u_{\overline{i\ell}} - 2u_{\overline{ij\ell}} =: \overline{u}_{ij,j\ell,i\ell}$	$\binom{n}{3}$
$\{\{i, j, \ell\}\}$	$u_{\overline{ij\ell}}$	$\binom{n}{3}$
$\{\{i, j\}, \{s, t\}, \{i, s\}\}$	$u_{\overline{ij}} + u_{\overline{st}} + u_{\overline{is}} - u_{\overline{ijs}} - u_{\overline{ist}} =: \overline{u}_{ij,st,is}$	$2 \binom{n}{2} \binom{n-2}{2}$
$\{\{i, j, s\}, \{i, t\}, \{j, t\}, \{s, t\}\}$	$u_{\overline{ijs}} + u_{\overline{it}} + u_{\overline{jt}} + u_{\overline{st}} - u_{\overline{ijt}} - u_{\overline{ist}} - u_{\overline{jst}} =: \overline{u}_{ijs,t}$	$(n-3) \binom{n}{3}$

Table 4: List of vertices of $\mathcal{M}_3^{0-1}(n)$. i, j, ℓ, s, t indicate distinct elements of N .

This yields a total number of vertices equal to $n \left(n + \frac{1}{6}(n-1)(n-2)(4n-3) \right)$.

The ten types of antichains are represented on Figures 1, 2 and 3.

4.2 Facets of $\text{conv}(\mathcal{M}_k^{0-1}(n))$

Concerning facets, the following result is noteworthy.

Theorem 2. Any facet of $\mathcal{M}_k(n)$ is a facet of $\mathcal{M}_k^{0-1}(n)$.

Proof. A facet of $\mathcal{M}_k(n)$ has dimension $d(n, k) - 1$ and corresponds to an inequality (12), say $\sum_{\substack{T \in \{\{i\}, S\} \\ |T| \leq k}} m^v(T) \geq 0$ for some $S \subseteq N, i \in S$. This facet, denoted by $F(S, i)$, is the set of points in $\mathcal{M}_k(n)$ satisfying the equality $\sum_{\substack{T \in \{\{i\}, S\} \\ |T| \leq k}} m^v(T) = 0$. In order to prove that the inequality

$\sum_{\substack{T \in \{\{i\}, S\} \\ |T| \leq k}} m^v(T) \geq 0$ also corresponds to a facet of $\mathcal{M}_k^{0-1}(n)$, it is enough to check that at least

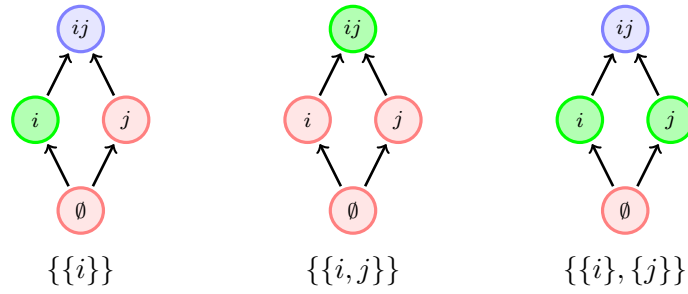


Figure 1: Representation of the three antichains having 1 or 2 elements i, j . For each solution, we represent the associated capacity. The green vertices correspond to the coalitions in the antichain, the red (resp. blue) vertices are the ones below (resp. above) the antichain. The capacity is thus equal to 1 (resp. 0) for the nodes in green or blue (resp. red).

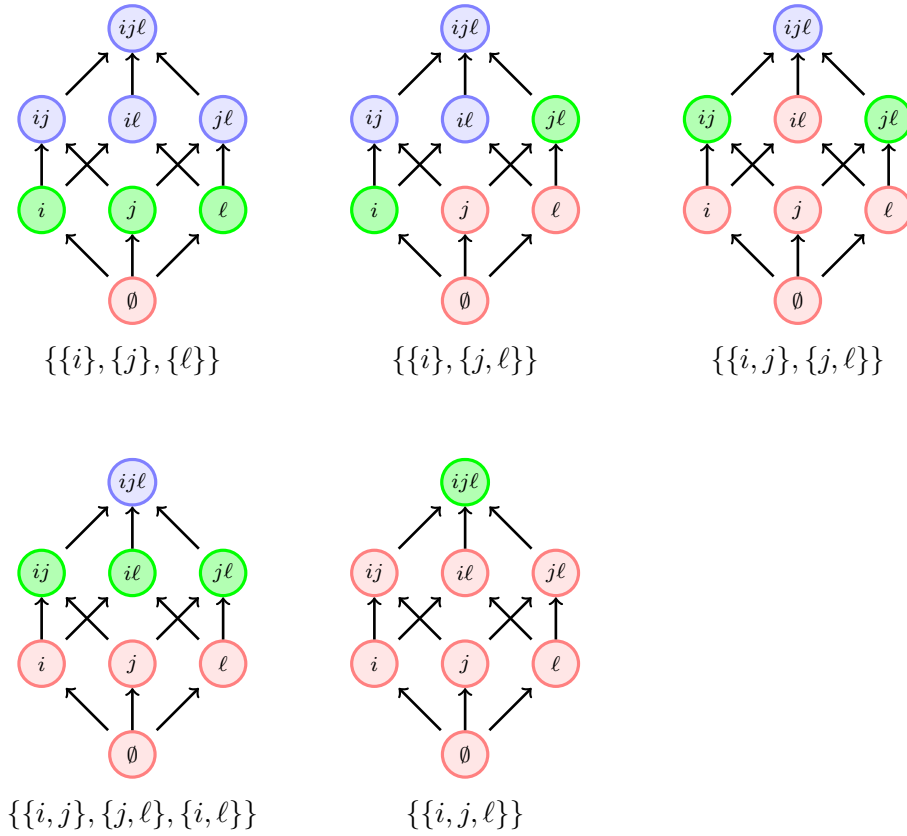


Figure 2: Representation of the five antichains having exactly three elements i, j, l . For each solution, we represent the associated capacity. The green vertices correspond to the coalitions in the antichain, the red (resp. blue) vertices are the ones below (resp. above) the antichain. The capacity is thus equal to 1 (resp. 0) for the nodes in green or blue (resp. red).

$d(n, k)$ vertices of $\mathcal{M}_k^{0-1}(n)$ satisfy the equality

$$\sum_{\substack{T \in [\{i\}, S] \\ |T| \leq k}} m^v(T) = 0 \quad (15)$$

Consider first the case $S = \{i\}$, $i \in N$. Observe that all $v = u_{\{j\}}$ with $j \neq i$ satisfy equality (15), which reduces to $m^v(\{i\}) = 0$, as well as all u_S with $2 \leq |S| \leq k$. This gives in total exactly $d(n, k)$ vertices satisfying the equality.

Take now $S \subseteq N$ with $|S| > 1$ and $i \in S$. Then the following vertices satisfy equality (15) as all terms on the left hand side are zero: $u_{\{j\}}$ with $j \neq i$, and more generally any u_L , $1 \leq |L| \leq k$, with $L \not\subseteq [\{i\}, S]$. This yields, letting $s := |S|$,

$$\begin{aligned} (n-1) + \left[\binom{n}{2} - \binom{s-1}{1} \right] + \left[\binom{n}{3} - \binom{s-1}{2} \right] + \cdots + \left[\binom{n}{k} - \binom{s-1}{k-1} \right] \\ = d(n, k) - \binom{s-1}{1} - \cdots - \binom{s-1}{k-1} \end{aligned}$$

vertices. It remains therefore to find $\binom{s-1}{1} + \cdots + \binom{s-1}{k-1}$ additional vertices. We claim that any vertex \bar{u}_T with $T \in [\{i\}, S]$, $2 \leq |T| \leq k$, satisfies (15), which yields exactly the number of missing vertices. Indeed, taking some $T_0 \in [\{i\}, S]$, $2 \leq |T_0| \leq k$, from (13) we obtain:

$$\begin{aligned} \sum_{\substack{T \in [\{i\}, S] \\ |T| \leq k}} m^{\bar{u}_{T_0}}(T) &= \sum_{T \in [\{i\}, T_0]} (-1)^{|T|+1} \\ &= \sum_{t=1}^{|T_0|} (-1)^{t+1} \binom{|T_0|-1}{t-1} = \sum_{t=0}^{|T_0|-1} (-1)^t \binom{|T_0|-1}{t} = 0. \end{aligned}$$

□

4.3 The case $n = 4$, $k = 3$

Using the software PORTA (POLYhedron Representation Transformation Algorithm) developed by Thomas Christof and Andreas Loebel¹, it is possible to obtain numerically all facets and vertices of $\text{conv}(\mathcal{M}_3^{0-1}(4))$, as well as those of $\mathcal{M}_3(4)$, and to confirm our theoretical results. PORTA permits to switch between the representation of a polyhedron by inequalities and the representation by vertices (and extreme rays, when a conic part exists). For a d -dimensional polyhedron, facets are valid inequalities of dimension $d - 1$, equivalently, containing at least d vertices.

$\mathcal{M}_3(4)$ and $\text{conv}(\mathcal{M}_3^{0-1}(4))$ are both 13-dimensional polytopes (see (2)). We start with the vertices of $\text{conv}(\mathcal{M}_3^{0-1}(4))$. We give below on Table 5 the instantiation of Table 4 for $n = 4$. The total number is 68 vertices. The set of facets determining $\mathcal{M}_3(4)$ is given by (12), and the total number of facets is $\sum_{i=1}^4 i \binom{4}{i} = 32$. Entering these 32 inequalities into PORTA together with the normalization equation $\sum_{S \subseteq N} m^v(S) = 1$ yields the representation by vertices. There are 303 vertices, among which only 68 correspond to 0-1-valued capacities, which are precisely those given in Table 4. See Appendix A for a complete list of the vertices.

Selecting these 68 vertices and entering them into PORTA again yields the representation by inequalities of $\text{conv}(\mathcal{M}_3^{0-1}(4))$: apart the equality corresponding to the normalization condition,

¹See <https://porta.zib.de/>.

antichain	vertex	number
$\{\{i\}\}$	$u_{\bar{i}}$	4
$\{\{i, j\}\}$	$u_{\bar{ij}}$	6
$\{\{i\}, \{j\}\}$	$u_{\bar{i}} + u_{\bar{j}} - u_{\bar{ij}}$	6
$\{\{i\}, \{j\}, \{l\}\}$	$u_{\bar{i}} + u_{\bar{j}} + u_{\bar{l}} - u_{\bar{ij}} - u_{\bar{il}} - u_{\bar{j\ell}} =: \bar{u}_{ij\ell}$	4
$\{\{i\}, \{j, \ell\}\}$	$u_{\bar{i}} + u_{\bar{j\ell}} - u_{\bar{ij\ell}} =: \bar{u}_{i,j\ell}$	12
$\{\{i, j\}, \{j, \ell\}\}$	$u_{\bar{ij}} + u_{\bar{j\ell}} - u_{\bar{ij\ell}} =: \bar{u}_{ij,j\ell}$	12
$\{\{i, j\}, \{j, \ell\}, \{i, \ell\}\}$	$u_{\bar{ij}} + u_{\bar{j\ell}} + u_{\bar{il}} - 2u_{\bar{ij\ell}} =: \bar{u}_{ij,j\ell,il}$	4
$\{\{i, j, \ell\}\}$	$u_{\bar{ij\ell}}$	4
$\{\{i, j\}, \{s, t\}, \{i, s\}\}$	$u_{\bar{ij}} + u_{\bar{st}} + u_{\bar{is}} - u_{\bar{ijs}} - u_{\bar{ist}} =: \bar{u}_{ij,st,is}$	12
$\{\{i, j, s\}, \{i, t\}, \{j, t\}, \{s, t\}\}$	$u_{\bar{ijs}} + u_{\bar{it}} + u_{\bar{jt}} + u_{\bar{st}} - u_{\bar{ijt}} - u_{\bar{ist}} - u_{\bar{jst}} =: \bar{u}_{ijs,t}$	4

Table 5: List of vertices of $\text{conv}(\mathcal{M}_3^{0-1}(4))$. i, j, ℓ, s, t indicate distinct elements of $\{1, 2, 3, 4\}$.

there are 222 facets, comprising the 32 facets defining $\mathcal{M}_3(4)$. The complete list is given in Appendix B.

