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# Subjective Expected Utility Through Stochastic Independence

Michel Grabisch\*    Benjamin Monet†    Vassili Vergopoulos‡

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## Abstract

This paper studies decision-making in the face of two stochastically independent sources of uncertainty. It characterizes axiomatically a Subjective Expected Utility representation of preferences where subjective beliefs consist of a product probability measure. The two key axioms in this characterization both involve some behavioral notions of stochastic independence. Our result can be understood as a purely subjective version of the Anscombe and Aumann (1963) theorem that avoids the controversial use of exogenous probabilities by appealing to stochastic independence. We also obtain an extension to Choquet Expected Utility representations.

**Keywords:** subjective probability, expected utility, stochastic independence, subjective independence, capacity, Choquet expectation.

**JEL classification:** D81.

## 1 Introduction

Several economic decisions under uncertainty involve multiple and stochastically independent sources of uncertainty. For instance, one may think of portfolio choice with independent assets or insurance premium computation of independent contracts. In strategic interactions too, each player forms beliefs about the profile of independent strategies of the other players. Randomization provides another example: economic agents often condition two uncertain actions on the outcome of an independent source of uncertainty. Finally,

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several decision-theoretic thought experiments of great conceptual interest involve multiple independent sources of uncertainty like the Ellsberg (1961) two-urn paradox or the Machina (2014) slightly bent coin problem.

This paper studies economic decisions in the presence of two stochastically independent sources of uncertainty. Its objective is to derive axiomatically a Subjective Expected Utility (SEU) representation of preferences. In principle, it would be possible to invoke the classical general theorems of Savage (1954) or Anscombe and Aumann (1963) and apply them to our particular case. However, this approach would fail to provide subjective beliefs in the specific form of a product probability measure and hence fail to account for the agent's belief in the stochastic independence of the two sources. In fact, we aim at formulating new simple axioms that take advantage of the specific structure that the two sources provide and hence obtain a new axiomatic derivation that is specifically adapted to cases of two independent sources of uncertainty.

The two key axioms of our approach both relate to the notion that the two sources of uncertainty are stochastically independent of each other. The first one, henceforth dubbed Stochastic Independence, essentially asserts that observing an event in one source does not affect the preference on acts whose outcomes only depend on the other source. The second key axiom, henceforth dubbed Subjective Independence, can be explained intuitively in terms of one of its consequences. Roughly, it essentially requires the indifference to randomization when the acts that are randomized are stochastically independent from the event on which they are randomized. The conjunction of Stochastic Independence and Subjective Independence turns out to be very restrictive. For instance, Section 2 shows that they rule out the Ellsberg paradoxical pattern of preferences. In fact, our main result, Theorem 7, shows that the two axioms (with the help of other ones that are familiar from the Savage theorem) characterize SEU representations where subjective beliefs take the form of a product probability measure.

Stochastic independence has received surprisingly little attention in the decision-theoretic literature. Most papers dealing with this notion assume a specific representation and characterize some form of stochastic independence within the postulated representation. A notable exception is Mongin (2020) where, as in our main theorem, stochastic independence is made a powerful tool for the derivation of the representation itself. (See also Mongin and Pivato (2015).) Our results resemble each other and point at similar conclusions: they essentially suggest that a sufficiently strong treatment of stochastic independence is almost sufficient for the SEU representation. Interestingly, this parallels the much more received idea that a sufficiently strong treatment of updating is almost sufficient for the SEU representation. (For instance, see Hammond (1988), Ghirardato (2002) or Bastianello et al. (2022)).

Another more specific feature of our main result is its close relationship to the Anscombe and Aumann (1963) theorem<sup>1</sup>. The latter provides a particularly simple and elegant derivation of SEU on an arbitrary state space. But it assumes exogenously given probabilities,

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<sup>1</sup>To be more precise, what we call the Anscombe-Aumann theorem throughout the paper is the account that Schmeidler (1989) provides.

and this goes against the behavioristic approach to probability. For this reason, the Savage purely behavioral approach to SEU is often judged to rely on a more solid conceptual basis. (See, for instance, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003).) But, as explicitly noted by Schmeidler (1989), the exogenous probabilities can be interpreted in a subjective way as arising from the Savage apparatus. This is exactly the plan that our Theorem 7 implements. Indeed, Schmeidler’s view requires the introduction of a second state space to which the Savage theorem is applied. From there, our axioms of Stochastic Independence and Subjective Independence imply the existence of preferences à la Anscombe-Aumann. Our SEU representation is finally obtained as a corollary of their theorem. Hence, we view our Theorem 7 as a purely subjective version of Anscombe-Aumann theorem that avoids the controversial use of exogenous probabilities by appealing to adequate notions of stochastic independence. The very possibility of obtaining such a result suggests that the assumption of exogenous probabilities can be made for mathematical simplicity and perhaps should not be the object of conceptual critique, at least when the introduction of a second, independent source is justifiable.

Furthermore, the Anscombe-Aumann theorem possesses the attractive feature of lending itself easily to generalizations. For instance, Schmeidler (1989) provides a generalization to Choquet Expected Utility (CEU) that accounts for the ambiguity an agent may perceive. Our second main result, Theorem 8, suggests that Theorem 7 retains this feature. Indeed, Theorem 8 axiomatically characterizes a version of CEU by weakening Subjective Independence. It can hence be seen as a purely subjective version of the Schmeidler theorem avoiding the use of exogenous probabilities.

The paper is structured as follows: Section 2 introduces Stochastic Independence and Subjective Independence in an informal way and applies them to the Ellsberg two-urn paradox. Section 3 introduces the framework and notation of the paper. It also puts emphasis on the main technical contribution of the paper, namely the purely subjective formulation of the Anscombe-Aumann mixture operation. Section 4 presents the axioms required for our results including Stochastic Independence and Subjective Independence. Section 5 presents our main result, the purely subjective version of the Anscombe-Aumann theorem. Section 6 provides the generalization of our main result to capacities and Choquet expectations. Finally, all the proofs are gathered in the appendix.

## 2 Revisiting the Ellsberg two-urn paradox

This section illustrates the results of the paper in the familiar context of the Ellsberg (1961) two-urn thought experiment. Suppose a Decision-Maker (DM) confronts two urns. Each urn contains 100 balls of two possible colors, red or black. The DM knows that Urn 2 contains 50 red balls and 50 black ones but he is ignorant of the proportions in Urn 1. Let  $\mathcal{S}_1 := \{R1, B1\}$  and  $\mathcal{S}_2 := \{R2, B2\}$  represent the two sources of uncertainty. Consider the following two bets on the colors of the balls drawn in the two urns:

$f$	R1	B1
R2	+\$10	\$0
B2	+\$10	\$0

$g$	R1	B1
R2	\$0	+\$10
B2	\$0	+\$10

The bet  $f$  (resp.  $g$ ) represents a bet on drawing a red (resp. black) ball in Urn 1. The two bets are ambiguous as their payments depend on the realization of events of unknown probabilities. Consider now the next pair of unambiguous bets:

$h$	R1	B1
R2	+\$10	+\$10
B2	\$0	\$0

$k$	R1	B1
R2	\$0	\$0
B2	+\$10	+\$10

The bet  $h$  (resp.  $k$ ) represents a bet on drawing a red (resp. black) ball in Urn 2. The DM is asked to rank these four acts in order of preference. The typical (ambiguity averse) Ellsberg ranking consists of the following:

$$h \sim k \succ g \sim f.$$

The DM may believe that the two sources of uncertainty in this experiment are stochastically independent of each other and hence may form preferences and beliefs that reflect such stochastic independence. Intuitively, stochastic independence will here mean that observing which state in  $\mathcal{S}_1$  obtains does not affect the preference rankings of alternatives (or acts) that only depend on  $\mathcal{S}_2$ .

**(Stochastic Independence)** For all alternatives  $a, b$ , if  $a$  is (weakly) preferred to  $b$  at every possible state in  $\mathcal{S}_1$ , then  $a$  is (weakly) preferred to  $b$  as well.

We now apply Stochastic Independence to the Ellsberg urns. First, we suppose that  $f \sim g$  and  $h \sim k$  as in the typical Ellsberg ranking. Second, we consider the following fifth bet denoted by  $l$ :

$l$	R1	B1
R2	+\$10	\$0
B2	\$0	+\$10

At R1, all three of  $h, k$  and  $l$  are indifferent. Indeed, both  $h(R1, \cdot)$  and  $l(R1, \cdot)$  represent a bet on R2 while  $k(R1, \cdot)$  represents a bet on B2. But the latter bet is indifferent to the former one since we have assumed  $h \sim k$ . Likewise, all three of  $h(B1, \cdot)$ ,  $k(B1, \cdot)$  and  $l(B1, \cdot)$  are indifferent. This relies again on the assumption  $h \sim k$ . Then, by Stochastic Independence, we obtain the indifference between  $h, k$  and  $l$ .

Furthermore, for two alternatives  $a, b$  and an event  $E_2$  in  $\mathcal{S}_2$ , the randomization of  $a$  and  $b$  on  $E_2$  describes the contingent plan consisting in choosing  $a$  if  $E_2$  obtains and  $b$  otherwise. Moreover, we say that some alternatives  $a$  and  $b$  are stochastically independent from some event  $E_2$  if observing that  $E_2$  holds does not affect the evaluation of  $a$  and  $b$ . This notion

remains vague and will only become formal in Section 3. In this context, the next principle asserts that the agent is indifferent to randomizing two indifferent alternatives on some event from which they are stochastically independent.

**(Indifference to Independent Randomization)** For all event  $E_2$  in  $\mathcal{S}_2$  and all indifferent alternatives  $a, b$ , if  $a$  and  $b$  are “stochastically independent” from  $E_2$ , then the randomization of  $a$  and  $b$  on  $E_2$  is indifferent to  $a$  and  $b$ .

Note that  $f$  and  $g$  only depend on  $\mathcal{S}_1$ . Hence, in some obvious sense, they are both stochastically independent from  $\{\text{R2}\}$ . Since  $f \sim g$ , Indifference to Independent Randomization implies that the randomization of  $f$  and  $g$  on  $\{\text{R2}\}$  is indifferent to each of  $f$  and  $g$ . However, note that this randomization is exactly  $l$ . Hence, the indifference between  $f, g$  and  $l$ . Combining the implications of Stochastic Independence and Indifference to Independent Randomization, we obtain the indifference between all of  $f, g, h$  and  $k$ , contradicting the Ellsberg pattern of preference.

Finally, the Ellsberg choices are often understood as violations of Savage’s (1954) Sure Thing Principle. But our analysis suggests that they can also be understood as incompatible with the conjunction of Stochastic Independence and Indifference to Independent Randomization. This raises the question of whether these principles can provide more generally a new derivation of subjective probability and expected utility maximization, which is the objective of our first main result, Theorem 7. This also raises the question of whether it is possible to account axiomatically for the Ellsberg choices through weaker versions of these principles, which is the objective of our second main result, Theorem 8.

A final remark is in order: it would have been equally possible to contradict the Ellsberg choices by invoking Stochastic Independence and its dual version obtained by exchanging the roles of the two sources of uncertainty  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . See Ceron and Vergopoulos (2021), or also Raiffa (1961) for an analysis of the Ellsberg on-urn experiment in terms of the two dual versions of Stochastic Independence. However, these two dual principles prove to be too weak for our main result, and this explains why we resort to Indifference to Independent Randomization. Furthermore, Proposition 6 clarifies the connections between Indifference to Independent Randomization (or rather the stronger axiom of Subjective Independence) and the dual version of Stochastic Independence.

### 3 Framework and notation

Consider two measurable state spaces  $(\mathcal{S}_1, \Sigma_1)$  and  $(\mathcal{S}_2, \Sigma_2)$ . Let  $\mathcal{S} := \mathcal{S}_1 \times \mathcal{S}_2$  denote their Cartesian product. A measurable rectangle is a subset of  $\mathcal{S}$  of the form  $E_1 \times E_2$  for  $E_1 \in \Sigma_1$  and  $E_2 \in \Sigma_2$ . Let  $\Sigma$  denote the product  $\sigma$ -algebra on  $\mathcal{S}$ , obtained as the smallest  $\sigma$ -algebra containing all measurable rectangles. Let also  $\Sigma_0$  denote the Boolean algebra generated by all measurable rectangles, obtained as the smallest collection of subsets stable by finite unions and complementation and containing all measurable rectangles. Clearly,  $\Sigma_0 \subseteq \Sigma$ .