Another interesting question is the volume of $\mathcal{M}_3(4)$ and $\text{conv}(\mathcal{M}_3^{0-1}(4))$. As these polyhedra are not simplicial nor order polytopes, there is no known closed form formula to compute their volume. We have used for the computation the softwares VINCI, developed by Benno Büeler and Andreas Enge² and LRS, developed by David Avis³ (see the survey paper [3] on the computation of volume of polytopes). The result is given below:

Volume of $\mathcal{M}_3(4)$	$V_1=0.000019927$
Volume of $\text{conv}(\mathcal{M}_3^{0-1}(4))$	$V_2=0.000019046$
ratio V_2/V_1	0.95581

Note that the value obtained for $\mathcal{M}_3(4)$ is very close to the theoretical value 0.0000192753 given in Table 2.

4.4 Back to the identification problem

Thanks to the complete characterization of $\mathcal{M}_3^{0-1}(n)$, we can have an explicit expression of the Choquet integral w.r.t. a capacity in $\text{conv}(\mathcal{M}_3^{0-1}(n))$. We have shown two important facts which support the validity of the approach: 1) The number of vertices is in $O(n^4)$, leading therefore to an optimization problem of polynomial complexity (both for the number of variables and the number of constraints) 2) The loss of generality, that is, the volume of $\mathcal{M}_3(n) \setminus \text{conv}(\mathcal{M}_3^{0-1}(n))$, seems to be small. The value for $n = 4$ is about 5% of the volume of $\mathcal{M}_3(4)$ and we conjecture that this value remains small for higher values of n .

We now consider the aggregation functions associated to the vertices. Table 6 gives the expression of the Choquet integral w.r.t. the vertices. Their expression is simple enough to be directly understood. In Table 7 we provide the corresponding importance and interaction indices for these vertices. We recall that the mean importance (Shapley value) of criterion i is given by

$$\phi_i = \sum_{S \subseteq N \setminus \{i\}} \frac{(n - |S| - 1)! |S|!}{n!} [\mu(S \cup \{i\}) - \mu(S)],$$

²See <https://www.multiprecision.org/vinci/>.

³See <http://cgm.cs.mcgill.ca/~avis/C/lrslib/>.

while the interaction index between criteria i and j is given by

$$I_{ij} = \sum_{S \subseteq N \setminus \{i,j\}} \frac{(n - |S| - 2)!|S|!}{(n - 1)!} [\mu(S \cup \{i, j\}) - \mu(S \cup \{i\}) - \mu(S \cup \{j\}) + \mu(S)],$$

and the interaction among criteria i, j, ℓ is

$$I_{ij\ell} = \sum_{S \subseteq N \setminus \{i,j,\ell\}} \frac{(n - |S| - 3)!|S|!}{(n - 2)!} [\mu(S \cup \{i, j, \ell\}) - \mu(S \cup \{i, j\}) - \mu(S \cup \{i, \ell\}) - \mu(S \cup \{j, \ell\}) + \mu(S \cup \{i\}) - \mu(S \cup \{j\}) + \mu(S \cup \{\ell\}) - \mu(S)].$$

The values of the importance index are conform to the intuition, i.e., symmetric expressions yield equal Shapley values (see Table 7). The interactions of order 2 are also simple to derive, given that the interaction of a pair of criteria is equal to $+1$ (resp., -1) for the min (resp., max). We note that the min and the max aggregation functions have exactly the same interaction of order 3. This directly derives from the expression $x_i \vee x_j \vee x_\ell = x_i \wedge x_j \wedge x_\ell + [x_i + x_j + x_\ell - x_i \wedge x_j - x_i \wedge x_\ell - x_j \wedge x_\ell]$, where the term in brackets corresponds to a Choquet integral w.r.t. a 2-additive capacity and has no interaction terms of order 3.

antichain	Choquet integral
$\{\{i\}\}$	$C_1(x) = x_i$
$\{\{i, j\}\}$	$C_2(x) = x_i \wedge x_j = \text{OS}_1^2(x_i, x_j)$
$\{\{i\}, \{j\}\}$	$C_3(x) = x_i + x_j - x_i \wedge x_j = x_i \vee x_j = \text{OS}_2^2(x_i, x_j)$
$\{\{i\}, \{j\}, \{\ell\}\}$	$C_4(x) = x_i + x_j + x_\ell - x_i \wedge x_j - x_i \wedge x_\ell - x_j \wedge x_\ell + x_i \wedge x_j \wedge x_\ell = x_i \vee x_j \vee x_\ell = \text{OS}_3^3(x_i, x_j, x_\ell)$
$\{\{i\}, \{j, \ell\}\}$	$C_5(x) = x_i + x_j \wedge x_\ell - x_i \wedge x_j \wedge x_\ell$
$\{\{i, j\}, \{j, \ell\}\}$	$C_6(x) = x_i \wedge x_j + x_j \wedge x_\ell - x_i \wedge x_j \wedge x_\ell$
$\{\{i, j\}, \{j, \ell\}, \{i, \ell\}\}$	$C_7(x) = x_i \wedge x_j + x_j \wedge x_\ell + x_j \wedge x_\ell - 2x_i \wedge x_j \wedge x_\ell = \text{OS}_2^3(x_i, x_j, x_\ell)$
$\{\{i, j, \ell\}\}$	$C_8(x) = x_i \wedge x_j \wedge x_\ell = \text{OS}_1^3(x_i, x_j, x_\ell)$
$\{\{i, j\}, \{s, t\}, \{i, s\}\}$	$C_9(x) = x_i \wedge x_j + x_s \wedge x_t + x_i \wedge x_s - x_i \wedge x_j \wedge x_s - x_i \wedge x_s \wedge x_t$
$\{\{i, j, s\}, \{i, t\}, \{j, t\}, \{s, t\}\}$	$C_{10}(x) = x_i \wedge x_j \wedge x_s + x_i \wedge x_t + x_j \wedge x_t + x_s \wedge x_t - x_i \wedge x_j \wedge x_t - x_i \wedge x_s \wedge x_t - x_j \wedge x_s \wedge x_t$

Table 6: List of vertices of $\mathcal{M}_3^{0-1}(n)$ and associated Choquet integrals. i, j, ℓ, s, t indicate distinct elements of N . $\text{OS}_q^p(x_1, \dots, x_p)$ is the q th order-statistic (i.e. the q th-smallest value) of a sequence x_1, \dots, x_p of p values.

The expression of the aggregation functions associated to the 10 types of vertices for the multilinear model is given in Table 8. We recall that the multilinear model is given by:

$$\text{MLE}_{m^v}(x) = \sum_{S \subseteq N} m^v(S) \prod_{i \in S} x_i.$$

The multi-linear model is monotone in $[0, 1]^N$.

Acknowledgment

The authors wish to thank Andreas Enge for his precious help in computing the volumes of $\text{conv}(\mathcal{M}_3^{0-1}(4))$ and $\mathcal{M}_3(4)$.

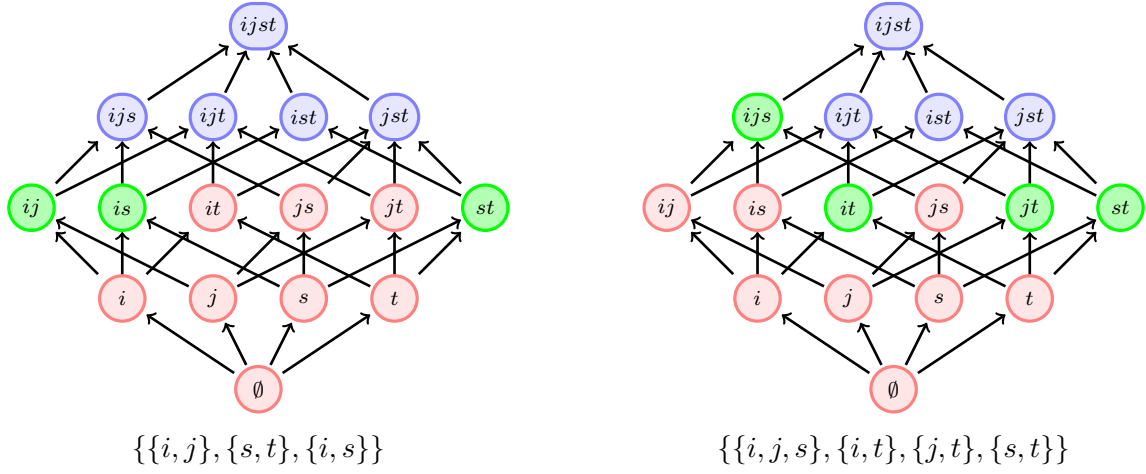


Figure 3: Representation of the two antichains having exactly four elements i, j, s, t . For each solution, we represent the associated capacity. The green vertices correspond to the coalitions in the antichain, the red (resp. blue) vertices are the ones below (resp. above) the antichain. The capacity is thus equal to 1 (resp. 0) for the nodes in green or blue (resp. red).