Consider also an outcome space  $\mathcal{X}$ . An act is a finitely-valued and measurable function from  $\mathcal{S}$  to  $\mathcal{X}$ . Let  $\mathcal{F}$  denote the set of all acts. Let also  $\mathcal{F}_0$  denote the set of all finitely-

valued and  $\Sigma_0$ -measurable acts. Clearly,  $\mathcal{F}_0 \subseteq \mathcal{F}$ . The DM is finally endowed with a preference relation  $\succsim$  applying to  $\mathcal{F}$ .

For  $i = 1, 2$ , let  $\mathcal{F}_i$  denote the set of all finitely-valued and measurable functions from  $\mathcal{S}_i$  to  $\mathcal{X}$ . Let  $\pi_i$  be the function from  $\mathcal{S}$  to  $\mathcal{S}_i$  defined by  $\pi_i(s) = s_i$  for all  $s \in \mathcal{S}$ . We define a binary relation  $\succsim_i$  on  $\mathcal{F}_i$  by setting, for all  $f_i, g_i \in \mathcal{F}_i$ ,

$$f_i \succsim_i g_i \iff f_i \circ \pi_i \succsim g_i \circ \pi_i. \quad (1)$$

Following the usual slight abuse of notation, we will identify every outcome  $x \in \mathcal{X}$  with the acts in  $\mathcal{F}$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_2$  constantly equal to  $x$ . Likewise, for  $i = 1, 2$ , we will identify each  $f_i \in \mathcal{F}_i$  with the act  $f_i \circ \pi_i \in \mathcal{F}$  and hence treat  $\mathcal{F}_1$  and  $\mathcal{F}_2$  as subsets of  $\mathcal{F}$ . We will also identify each  $E_i \in \Sigma_i$  with  $E_i \times \mathcal{S}_{-i} \in \Sigma$  and hence treat  $\Sigma_1$  and  $\Sigma_2$  as subsets of  $\Sigma$ .

Consider any two acts  $f, g \in \mathcal{F}$  and event  $E \in \Sigma$ . Then,  $f_E g$  denotes the element of  $\mathcal{F}$  which is equal to  $f$  on  $E$  and equal to  $g$  on  $\mathcal{S} \setminus E$ . For instance, and by the previous paragraph, for  $f_1, g_1 \in \mathcal{F}_1$  and  $E_2 \subseteq \mathcal{S}_2$ ,  $f_{1E_2} g_1$  represents the more cumbersome  $f_1 \circ \pi_{1\mathcal{S}_1 \times E_2} g_1 \circ \pi_1$ .

A subset  $E_2 \in \Sigma_2$  is said to be *null* if  $f_{2E_2} g_2 \sim_2 g_2$  for all  $f_2, g_2 \in \mathcal{F}_2$ . Otherwise,  $E_2$  is said to be *nonnull*.

Consider a  $\sigma$ -algebra  $\mathcal{B}_2$  included in  $\Sigma_2$  and a  $\Sigma_2$ -measurable partition  $\Pi_2$  of  $\mathcal{S}_2$ . We say that  $\mathcal{B}_2$  is *ordinally (stochastically) independent from*  $\Pi_2$  if, for all  $f_2, g_2, h_2 \in \mathcal{F}_2$  such that  $f_2$  and  $g_2$  are  $\mathcal{B}_2$ -measurable and all nonnull  $E_2 \in \Pi_2$ , we have

$$f_2 \succsim_2 g_2 \iff f_{2E_2} h_2 \succsim_2 g_{2E_2} h_2. \quad (2)$$

Additionally, in the case where  $\Pi_2$  is the partition generated by some event  $E_2 \in \Sigma_2$  (that is,  $\Pi_2 = \{E_2, \mathcal{S}_2 \setminus E_2\}$ ), we simply say that  $\mathcal{B}_2$  is *ordinally independent from*  $E_2$ .

In Formula (2), the ranking  $f_{2E_2} h_2 \succsim_2 g_{2E_2} h_2$  means that the DM prefers  $f_2$  to  $g_2$  conditional on  $E_2$ . Hence, the definition implies that the DM prefers  $f_2$  to  $g_2$  conditional on  $E_2$  if and only if he already prefers  $f_2$  to  $g_2$  *ex ante*; that is, learning that some event in  $\Pi_2$  holds true does not affect the DM's preferences relative to  $\mathcal{B}_2$ -measurable acts.

Consider a countably additive probability measure  $P_2$  on  $(\mathcal{S}_2, \Sigma_2)$ . Let  $\mathcal{B}_2$  be a  $\sigma$ -algebra in  $\Sigma_2$  and  $\Pi_2$  be a  $\Sigma_2$ -measurable partition of  $\mathcal{S}_2$ . We say that  $\mathcal{B}_2$  is *cardinally (stochastically) independent from*  $\Pi_2$  if  $P_2(E_2 \cap F_2) = P_2(E_2) \cdot P_2(F_2)$  for every  $E_2 \in \mathcal{B}_2$  and  $F_2 \in \Pi_2$ .

Consider now a nonconstant function  $u$  from  $\mathcal{X}$  to  $\mathbb{R}$  and a countably additive probability measure  $P_2$  on  $(\mathcal{S}_2, \Sigma_2)$ . We say that  $(u, P_2)$  provides a Subjective Expected Utility (SEU) representation of  $\succsim_2$  if for all  $f_2, g_2 \in \mathcal{F}_2$ ,

$$f_2 \succsim_2 g_2 \iff \int_{\mathcal{S}_2} u \circ f_2 dP_2 \geq \int_{\mathcal{S}_2} u \circ g_2 dP_2.$$

Suppose  $(u, P_2)$  provides an SEU representation of  $\succsim_2$ . In general, the cardinal and ordinal notions of stochastic independence do not correspond to each other. That is, the cardinal

stochastic independence of some  $\sigma$ -algebra  $\mathcal{B}_2$  from some partition  $\Pi_2$  does imply the ordinal stochastic independence of  $\mathcal{B}_2$  from  $\Pi_2$ , but the converse implication might fail. Indeed, cardinal stochastic independence requires the DM's expectation of  $\mathcal{B}_2$ -measurable acts to remain unchanged after updating beliefs on some event in  $\Pi_2$ . Meanwhile, ordinal stochastic independence requires the DM's preferences on  $\mathcal{B}_2$ -measurable acts to remain unchanged after updating beliefs on some event in  $\Pi_2$ . Clearly, the latter will not imply the former in the cases where  $\mathcal{B}_2$  is "too poor" (typically when  $\mathcal{B}_2$  is finite): it might be that preferences remain unchanged, not because the expectations do so, but simply because the changes in expectations are so small that they do not result in preference reversals. Hence, the correspondence between cardinal and ordinal stochastic independence requires a condition of richness on the  $\sigma$ -algebra that we present next.

A  $\sigma$ -algebra  $\mathcal{B}_2$  in  $\Sigma_2$  is said to be *admissible* if for all  $\mathcal{B}_2$ -measurable  $f_2, g_2, h_2 \in \mathcal{F}_2$  such that  $f_2 \succ_2 g_2$ , there exists a finite  $\mathcal{B}_2$ -measurable partition  $\Pi_2$  of  $\mathcal{S}_2$  such that  $f_2 \succ h_{2E_2}g_2$  and  $h_{2E_2}f_2 \succ g_2$  for all  $E_2 \in \Pi_2$ . In other words,  $\mathcal{B}_2$  is admissible if the restriction of  $\succsim_2$  to  $\mathcal{B}_2$ -measurable acts satisfies Savage's (1954) P6 (Small Event Continuity).

**Proposition 1** *Consider a nonconstant function  $u$  from  $\mathcal{X}$  to  $\mathbb{R}$  and a countably additive probability measure  $P_2$  on  $(S_2, \Sigma_2)$  providing an SEU representation of  $\succsim_2$ . Let  $\mathcal{B}_2$  be an admissible  $\sigma$ -algebra in  $\Sigma_2$  and  $\Pi_2$  be a  $\Sigma_2$ -measurable partition  $\Pi_2$  of  $\mathcal{S}_2$ . Then, the following are equivalent:*

- (i)  $\mathcal{B}_2$  is *cardinally stochastically independent* from  $\Pi_2$ ,
- (ii)  $\mathcal{B}_2$  is *ordinally stochastically independent* from  $\Pi_2$ .

Fix an act  $f \in \mathcal{F}$  and a  $\sigma$ -algebra  $\mathcal{B}_2$  in  $\Sigma_2$ . We say that  $f$  is *marginally  $\mathcal{B}_2$ -measurable* if the function  $f(s_1, \cdot)$  is  $\mathcal{B}_2$ -measurable for every  $s_1 \in \mathcal{S}_1$ . Consider now a countably additive probability measure  $P_2$  on  $(S_2, \Sigma_2)$ . Fix an act  $f \in \mathcal{F}$  and an event  $E_2 \in \Sigma_2$ . We say that  $f$  is *cardinally (stochastically) independent from  $E_2$*  if there exists a  $\sigma$ -algebra in  $\Sigma_2$  that is cardinally stochastically independent from  $E_2$  and makes  $f$  marginally measurable. We say that  $f$  is *ordinally (stochastically) independent from  $E_2$*  if there exists an admissible  $\sigma$ -algebra in  $\Sigma_2$  that is ordinally stochastically independent from  $E_2$  and makes  $f$  marginally measurable. The presence of the admissibility requirement is a peculiar feature of this definition, which the next corollary justifies. See also the next section.

Furthermore, let  $\Delta$  denote the set of all finitely-supported probability distributions on  $\mathcal{X}$ . Suppose the pair  $(u, P_2)$  provides an SEU representation of  $\succsim_2$ . We define a mapping  $\Phi$  from  $\mathcal{F}$  to  $\Delta^{\mathcal{S}_1}$  by letting  $\Phi(f)(s_1)$  denote the probability distribution of  $f(s_1, \cdot)$  under  $P_2$  for all  $f \in \mathcal{F}$  and  $s_1 \in \mathcal{S}_1$ .

**Corollary 2** *Consider a nonconstant function  $u$  from  $\mathcal{X}$  to  $\mathbb{R}$  and a countably additive probability measure  $P_2$  on  $(S_2, \Sigma_2)$  providing an SEU representation of  $\succsim_2$ . Let  $E_2 \in \Sigma_2$  and  $f, g \in \mathcal{F}$  be such that  $f$  and  $g$  are ordinally stochastically independent from  $E_2$ . Then,*

$$\Phi(f_{E_2}g) = P_2(E_2) \cdot \Phi(f) + P_2(E_2^c) \cdot \Phi(g).$$

Corollary 2 holds the key to our approach. To understand why, it is helpful to remind that the Anscombe and Aumann (1963) framework defines acts as functions from a state space to a lottery space. Then, the set of acts automatically inherits a mixing operation from the familiar mixing operation on lotteries. Such act mixing is of crucial importance in the formulation of the Anscombe-Aumann axioms and permits a simplified derivation of subjective probability and expected utility maximization. In contrast, a framework à la Savage (1954) only allows one to splice two acts on some event and, in general, the mixing and splicing operations do not correspond to each other.

Here, one can think of each  $\Phi(f)$  and  $\Phi(g)$  as the Anscombe-Aumann acts from  $\mathcal{S}_1$  to  $\Delta$  induced by  $f$  and  $g$ . Then,  $P_2(E_2) \cdot \Phi(f) + P_2(E_2^c) \cdot \Phi(g)$  represents a mixing of these induced acts. Hence, Corollary 2 essentially shows that the mixing of the induced acts is induced by the Savagean splicing  $f_{E_2}g$  of the initial acts  $f$  and  $g$  on  $E_2$  when  $f$  and  $g$  are ordinally stochastically independent from  $E_2$ . In other words, the corollary identifies the purely behavioral conditions under which the Savage splicing corresponds to the Anscombe-Aumann mixing. This correspondence is finally key in reformulating the Anscombe-Aumann axioms in purely subjective terms not relying on objective probabilities.