Ft	ϕ_i	ϕ_j	ϕ_ℓ	I_{ij}	$I_{i\ell}$	$I_{j\ell}$	$I_{ij\ell}$							
C_4	1/3	1/3	1/3	-1/2	-1/2	-1/2	+1							
C_5	2/3	1/6	1/6	-1/2	1/2	-1/2	-1							
C_6	1/6	2/3	1/6	1/2	-1/2	1/2	-1							
C_7	1/3	1/3	1/3	0	0	0	-2							
C_8	1/3	1/3	1/3	1/2	1/2	1/2	+1							
Ft	ϕ_i	ϕ_j	ϕ_s	ϕ_t	I_{ij}	I_{is}	I_{it}	I_{js}	I_{jt}	I_{st}	I_{ijs}	I_{ijt}	I_{ist}	I_{jst}
C_9	1/3	1/6	1/3	1/6	1/2	0	-1/2	-1/2	0	1/2	-1	0	-1	0
C_{10}	1/6	1/6	1/6	1/2	0	0	0	0	0	0	1	-1	-1	-1

Table 7: Importance and interaction indices of the vertices.

antichain	Multilinear Model
$\{\{i\}\}$	$M_1(x) = x_i$
$\{\{i, j\}\}$	$M_2(x) = x_i x_j$
$\{\{i\}, \{j\}\}$	$M_3(x) = x_i + x_j - x_i x_j$
$\{\{i\}, \{j\}, \{\ell\}\}$	$M_4(x) = x_i + x_j + x_\ell - x_i x_j - x_i x_\ell - x_j x_\ell + x_i x_j x_\ell$
$\{\{i\}, \{j, \ell\}\}$	$M_5(x) = x_i + x_j x_\ell - x_i x_j x_\ell$
$\{\{i, j\}, \{j, \ell\}\}$	$M_6(x) = x_i x_j + x_j x_\ell - x_i x_j x_\ell$
$\{\{i, j\}, \{j, \ell\}, \{i, \ell\}\}$	$M_7(x) = x_i x_j + x_j x_\ell + x_j x_\ell - 2 x_i x_j x_\ell$
$\{\{i, j, \ell\}\}$	$M_8(x) = x_i x_j x_\ell$
$\{\{i, j\}, \{s, t\}, \{i, s\}\}$	$M_9(x) = x_i x_j + x_s x_t + x_i x_s - x_i x_j x_s - x_i x_s x_t$
$\{\{i, j, s\}, \{i, t\}, \{j, t\}, \{s, t\}\}$	$M_{10}(x) = x_i x_j x_s + x_i x_t + x_j x_t + x_s x_t - x_i x_j x_t - x_i x_s x_t - x_j x_s x_t$

Table 8: List of vertices of $\mathcal{M}_3^{0-1}(n)$ and associated Multilinear Models. i, j, ℓ, s, t indicate distinct elements of N .

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A List of vertices of $\mathcal{M}_3(4)$

Coordinates are given in terms of the Möbius transform. We have used the following shorthand: m_2 for $m^v(\{2\})$, m_{12} for $m^v(\{1,2\})$, etc. Vertex numbers in bold indicates the 0-1-valued vertices, i.e., those of $\mathcal{M}_3^{0-1}(4)$.

	m_1	m_2	m_3	m_4	m_{12}	m_{13}	m_{14}	m_{23}	m_{24}	m_{34}	m_{123}	m_{124}	m_{134}	m_{234}
(1)	0	0	0	0	0	1/3	1/3	1/3	1/3	1/3	1/3	1/3	-2/3	-2/3
(2)	0	0	0	0	1/3	0	1/3	1/3	1/3	1/3	1/3	-2/3	1/3	-2/3
(3)	0	0	0	0	1/3	1/3	0	1/3	1/3	1/3	-2/3	1/3	1/3	-2/3
(4)	0	0	0	0	1/3	1/3	1/3	0	1/3	1/3	1/3	-2/3	-2/3	1/3
(5)	0	0	0	0	1/3	1/3	1/3	1/3	0	1/3	-2/3	1/3	-2/3	1/3
(6)	0	0	0	0	1/3	1/3	1/3	1/3	1/3	0	-2/3	-2/3	1/3	1/3
(7)	0	0	0	1/3	1/3	1/3	0	1/3	0	0	-2/3	-1/3	1/3	1/3
(8)	0	0	0	1/3	1/3	1/3	0	1/3	0	0	-2/3	1/3	-1/3	1/3
(9)	0	0	0	1/3	1/3	1/3	0	1/3	0	0	-2/3	1/3	1/3	-1/3
(10)	0	0	0	1/2	0	1/2	0	1/2	0	0	-1/2	1/2	-1/2	0
(11)	0	0	0	1/2	0	1/2	0	1/2	0	0	-1/2	1/2	0	-1/2
(12)	0	0	0	1/2	0	1/2	0	1/2	0	0	0	1/2	-1/2	-1/2
(13)	0	0	0	1/2	1/2	0	0	1/2	0	0	-1/2	-1/2	1/2	0
(14)	0	0	0	1/2	1/2	0	0	1/2	0	0	-1/2	0	1/2	-1/2
(15)	0	0	0	1/2	1/2	0	0	1/2	0	0	0	-1/2	1/2	-1/2
(16)	0	0	0	1/2	1/2	1/2	0	0	0	0	-1/2	-1/2	0	1/2
(17)	0	0	0	1/2	1/2	1/2	0	0	0	0	-1/2	0	-1/2	1/2
(18)	0	0	0	1/2	1/2	1/2	0	0	0	0	0	-1/2	-1/2	1/2
(19)	0	0	1/3	0	1/3	0	1/3	0	1/3	0	-1/3	-2/3	1/3	1/3
(20)	0	0	1/3	0	1/3	0	1/3	0	1/3	0	1/3	-2/3	-1/3	1/3
(21)	0	0	1/3	0	1/3	0	1/3	0	1/3	0	1/3	-2/3	1/3	-1/3
(22)	0	0	1/2	0	0	0	1/2	0	1/2	0	1/2	-1/2	-1/2	0
(23)	0	0	1/2	0	0	0	1/2	0	1/2	0	1/2	-1/2	0	-1/2
(24)	0	0	1/2	0	0	0	1/2	0	1/2	0	1/2	0	-1/2	-1/2
(25)	0	0	1/2	0	1/2	0	0	0	1/2	0	-1/2	-1/2	1/2	0
(26)	0	0	1/2	0	1/2	0	0	0	1/2	0	-1/2	0	1/2	-1/2
(27)	0	0	1/2	0	1/2	0	0	0	1/2	0	0	-1/2	1/2	-1/2
(28)	0	0	1/2	0	1/2	0	1/2	0	0	0	-1/2	-1/2	0	1/2
(29)	0	0	1/2	0	1/2	0	1/2	0	0	0	-1/2	0	-1/2	1/2
(30)	0	0	1/2	0	1/2	0	1/2	0	0	0	0	-1/2	-1/2	1/2
(31)	0	0	1/2	1/2	1/2	0	0	0	0	-1/2	-1/2	-1/2	1/2	1/2
(32)	0	0	1/2	1/2	1/2	0	0	0	0	-1/2	-1/2	0	0	1/2
(33)	0	0	1/2	1/2	1/2	0	0	0	0	-1/2	-1/2	0	1/2	0
(34)	0	0	1/2	1/2	1/2	0	0	0	0	-1/2	0	-1/2	0	1/2
(35)	0	0	1/2	1/2	1/2	0	0	0	0	-1/2	0	-1/2	1/2	0
(36)	0	0	1/2	1/2	1/2	0	0	0	1/2	-1/2	-1/2	-1/2	1/2	0
(37)	0	0	1/2	1/2	1/2	0	0	1/2	0	-1/2	-1/2	-1/2	1/2	0
(38)	0	0	1/2	1/2	1/2	0	1/2	0	0	-1/2	-1/2	-1/2	0	1/2
(39)	0	0	1/2	1/2	1/2	1/2	0	0	0	-1/2	-1/2	-1/2	0	1/2
(40)	0	0	2/3	2/3	2/3	0	0	0	0	-2/3	-2/3	-1/3	1/3	1/3
(41)	0	0	2/3	2/3	2/3	0	0	0	0	-2/3	-1/3	-2/3	1/3	1/3
(42)	0	1/3	0	0	0	1/3	1/3	0	0	1/3	-1/3	1/3	-2/3	1/3
(43)	0	1/3	0	0	0	1/3	1/3	0	0	1/3	1/3	-1/3	-2/3	1/3
(44)	0	1/3	0	0	0	1/3	1/3	0	0	1/3	1/3	1/3	-2/3	-1/3
(45)	0	1/2	0	0	0	0	1/2	0	0	1/2	1/2	-1/2	-1/2	0
(46)	0	1/2	0	0	0	0	1/2	0	0	1/2	1/2	-1/2	0	-1/2
(47)	0	1/2	0	0	0	0	1/2	0	0	1/2	1/2	0	-1/2	-1/2
(48)	0	1/2	0	0	0	1/2	0	0	0	1/2	-1/2	1/2	-1/2	0

	m_1	m_2	m_3	m_4	m_{12}	m_{13}	m_{14}	m_{23}	m_{24}	m_{34}	m_{123}	m_{124}	m_{134}	m_{234}
(49)	0	1/2	0	0	0	1/2	0	0	0	1/2	-1/2	1/2	0	-1/2
(50)	0	1/2	0	0	0	1/2	0	0	0	1/2	0	1/2	-1/2	-1/2
(51)	0	1/2	0	0	0	1/2	1/2	0	0	0	-1/2	-1/2	0	1/2
(52)	0	1/2	0	0	0	1/2	1/2	0	0	0	-1/2	0	-1/2	1/2
(53)	0	1/2	0	0	0	1/2	1/2	0	0	0	0	-1/2	-1/2	1/2
(54)	0	1/2	0	1/2	0	1/2	0	0	-1/2	0	-1/2	0	0	1/2
(55)	0	1/2	0	1/2	0	1/2	0	0	-1/2	0	-1/2	1/2	-1/2	1/2
(56)	0	1/2	0	1/2	0	1/2	0	0	-1/2	0	-1/2	1/2	0	0
(57)	0	1/2	0	1/2	0	1/2	0	0	-1/2	0	0	0	-1/2	1/2
(58)	0	1/2	0	1/2	0	1/2	0	0	-1/2	0	0	1/2	-1/2	0
(59)	0	1/2	0	1/2	0	1/2	0	0	-1/2	1/2	-1/2	1/2	-1/2	0
(60)	0	1/2	0	1/2	0	1/2	0	1/2	-1/2	0	-1/2	1/2	-1/2	0
(61)	0	1/2	0	1/2	0	1/2	1/2	0	-1/2	0	-1/2	0	-1/2	1/2
(62)	0	1/2	0	1/2	1/2	1/2	0	0	-1/2	0	-1/2	0	-1/2	1/2
(63)	0	1/2	1/2	0	0	0	1/2	-1/2	0	0	0	-1/2	0	1/2
(64)	0	1/2	1/2	0	0	0	1/2	-1/2	0	0	0	0	-1/2	1/2
(65)	0	1/2	1/2	0	0	0	1/2	-1/2	0	0	1/2	-1/2	-1/2	1/2
(66)	0	1/2	1/2	0	0	0	1/2	-1/2	0	0	1/2	-1/2	0	0
(67)	0	1/2	1/2	0	0	0	1/2	-1/2	0	0	1/2	0	-1/2	0
(68)	0	1/2	1/2	0	0	0	1/2	-1/2	0	1/2	1/2	-1/2	-1/2	0
(69)	0	1/2	1/2	0	0	0	1/2	-1/2	1/2	0	1/2	-1/2	-1/2	0
(70)	0	1/2	1/2	0	0	1/2	1/2	-1/2	0	0	0	-1/2	-1/2	1/2
(71)	0	1/2	1/2	0	1/2	0	1/2	-1/2	0	0	0	-1/2	-1/2	1/2
(72)	0	2/3	0	2/3	0	2/3	0	0	-2/3	0	-2/3	1/3	-1/3	1/3
(73)	0	2/3	0	2/3	0	2/3	0	0	-2/3	0	-1/3	1/3	-2/3	1/3
(74)	0	2/3	2/3	0	0	0	2/3	-2/3	0	0	1/3	-2/3	-1/3	1/3
(75)	0	2/3	2/3	0	0	0	2/3	-2/3	0	0	1/3	-1/3	-2/3	1/3
(76)	1/3	0	0	0	0	0	0	1/3	1/3	1/3	-1/3	1/3	1/3	-2/3
(77)	1/3	0	0	0	0	0	0	1/3	1/3	1/3	1/3	-1/3	1/3	-2/3
(78)	1/3	0	0	0	0	0	0	1/3	1/3	1/3	1/3	1/3	-1/3	-2/3
(79)	1/2	0	0	0	0	0	0	0	1/2	1/2	1/2	-1/2	-1/2	0
(80)	1/2	0	0	0	0	0	0	0	1/2	1/2	1/2	-1/2	0	-1/2
(81)	1/2	0	0	0	0	0	0	0	1/2	1/2	1/2	0	-1/2	-1/2
(82)	1/2	0	0	0	0	0	0	1/2	0	1/2	-1/2	1/2	-1/2	0
(83)	1/2	0	0	0	0	0	0	1/2	0	1/2	-1/2	1/2	0	-1/2
(84)	1/2	0	0	0	0	0	0	1/2	0	1/2	0	1/2	-1/2	-1/2
(85)	1/2	0	0	0	0	0	0	1/2	1/2	0	-1/2	-1/2	1/2	0
(86)	1/2	0	0	0	0	0	0	1/2	1/2	0	-1/2	0	1/2	-1/2
(87)	1/2	0	0	0	0	0	0	1/2	1/2	0	0	-1/2	1/2	-1/2
(88)	1/2	0	0	1/2	0	0	-1/2	1/2	0	0	-1/2	0	1/2	0
(89)	1/2	0	0	1/2	0	0	-1/2	1/2	0	0	-1/2	1/2	0	0
(90)	1/2	0	0	1/2	0	0	-1/2	1/2	0	0	-1/2	1/2	1/2	-1/2
(91)	1/2	0	0	1/2	0	0	-1/2	1/2	0	0	0	0	1/2	-1/2
(92)	1/2	0	0	1/2	0	0	-1/2	1/2	0	0	0	1/2	0	-1/2
(93)	1/2	0	0	1/2	0	0	-1/2	1/2	0	1/2	-1/2	1/2	0	-1/2
(94)	1/2	0	0	1/2	0	0	-1/2	1/2	1/2	0	-1/2	0	1/2	-1/2
(95)	1/2	0	0	1/2	0	1/2	-1/2	1/2	0	0	-1/2	1/2	0	-1/2
(96)	1/2	0	0	1/2	1/2	0	-1/2	1/2	0	0	-1/2	0	1/2	-1/2
(97)	1/2	0	1/2	0	0	-1/2	0	0	1/2	0	0	-1/2	1/2	0
(98)	1/2	0	1/2	0	0	-1/2	0	0	1/2	0	0	0	1/2	-1/2
(99)	1/2	0	1/2	0	0	-1/2	0	0	1/2	0	1/2	-1/2	0	0
(100)	1/2	0	1/2	0	0	-1/2	0	0	1/2	0	1/2	-1/2	1/2	-1/2
(101)	1/2	0	1/2	0	0	-1/2	0	0	1/2	0	1/2	0	0	-1/2