The conclusion of Corollary 2 is straightforward if  $f$  and  $g$  are assumed to be cardinally independent from  $E_2$ . Hence, the proof consists in showing that the ordinal independence of  $f$  and  $g$  from  $E_2$  implies their cardinal independence from  $E_2$ . As suggested by Proposition 1, one needs then to include the admissibility requirement in the definition of ordinal independence.

Finally, Ghirardato et al. (2003) provide a different purely subjective approach to the Anscombe-Aumann mixing of acts. They first define a mixing operation on the set  $\mathcal{X}$  of outcomes and then extend pointwise the operation to all acts. This method requires a sufficiently rich set  $\mathcal{X}$ . For instance, there must exist an outcome in  $\mathcal{X}$  that can be interpreted as the mixture between any two outcomes  $x, y \in \mathcal{X}$  with respect to any weight  $\lambda \in [0, 1]$ . In contrast, the mixture between  $x$  and  $y$  of weight  $\lambda$  is not here necessarily another outcome, but rather a binary act  $x_{E_2}y$  with  $P_2(E_2) = \lambda$ . Hence, in our approach, the set  $\mathcal{X}$  of outcomes remains arbitrary. On the other hand, the second source of uncertainty  $\mathcal{S}_2$  must be sufficiently rich for such an event  $E_2$  to exist in the first place.

## 4 Axioms

We say that  $\succsim$  is *complete* if  $f \succsim g$  or  $g \succsim f$  for all  $f, g \in \mathcal{F}$ . We say that it is *transitive* if, for all  $f, g, h \in \mathcal{F}$ ,  $f \succsim g$  and  $g \succsim h$  implies  $f \succsim h$ . We say that it is *nontrivial* if there exist  $x, y \in \mathcal{X}$  such that  $x \succ y$ .

**(A1—Weak Order)**  $\succsim$  is complete, transitive and nontrivial.

Next, we will rely on a marginal version of Savage's (1954) Sure Thing Principle that only applies to the marginal preference  $\succsim_2$ . Proposition 5 explains how it is possible to dispense with this axiom.

**(A2—Sure Thing Principle)** For all  $f_2, g_2, h_2, k_2 \in \mathcal{F}_2$  and  $E_2 \in \Sigma_2$ ,  $f_2 h_2 \succsim_2 g_2 h_2$  if and only if  $f_2 k_2 \succsim_2 g_2 k_2$ .

We will also rely on an  $\mathcal{S}_2$ -marginal version of Savage's (1954) Eventwise Monotonicity axiom. As such, A3 simultaneously captures a form of monotonicity and a form of state independence of tastes over outcomes. Given the terminology of Section 3, it also requires the trivial  $\sigma$ -algebra  $\{\emptyset, \mathcal{S}_2\}$  to be ordinally independent from every finite  $\Sigma_2$ -measurable partition of  $\mathcal{S}_2$ .

**(A3—Eventwise Monotonicity)** For all  $x, y \in \mathcal{X}$ , all  $h_2 \in \mathcal{F}_2$  and all nonnull  $E_2 \in \Sigma_2$ ,  $x \succsim_2 y$  if and only if  $x_{E_2} h_2 \succsim_2 y_{E_2} h_2$ .

The following is an  $\mathcal{S}_2$ -marginal version of Savage's (1954) Weak Comparative Probability axiom.

**(A4—Weak Comparative Probability)** For all  $x^*, x, y^*, y \in \mathcal{X}$  such that  $x^* \succ_2 x$  and  $y^* \succ_2 y$ , and all  $E_2, F_2 \in \Sigma_2$ ,  $x^*_{E_2} x \succsim_2 x^*_{F_2} x$  if and only if  $y^*_{E_2} y \succsim_2 y^*_{F_2} y$ .

A sequence  $\{E^n, n \geq 1\}$  of events in  $\Sigma$  is said to be *decreasing* if  $E^n \supseteq E^{n+1}$  for every  $n \geq 1$ . It is said to be *vanishing* if  $\bigcap_{n \geq 1} E^n = \emptyset$ . Axiom A5 below is absent from the Savage (1954) theorem and is in fact due to Arrow (1970). It delivers the countable additivity of the subjective probability. Countable additivity is really optional within the Savage framework. But it becomes natural in our approach for the following technical reason: it essentially allows us to have preference  $\succsim$  defined over the domain of  $\Sigma$ -measurable acts (instead of acts measurable with respect to the more restrictive product Boolean algebra of  $\Sigma_1$  and  $\Sigma_2$ ) while still having the subjective probability measure on  $\Sigma$  entirely determined by its marginal probability measures on  $(\mathcal{S}_1, \Sigma_1)$  and  $(\mathcal{S}_2, \Sigma_2)$ . Countable additivity also plays a crucial role in obtaining the existence of a sufficiently rich  $\sigma$ -subalgebra that is cardinally independent from any given subset. See Lemma 13.

**(A5—Monotone Continuity)** For all  $f, g \in \mathcal{F}$  such that  $f \succ g$ , all  $x \in \mathcal{X}$  and all vanishing decreasing sequence  $\{E^n, n \geq 1\}$  of events in  $\Sigma$ , there exists  $N \geq 1$  such that  $x_{E^N} f \succ g$  and  $f \succ x_{E^N} g$ .

The next axiom is a version of Savage's (1954) Small Event Continuity axiom. The latter essentially requires the existence of arbitrarily fine partitions of the state space. But note that A6 is slightly stronger than the  $\mathcal{S}_2$ -marginal version of Savage's axiom as it applies to all acts in  $\mathcal{F}$ .

**(A6—Small Event Continuity)** For all  $f, g, h \in \mathcal{F}$  such that  $f \succ g$ , there exists a finite  $\Sigma_2$ -measurable partition  $\Pi_2$  of  $\mathcal{S}_2$  such that  $f \succ h_{E_2} g$  and  $h_{E_2} f \succ g$  for all  $E_2 \in \Pi_2$ .

The following axiom is a formal expression of the principle of Stochastic Independence of  $\mathcal{S}_2$  from  $\mathcal{S}_1$  met in Section 2 and captures a form of stochastic independence between the two sources of uncertainty.

**(A7—Stochastic Independence)** For all  $f, g \in \mathcal{F}$ , if  $f(s_1, \cdot) \succsim_2 g(s_1, \cdot)$  for every  $s_1 \in \mathcal{S}_1$ , then  $f \succsim g$ .

The next results clarify the connection between A7 and stochastic independence. Consider two countably additive probability measures  $P_1$  on  $(\mathcal{S}_1, \Sigma_1)$  and  $P_2$  on  $(\mathcal{S}_2, \Sigma_2)$ . Then,

$P_1 \otimes P_2$  denotes the independent product of  $P_1$  and  $P_2$ ; that is, the unique countably additive probability measure on  $\Sigma$  satisfying  $(P_1 \otimes P_2)(E_1 \times E_2) = P_1(E_1) \cdot P_2(E_2)$  for all  $E_1 \in \Sigma_1$  and  $E_2 \in \Sigma_2$ . A countably additive probability measure  $P$  on  $\Sigma$  is said to be the *independent product* of its marginals if  $P = P_1 \otimes P_2$ , where  $P_1$  and  $P_2$  are the marginals of  $P$  on  $\Sigma_1$  and  $\Sigma_2$  respectively.

Consider now a nonconstant function  $u$  from  $\mathcal{X}$  to  $\mathbb{R}$  and a countably additive probability measure  $P$  on  $(S, \Sigma)$ . We say that  $(u, P)$  provides a Subjective Expected Utility (SEU) representation of  $\succsim$  if for all  $f, g \in \mathcal{F}$ ,

$$f \succsim g \iff \int_S u \circ f \, dP \geq \int_S u \circ g \, dP.$$

A probability measure  $P_2$  on  $(S_2, \Sigma_2)$  is said to be *nonatomic* if for all  $E_2 \in \Sigma_2$  and all  $\lambda \in \mathbb{R}$  such that  $0 \leq \lambda \leq P_2(E_2)$ , there exists  $F_2 \in \Sigma_2$  such that  $F_2 \subseteq E_2$  and  $P_2(F_2) = \lambda$ .

**Proposition 3** *Suppose the pair  $(u, P)$  provides an SEU representation of  $\succsim$  with  $u$  nonconstant and  $P$  countably additive on  $\Sigma$ . Suppose also the marginal of  $P$  on  $\Sigma_2$  is nonatomic. Then, the following are equivalent:*

- (i)  $\succsim$  satisfies A7,
- (ii)  $P$  is the independent product of its marginal probability measures on  $\Sigma_1$  and  $\Sigma_2$ .

Under the additional assumption of an SEU representation on  $\mathcal{S}$  with nonatomicity on  $\mathcal{S}_2$ , Proposition 3 shows that A7 characterizes precisely the independent product probability measures and hence the situations where the agent believes in the stochastic independence of the two sources of uncertainty in the usual probabilistic sense. The next proposition provides another way to relate explicitly A7 to stochastic independence in the absence of additional assumptions.

**Proposition 4** *If  $\succsim$  satisfies A7, then, for all  $f_2, g_2, h_2 \in \mathcal{F}_2$  and all  $E_1 \in \Sigma_1$ ,  $f_2 \succsim_2 g_2$  implies  $f_{2E_1} h_2 \succsim g_{2E_1} h_2$ .*

In Proposition 4, the ranking  $f_{2E_1} h_2 \succsim g_{2E_1} h_2$  means that the DM prefers  $f_2$  to  $g_2$  conditional on  $E_1$ . Hence, A7 implies that the DM prefers  $f_2$  to  $g_2$  conditional on  $E_1$  whenever he already prefers  $f_2$  to  $g_2$  *ex ante*; that is, learning that some event in  $\Sigma_1$  holds true does not affect the DM's preferences relative to acts in  $\mathcal{F}_2$ . By analogy with the language of Section 3, we summarize this by saying that  $\Sigma_2$  is *weakly* ordinally independent from  $\Sigma_1$ . Here, stochastic independence is weak because we only have an implication instead of an equivalence. The full equivalence would require the following strict version of A7 dealing with strict preference rankings:

**(A7\*—Strict Stochastic Independence)** For all  $f, g \in \mathcal{F}$ , if  $f(s_1, \cdot) \succsim_2 g(s_1, \cdot)$  for every  $s_1 \in \mathcal{S}_1$ , then  $f \succsim g$ . If additionally  $f(s_1, \cdot) \succ_2 g(s_1, \cdot)$  for every  $s_1$  in some nonnull subset in  $\Sigma_1$ , then  $f \succ g$ .

Our formulation of Choquet Expected Utility preferences below in Theorem 8, or Ergin and Gul’s (2009) Second-Order Probabilistically Sophisticated Expected Utility representation (their Theorem 3) represent cases where A7 is satisfied, and yet there is no clear sense according to which beliefs reflect the stochastic independence between the two sources. (See also our discussion of Ergin and Gul below Theorem 7.) In this respect, one reason why A7 is weak is that it does *not* incorporate the following dual version:

**( $\overline{\text{A7}}$ —Dual Stochastic Independence)** For all  $f, g \in \mathcal{F}$ , if  $f(\cdot, s_2) \succsim_1 g(\cdot, s_2)$  for every  $s_2 \in \mathcal{S}_2$ , then  $f \succsim g$ .

$\overline{\text{A7}}$  represents the dual version of the axiom A7 of Stochastic Independence. Hence, by a symmetric version of Proposition 4, it essentially requires the weak ordinal independence of  $\Sigma_1$  from  $\Sigma_2$ . As it turns out, A7 and  $\overline{\text{A7}}$  enter in general in competition with one another under ambiguity. For instance, in the context of the example from Section 2, A7 implies the indifference between  $h$ ,  $k$  and  $l$ . Meanwhile,  $\overline{\text{A7}}$  implies the indifference between  $f$ ,  $g$  and  $l$ . As a result, the conjunction of the two forces the indifference between  $f$ ,  $g$ ,  $h$  and  $k$  and ends up contradicting the Ellsberg choices.