	m_1	m_2	m_3	m_4	m_{12}	m_{13}	m_{14}	m_{23}	m_{24}	m_{34}	m_{123}	m_{124}	m_{134}	m_{234}
(102)	1/2	0	1/2	0	0	-1/2	0	0	1/2	1/2	1/2	-1/2	0	-1/2
(103)	1/2	0	1/2	0	0	-1/2	0	1/2	1/2	0	0	-1/2	1/2	-1/2
(104)	1/2	0	1/2	0	0	-1/2	1/2	0	1/2	0	1/2	-1/2	0	-1/2
(105)	1/2	0	1/2	0	1/2	-1/2	0	0	1/2	0	0	-1/2	1/2	-1/2
(106)	1/2	1/2	0	0	-1/2	0	0	0	0	1/2	0	1/2	-1/2	0
(107)	1/2	1/2	0	0	-1/2	0	0	0	0	1/2	0	1/2	0	-1/2
(108)	1/2	1/2	0	0	-1/2	0	0	0	0	1/2	1/2	0	-1/2	0
(109)	1/2	1/2	0	0	-1/2	0	0	0	0	1/2	1/2	0	0	-1/2
(110)	1/2	1/2	0	0	-1/2	0	0	0	0	1/2	1/2	1/2	-1/2	-1/2
(111)	1/2	1/2	0	0	-1/2	0	0	0	1/2	1/2	1/2	0	-1/2	-1/2
(112)	1/2	1/2	0	0	-1/2	0	0	1/2	0	1/2	0	1/2	-1/2	-1/2
(113)	1/2	1/2	0	0	-1/2	0	1/2	0	0	1/2	1/2	0	-1/2	-1/2
(114)	1/2	1/2	0	0	-1/2	1/2	0	0	0	1/2	0	1/2	-1/2	-1/2
(115)	1/2	1/2	1/2	1/2	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	1/2	1/2	1/2	1/2
(116)	2/3	0	0	2/3	0	0	-2/3	2/3	0	0	-2/3	1/3	1/3	-1/3
(117)	2/3	0	0	2/3	0	0	-2/3	2/3	0	0	-1/3	1/3	1/3	-2/3
(118)	2/3	0	2/3	0	0	-2/3	0	0	2/3	0	1/3	-2/3	1/3	-1/3
(119)	2/3	0	2/3	0	0	-2/3	0	0	2/3	0	1/3	-1/3	1/3	-2/3
(120)	2/3	2/3	0	0	-2/3	0	0	0	0	2/3	1/3	1/3	-2/3	-1/3
(121)	2/3	2/3	0	0	-2/3	0	0	0	0	2/3	1/3	1/3	-1/3	-2/3
(122)	0	0	0	0	0	0	0	0	0	0	0	0	0	1
(123)	0	0	0	0	0	0	0	0	0	0	0	0	1	0
(124)	0	0	0	0	0	0	0	0	0	0	0	1	0	0
(125)	0	0	0	0	0	0	0	0	0	0	1	0	0	0
(126)	0	0	0	0	0	0	0	0	0	1	0	0	0	0
(127)	0	0	0	0	0	0	0	0	1	0	0	0	0	0
(128)	0	0	0	0	0	0	0	1	0	0	0	0	0	0
(129)	0	0	0	0	0	0	1	0	0	0	0	0	0	0
(130)	0	0	0	0	0	1	0	0	0	0	0	0	0	0
(131)	0	0	0	0	1	0	0	0	0	0	0	0	0	0
(132)	0	0	0	1/2	0	1/2	0	1	0	0	-1/2	1/2	-1/2	-1/2
(133)	0	0	0	1/2	0	1	0	1/2	0	0	-1/2	1/2	-1/2	-1/2
(134)	0	0	0	1/2	1/2	0	0	1	0	0	-1/2	-1/2	1/2	-1/2
(135)	0	0	0	1/2	1/2	1	0	0	0	0	-1/2	-1/2	-1/2	1/2
(136)	0	0	0	1/2	1	0	0	1/2	0	0	-1/2	-1/2	1/2	-1/2
(137)	0	0	0	1/2	1	1/2	0	0	0	0	-1/2	-1/2	-1/2	1/2
(138)	0	0	0	1	0	0	0	0	0	0	0	0	0	0
(139)	0	0	1/2	0	0	0	1/2	0	1	0	1/2	-1/2	-1/2	-1/2
(140)	0	0	1/2	0	0	0	1	0	1/2	0	1/2	-1/2	-1/2	-1/2
(141)	0	0	1/2	0	1/2	0	0	0	1	0	-1/2	-1/2	1/2	-1/2
(142)	0	0	1/2	0	1/2	0	1	0	0	0	-1/2	-1/2	-1/2	1/2
(143)	0	0	1/2	0	1	0	0	0	1/2	0	-1/2	-1/2	1/2	-1/2
(144)	0	0	1/2	0	1	0	1/2	0	0	0	-1/2	-1/2	-1/2	1/2
(145)	0	0	1/2	1/2	1	0	0	0	0	-1/2	-1/2	-1/2	0	1/2
(146)	0	0	1/2	1/2	1	0	0	0	0	-1/2	-1/2	-1/2	1/2	0
(147)	0	0	2/3	2/3	1	0	0	0	0	-2/3	-2/3	-2/3	1/3	1/3
(148)	0	0	1	0	0	0	0	0	0	0	0	0	0	0
(149)	0	1/2	0	0	0	0	1/2	0	0	1	1/2	-1/2	-1/2	-1/2
(150)	0	1/2	0	0	0	0	1	0	0	1/2	1/2	-1/2	-1/2	-1/2
(151)	0	1/2	0	0	0	1/2	0	0	0	1	-1/2	1/2	-1/2	-1/2
(152)	0	1/2	0	0	0	1/2	1	0	0	0	-1/2	-1/2	-1/2	1/2

	m_1	m_2	m_3	m_4	m_{12}	m_{13}	m_{14}	m_{23}	m_{24}	m_{34}	m_{123}	m_{124}	m_{134}	m_{234}
(153)	0	1/2	0	0	0	1	0	0	0	1/2	-1/2	1/2	-1/2	-1/2
(154)	0	1/2	0	0	0	1	1/2	0	0	0	-1/2	-1/2	-1/2	1/2
(155)	0	1/2	0	1/2	0	1	0	0	-1/2	0	-1/2	0	-1/2	1/2
(156)	0	1/2	0	1/2	0	1	0	0	-1/2	0	-1/2	1/2	-1/2	0
(157)	0	1/2	1/2	0	0	0	1	-1/2	0	0	0	-1/2	-1/2	1/2
(158)	0	1/2	1/2	0	0	0	1	-1/2	0	0	1/2	-1/2	-1/2	0
(159)	0	2/3	0	2/3	0	1	0	0	-2/3	0	-2/3	1/3	-2/3	1/3
(160)	0	2/3	2/3	0	0	0	1	-2/3	0	0	1/3	-2/3	-2/3	1/3
(161)	0	1	0	0	0	0	0	0	0	0	0	0	0	0
(162)	1/2	0	0	0	0	0	0	0	1/2	1	1/2	-1/2	-1/2	-1/2
(163)	1/2	0	0	0	0	0	0	0	1	1/2	1/2	-1/2	-1/2	-1/2
(164)	1/2	0	0	0	0	0	0	1/2	0	1	-1/2	1/2	-1/2	-1/2
(165)	1/2	0	0	0	0	0	0	1/2	1	0	-1/2	-1/2	1/2	-1/2
(166)	1/2	0	0	0	0	0	0	1	0	1/2	-1/2	1/2	-1/2	-1/2
(167)	1/2	0	0	0	0	0	0	1	1/2	0	-1/2	-1/2	1/2	-1/2
(168)	1/2	0	0	1/2	0	0	-1/2	1	0	0	-1/2	0	1/2	-1/2
(169)	1/2	0	0	1/2	0	0	-1/2	1	0	0	-1/2	1/2	0	-1/2
(170)	1/2	0	1/2	0	0	-1/2	0	0	1	0	0	-1/2	1/2	-1/2
(171)	1/2	0	1/2	0	0	-1/2	0	0	1	0	1/2	-1/2	0	-1/2
(172)	1/2	1/2	0	0	-1/2	0	0	0	0	1	0	1/2	-1/2	-1/2
(173)	1/2	1/2	0	0	-1/2	0	0	0	0	1	1/2	0	-1/2	-1/2
(174)	2/3	0	0	2/3	0	0	-2/3	1	0	0	-2/3	1/3	1/3	-2/3
(175)	2/3	0	2/3	0	0	-2/3	0	0	1	0	1/3	-2/3	1/3	-2/3
(176)	2/3	2/3	0	0	-2/3	0	0	0	0	1	1/3	1/3	-2/3	-2/3
(177)	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(178)	0	0	0	0	0	0	1/2	1/2	1/2	1/2	1/2	-1/2	0	-1
(179)	0	0	0	0	0	0	1/2	1/2	1/2	1/2	1/2	0	-1/2	-1
(180)	0	0	0	0	0	1/2	0	1/2	1/2	1/2	-1/2	1/2	0	-1
(181)	0	0	0	0	0	1/2	0	1/2	1/2	1/2	0	1/2	-1/2	-1
(182)	0	0	0	0	0	1/2	1/2	0	1/2	1/2	1/2	-1/2	-1	0
(183)	0	0	0	0	0	1/2	1/2	0	1/2	1/2	1/2	0	-1	-1/2
(184)	0	0	0	0	0	1/2	1/2	1/2	0	1/2	-1/2	1/2	-1	0
(185)	0	0	0	0	0	1/2	1/2	1/2	0	1/2	0	1/2	-1	-1/2
(186)	0	0	0	0	1/2	0	0	1/2	1/2	1/2	-1/2	0	1/2	-1
(187)	0	0	0	0	1/2	0	0	1/2	1/2	1/2	0	-1/2	1/2	-1
(188)	0	0	0	0	1/2	0	1/2	0	1/2	1/2	1/2	-1	-1/2	0
(189)	0	0	0	0	1/2	0	1/2	0	1/2	1/2	1/2	-1	0	-1/2
(190)	0	0	0	0	1/2	0	1/2	1/2	1/2	0	-1/2	-1	1/2	0
(191)	0	0	0	0	1/2	0	1/2	1/2	1/2	0	0	-1	1/2	-1/2
(192)	0	0	0	0	1/2	1/2	0	1/2	0	1/2	-1	1/2	-1/2	0
(193)	0	0	0	0	1/2	1/2	0	1/2	0	1/2	-1	1/2	0	-1/2
(194)	0	0	0	0	1/2	1/2	0	1/2	1/2	0	-1	-1/2	1/2	0
(195)	0	0	0	0	1/2	1/2	0	1/2	1/2	0	-1	0	1/2	-1/2
(196)	0	0	0	0	1/2	1/2	1/2	0	0	1/2	-1/2	0	-1	1/2
(197)	0	0	0	0	1/2	1/2	1/2	0	0	1/2	0	-1/2	-1	1/2
(198)	0	0	0	0	1/2	1/2	1/2	0	1/2	0	-1/2	-1	0	1/2
(199)	0	0	0	0	1/2	1/2	1/2	0	1/2	0	0	-1	-1/2	1/2
(200)	0	0	0	0	1/2	1/2	1/2	1/2	0	0	-1	-1/2	0	1/2
(201)	0	0	0	0	1/2	1/2	1/2	1/2	0	0	-1	0	-1/2	1/2
(202)	0	0	0	0	0	0	1	1/2	1/2	1/2	1/2	-1/2	-1/2	-1
(203)	0	0	0	0	0	1/2	1/2	0	1	1/2	1/2	-1/2	-1	-1/2

	m_1	m_2	m_3	m_4	m_{12}	m_{13}	m_{14}	m_{23}	m_{24}	m_{34}	m_{123}	m_{124}	m_{134}	m_{234}
(204)	0	0	0	0	0	1/2	1/2	1	0	1/2	-1/2	1/2	-1	-1/2
(205)	0	0	0	0	0	1	0	1/2	1/2	1/2	-1/2	1/2	-1/2	-1
(206)	0	0	0	0	1/2	0	1/2	0	1/2	1	1/2	-1	-1/2	-1/2
(207)	0	0	0	0	1/2	0	1/2	1	1/2	0	-1/2	-1	1/2	-1/2
(208)	0	0	0	0	1/2	1/2	0	1/2	0	1	-1	1/2	-1/2	-1/2
(209)	0	0	0	0	1/2	1/2	0	1/2	1	0	-1	-1/2	1/2	-1/2
(210)	0	0	0	0	1/2	1/2	1	1/2	0	0	-1	-1/2	-1/2	1/2
(211)	0	0	0	0	1/2	1	1/2	0	1/2	0	-1/2	-1	-1/2	1/2
(212)	0	0	0	0	1	0	0	1/2	1/2	1/2	-1/2	-1/2	1/2	-1
(213)	0	0	0	0	1	1/2	1/2	0	0	1/2	-1/2	-1/2	-1	1/2
(214)	0	0	1/2	1/2	1	0	0	0	1/2	-1/2	-1/2	-1	1/2	0
(215)	0	0	1/2	1/2	1	0	0	1/2	0	-1/2	-1	-1/2	1/2	0
(216)	0	0	1/2	1/2	1	0	1/2	0	0	-1/2	-1/2	-1	0	1/2
(217)	0	0	1/2	1/2	1	1/2	0	0	0	-1/2	-1	-1/2	0	1/2
(218)	0	1/2	0	1/2	0	1	0	0	-1/2	1/2	-1/2	1/2	-1	0
(219)	0	1/2	0	1/2	0	1	0	1/2	-1/2	0	-1	1/2	-1/2	0
(220)	0	1/2	0	1/2	0	1	1/2	0	-1/2	0	-1/2	0	-1	1/2
(221)	0	1/2	0	1/2	1/2	1	0	0	-1/2	0	-1	0	-1/2	1/2
(222)	0	1/2	1/2	0	0	0	1	-1/2	0	1/2	1/2	-1/2	-1	0
(223)	0	1/2	1/2	0	0	0	1	-1/2	1/2	0	1/2	-1	-1/2	0
(224)	0	1/2	1/2	0	0	1/2	1	-1/2	0	0	0	-1/2	-1	1/2
(225)	0	1/2	1/2	0	1/2	0	1	-1/2	0	0	0	-1	-1/2	1/2
(226)	1/2	0	0	1/2	0	0	-1/2	1	0	1/2	-1/2	1/2	0	-1
(227)	1/2	0	0	1/2	0	0	-1/2	1	1/2	0	-1/2	0	1/2	-1
(228)	1/2	0	0	1/2	0	1/2	-1/2	1	0	0	-1	1/2	0	-1/2
(229)	1/2	0	0	1/2	1/2	0	-1/2	1	0	0	-1	0	1/2	-1/2
(230)	1/2	0	1/2	0	0	-1/2	0	0	1	1/2	1/2	-1/2	0	-1
(231)	1/2	0	1/2	0	0	-1/2	0	1/2	1	0	0	-1/2	1/2	-1
(232)	1/2	0	1/2	0	0	-1/2	1/2	0	1	0	1/2	-1	0	-1/2
(233)	1/2	0	1/2	0	1/2	-1/2	0	0	1	0	0	-1	1/2	-1/2
(234)	1/2	1/2	0	0	-1/2	0	0	0	1/2	1	1/2	0	-1/2	-1
(235)	1/2	1/2	0	0	-1/2	0	0	1/2	0	1	0	1/2	-1/2	-1
(236)	1/2	1/2	0	0	-1/2	0	1/2	0	0	1	1/2	0	-1	-1/2
(237)	1/2	1/2	0	0	-1/2	1/2	0	0	0	1	0	1/2	-1	-1/2
(238)	0	0	0	0	0	0	0	0	1	1	0	0	0	-1
(239)	0	0	0	0	0	0	0	1	0	1	0	0	0	-1
(240)	0	0	0	0	0	0	0	1	1	0	0	0	0	-1
(241)	0	0	0	0	0	0	1	0	0	1	0	0	-1	0
(242)	0	0	0	0	0	0	1	0	1	0	0	-1	0	0
(243)	0	0	0	0	0	1	0	0	0	1	0	0	-1	0
(244)	0	0	0	0	0	1	0	1	0	0	-1	0	0	0
(245)	0	0	0	0	0	1	1	0	0	0	0	0	-1	0
(246)	0	0	0	0	1	0	0	0	1	0	0	-1	0	0
(247)	0	0	0	0	1	0	0	1	0	0	-1	0	0	0
(248)	0	0	0	0	1	0	1	0	0	0	0	-1	0	0
(249)	0	0	0	0	1	1	0	0	0	0	-1	0	0	0
(250)	0	0	0	1/2	0	1	0	1	0	0	-1	1/2	-1/2	-1/2
(251)	0	0	0	1/2	1	0	0	1	0	0	-1	-1/2	1/2	-1/2
(252)	0	0	0	1/2	1	1	0	0	0	0	-1	-1/2	-1/2	1/2
(253)	0	0	0	1	0	0	0	1	0	0	0	0	0	-1
(254)	0	0	0	1	0	1	0	0	0	0	0	0	-1	0