Now, if one is willing to reinforce A7, it is possible to use our specific Cartesian structure to dispense with the Sure Thing Principle (A2), as we now explain. We will say that *a fair coin exists in  $\mathcal{S}_1$*  if there exists an event  $E_1 \in \Sigma_1$  such that  $x_{E_1}y \sim_1 y_{E_1}x$  for all  $x, y \in \mathcal{X}$ . This represents a plausible, yet restrictive, condition of richness of  $\mathcal{S}_1$ .

**Proposition 5** *Suppose a fair coin exists in  $\mathcal{S}_1$ . If  $\succsim$  satisfies A1, A7\* and  $\overline{\text{A7}}$ , then it also satisfies A2.*

Proposition 5 shows how A2 is implied by the axioms of stochastic independence A7\* and  $\overline{\text{A7}}$  and the existence of a fair coin. Interestingly, the proof of this result consists of a generalization of the derivation of the Sure Thing Principle due to Raiffa (1961) in the specific context of the Ellsberg (1961) one-urn paradox. It is hence possible to use this proposition to remove A2 from our Theorem 7.

Finally, A7 asserts that there are no complementarities between the uncertainty on  $\mathcal{S}_2$  conditional on different elements of  $\mathcal{S}_1$ . For instance, and as a result, these never provide a hedge for each other. A7 takes the form of a standard statewise dominance principle and therefore may appear to be normatively appealing. For instance, in the domain of Anscombe–Aumann acts that is induced by  $\mathcal{F}$  (see Lemma 12 in Appendix B), A7 expresses not only the existence of well-defined preferences, but also their statewise monotonicity. The two are often justified on normative grounds. However, see Bommier (2017) or Monet and Vergopoulos (2022) for normative arguments under ambiguity against monotonicity.

Both in normative and descriptive applications, the plausibility of A7 depends on the timing of resolution of uncertainty. Suppose that the uncertainty attached to  $\mathcal{S}_1$  resolves prior to that attached to  $\mathcal{S}_2$ . Then, statewise dominance with respect to  $\mathcal{S}_1$  as captured by A7 becomes indeed relevant. If however this timing is reversed, then A7 loses part of its appeal as what happens on each state in  $\mathcal{S}_1$  becomes a counterfactual consideration.

Our final axiom is a purely subjective version of the Independence axiom of Marschak (1950), Samuelson (1952) and Anscombe and Aumann (1963) that avoids the use of exogenous probabilities by relying on the notion of stochastic independence.

**(A8—Subjective Independence)** For all nonnull  $E_2 \in \Sigma_2$  and all  $f, g, h \in \mathcal{F}$  that are ordinally stochastically independent from  $E_2$ ,  $f \succsim g$  if and only if  $f_{E_2}h \succsim g_{E_2}h$ .

As usual with the classical independence axioms, A8 captures the idea that there is no complementarity between outcomes obtained on disjoint events in  $\mathcal{S}_2$ . Put differently, the uncertain outcomes induced by some act  $f$  on some event  $E_2$  do not provide a hedge for the uncertain outcomes of  $f$  on the complementary event  $\mathcal{S}_2 \setminus E_2$ . In the classical independence axioms, this idea takes the form of a requirement supposed to hold universally. In our case, the requirement must be restricted by ordinal stochastic independence. Without this restriction, A8 would prove to be too strong and unnecessary for our SEU representation. (Incidentally, the admissibility requirement featured in the definition of ordinal independence is also key for the necessity of A8.)

Just like the classical independence axioms, A8 entails in particular a form of Indifference to Randomization already met informally in the analysis of the Ellsberg paradox from Section 2. Indeed, suppose  $f, g \in \mathcal{F}$  are ordinally independent from  $E_2$  with  $f \sim g$ . Then, A8 implies  $f_{E_2}g \sim f \sim g$ . Hence, the agent is indifferent to randomizing  $f$  and  $g$  on every event from which both  $f$  and  $g$  are ordinally independent. To further illustrate, suppose that  $f$  has “good” outcomes on  $E_2$  and “bad” ones outside  $E_2$ . Likewise, suppose that  $g$  has “bad” outcomes on  $E_2$  and “good” ones outside  $E_2$ . Then, it is tempting to randomize  $f$  and  $g$  on  $E_2$  and implement  $f_{E_2}g$  since this act eliminates the “bad” outcomes and only results in “good” ones. But this is typically a situation where  $f$  and  $g$  are not ordinally independent from  $E_2$ . When they are, then the “good” and “bad” outcomes of  $f$  and  $g$  are equally dispersed on  $E_2$  and its complement, and there can be no incentive to randomize.

Furthermore, the restriction by ordinal stochastic independence confers to the logic of the Independence axiom an entirely novel interpretation. Indeed, suppose  $f, g$  and  $h$  are ordinally independent from  $E_2$ . This implies the following: after observing the true state in  $\mathcal{S}_1$  and updating preferences between  $f$  and  $g$  accordingly, further observing  $E_2$  and updating preferences between  $f$  and  $g$  accordingly will not result in reversals. More formally, for all  $s_1 \in \mathcal{S}_1$

$$f(s_1, \cdot) \succsim_2 g(s_1, \cdot) \iff f(s_1, \cdot)_{E_2}h(s_1, \cdot) \succsim_2 g(s_1, \cdot)_{E_2}h(s_1, \cdot).$$

In short, and more intuitively,  $f$  and  $g$  will become stochastically independent from  $E_2$  *ex post*; that is, once the true state in  $\mathcal{S}_1$  is observed. Then, A8 requires the following equivalence:

$$f \succsim g \iff f_{E_2}h \succsim g_{E_2}h.$$

Put differently, A8 requires the agent to acknowledge the stochastic independence of  $f$  and  $g$  from  $E_2$  *ex ante*; that is, prior to the resolution of uncertainty in  $\mathcal{S}_1$ . Indeed, the ranking  $f_{E_2}h \succsim g_{E_2}h$  can be interpreted as a preference for  $f$  over  $g$  conditional on  $E_2$ . Then, the equivalence  $f \succsim g$  if and only if  $f_{E_2}h \succsim g_{E_2}h$  means that  $f$  is preferred to  $g$  if and

only if it is also preferred conditional on  $E_2$ . In other words, learning  $E_2$  does not change the preference, which is the distinctive feature for the stochastic independence of  $f$  and  $g$  from  $E_2$ . Thus, A8 requires a form of consistency between the perceptions of stochastic independence *ex ante* and *ex post*.

Now, consistency in the perception of stochastic independence as just depicted has a normative flavor similar to the requirement of Dynamic Consistency. See Machina (1989) or Ghirardato (2002). But both its normative and descriptive plausibility depends on the timing of resolution of uncertainty. Because it relies on a reasoning contingent on  $\mathcal{S}_1$ , A8 is just like A7 most relevant when it is known by the agent that  $\mathcal{S}_1$  resolves prior to  $\mathcal{S}_2$ , and may appear counterfactual under the reversed timing. Furthermore, one may expect violations of A8 under ambiguity as already suggested in Section 2. Indeed, stochastic independence *ex post* is purely defined in terms of the marginal preference  $\succsim_2$  on  $\mathcal{S}_2$  and cannot reflect the ambiguity perceived on  $\mathcal{S}_1$ . Now, by requiring consistency between stochastic independence *ex ante* and *ex post*, A8 essentially forbids  $\succsim$  to reveal any form of ambiguity aversion.

Finally, the next proposition relates explicitly A8 to the idea of stochastic independence between the two sources of uncertainty. It presents a consequence of A8 that is directly comparable to the defining property of ordinal independence given in Section 3. Interestingly, this consequence of A8 turns out to be the symmetric counterpart of the consequence of A7 obtained in Proposition 4.

**Proposition 6** *Suppose there exists an admissible  $\sigma$ -algebra that is ordinally stochastically independent from some nonnull event  $E_2 \in \Sigma_2$ . If  $\succsim$  satisfies A8, then, for all  $f_1, g_1, h_1 \in \mathcal{F}_1$ ,  $f_1 \succsim_1 g_1$  if and only if  $f_1 E_2 h_1 \succsim g_1 E_2 h_1$ .*

Similarly to Proposition 4, Proposition 6 shows that learning that some event  $E_2$  holds true does not affect the DM's preferences relative to acts in  $\mathcal{F}_1$ . Note however that this conclusion requires a form of richness of  $\Sigma_2$  relative to  $E_2$ . By analogy with the language of Section 3, we say that  $\Sigma_1$  is ordinally independent from  $\Sigma_2$  to qualify the conclusion of the proposition. Finally, Propositions 4 and 6 hence suggest together that A7 and A8 embody two different and complementary axioms of stochastic independence. For instance, though our formulation of Choquet Expected Utility preferences in Theorem 8 below, or Ergin and Gul's (2009) Second-Order Probabilistically Sophisticated Expected Utility representation (their Theorem 3) satisfy A7, they fail A8.

## 5 Main result

We now come to our main result:

**Theorem 7** *A binary relation  $\succsim$  on  $\mathcal{F}$  satisfies Axioms A1–A8 if and only if there exist two countably additive probability measures  $P_1$  on  $\Sigma_1$  and  $P_2$  on  $\Sigma_2$ , with  $P_2$  nonatomic,*

and a nonconstant function  $u$  from  $\mathcal{X}$  to  $\mathbb{R}$  such that for all  $f, g \in \mathcal{F}$ ,

$$f \succsim g \iff \int_{\mathcal{S}} u \circ f \, dP_1 \otimes P_2 \geq \int_{\mathcal{S}} u \circ g \, dP_1 \otimes P_2.$$

Moreover,  $P_1$  and  $P_2$  are unique, and  $u$  is unique up to positive affine transformations.

Theorem 7 provides an axiomatic characterization of subjective probability and expected utility maximization in the spirit of the classical characterizations of Savage (1954) and Anscombe and Aumann (1963). As usual in these representations, the DM is characterized by a utility function representing his tastes for certain outcomes and a subjective probability representing his beliefs on uncertain events. But a distinctive feature of our approach lies in that it explicitly assumes two sources of uncertainty and derives a subjective probability in the form of a product probability measure. The theorem is hence adapted to situations where the DM subjectively believes that the two sources of uncertainty are stochastically independent of each other. Contrarily to the Anscombe-Aumann theorem, it does not assume exogenous probabilities on the one source and is thus relevant to situations where the DM formulates subjective probability judgements on the two sources.

In terms of its axioms, the theorem invokes two different and complementary novel axioms relying on the notion of stochastic independence, namely Stochastic Independence (A7) and Subjective Independence (A8). More precisely, A7 captures the belief that the two sources of uncertainty are stochastically independent of each other while A8 requires a form of consistency between the *ex ante* and *ex post* perception of stochastic independence. In fact, Section 2 has already shown that these two axioms are incompatible with sensitivity to ambiguity as exemplified by the Ellsberg (1961) two-urn paradox. Theorem 7 goes further by showing that the two axioms provide together a genuine full derivation of subjective probability and expected utility.

Interestingly, the background framework of Theorem 7 can be seen as a particular case of the framework of Savage (1954) corresponding to the case where the state space has a product structure. Hence, it could be possible to obtain the same representation as that of Theorem 7 by applying a version of the Savage theorem and adding a further axiom forcing the subjective probability to be a product measure. For instance, Proposition 3 below partially implements this idea. However, this is not the approach undertaken in Theorem 7. We rather exploit the specific structure of the Cartesian product to formulate the two key axioms of Stochastic Independence and Subjective Independence and hence obtain a new derivation of subjective probability and expected utility. There are of course both advantages and disadvantages to these two key axioms. On the one hand, they considerably simplify the derivation of the expected utility representation. On the other hand, they appear to be too restrictive to account satisfactorily for situations where the DM believes in the correlation or stochastic dependence between the two sources of uncertainty.

Furthermore, the alternative approach to expected utility embodied by Theorem 7 lies somewhere in between the classical theorems of Savage and Ascombe-Aumann and retains

some attractive features of each. Like the former and unlike the latter, Theorem 7 is purely subjective in that it does not suppose a notion of exogenously given probabilities. In fact, the Savage theorem dispenses with such objective probabilities at the cost of a nonatomic probability measure. The state space must hence be infinite in this approach. In contrast, nonatomicity is only required here for  $P_2$ , and this leaves the possibility of a totally arbitrary probability space  $(\mathcal{S}_1, \Sigma_1, P_1)$ , a typical feature of the Anscombe-Aumann theorem. On a different note, the Anscombe-Aumann assumption of exogenously given probabilities has proved to be greatly amenable to mathematical treatment. For instance, Schmeidler (1989), and many others after him, used them to derive generalizations of SEU accounting for the Knightian uncertainty, or ambiguity, an agent may perceive as in the Choquet Expected Utility (CEU) model of preference. In the next section, we suggest that Theorem 7 retains this feature by providing its generalization to a version of CEU.