	m_1	m_2	m_3	m_4	m_{12}	m_{13}	m_{14}	m_{23}	m_{24}	m_{34}	m_{123}	m_{124}	m_{134}	m_{234}
(255)	0	0	0	1	1	0	0	0	0	0	0	-1	0	0
(256)	0	0	1/2	0	0	0	1	0	1	0	1/2	-1	-1/2	-1/2
(257)	0	0	1/2	0	1	0	0	0	1	0	-1/2	-1	1/2	-1/2
(258)	0	0	1/2	0	1	0	1	0	0	0	-1/2	-1	-1/2	1/2
(259)	0	0	1	0	0	0	0	0	1	0	0	0	0	-1
(260)	0	0	1	0	0	0	1	0	0	0	0	0	-1	0
(261)	0	0	1	0	1	0	0	0	0	0	-1	0	0	0
(262)	0	0	1	1	0	0	0	0	0	-1	0	0	0	0
(263)	0	1/2	0	0	0	0	1	0	0	1	1/2	-1/2	-1	-1/2
(264)	0	1/2	0	0	0	1	0	0	0	1	-1/2	1/2	-1	-1/2
(265)	0	1/2	0	0	0	1	1	0	0	0	-1/2	-1/2	-1	1/2
(266)	0	1	0	0	0	0	0	0	0	1	0	0	0	-1
(267)	0	1	0	0	0	0	1	0	0	0	0	-1	0	0
(268)	0	1	0	0	0	1	0	0	0	0	-1	0	0	0
(269)	0	1	0	1	0	0	0	0	-1	0	0	0	0	0
(270)	0	1	1	0	0	0	0	-1	0	0	0	0	0	0
(271)	1/2	0	0	0	0	0	0	0	1	1	1/2	-1/2	-1/2	-1
(272)	1/2	0	0	0	0	0	0	1	0	1	-1/2	1/2	-1/2	-1
(273)	1/2	0	0	0	0	0	0	1	1	0	-1/2	-1/2	1/2	-1
(274)	1	0	0	0	0	0	0	0	0	1	0	0	-1	0
(275)	1	0	0	0	0	0	0	0	1	0	0	-1	0	0
(276)	1	0	0	0	0	0	0	1	0	0	-1	0	0	0
(277)	1	0	0	1	0	0	-1	0	0	0	0	0	0	0
(278)	1	0	1	0	0	-1	0	0	0	0	0	0	0	0
(279)	1	1	0	0	-1	0	0	0	0	0	0	0	0	0
(280)	0	0	0	0	0	0	1	1	0	1	0	0	-1	-1
(281)	0	0	0	0	0	0	1	1	1	0	0	-1	0	-1
(282)	0	0	0	0	0	1	0	0	1	1	0	0	-1	-1
(283)	0	0	0	0	0	1	0	1	1	0	-1	0	0	-1
(284)	0	0	0	0	0	1	1	0	1	0	0	-1	-1	0
(285)	0	0	0	0	0	1	1	1	0	0	-1	0	-1	0
(286)	0	0	0	0	1	0	0	0	1	1	0	-1	0	-1
(287)	0	0	0	0	1	0	0	1	0	1	-1	0	0	-1
(288)	0	0	0	0	1	0	1	0	0	1	0	-1	-1	0
(289)	0	0	0	0	1	0	1	1	0	0	-1	-1	0	0
(290)	0	0	0	0	1	1	0	0	0	1	-1	0	-1	0
(291)	0	0	0	0	1	1	0	0	1	0	-1	-1	0	0
(292)	0	0	0	0	0	0	1	0	1	1	1	-1	-1	-1
(293)	0	0	0	0	0	1	0	1	0	1	-1	1	-1	-1
(294)	0	0	0	0	1	0	0	1	1	0	-1	-1	1	-1
(295)	0	0	0	0	1	1	1	0	0	0	-1	-1	-1	1
(296)	0	1	1	1	0	0	0	-1	-1	-1	0	0	0	1
(297)	1	0	1	1	0	-1	-1	0	0	-1	0	0	1	0
(298)	1	1	0	1	-1	0	-1	0	-1	0	0	1	0	0
(299)	1	1	1	0	-1	-1	0	-1	0	0	1	0	0	0
(300)	0	0	0	0	0	0	0	1	1	1	0	0	0	-2
(301)	0	0	0	0	0	1	1	0	0	1	0	0	-2	0
(302)	0	0	0	0	1	0	1	0	1	0	0	-2	0	0
(303)	0	0	0	0	1	1	0	1	0	0	-2	0	0	0

B List of facets of $\mathcal{M}_3^{0-1}(4)$

As in Appendix A, we have used the following shorthand: m_2 for $m^v(\{2\})$, m_{12} for $m^v(\{1, 2\})$, etc. Note that m_1 has been eliminated by the normalization equation $m_1 = 1 - m_2 - m_3 - m_4 - m_{12} - \dots - m_{234}$. Facets numbers in bold indicate the 32 facets belonging to $\mathcal{M}_3(4)$.

$$\begin{aligned}
 (1) - m_2 &\leq 0 \\
 (2) - m_3 &\leq 0 \\
 (3) - m_4 &\leq 0 \\
 (4) - m_2 - m_{12} &\leq 0 \\
 (5) - m_2 - m_{23} &\leq 0 \\
 (6) - m_2 - m_{24} &\leq 0 \\
 (7) - m_3 - m_{13} &\leq 0 \\
 (8) - m_3 - m_{23} &\leq 0 \\
 (9) - m_3 - m_{34} &\leq 0 \\
 (10) - m_4 - m_{14} &\leq 0 \\
 (11) - m_4 - m_{24} &\leq 0 \\
 (12) - m_4 - m_{34} &\leq 0 \\
 (13) - m_2 - m_{12} - m_{23} - m_{123} &\leq 0 \\
 (14) - m_2 - m_{12} - m_{24} - m_{124} &\leq 0 \\
 (15) - m_2 - m_{23} - m_{24} - m_{234} &\leq 0 \\
 (16) - m_3 - m_{13} - m_{23} - m_{123} &\leq 0 \\
 (17) - m_3 - m_{13} - m_{34} - m_{134} &\leq 0 \\
 (18) - m_3 - m_{23} - m_{34} - m_{234} &\leq 0 \\
 (19) - m_4 - m_{14} - m_{24} - m_{124} &\leq 0 \\
 (20) - m_4 - m_{14} - m_{34} - m_{134} &\leq 0 \\
 (21) - m_4 - m_{24} - m_{34} - m_{234} &\leq 0 \\
 (22) - m_2 - m_3 - m_{12} - m_{23} - m_{34} &\leq 0 \\
 (23) - m_2 - m_3 - m_{13} - m_{23} - m_{24} &\leq 0 \\
 (24) - m_2 - m_4 - m_{12} - m_{24} - m_{34} &\leq 0 \\
 (25) - m_2 - m_4 - m_{14} - m_{23} - m_{24} &\leq 0 \\
 (26) - m_3 - m_4 - m_{13} - m_{24} - m_{34} &\leq 0 \\
 (27) - m_3 - m_4 - m_{14} - m_{23} - m_{34} &\leq 0 \\
 (28) - m_2 - m_3 - m_4 - m_{12} - m_{23} - m_{24} - m_{234} &\leq 0 \\
 (29) - m_2 - m_3 - m_4 - m_{13} - m_{23} - m_{34} - m_{234} &\leq 0 \\
 (30) - m_2 - m_3 - m_4 - m_{14} - m_{24} - m_{34} - m_{234} &\leq 0 \\
 (31) - m_2 - m_{12} - m_{23} - m_{24} - m_{123} - m_{124} - m_{234} &\leq 0 \\
 (32) - m_3 - m_{13} - m_{23} - m_{34} - m_{123} - m_{134} - m_{234} &\leq 0 \\
 (33) - m_4 - m_{14} - m_{24} - m_{34} - m_{124} - m_{134} - m_{234} &\leq 0 \\
 (34) - m_2 - m_3 - m_4 - m_{12} - m_{13} - m_{23} - m_{24} - m_{123} - m_{234} &\leq 0 \\
 (35) - m_2 - m_3 - m_4 - m_{12} - m_{13} - m_{23} - m_{34} - m_{123} - m_{234} &\leq 0 \\
 (36) - m_2 - m_3 - m_4 - m_{12} - m_{14} - m_{23} - m_{24} - m_{124} - m_{234} &\leq 0 \\
 (37) - m_2 - m_3 - m_4 - m_{12} - m_{14} - m_{24} - m_{34} - m_{124} - m_{234} &\leq 0 \\
 (38) - m_2 - m_3 - m_4 - m_{13} - m_{14} - m_{23} - m_{34} - m_{134} - m_{234} &\leq 0 \\
 (39) - m_2 - m_3 - m_4 - m_{13} - m_{14} - m_{24} - m_{34} - m_{134} - m_{234} &\leq 0
 \end{aligned}$$

$$\begin{aligned}
(40) & -2m_2 - m_{12} - m_{23} - m_{24} && \leq 0 \\
(41) & -2m_3 - m_{13} - m_{23} - m_{34} && \leq 0 \\
(42) & -2m_4 - m_{14} - m_{24} - m_{34} && \leq 0 \\
(43) & -2m_2 - m_3 - m_{12} - m_{13} - m_{23} - m_{24} - m_{123} - m_{234} && \leq 0 \\
(44) & -2m_2 - m_3 - m_{12} - m_{23} - m_{24} - m_{34} - m_{123} - m_{234} && \leq 0 \\
(45) & -2m_2 - m_4 - m_{12} - m_{14} - m_{23} - m_{24} - m_{124} - m_{234} && \leq 0 \\
(46) & -2m_2 - m_4 - m_{12} - m_{23} - m_{24} - m_{34} - m_{124} - m_{234} && \leq 0 \\
(47) & -m_2 - 2m_3 - m_{12} - m_{13} - m_{23} - m_{34} - m_{123} - m_{234} && \leq 0 \\
(48) & -m_2 - 2m_3 - m_{13} - m_{23} - m_{24} - m_{34} - m_{123} - m_{234} && \leq 0 \\
(49) & -m_2 - m_3 - m_4 - m_{12} - 2m_{23} - m_{24} - m_{123} - m_{234} && \leq 0 \\
(50) & -m_2 - m_3 - m_4 - m_{12} - m_{23} - 2m_{24} - m_{124} - m_{234} && \leq 0 \\
(51) & -m_2 - m_3 - m_4 - m_{13} - 2m_{23} - m_{34} - m_{123} - m_{234} && \leq 0 \\
(52) & -m_2 - m_3 - m_4 - m_{13} - m_{23} - 2m_{34} - m_{134} - m_{234} && \leq 0 \\
(53) & -m_2 - m_3 - m_4 - m_{14} - 2m_{24} - m_{34} - m_{124} - m_{234} && \leq 0 \\
(54) & -m_2 - m_3 - m_4 - m_{14} - m_{24} - 2m_{34} - m_{134} - m_{234} && \leq 0 \\
(55) & -m_2 - 2m_4 - m_{12} - m_{14} - m_{24} - m_{34} - m_{124} - m_{234} && \leq 0 \\
(56) & -m_2 - 2m_4 - m_{14} - m_{23} - m_{24} - m_{34} - m_{124} - m_{234} && \leq 0 \\
(57) & -2m_3 - m_4 - m_{13} - m_{14} - m_{23} - m_{34} - m_{134} - m_{234} && \leq 0 \\
(58) & -2m_3 - m_4 - m_{13} - m_{23} - m_{24} - m_{34} - m_{134} - m_{234} && \leq 0 \\
(59) & -m_3 - 2m_4 - m_{13} - m_{14} - m_{24} - m_{34} - m_{134} - m_{234} && \leq 0 \\
(60) & -m_3 - 2m_4 - m_{14} - m_{23} - m_{24} - m_{34} - m_{134} - m_{234} && \leq 0 \\
(61) & -m_2 - m_3 - m_4 - m_{12} - m_{13} - 2m_{23} - m_{24} - 2m_{123} - m_{234} && \leq 0 \\
(62) & -m_2 - m_3 - m_4 - m_{12} - m_{13} - 2m_{23} - m_{34} - 2m_{123} - m_{234} && \leq 0 \\
(63) & -m_2 - m_3 - m_4 - m_{12} - m_{14} - m_{23} - 2m_{24} - 2m_{124} - m_{234} && \leq 0 \\
(64) & -m_2 - m_3 - m_4 - m_{12} - m_{14} - 2m_{24} - m_{34} - 2m_{124} - m_{234} && \leq 0 \\
(65) & -m_2 - m_3 - m_4 - m_{12} - 2m_{23} - m_{24} - m_{34} - m_{123} - 2m_{234} && \leq 0 \\
(66) & -m_2 - m_3 - m_4 - m_{12} - m_{23} - 2m_{24} - m_{34} - m_{124} - 2m_{234} && \leq 0 \\
(67) & -m_2 - m_3 - m_4 - m_{13} - m_{14} - m_{23} - 2m_{34} - 2m_{134} - m_{234} && \leq 0 \\
(68) & -m_2 - m_3 - m_4 - m_{13} - m_{14} - m_{24} - 2m_{34} - 2m_{134} - m_{234} && \leq 0 \\
(69) & -m_2 - m_3 - m_4 - m_{13} - 2m_{23} - m_{24} - m_{34} - m_{123} - 2m_{234} && \leq 0 \\
(70) & -m_2 - m_3 - m_4 - m_{13} - m_{23} - m_{24} - 2m_{34} - m_{134} - 2m_{234} && \leq 0 \\
(71) & -m_2 - m_3 - m_4 - m_{14} - m_{23} - 2m_{24} - m_{34} - m_{124} - 2m_{234} && \leq 0 \\
(72) & -m_2 - m_3 - m_4 - m_{14} - m_{23} - m_{24} - 2m_{34} - m_{134} - 2m_{234} && \leq 0 \\
(73) & -2m_2 - m_3 - m_{12} - m_{13} - 2m_{23} - m_{24} - 2m_{123} - m_{234} && \leq 0 \\
(74) & -2m_2 - m_3 - m_{12} - 2m_{23} - m_{24} - m_{34} - m_{123} - 2m_{234} && \leq 0 \\
(75) & -2m_2 - m_4 - m_{12} - m_{14} - m_{23} - 2m_{24} - 2m_{124} - m_{234} && \leq 0 \\
(76) & -2m_2 - m_4 - m_{12} - m_{23} - 2m_{24} - m_{34} - m_{124} - 2m_{234} && \leq 0 \\
(77) & -m_2 - 2m_3 - m_{12} - m_{13} - 2m_{23} - m_{34} - 2m_{123} - m_{234} && \leq 0 \\
(78) & -m_2 - 2m_3 - m_{13} - 2m_{23} - m_{24} - m_{34} - m_{123} - 2m_{234} && \leq 0 \\
(79) & -m_2 - 2m_4 - m_{12} - m_{14} - 2m_{24} - m_{34} - 2m_{124} - m_{234} && \leq 0 \\
(80) & -m_2 - 2m_4 - m_{14} - m_{23} - 2m_{24} - m_{34} - m_{124} - 2m_{234} && \leq 0 \\
(81) & -2m_3 - m_4 - m_{13} - m_{14} - m_{23} - 2m_{34} - 2m_{134} - m_{234} && \leq 0 \\
(82) & -2m_3 - m_4 - m_{13} - m_{23} - m_{24} - 2m_{34} - m_{134} - 2m_{234} && \leq 0
\end{aligned}$$

$$\begin{aligned}
(83) & -m_3 - 2m_4 - m_{13} - m_{14} - m_{24} - 2m_{34} - 2m_{134} - m_{234} && \leq 0 \\
(84) & -m_3 - 2m_4 - m_{14} - m_{23} - m_{24} - 2m_{34} - m_{134} - 2m_{234} && \leq 0 \\
(85) & -m_2 - m_3 - m_4 - m_{12} - m_{13} - 2m_{23} - m_{24} - m_{34} - 2m_{123} - 2m_{234} && \leq 0 \\
(86) & -m_2 - m_3 - m_4 - m_{12} - m_{14} - m_{23} - 2m_{24} - m_{34} - 2m_{124} - 2m_{234} && \leq 0 \\
(87) & -m_2 - m_3 - m_4 - m_{13} - m_{14} - m_{23} - m_{24} - 2m_{34} - 2m_{134} - 2m_{234} && \leq 0 \\
(88) & -2m_2 - 2m_3 - m_{12} - m_{13} - 2m_{23} - m_{24} - m_{34} - 2m_{123} - 2m_{234} && \leq 0 \\
(89) & -2m_2 - 2m_4 - m_{12} - m_{14} - m_{23} - 2m_{24} - m_{34} - 2m_{124} - 2m_{234} && \leq 0 \\
(90) & -2m_3 - 2m_4 - m_{13} - m_{14} - m_{23} - m_{24} - 2m_{34} - 2m_{134} - 2m_{234} && \leq 0 \\
(91) & m_4 + m_{23} + m_{34} + m_{134} + m_{234} && \leq 1 \\
(92) & m_4 + m_{23} + m_{24} + m_{124} + m_{234} && \leq 1 \\
(93) & m_4 + m_{13} + m_{34} + m_{134} + m_{234} && \leq 1 \\
(94) & m_4 + m_{13} + m_{14} + m_{124} + m_{134} && \leq 1 \\
(95) & m_4 + m_{12} + m_{24} + m_{124} + m_{234} && \leq 1 \\
(96) & m_4 + m_{12} + m_{14} + m_{124} + m_{134} && \leq 1 \\
(97) & m_3 + m_{24} + m_{34} + m_{134} + m_{234} && \leq 1 \\
(98) & m_3 + m_{23} + m_{24} + m_{123} + m_{234} && \leq 1 \\
(99) & m_3 + m_{14} + m_{34} + m_{134} + m_{234} && \leq 1 \\
(100) & m_3 + m_{13} + m_{14} + m_{123} + m_{134} && \leq 1 \\
(101) & m_3 + m_{12} + m_{23} + m_{123} + m_{234} && \leq 1 \\
(102) & m_3 + m_{12} + m_{13} + m_{123} + m_{134} && \leq 1 \\
(103) & m_2 + m_{24} + m_{34} + m_{124} + m_{234} && \leq 1 \\
(104) & m_2 + m_{23} + m_{34} + m_{123} + m_{234} && \leq 1 \\
(105) & m_2 + m_{14} + m_{24} + m_{124} + m_{234} && \leq 1 \\
(106) & m_2 + m_{13} + m_{23} + m_{123} + m_{234} && \leq 1 \\
(107) & m_2 + m_{12} + m_{14} + m_{123} + m_{124} && \leq 1 \\
(108) & m_2 + m_{12} + m_{13} + m_{123} + m_{124} && \leq 1 \\
(109) & m_4 + m_{23} + m_{24} + m_{34} + m_{124} + m_{134} + m_{234} && \leq 1 \\
(110) & m_4 + m_{13} + m_{14} + m_{34} + m_{124} + m_{134} + m_{234} && \leq 1 \\
(111) & m_4 + m_{12} + m_{14} + m_{24} + m_{124} + m_{134} + m_{234} && \leq 1 \\
(112) & m_3 + m_{23} + m_{24} + m_{34} + m_{123} + m_{134} + m_{234} && \leq 1 \\
(113) & m_3 + m_{13} + m_{14} + m_{34} + m_{123} + m_{134} + m_{234} && \leq 1 \\
(114) & m_3 + m_{12} + m_{13} + m_{23} + m_{123} + m_{134} + m_{234} && \leq 1 \\
(115) & m_2 + m_{23} + m_{24} + m_{34} + m_{123} + m_{124} + m_{234} && \leq 1 \\
(116) & m_2 + m_{12} + m_{14} + m_{24} + m_{123} + m_{124} + m_{234} && \leq 1 \\
(117) & m_2 + m_{12} + m_{13} + m_{23} + m_{123} + m_{124} + m_{234} && \leq 1 \\
(118) & m_2 + m_3 + m_4 + m_{23} + m_{24} + m_{34} + m_{234} && \leq 1 \\
(119) & m_3 + m_4 + m_{14} + m_{23} + m_{34} + m_{123} + m_{124} + m_{134} + m_{234} && \leq 1 \\
(120) & m_3 + m_4 + m_{13} + m_{24} + m_{34} + m_{123} + m_{124} + m_{134} + m_{234} && \leq 1 \\
(121) & m_2 + m_4 + m_{14} + m_{23} + m_{24} + m_{123} + m_{124} + m_{134} + m_{234} && \leq 1 \\
(122) & m_2 + m_4 + m_{12} + m_{24} + m_{34} + m_{123} + m_{124} + m_{134} + m_{234} && \leq 1 \\
(123) & m_2 + m_3 + m_{13} + m_{23} + m_{24} + m_{123} + m_{124} + m_{134} + m_{234} && \leq 1 \\
(124) & m_2 + m_3 + m_{12} + m_{23} + m_{34} + m_{123} + m_{124} + m_{134} + m_{234} && \leq 1 \\
(125) & m_2 + m_3 + m_4 + m_{14} + m_{23} + m_{24} + m_{34} + m_{124} + m_{134} + m_{234} && \leq 1
\end{aligned}$$

- (126) $m_2 + m_3 + m_4 + m_{13} + m_{23} + m_{24} + m_{34} + m_{123} + m_{134} + m_{234} \leq 1$
- (127) $m_2 + m_3 + m_4 + m_{12} + m_{23} + m_{24} + m_{34} + m_{123} + m_{124} + m_{234} \leq 1$
- (128) $m_2 + m_3 + m_4 + m_{13} + m_{14} + m_{23} + m_{24} + m_{34} + m_{123} + m_{124} + m_{134} + m_{234} \leq 1$
- (129) $m_2 + m_3 + m_4 + m_{12} + m_{14} + m_{23} + m_{24} + m_{34} + m_{123} + m_{124} + m_{134} + m_{234} \leq 1$
- (130) $m_2 + m_3 + m_4 + m_{12} + m_{13} + m_{23} + m_{24} + m_{34} + m_{123} + m_{124} + m_{134} + m_{234} \leq 1$
- (131) $m_2 + m_3 + m_4 + m_{12} + m_{13} + m_{14} + m_{23} + m_{24} + m_{34} + m_{123} + m_{124} + m_{134} + m_{234} \leq 1$
- (132) $-m_2 + m_3 + m_4 + m_{14} + m_{34} + m_{134} + m_{234} \leq 1$
- (133) $-m_2 + m_3 + m_4 + m_{13} + m_{34} + m_{134} + m_{234} \leq 1$
- (134) $m_4 - m_{12} + m_{23} + m_{24} + m_{34} + m_{134} + m_{234} \leq 1$
- (135) $m_4 - m_{12} + m_{13} + m_{14} + m_{34} + m_{134} + m_{234} \leq 1$
- (136) $m_4 - m_{13} + m_{23} + m_{24} + m_{34} + m_{124} + m_{234} \leq 1$
- (137) $m_4 + m_{13} + m_{14} - m_{23} + m_{34} + m_{124} + m_{134} \leq 1$
- (138) $m_4 + m_{12} - m_{13} + m_{14} + m_{24} + m_{124} + m_{234} \leq 1$
- (139) $m_4 + m_{12} + m_{14} - m_{23} + m_{24} + m_{124} + m_{134} \leq 1$
- (140) $m_3 - m_{12} + m_{23} + m_{24} + m_{34} + m_{134} + m_{234} \leq 1$
- (141) $m_3 - m_{12} + m_{13} + m_{14} + m_{34} + m_{134} + m_{234} \leq 1$
- (142) $m_3 - m_{14} + m_{23} + m_{24} + m_{34} + m_{123} + m_{234} \leq 1$
- (143) $m_3 + m_{13} + m_{14} - m_{24} + m_{34} + m_{123} + m_{134} \leq 1$
- (144) $m_3 + m_{12} + m_{13} - m_{14} + m_{23} + m_{123} + m_{234} \leq 1$
- (145) $m_3 + m_{12} + m_{13} + m_{23} - m_{24} + m_{123} + m_{134} \leq 1$
- (146) $m_2 - m_3 + m_4 + m_{14} + m_{24} + m_{124} + m_{234} \leq 1$
- (147) $m_2 - m_3 + m_4 + m_{12} + m_{24} + m_{124} + m_{234} \leq 1$
- (148) $m_2 - m_{13} + m_{23} + m_{24} + m_{34} + m_{124} + m_{234} \leq 1$
- (149) $m_2 - m_{14} + m_{23} + m_{24} + m_{34} + m_{123} + m_{234} \leq 1$
- (150) $m_2 + m_{12} - m_{13} + m_{14} + m_{24} + m_{124} + m_{234} \leq 1$
- (151) $m_2 + m_{12} + m_{14} + m_{24} - m_{34} + m_{123} + m_{124} \leq 1$
- (152) $m_2 + m_{12} + m_{13} - m_{14} + m_{23} + m_{123} + m_{234} \leq 1$
- (153) $m_2 + m_{12} + m_{13} + m_{23} - m_{34} + m_{123} + m_{124} \leq 1$
- (154) $m_2 + m_3 - m_4 + m_{13} + m_{23} + m_{123} + m_{234} \leq 1$
- (155) $m_2 + m_3 - m_4 + m_{12} + m_{23} + m_{123} + m_{234} \leq 1$
- (156) $m_4 - m_{12} + m_{24} + m_{34} - m_{123} + m_{134} + m_{234} \leq 1$
- (157) $m_4 - m_{12} + m_{23} + m_{34} - m_{124} + m_{134} + m_{234} \leq 1$
- (158) $m_4 - m_{12} + m_{14} + m_{34} - m_{123} + m_{134} + m_{234} \leq 1$
- (159) $m_4 - m_{12} + m_{13} + m_{34} - m_{124} + m_{134} + m_{234} \leq 1$
- (160) $m_4 - m_{13} + m_{24} + m_{34} - m_{123} + m_{124} + m_{234} \leq 1$
- (161) $m_4 - m_{13} + m_{23} + m_{24} + m_{124} - m_{134} + m_{234} \leq 1$
- (162) $m_4 - m_{13} + m_{14} + m_{24} - m_{123} + m_{124} + m_{234} \leq 1$
- (163) $m_4 + m_{14} - m_{23} + m_{34} - m_{123} + m_{124} + m_{134} \leq 1$
- (164) $m_4 + m_{14} - m_{23} + m_{24} - m_{123} + m_{124} + m_{134} \leq 1$
- (165) $m_4 + m_{13} + m_{14} - m_{23} + m_{124} + m_{134} - m_{234} \leq 1$
- (166) $m_4 + m_{12} - m_{13} + m_{24} + m_{124} - m_{134} + m_{234} \leq 1$
- (167) $m_4 + m_{12} + m_{14} - m_{23} + m_{124} + m_{134} - m_{234} \leq 1$
- (168) $m_3 - m_{12} + m_{24} + m_{34} - m_{123} + m_{134} + m_{234} \leq 1$