In comparison to the Savage theorem, it may finally appear that the main weakness of Theorem 7 lies in its explicit use of two sources of uncertainty. Note however that, when commenting on de Finetti's divisibility assumption and his own P6, Savage (1954)[p. 33] writes: "Such a postulate could be made relatively acceptable by observing that it will obtain if, for example, in all the world there is a coin that the person is firmly convinced is fair". In our view, this amounts to factorizing the state space  $\mathcal{S}$  into a product  $\mathcal{S}_1 \times \mathcal{S}_2$  with  $\mathcal{S}_1$  representing a first source of uncertainty of primary interest and  $\mathcal{S}_2$  representing the outcome of infinitely many tosses of the fair coin. In this sense, the assumption of a product structure for the state space is already implicitly present in Savage's construction. Furthermore, the assumption of a product state space is not as restrictive as it may appear. Indeed, the second source does not literally need to exist and can simply be imagined by the DM as a device allowing him to structure and discipline his preferences, and hence make better decisions. This is for instance how we understand Raiffa's (1961) assumption of a fair coin in his analysis of the Ellsberg one-urn paradox.

Furthermore, the assumption that the second source of uncertainty  $\mathcal{S}_2$  carries a probability measure may appear to be excessively restrictive. This is admittedly the case if it is supposed that  $\mathcal{S}_2$  is exogenously imposed to the agent. The agent may then have difficulties in formulating probability judgements on  $\mathcal{S}_2$ . If however he can freely choose  $\mathcal{S}_2$  or simply freely imagine it as explained in the previous paragraph, then this assumption becomes more acceptable. It simply means that the agent freely chooses or imagines an unambiguous source of uncertainty. If  $\mathcal{S}_2$  simply represents a randomizing device, then the assumption of a probability measure on it also becomes quite standard in the light of the literature. See for instance Klibanoff (2001), Eichberger et al. (2016) or Battigalli et al. (2017).

Likewise, the assumption that the probability measure  $P_2$  on  $\mathcal{S}_2$  is nonatomic is also quite restrictive in applications and forbids for instance the case of a finite  $\mathcal{S}_2$ . As already suggested, Savage overcomes the difficulty by supposing that  $\mathcal{S}_2$  represents the outcome of a fair coin that can be tossed infinitely many times. At a more pragmatic level, the nonatomicity of  $P_2$  turns out to be a very convenient assumption. Indeed, in the Anscombe-Aumann framework, the randomizing device must be sufficiently rich for the mixing of any two acts with respect to any probability value to be meaningful. Nonatomicity on  $\mathcal{S}_2$  is

exactly what allows  $\mathcal{S}_2$  to be sufficiently rich in this sense and hence what allows us to endogenize the Anscombe-Aumann objective probabilities.

The product structure of the state space and the representation obtained in Theorem 7 can also be of relevance beyond the specific context of choice under uncertainty. For instance, they can also be interpreted in the context of intertemporal choice under uncertainty. The elements of  $\mathcal{S}_2$  represent then different points in time, and  $P_2$  encodes an abstract discounting function. This is similar to Bastianello and Faro (2022). Note that nonatomicity on  $\mathcal{S}_2$  is inconsistent with their assumption of discrete time, but remains compatible with the assumption of continuous time. For another context, think of social choice under uncertainty à la Mongin and Pivato (2015, 2020). Then,  $\mathcal{S}_2$  represents a collection of individuals, and  $P_2$  encodes the utilitarian weights assigned to their preferences. Because of nonatomicity, the collection of individuals must be infinite. This may appear to be an unrealistic assumption. But it has methodological advantages and has received some attention in the literature. See Zhou (1997) or Askoura and Billot (2021).

Theorem 7 is also comparable to more recent results of Mongin and Pivato (2015) and Mongin (2020). Like us, they derive axiomatically expected utility representations on two stochastically independent sources of uncertainty. But they do so in a different framework where acts are real-valued functions and state spaces are finite and rely on topological methods. In spite of the similarities in the representations obtained, the axioms and arguments are very different. For instance, our second source of uncertainty must necessarily be infinite, while Mongin and Pivato's second source must necessarily be finite. Each of these properties may appear to be too restrictive for a general theory and hence each of the two approaches may prove to be useful in different specific economic applications.

On a different note, Theorem 7 and our whole methodological approach are closely related to those of Ergin and Gul (2009) and Webb (2017). Just like ours, Ergin and Gul's main result provides a purely subjective construction of the Anscombe-Aumann framework. However, their approach does not provide the adequate tools to formulate the Anscombe-Aumann mixing of acts in purely subjective terms. For this reason, they essentially treat their Anscombe-Aumann preferences as if they were Savagean ones. For instance, they apply the (Comonotonic) Sure Thing Principle or Small Event Continuity axioms (on  $\mathcal{S}_1$ ) which are typical of the Savage framework and practically force each source of uncertainty to carry a nonatomic probability and hence be infinite. In contrast, and as already explained in the discussion of Corollary 2, our notion of ordinal stochastic independence provides a purely subjective formulation of the mixing of acts. Arguments that are typical of the Anscombe-Aumann framework can then be adapted to our setting. Theorem 7 and 8 exploit precisely this possibility.

Another major difference between the work of Ergin and Gul and ours lies in the significance to be accorded to the structure of independent product obtained for both their notion of beliefs and ours. In our Theorem 7, the agent clearly believes that the two sources of uncertainty are stochastically independent of each other. This is captured axiomatically by the combination of A7 and A8 and results in the representation in a product probability measure. In contrast, in Ergin and Gul's results, the product structure of beliefs is rather a

tricky artefact delivering the uniqueness of belief in the form of a measure on the product state space. For instance, in their Theorems 1 and 3, the representation obtained only depends on the marginals of the probability measure  $P$  on  $\mathcal{S}_1 \times \mathcal{S}_2$ . Hence, the fact that  $P$  is the independent product of its marginals does not reflect a meaningful property of preference. Alternatively, the measure  $P$  does not represent the comparative likelihood relation on events.<sup>2</sup>

We now briefly sketch the proof of Theorem 7. In Step 1, an application of the Savage theorem to  $\succsim_2$  yields a pair  $(u, P_2)$  providing an SEU representation of  $\succsim_2$ . The probability measure is nonatomic and, thanks to A5, countably additive. As for Step 2, let  $\mathcal{A}$  denote the set of all Anscombe-Aumann acts from  $\mathcal{S}_1$  to the set  $\Delta$  of finitely-supported probability measures on  $\mathcal{X}$ ; that is, the set of all finitely-valued and measurable functions from  $(\mathcal{S}_1, \Sigma_1)$  to  $\Delta$ . Recall also the mapping  $\Phi$  whose definition is given in Section 3. By the nonatomicity of  $P_2$ ,  $\Phi$  defines an onto mapping from  $\mathcal{F}_0$  to  $\mathcal{A}$ . In other words, each act  $f \in \mathcal{F}_0$  induces an Anscombe-Aumann act  $\Phi(f) \in \mathcal{A}$  and every Anscombe-Aumann act  $\alpha \in \mathcal{A}$  can be obtained from some act  $f \in \mathcal{F}_0$  satisfying  $\alpha = \Phi(f)$ . Step 3 uses A7 to show the existence of a nontrivial and monotonic weak order  $\succsim_{\mathcal{A}}$  on  $\mathcal{A}$  such that for all  $f, g \in \mathcal{F}_0$ ,

$$f \succsim g \iff \Phi(f) \succsim_{\mathcal{A}} \Phi(g).$$

Step 4 uses A6 and A8 to show that  $\succsim_{\mathcal{A}}$  further satisfies the Anscombe-Aumann Continuity and Independence axioms respectively. Hence, their theorem provides a probability measure  $P_1$  on  $\Sigma_1$  (which, by A5, is also countably additive) such that the pair  $(u, P_1)$  provides an expected utility representation of  $\succsim_{\mathcal{A}}$ . This is sufficient for our representation on the restricted domain  $\mathcal{F}_0$ . Finally, Step 5 extends the representation from  $\mathcal{F}_0$  to  $\mathcal{F}$  using a familiar argument due to Savage (1954) in the extension of his representation from finitely-valued acts to infinitely-valued ones.

This proof sketch clearly suggests that Theorem 7 can rightly be considered to be a purely subjective version of the Anscombe-Aumann theorem. This is not merely to say that there is a resemblance between the two theorems. This is rather to stress that Theorem 7 is obtained as a corollary of the Anscombe-Aumann theorem, one where the assumption of exogenously given objective probabilities is essentially replaced with the two purely subjective axioms of stochastic independence. This point has some conceptual importance. Indeed, though Anscombe-Aumann achieved a considerable simplification of the Savage construction, their use of objective probabilities has been criticized for going against the behavioristic approach to probability. See for instance Mongin (2020). For this reason, the Savage purely behavioral approach to Subjective Expected Utility (SEU) is often judged to rely on a more solid conceptual basis. In this context, Theorem 7 shows that objective probabilities are not a necessary ingredient of the Anscombe-Aumann theorem and that one can very well dispense with them thanks to considerations of stochastic independence.

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<sup>2</sup>To illustrate, suppose  $\mathcal{S}_1 = \mathcal{S}_2 = [0, 1]$  and let  $P$  be the independent product of two copies of the Lebesgue measure  $[0, 1]$ . Fix  $x, y \in \mathcal{X}$  with  $x \succ y$ . Let  $f = x_{[0, 1/2]} \times_{\mathcal{S}_2} y$  and  $g = x_{\mathcal{S}_1} \times_{[0, 1/2]} y$ . In the context of Ergin and Gul's Theorem 3, we obtain  $P(E) = P(F)$  event though  $f \succ g$  at least if there is sufficient second-order risk aversion in their sense.

Hence, the use of objective probabilities à la Anscombe-Aumann is purely a matter of mathematical convenience and should not be a matter of conceptual dispute.

## 6 Purely subjective Choquet preferences

In this section, we show that the axiomatic derivation of subjective expected utility of Theorem 7 lends itself easily to generalizations accounting for the ambiguity a DM may perceive as exemplified in the Ellsberg paradox. We present a generalization to Choquet expected utility à la Schmeidler (1989). But we first have to weaken A5 into an  $\mathcal{S}_2$ -marginal version and restrict A8 to marginally comonotonic acts.

**(A5'—Weak Monotone Continuity)** For all  $f_2, g_2 \in \mathcal{F}_2$  such that  $f_2 \succ_2 g_2$ , all  $x \in \mathcal{X}$  and all vanishing decreasing sequence  $\{E_2^n, n \geq 1\}$  of events in  $\Sigma_2$ , there exists  $N \geq 1$  such that  $x_{E_2^N} f_2 \succ_2 g_2$  and  $f_2 \succ_2 x_{E_2^N} g_2$ .

We say that a pair  $\{f, g\}$  of two acts  $f, g \in \mathcal{F}$  is *marginally comonotonic on  $\mathcal{S}_1$*  if there are no  $s_1, s'_1 \in \mathcal{S}_1$  such that  $f(s_1, \cdot) \succ_2 f(s'_1, \cdot)$  and  $g(s'_1, \cdot) \succ_2 g(s_1, \cdot)$ . For instance, suppose  $f = f_1 \circ \pi_1$  and  $g = g_1 \circ \pi_1$  for some  $f_1, g_1 \in \mathcal{F}_1$ . Then, the pair  $\{f, g\}$  is marginally comonotonic on  $\mathcal{S}_1$  if and only if the pair  $\{f_1, g_1\}$  is comonotonic in the usual sense of Schmeidler (1989). Consider the following weak version of Subjective Independence.

**(A8'—Subjective Comonotonic Independence)** For all nonnull  $E_2 \in \Sigma_2$  and all  $f, g, h \in \mathcal{F}$  that are ordinally stochastically independent from  $E_2$ , if the pairs  $\{f, h\}$  and  $\{g, h\}$  are marginally comonotonic on  $\mathcal{S}_1$ , then  $f \succsim g$  if and only if  $f_{E_2} h \succsim g_{E_2} h$ .

Axiom A8' restricts the logic of Subjective Independence to pairs of acts that are marginally comonotonic on  $\mathcal{S}_1$ . For instance, suppose that  $f$  and  $g$  are ordinally independent from  $E_2$  and that  $\{f, g\}$  is marginally comonotonic on  $\mathcal{S}_1$ . Suppose also  $f \sim g$ . Then, under A8', we have  $f_{E_2} g \sim f \sim g$ . Hence, preferences may exhibit a nonindifference to randomization, but only if the randomized acts are not marginally comonotonic on  $\mathcal{S}_1$ . In this case, the randomization between  $f$  and  $g$  may become more attractive than each of  $f$  and  $g$  by providing a hedge to the potential losses of either  $f$  or  $g$  and thereby reducing the exposure to ambiguity. Put differently the DM may believe that  $f$  and  $g$  are ordinally independent from  $E_2$  *ex post* (i.e. after the resolution of  $\mathcal{S}_1$ ) and yet refuse to acknowledge this ordinal independence *ex ante* (i.e. before the resolution of  $\mathcal{S}_1$ ) precisely because of the failure of marginal comonotonicity and the potential hedge it provides. Hence, the (naturally arguable) core idea implicit in A8' is that potential hedges on  $\mathcal{S}_1$  provide a reason to have different perceptions of stochastic independence *ex ante* and *ex post*.

The Ellsberg two-urn paradox studied in Section 2 provides of course a typical illustration of this logic. Suppose first that A7 holds. Then, as already explained there, we have the indifference between  $h, k$  and  $l$ . Now, suppose the Ellsberg (ambiguity averse) pattern of preference. In particular, we have  $h \succ f$  and  $k \succ g$ . Then, it follows that  $l \succ f$  and  $l \succ g$ . However,  $l$  is the randomization between  $f$  and  $g$  on event  $E_2 = \{R2\}$ . Furthermore,  $f$  and  $g$  are trivially ordinally independent from  $E_2$  since their payments only depend on  $\mathcal{S}_1$ .

But note that  $\{f, g\}$  is not marginally comonotonic on  $\mathcal{S}_1$ . Indeed, when moving from  $R1$  to  $B1$ ,  $f$  yields a lower payment while  $g$  yields a higher one. Overall, the Ellsberg pattern of preference remains consistent with  $A8'$ .

A *capacity* on  $(\mathcal{S}_1, \Sigma_1)$  is a function  $v_1$  from  $\Sigma_1$  to  $[0, 1]$  such that  $v_1(\mathcal{S}_1) = 1$ ,  $v_1(\emptyset) = 0$ , and  $v_1(E_1) \geq v_1(F_1)$  for all  $E_1, F_1 \in \Sigma_1$  such that  $F_1 \subseteq E_1$ .

Consider a capacity  $v_1$  on  $(\mathcal{S}_1, \Sigma_1)$  and a bounded and measurable function  $F_1$  from  $\mathcal{S}_1$  to  $\mathbb{R}$ . For all  $x \in \mathbb{R}$ , the set  $\{F_1 \geq x\}$  of all  $s_1 \in \mathcal{S}_1$  such that  $F_1(s_1) \geq x$  lies in  $\Sigma_1$ . The *Choquet integral* of  $F_1$  with respect to  $v_1$  is then defined in the following way:

$$\int_{\mathcal{S}_1} F_1(s_1) dv_1(s_1) = \int_{-\infty}^0 \{v_1[\{F_1 \geq x\}] - 1\} dx + \int_0^{+\infty} v_1[\{F_1 \geq x\}] dx.$$

**Theorem 8** *A binary relation  $\succsim$  on  $\mathcal{F}$  satisfies Axioms  $A1$ – $A4$ ,  $A5'$ ,  $A6$ – $A7$  and  $A8'$  if and only if there exist a capacity  $v_1$  on  $(\mathcal{S}_1, \Sigma_1)$ , a nonatomic and countably additive probability measure  $P_2$  on  $(\mathcal{S}_2, \Sigma_2)$ , and a nonconstant function  $u$  from  $\mathcal{X}$  to  $\mathbb{R}$  such that for all  $f, g \in \mathcal{F}$ ,*

$$f \succsim g \iff \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ f(s_1, s_2) dP_2(s_2) dv_1(s_1) \geq \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ g(s_1, s_2) dP_2(s_2) dv_1(s_1).$$

*Moreover,  $v_1$  and  $P_2$  are unique, and  $u$  is unique up to positive affine transformations.*

Theorem 8 provides a generalization of Theorem 7 to Schmeidler's (1989) Choquet expected utility model. The DM is characterized by a utility function representing his tastes for certain outcomes and a subjective countably additive probability on  $\mathcal{S}_2$  representing his beliefs on uncertain events from  $\mathcal{S}_2$ . But he is furthermore characterized by a capacity on  $\mathcal{S}_1$  representing his ambiguous beliefs on uncertain events from  $\mathcal{S}_1$  and allowing for ambiguity-sensitive choices as in the Ellsberg (1961) paradox.

It is instructive to compare the approach to ambiguity in Theorem 8 to the *source method* of Abdellaoui et al. (2011). Building on earlier work of Fox and Tversky (1995), these authors analyze uncertainty in terms of its various sources. Such a source is uniform if the agent holds probabilistic beliefs on all of its events. For each uniform source, a source function converts a subjective probability value into a willingness to bet. Hence, ambiguity is captured by graphs on the probability interval, rather than measures on algebras, as is the case of Theorem 8. For instance, their empirical evidence suggests that a source function on an unambiguous urn leaves the probabilities unchanged. Meanwhile, on an ambiguous urn, the willingness to bet on an event of probability  $p$  is significantly lower than  $p$  at least when  $p$  is high enough. Now, by nature or at least in its current stage, the source method does not take a clear stand on the issue of stochastic dependence or independence between sources. In particular, it is not clear how it evaluates acts, which, like  $l$  in Section 2, depend on two sources. Put differently, the combination of two sources  $\mathcal{S}_1$  and  $\mathcal{S}_2$  defines another source  $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ , and the connection between the source function on  $\mathcal{S}$  and

those on each of  $\mathcal{S}_1$  and  $\mathcal{S}_2$  remains to be explored. What complicates the matter is that  $\mathcal{S}$  cannot be uniform even if both  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are because it seems to us that the uniformity of  $\mathcal{S}$  would imply source indifference and, for instance, contradict the Ellsberg choices. In contrast, Theorem 8 and the specific order of integration that it incorporates take a stand on the issue of evaluating acts depending on the two sources. Needless to say, this stand remains open to criticism. See, for instance, Monet and Vergopoulos (2022) for the motivation and characterization of the dual order of integration.

The representation obtained in Theorem 8 is very related to that Ergin and Gul (2009) obtain in their Theorem 4. They essentially obtain the particular case of Theorem 8 where the capacity is a monotone increasing continuous transformation of a probability measure on  $\mathcal{S}_1$ . This property originates in their assumption that the agent is probabilistically sophisticated on each source of uncertainty. It is however too restrictive for some applications where, as in the Ellsberg paradox, the ambiguity an agent perceives makes him depart from both SEU and probabilistic sophistication.

Other characterizations of Choquet expected utility include prominently Gilboa (1987), Schmeidler (1989) and Sarin and Wakker (1992). In fact, the relationship between Theorem 8 and the contributions of Gilboa (1987) and Schmeidler (1989) is entirely analogous to that between Theorem 7 and the contributions of Savage (1954) and Anscombe and Aumann (1963). For instance, Theorem 8 can be understood as a purely subjective version of the Schmeidler (1989) theorem where a capacity and Choquet expectation on an arbitrary set  $\mathcal{S}_1$  are derived in a purely subjective way. But for brevity, we will not paraphrase the discussion already presented in the previous section.

Also, several classical tools from the ambiguity literature are formulated within the Anscombe and Aumann (1963) framework and in terms of the mixing of acts that it allows. These include for instance the axiom of Uncertainty Aversion of Schmeidler (1989) and Gilboa and Schmeidler (1989), or the unambiguous preference relation of Ghirardato, Maccheroni, and Marinacci (2004). We conjecture that our stochastic independence technology, as embodied by Corollary 2, can also be used to obtain purely subjective versions of many of these tools. If confirmed, this would mean that it is equally possible to obtain a purely subjective version of the MaxMin model of Gilboa and Schmeidler (1989) or of the construction of the unambiguous preference relation.

Now, these remarks raise the following question: could our technology also allow one to obtain purely subjective axiomatic foundations for the so-called KMM preferences of Klibanoff et al. (2005)? The question is of importance because these preferences seem to impose themselves when it comes to economic applications of ambiguity. But most axiomatic treatments of KMM preferences involve a framework richer than that of Anscombe-Aumann. See Klibanoff et al. (2005) or Seo (2009). For instance, Seo (2009) uses a set of lotteries of Anscombe-Aumann acts as a domain of preference. This allows the agent to perform randomizations both prior to the resolution of the ambiguous source and after its resolution. Hence, it might be possible to obtain a purely subjective version of his KMM-like representation by supposing a third state space to perform randomization *ex ante*. In fact, it might even be possible to use a single and sufficiently rich space  $\mathcal{S}_2$  to obtain two

stochastically independent subalgebras and use them respectively for simulating *ex ante* and *ex post* randomizations. Meanwhile, Denti and Pomatto (2022) obtain specifically *identifiable* KMM representations. Note that it equally seems possible to obtain purely subjective versions of such identifiable representations as these are formulated in terms of axioms involving the mixing operation of acts.

To conclude, as already explained in the previous section, the assumption of a product state space is already and implicitly present in Savage’s justification of his P6. Yet Savage does not need to formulate this product structure explicitly because his construction provides probabilities for all events (in each of the two implicit sources of uncertainty) through the same technology. In contrast, Theorem 7 provides a differential treatment for the two sources of uncertainty: probabilities on  $\mathcal{S}_2$  are obtained through the Sure Thing Principle (A2) while probabilities on  $\mathcal{S}_1$  are obtained through Subjective Independence (A8). A merit of the explicit assumption of the product structure is the possibility of such a differential treatment of probabilities which, as demonstrated by Theorem 8, lends itself easily to generalizations to ambiguity and Knightian uncertainty. Our point here is closely related to the contribution of Sarin and Wakker (1992). But these authors specifically obtain a version of CEU, and it is not clear how to adapt their arguments to other models of ambiguity like MaxMin.<sup>3</sup> In contrast, our approach makes it relatively straightforward to obtain extensions to such other models.

## Appendix A — Proofs of Propositions and Corollary

**Lemma 9** *Consider a nonconstant function  $u$  from  $\mathcal{X}$  to  $\mathbb{R}$  and a countably additive probability measure  $P_2$  on  $(S_2, \Sigma_2)$  providing an SEU representation of  $\succsim_2$ . Let  $\mathcal{B}_2$  be a  $\sigma$ -algebra in  $\Sigma_2$ . Then, the following are equivalent:*

- (i) *The restriction of  $P_2$  to  $\mathcal{B}_2$  is nonatomic,*
- (ii)  *$\mathcal{B}_2$  is admissible.*

*Proof.* Suppose first that  $\mathcal{B}_2$  is admissible. The restriction of  $\succsim_2$  to  $\mathcal{B}_2$ -measurable acts has an SEU representation  $(u, P_2^{\mathcal{B}_2})$  where  $P_2^{\mathcal{B}_2}$  is the restriction of  $P_2$  to  $\mathcal{B}_2$ . Meanwhile, since  $\mathcal{B}_2$  is admissible, this restriction satisfies all the Savage (1954) axioms. Hence, there exists a pair  $(v, Q_2^{\mathcal{B}_2})$  providing an SEU representation of this restriction. Here,  $Q_2^{\mathcal{B}_2}$  is a nonatomic countably additive probability measure on  $(S_2, \mathcal{B}_2)$ . By uniqueness in the Savage theorem, we must have  $P_2^{\mathcal{B}_2} = Q_2^{\mathcal{B}_2}$ . Hence, the restriction of  $P_2$  to  $\mathcal{B}_2$  is nonatomic.