- (169) $m_3 - m_{12} + m_{23} + m_{34} - m_{124} + m_{134} + m_{234} \leq 1$
- (170) $m_3 - m_{12} + m_{14} + m_{34} - m_{123} + m_{134} + m_{234} \leq 1$
- (171) $m_3 - m_{12} + m_{13} + m_{34} - m_{124} + m_{134} + m_{234} \leq 1$
- (172) $m_3 - m_{14} + m_{23} + m_{34} + m_{123} - m_{124} + m_{234} \leq 1$
- (173) $m_3 - m_{14} + m_{23} + m_{24} + m_{123} - m_{134} + m_{234} \leq 1$
- (174) $m_3 + m_{13} - m_{14} + m_{23} + m_{123} - m_{124} + m_{234} \leq 1$
- (175) $m_3 + m_{13} - m_{24} + m_{34} + m_{123} - m_{124} + m_{134} \leq 1$
- (176) $m_3 + m_{13} + m_{23} - m_{24} + m_{123} - m_{124} + m_{134} \leq 1$
- (177) $m_3 + m_{13} + m_{14} - m_{24} + m_{123} + m_{134} - m_{234} \leq 1$
- (178) $m_3 + m_{12} - m_{14} + m_{23} + m_{123} - m_{134} + m_{234} \leq 1$
- (179) $m_3 + m_{12} + m_{13} - m_{24} + m_{123} + m_{134} - m_{234} \leq 1$
- (180) $m_2 - m_{13} + m_{24} + m_{34} - m_{123} + m_{124} + m_{234} \leq 1$
- (181) $m_2 - m_{13} + m_{23} + m_{24} + m_{124} - m_{134} + m_{234} \leq 1$
- (182) $m_2 - m_{13} + m_{14} + m_{24} - m_{123} + m_{124} + m_{234} \leq 1$
- (183) $m_2 - m_{14} + m_{23} + m_{34} + m_{123} - m_{124} + m_{234} \leq 1$
- (184) $m_2 - m_{14} + m_{23} + m_{24} + m_{123} - m_{134} + m_{234} \leq 1$
- (185) $m_2 + m_{13} - m_{14} + m_{23} + m_{123} - m_{124} + m_{234} \leq 1$
- (186) $m_2 + m_{12} - m_{13} + m_{24} + m_{124} - m_{134} + m_{234} \leq 1$
- (187) $m_2 + m_{12} - m_{14} + m_{23} + m_{123} - m_{134} + m_{234} \leq 1$
- (188) $m_2 + m_{12} + m_{24} - m_{34} + m_{123} + m_{124} - m_{134} \leq 1$
- (189) $m_2 + m_{12} + m_{23} - m_{34} + m_{123} + m_{124} - m_{134} \leq 1$
- (190) $m_2 + m_{12} + m_{14} - m_{34} + m_{123} + m_{124} - m_{234} \leq 1$
- (191) $m_2 + m_{12} + m_{13} - m_{34} + m_{123} + m_{124} - m_{234} \leq 1$
- (192) $m_4 - m_{12} + m_{34} - m_{123} - m_{124} + m_{134} + m_{234} \leq 1$
- (193) $m_4 - m_{13} + m_{24} - m_{123} + m_{124} - m_{134} + m_{234} \leq 1$
- (194) $m_4 + m_{14} - m_{23} - m_{123} + m_{124} + m_{134} - m_{234} \leq 1$
- (195) $m_3 - m_{12} + m_{34} - m_{123} - m_{124} + m_{134} + m_{234} \leq 1$
- (196) $m_3 - m_{14} + m_{23} + m_{123} - m_{124} - m_{134} + m_{234} \leq 1$
- (197) $m_3 + m_{13} - m_{24} + m_{123} - m_{124} + m_{134} - m_{234} \leq 1$
- (198) $m_2 - m_{13} + m_{24} - m_{123} + m_{124} - m_{134} + m_{234} \leq 1$
- (199) $m_2 - m_{14} + m_{23} + m_{123} - m_{124} - m_{134} + m_{234} \leq 1$
- (200) $m_2 + m_{12} - m_{34} + m_{123} + m_{124} - m_{134} - m_{234} \leq 1$
- (201) $-m_2 + m_3 + m_4 - m_{12} + m_{14} + m_{34} - m_{123} + m_{134} + m_{234} \leq 1$
- (202) $-m_2 + m_3 + m_4 - m_{12} + m_{13} + m_{34} - m_{124} + m_{134} + m_{234} \leq 1$
- (203) $m_2 - m_3 + m_4 - m_{13} + m_{14} + m_{24} - m_{123} + m_{124} + m_{234} \leq 1$
- (204) $m_2 - m_3 + m_4 + m_{12} - m_{13} + m_{24} + m_{124} - m_{134} + m_{234} \leq 1$
- (205) $m_2 + m_3 - m_4 + m_{13} - m_{14} + m_{23} + m_{123} - m_{124} + m_{234} \leq 1$
- (206) $m_2 + m_3 - m_4 + m_{12} - m_{14} + m_{23} + m_{123} - m_{134} + m_{234} \leq 1$
- (207) $2m_2 + 2m_3 + 2m_4 + m_{12} + m_{13} + m_{14} + 2m_{23} + 2m_{24} + 2m_{34} + 2m_{123} + 2m_{124} + 2m_{134} + 2m_{234} \leq 2$
- (208) $2m_3 + 2m_4 + m_{13} + m_{14} + m_{23} + m_{24} + 2m_{34} + 2m_{134} + 2m_{234} \leq 2$
- (209) $2m_2 + 2m_4 + m_{12} + m_{14} + m_{23} + 2m_{24} + m_{34} + 2m_{124} + 2m_{234} \leq 2$
- (210) $2m_2 + 2m_3 + m_{12} + m_{13} + 2m_{23} + m_{24} + m_{34} + 2m_{123} + 2m_{234} \leq 2$
- (211) $m_2 + 2m_3 + 2m_4 + m_{13} + m_{14} + m_{23} + 2m_{24} + 2m_{34} + m_{124} + 2m_{134} + 2m_{234} \leq 2$

- (212) $m_2 + 2m_3 + 2m_4 + m_{13} + m_{14} + 2m_{23} + m_{24} + 2m_{34} + m_{123} + 2m_{134} + 2m_{234} \leq 2$
- (213) $2m_2 + m_3 + 2m_4 + m_{12} + m_{14} + m_{23} + 2m_{24} + 2m_{34} + 2m_{124} + m_{134} + 2m_{234} \leq 2$
- (214) $2m_2 + m_3 + 2m_4 + m_{12} + m_{14} + 2m_{23} + 2m_{24} + m_{34} + m_{123} + 2m_{124} + 2m_{234} \leq 2$
- (215) $2m_2 + 2m_3 + m_4 + m_{12} + m_{13} + 2m_{23} + m_{24} + 2m_{34} + 2m_{123} + m_{134} + 2m_{234} \leq 2$
- (216) $2m_2 + 2m_3 + m_4 + m_{12} + m_{13} + 2m_{23} + 2m_{24} + m_{34} + 2m_{123} + m_{124} + 2m_{234} \leq 2$
- (217) $m_2 + 2m_3 + 2m_4 + m_{12} + m_{13} + m_{14} + m_{23} + 2m_{24} + 2m_{34} + m_{123} + m_{124} + 2m_{134} + 2m_{234} \leq 2$
- (218) $m_2 + 2m_3 + 2m_4 + m_{12} + m_{13} + m_{14} + 2m_{23} + m_{24} + 2m_{34} + m_{123} + m_{124} + 2m_{134} + 2m_{234} \leq 2$
- (219) $2m_2 + m_3 + 2m_4 + m_{12} + m_{13} + m_{14} + m_{23} + 2m_{24} + 2m_{34} + m_{123} + 2m_{124} + m_{134} + 2m_{234} \leq 2$
- (220) $2m_2 + m_3 + 2m_4 + m_{12} + m_{13} + m_{14} + 2m_{23} + 2m_{24} + m_{34} + m_{123} + 2m_{124} + m_{134} + 2m_{234} \leq 2$
- (221) $2m_2 + 2m_3 + m_4 + m_{12} + m_{13} + m_{14} + 2m_{23} + m_{24} + 2m_{34} + 2m_{123} + m_{124} + m_{134} + 2m_{234} \leq 2$
- (222) $2m_2 + 2m_3 + m_4 + m_{12} + m_{13} + m_{14} + 2m_{23} + 2m_{24} + m_{34} + 2m_{123} + m_{124} + m_{134} + 2m_{234} \leq 2$