Next, suppose that the restriction of  $P_2$  to  $\mathcal{B}_2$  is nonatomic. Then, the restriction of  $\succsim_2$  to  $\mathcal{B}_2$ -measurable acts has an SEU representation à la Savage. By his theorem, this restriction satisfies his P6. Hence,  $\mathcal{B}_2$  is admissible.  $\square$

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<sup>3</sup>Incidentally, note that the examples of Sarin and Wakker (1992) demonstrate that our Choquet representation in Theorem 8 is *not* a particular case of their Choquet representation.

Proof of Proposition 1. Suppose (i). Fix a nonnull  $E_2 \in \Pi_2$  and let  $f_2, g_2, h_2 \in \mathcal{F}_2$  be such that  $f_2$  and  $g_2$  are  $\mathcal{B}_2$ -measurable. Then, the SEU representation yields

$$f_{2E_2}h_2 \succsim_2 g_{2E_2}h_2 \iff \mathbb{E}_{P_2}[u \circ f_2|E_2] \geq \mathbb{E}_{P_2}[u \circ g_2|E_2].$$

However, since  $f_2$  and  $g_2$  are  $\mathcal{B}_2$ -measurable, item (i) implies  $\mathbb{E}_{P_2}[u \circ f_2|E_2] = \mathbb{E}_{P_2}[u \circ f_2]$  and  $\mathbb{E}_{P_2}[u \circ g_2|E_2] = \mathbb{E}_{P_2}[u \circ g_2]$ . Using again the SEU representation, we obtain

$$f_{2E_2}h_2 \succsim_2 g_{2E_2}h_2 \iff f_2 \succsim g_2.$$

Suppose (ii). Fix a nonnull  $G_2 \in \Pi_2$ . Consider the restriction of  $\succsim_2$  to  $\mathcal{B}_2$ -measurable acts. It has an SEU representation with respect to  $(u, P_2^{\mathcal{B}_2})$  where  $P_2^{\mathcal{B}_2}$  is the restriction of  $P_2$  to  $\mathcal{B}_2$ , which is nonatomic by Lemma 9. Meanwhile, by (ii), this restriction also has an SEU representation with respect to  $(u, P_2^{\mathcal{B}_2}(\cdot|G_2))$ , where  $P_2^{\mathcal{B}_2}(\cdot|G_2)$  the restriction to  $\mathcal{B}_2$  of  $P_2(\cdot|G_2)$ . By uniqueness in the Savage theorem, we must have  $P_2^{\mathcal{B}_2} = P_2(\cdot|G_2)$ . Hence,  $P_2(E_2) = P_2(E_2|G_2)$  and therefore  $P_2(E_2 \cap G_2) = P_2(E_2) \cdot P_2(G_2)$  for all  $E_2 \in \mathcal{B}_2$  and all nonnull  $G_2 \in \Pi_2$ . If  $G_2$  is null, the result is straightforward.  $\square$

Proof of Corollary 2. Fix two acts  $f, g \in \mathcal{F}$ . Let  $\mathcal{B}_2$  be an admissible  $\sigma$ -algebra in  $\Sigma_2$  that is ordinally independent from  $E_2$  and makes  $f$  and  $g$  marginally measurable. Then,

$$\begin{aligned} \Phi(f_{E_2}g)(s_1)(x) &= P_2[\{f_{E_2}g(s_1, \cdot) = x\}] \\ &= P_2(E_2) \cdot P_2[\{f(s_1, \cdot) = x\}|E_2] + (P_2(E_2^c) \cdot P_2[\{g(s_1, \cdot) = x\}|E_2^c]) \\ &= P_2(E_2) \cdot P_2[\{f(s_1, \cdot) = x\}] + P_2(E_2^c) \cdot P_2[\{g(s_1, \cdot) = x\}] \\ &= P_2(E_2) \cdot \Phi(f)(s_1)(x) + P_2(E_2^c) \cdot \Phi(g)(s_1)(x), \end{aligned}$$

for all  $s_1 \in \mathcal{S}_1$  and  $x \in \mathcal{X}$ . Here, the first and last equalities are by the definition of  $\Phi$ , and the third one is by Proposition 1.  $\square$

Proof of Proposition 3. Suppose (i). Let  $\succsim^\ell$  and  $\succsim_2^\ell$  be the qualitative probabilities induced by  $\succsim$  and  $\succsim_2$  respectively. By Savage (1954), there exists a unique nonatomic probability measure  $P_2$  weakly agreeing with  $\succsim_2^\ell$  in the following sense: For all  $E_2, F_2 \in \Sigma_2$ ,  $E_2 \succsim_2^\ell F_2$  implies  $P_2(E_2) \geq P_2(F_2)$ . So fix a nonnull  $E_1 \in \Sigma_1$ . An easy consequence of A7 is the following property: For all  $E_2, F_2 \in \Sigma_2$ ,  $E_2 \succsim_2^\ell F_2$  implies  $E_1 \times E_2 \succsim^\ell E_1 \times F_2$ . Then, we obtain that the Bayesian update of  $P$  on  $E_1 \times \mathcal{S}_2$  weakly agrees with  $\succsim_2^\ell$  and is therefore equal to  $P_2$ . As this holds for all nonnull  $E_1 \in \Sigma_1$ , we obtain that  $P$  is the independent product of its marginal on  $\mathcal{S}_1$  and  $P_2$ . Finally, (ii) implies (i) by a straightforward application of the Fubini theorem.  $\square$

Proof of Proposition 4. Let  $f_2, g_2, h_2 \in \mathcal{F}_2$  and  $E_1 \in \Sigma_1$  be such that  $f_2 \succsim_2 g_2$ . Let  $f, g \in \mathcal{F}$  be defined by  $f = f_{2E_1}h_2$  and  $g = g_{2E_1}h_2$ . We have  $f(s_1, \cdot) = h_2 = g(s_1, \cdot)$  for all  $s_1 \notin E_1$  and  $f(s_1, \cdot) = f_2 \succsim_2 g_2 = g(s_1, \cdot)$  for all  $s_1 \in E_1$ . A7 yields  $f \succsim g$ .  $\square$

Proof of Proposition 5. We proceed by means of contradiction. Suppose  $f_2, g_2, h_2, k_2 \in \mathcal{F}_2$  and  $E_2 \in \Sigma_2$  are such that  $f_{2E_2}h_2 \succsim_2 g_{2E_2}h_2$  and  $g_{2E_2}k_2 \succ_2 f_{2E_2}k_2$ . Let  $E_1^* \in \Sigma_1$  be a fair coin. Then, define  $f \in \mathcal{F}$  by setting  $f(s_1, s_2) = (f_{2E_2}h_2)(s_2)$  if  $s_1 \in E_1^*$  and  $f(s_1, s_2) = (g_{2E_2}k_2)(s_2)$  if  $s_1 \notin E_1^*$ . Likewise, define  $g \in \mathcal{F}$  by setting  $g(s_1, s_2) = (g_{2E_2}h_2)(s_2)$  if  $s_1 \in E_1^*$  and  $g(s_1, s_2) = (f_{2E_2}k_2)(s_2)$  if  $s_1 \notin E_1^*$ .

Suppose  $E_1^*$  is null. For all  $x, y \in \mathcal{X}$ ,  $x_{E_1^*}y = y$  on the complement of  $E_1^* \times \mathcal{S}_2$ . So  $x_{E_1^*}y \sim y$ . Likewise,  $y_{E_1^*}x = x$  on the complement of  $E_1^* \times \mathcal{S}_2$ . So  $y_{E_1^*}x \sim x$ . But  $E_1^*$  is a fair coin so we obtain  $x \sim y$ , thereby contradicting A1. So  $E_1^*$  is nonnull. By symmetry,  $E_1^{*c}$  is nonnull as well.

Now, fix  $s_2 \in \mathcal{S}_2$ . Consider first the case where  $s_2 \in E_2$ . Then,  $f(\cdot, s_2) = f_{2E_2}h_2$  and  $g(\cdot, s_2) = g_{2E_2}h_2$ . Since  $E_1^*$  is a fair coin, we obtain  $f(\cdot, s_2) \sim_1 g(\cdot, s_2)$ . Consider now the case where  $s_2 \notin E_2$ . Then,  $f(\cdot, s_2) = f_{2E_2}k_2$  and  $g(\cdot, s_2) = g_{2E_2}k_2$ . Therefore, we obtain  $f(\cdot, s_2) \sim_1 g(\cdot, s_2)$ . By A7, we obtain  $f \sim g$ .

Meanwhile, fix  $s_1 \in \mathcal{S}_1$ . Consider first the case where  $s_1 \in E_1^*$ . Then,  $f(s_1, \cdot) = f_{2E_2}h_2$  and  $g(s_1, \cdot) = g_{2E_2}h_2$ . So, by assumption,  $f(s_1, \cdot) \succsim_2 g(s_1, \cdot)$ . Consider now the case where  $s_1 \notin E_1^*$ . Then,  $f(s_1, \cdot) = f_{2E_2}k_2$  and  $g(s_1, \cdot) = g_{2E_2}k_2$ . So, by assumption,  $f(s_1, \cdot) \succ_2 g(s_1, \cdot)$ . Thus, we have  $f(s_1, \cdot) \succsim_2 g(s_1, \cdot)$  for every  $s_1 \in \mathcal{S}_1$  and  $f(s_1, \cdot) \succ_2 g(s_1, \cdot)$  for every  $s_1 \in E_1^{*c}$  with  $E_1^{*c}$  nonnull. By A7\*, we obtain  $f \succ g$ , which contradicts the conclusion of the previous paragraph.  $\square$

Proof of Proposition 6. Let  $\mathcal{B}_2$  be an admissible  $\sigma$ -algebra that is ordinally independent from  $E_2$ . Let  $f_1, g_1, h_1 \in \mathcal{F}_1$ . Define  $f, g, h \in \mathcal{F}$  by setting  $f = f_1 \circ \pi_1$ ,  $g = g_1 \circ \pi_1$  and  $h = h_1 \circ \pi_1$ . Then, all of  $f, g$  and  $h$  are marginally  $\mathcal{B}_2$ -measurable, hence ordinally independent from  $E_2$ . Therefore, A8 implies  $f \succsim g$  if and only if  $f_{E_2}h \succsim g_{E_2}h$ . The result readily follows by Formula (1).  $\square$

## Appendix B — Proof of Theorems 7 and 8

**Lemma 10** *Suppose  $\succsim$  satisfies Axioms A1—A4, A5' and A6. Then, there exists a nonatomic and countably additive probability measure  $P_2$  on  $(\mathcal{S}_2, \Sigma_2)$  and a nonconstant function  $u$  from  $\mathcal{X}$  to  $\mathbb{R}$  such that for all  $f_2, g_2 \in \mathcal{F}_2$ ,*

$$f_2 \succsim_2 g_2 \iff \mathbb{E}_{P_2}[u \circ f_2] \geq \mathbb{E}_{P_2}[u \circ g_2]. \quad (3)$$

*Moreover,  $P_2$  is unique, and  $u$  is unique up to positive affine transformations.*

*Proof.* Note that  $\succsim_2$  satisfies all the axioms P1—P6 of Savage (1954). Then, by the Savage theorem (on finitely-valued acts), there exist a nonatomic and finitely additive probability measure  $P_2$  on  $(\mathcal{S}_2, \Sigma_2)$  and a nonconstant function  $u$  from  $\mathcal{X}$  to  $\mathbb{R}$  providing the representation (3). Still by Savage's theorem,  $P_2$  is unique, and  $u$  is unique up to positive affine transformations. Moreover, A5' implies that  $\succsim_2$  satisfies the Arrow (1970)[Chapter 2] axiom of Monotone Continuity. By his Lemma 3,  $P_2$  must be countably additive.  $\square$

Let  $\Delta$  denote the set of all finitely-supported probability distributions on  $\mathcal{X}$ . Let  $\mathcal{A}$  denote the set of all finitely-valued and measurable functions from  $\mathcal{S}_1$  to  $\Delta$ . Let  $P_2$  be the probability measure from Lemma 10. We define a mapping  $\Phi$  from  $\mathcal{F}$  to  $\Delta^{\mathcal{S}_1}$  by letting  $\Phi(f)(s_1)$  denote the probability distribution of  $f(s_1, \cdot)$  under  $P_2$  for all  $f \in \mathcal{F}$  and  $s_1 \in \mathcal{S}_1$ . Hence, by construction, we have, for all  $f \in \mathcal{F}$  and  $s_1 \in \mathcal{S}_1$ ,

$$\mathbb{E}_{P_2}[v \circ f(s_1, \cdot)] = \mathbb{E}_{\Phi(f)(s_1)}[v], \quad (4)$$

for all function  $v$  from  $\mathcal{X}$  to  $\mathbb{R}$ .

**Lemma 11** *The mapping  $\Phi$  is onto from  $\mathcal{F}_0$  to  $\mathcal{A}$ .*

*Proof.* Fix  $f \in \mathcal{F}_0$  and let  $\alpha = \Phi(f)$ . There exist a finite  $\Sigma_1$ -measurable partition  $\Pi_1 = \{E_1^1, \dots, E_1^N\}$  of  $\mathcal{S}_1$  and a finite  $\Sigma_2$ -measurable partition  $\Pi_2 = \{E_2^1, \dots, E_2^M\}$  of  $\mathcal{S}_2$  such that  $f$  is constant on  $E_1^n \times E_2^m$  for all  $n \in [1 \dots N]$  and  $m \in [1 \dots M]$ . Then,  $\alpha$  is constant on each subset in  $\Pi_1$  and hence an element of  $\mathcal{A}$ .

Fix  $\alpha \in \mathcal{A}$ . Then, there exist a finite measurable partition  $\{E_1^1, \dots, E_1^N\}$  of  $(\mathcal{S}_1, \Sigma_1)$  and a collection  $\{l_1, \dots, l_N\}$  of elements of  $\Delta$  such that  $\alpha$  is equal to  $l_n$  on  $E_1^n$  for every  $n \in [1 \dots N]$ . But furthermore, by the nonatomicity of  $P_2$ , we can find for every  $n \in [1 \dots N]$  an act  $f_2^n \in \mathcal{F}_2$  whose probability distribution under  $P_2$  is  $l_n$ . Then, define a function  $f$  from  $\mathcal{S}$  to  $\mathcal{X}$  by setting for all  $(s_1, s_2) \in \mathcal{S}$

$$f(s_1, s_2) = f_2^n(s_2) \quad \text{if } s_1 \in E_1^n.$$

Clearly,  $f$  is an element of  $\mathcal{F}_0$ . Moreover, fix  $s_1 \in \mathcal{S}_1$  and let  $n \in [1 \dots N]$  be such that  $s_1 \in E_1^n$ . Then,  $f(s_1, \cdot) = f_2^n$ . So by construction  $\Phi(f)(s_1) = l_n$ . Then,  $\Phi(f)(s_1) = \alpha(s_1)$  and therefore  $\Phi(f) = \alpha$ .  $\square$

Consider a binary relation  $\succsim_{\mathcal{A}}$  on  $\mathcal{A}$ . It is said to be *complete* if  $\alpha \succsim_{\mathcal{A}} \beta$  or  $\beta \succsim_{\mathcal{A}} \alpha$  for all  $\alpha, \beta \in \mathcal{A}$ . It is said to be *transitive* if, for all  $\alpha, \beta, \gamma \in \mathcal{A}$ ,  $\alpha \succsim_{\mathcal{A}} \beta$  and  $\beta \succsim_{\mathcal{A}} \gamma$  implies  $\alpha \succsim_{\mathcal{A}} \gamma$ . It is said to be *nontrivial* if there exist  $l, m \in \Delta$  such that  $l \succ_{\mathcal{A}} m$ . It is said to be *monotonic* if for all  $\alpha, \beta \in \mathcal{A}$  such that  $\alpha(s_1) \succsim_{\mathcal{A}} \beta(s_1)$  for all  $s_1 \in \mathcal{S}_1$ ,  $\alpha \succsim_{\mathcal{A}} \beta$ .

**Lemma 12** *Suppose  $\succsim$  satisfies Axioms Axioms A1–A4, A5' and A6–A7. Then, there exists a nontrivial, complete, transitive and monotonic binary relation  $\succsim_{\mathcal{A}}$  on  $\mathcal{A}$  such that, for all  $f, g \in \mathcal{F}_0$ ,*

$$f \succsim g \iff \Phi(f) \succsim_{\mathcal{A}} \Phi(g).$$

*Proof.* We first show that  $\Phi(f) = \Phi(g)$  implies  $f \sim g$  for all  $f, g \in \mathcal{F}_0$ . If  $\Phi(f) = \Phi(g)$ , then an application of Formula (4) to the case  $v = u$  yields  $\mathbb{E}_{P_2}[u \circ f(s_1, \cdot)] = \mathbb{E}_{P_2}[u \circ g(s_1, \cdot)]$  for all  $s_1 \in \mathcal{S}_1$ . Then, by Formula (3),  $f(s_1, \cdot) \sim_2 g(s_1, \cdot)$  for all  $s_1 \in \mathcal{S}_1$ , and A7 implies  $f \sim g$ . Then, define  $\succsim_{\mathcal{A}}$  on  $\mathcal{A}$  by setting for all  $\alpha, \beta \in \mathcal{A}$

$$\alpha \succsim_{\mathcal{A}} \beta \iff \text{there exist } f, g \in \mathcal{F} \text{ such that } \Phi(f) = \alpha, \Phi(g) = \beta \text{ and } f \succsim g.$$

It is straightforward to check nontriviality. Completeness follows from Lemma 11 and A1. Transitivity follows from the previous paragraph. Suppose now that  $\alpha, \beta \in \mathcal{A}$  are such that  $\alpha(s_1) \lesssim_{\mathcal{A}} \beta(s_1)$  for all  $s_1 \in \mathcal{S}_1$ . Let  $f, g \in \mathcal{F}_0$  be such that  $\Phi(f) = \alpha$  and  $\Phi(g) = \beta$ . Then,  $\Phi(f(s_1, \cdot) \circ \pi_2) = \alpha(s_1)$  and  $\Phi(g(s_1, \cdot) \circ \pi_2) = \beta(s_1)$  for all  $s_1 \in \mathcal{S}_1$ . Hence, we have  $f(s_1, \cdot) \lesssim_2 g(s_1, \cdot)$  for all  $s_1 \in \mathcal{S}_1$ , and A7 implies  $f \lesssim g$ . This finally yields  $\alpha \lesssim_{\mathcal{A}} \beta$ .  $\square$

In the next preliminary lemma, we say that two events  $E_2, F_2 \in \Sigma_2$  are *cardinally independent* if  $P_2(E_2 \cap F_2) = P_2(E_2) \cdot P_2(F_2)$ . Moreover, fix a subset  $E_2 \in \Sigma_2$  and an  $N$ -fold measurable partition  $\Pi_2 = \{E_2^1, \dots, E_2^N\}$  of  $E_2$ . Then, we say that  $\Pi_2$  is an *equipartition* of  $E_2$  if  $P_2(E_2^n)$  is independent of  $n \in [1 \dots N]$ .

**Lemma 13** *Consider any cardinally independent  $E_2, F_2 \in \Sigma_2$ . Then, there exists a binary  $\Sigma_2$ -measurable equipartition  $\{F_2^1, F_2^2\}$  of  $F_2$  such that  $E_2$  and  $F_2^1$  are cardinally independent and  $E_2$  and  $F_2^2$  are cardinally independent.*

*Proof.* By the nonatomicity of  $P_2$ , there exist  $G_2, H_2 \in \Sigma_2$  such that  $G_2 \subseteq F_2 \cap E_2$  and  $H_2 \subseteq F_2 \cap E_2^c$  and such that  $P_2(G_2) = P(F_2) \cdot P_2(E_2)/2$  and  $P_2(H_2) = P(F_2) \cdot P_2(E_2^c)/2$ . Now, set  $F_2^1 = G_2 \cup H_2$ . Clearly,  $F_2^1$  lies in  $\Sigma_2$  and satisfies  $P(F_2^1) = P_2(F_2)/2$ . Moreover,  $P_2(F_2^1 \cap E_2) = P_2(G_2) = P(F_2^1) \cdot P_2(E_2)$ . Hence,  $E_2$  and  $F_2^1$  are cardinally independent. It is then sufficient to set  $F_2^2 = F_2 \setminus F_2^1$  to conclude.  $\square$

An event  $E_2 \in \Sigma_2$  is called an *atom* of  $P_2$  if  $P_2(E_2) > 0$  and, for all  $F_2 \in \Sigma_2$ ,  $P_2(F_2 \cap E_2) = 0$  or  $P_2(F_2^c \cap E_2) = 0$ . A standard result (which requires  $\Sigma_2$  to be a  $\sigma$ -algebra) asserts the following:  $P_2$  is nonatomic if and only if  $P_2$  has no atoms. Similar definitions and results can easily be adapted to the restriction of  $P_2$  to any  $\sigma$ -subalgebra of  $\Sigma_2$ .

**Lemma 14** *Consider any  $E_2 \in \Sigma_2$ . Then, there exists a  $\sigma$ -algebra  $\mathcal{B}_2$  in  $\Sigma_2$  that is cardinally independent from  $E_2$  and such that the restriction of  $P_2$  to  $\mathcal{B}_2$  is nonatomic.*

*Proof.* Fix  $E_2 \in \Sigma_2$  and let  $\Pi_0 = \{\mathcal{S}_2\}$  denote the trivial partition of  $\mathcal{S}_2$ . Iterated applications of Lemma 13 provide a sequence  $\{\Pi_n, n \in \mathbb{N}\}$  of  $\Sigma_2$ -measurable partitions of  $\mathcal{S}_2$  such that, for all  $n \in \mathbb{N}$ ,

- (i) For all  $F_2 \in \Pi_n$ ,  $F_2$  and  $E_2$  are cardinally independent,
- (ii)  $\Pi_n$  is a  $2^n$ -fold equipartition of  $\mathcal{S}_2$ ,
- (iii)  $\Pi_{n+1}$  refines  $\Pi_n$ .

Now, define  $\mathcal{P}$  as the collection of all subsets of  $\mathcal{S}_2$  that are either empty, or an element of some partition  $\Pi_n$  for  $n \in \mathbb{N}$ . Let  $\Lambda_2$  denote the smallest  $\lambda$ -system containing  $\mathcal{P}$  and, likewise,  $\mathcal{B}_2$  denote the smallest  $\sigma$ -algebra containing  $\mathcal{P}$ . By (iii),  $\mathcal{P}$  is stable under finite intersection and hence a  $\pi$ -system. Then, the Monotone Class Theorem (for instance, see Kallenberg (2002)) yields  $\mathcal{B}_2 = \Lambda_2$ . But furthermore let  $E_2^\perp \subseteq \Sigma_2$  denote the collection of all subsets in  $\Sigma_2$  that are cardinally independent

from  $E_2$ . Clearly,  $E_2^\perp$  is a  $\lambda$ -system which, by (i), contains  $\mathcal{P}$ . Therefore,  $\Lambda_2 \subseteq E_2^\perp$  and  $\mathcal{B}_2$  is hence cardinally independent from  $E_2$ .

We now prove by contradiction that the restriction of  $P_2$  to  $\mathcal{B}_2$  has no atoms. Suppose there exists an atom  $E_2^0 \in \mathcal{B}_2$ . We will first show the following by induction: For all  $n \geq 0$ , there exists  $F_2^n \in \Pi_n$  such that

$$P_2(F_2^n \cap E_2^0) = P_2(E_2^0).$$

Suppose first  $n = 0$ . It is then sufficient to take  $F_2^0 = \mathcal{S}_2$ . Suppose now the result at rank  $n$ , and let  $F_2^n \in \Pi_n$  such that  $P_2(F_2^n \cap E_2^0) = P_2(E_2^0)$ . By (ii) and (iii), we can find  $G_2^{n+1}, H_2^{n+1} \in \Pi_{n+1}$  such that  $F_2^n = G_2^{n+1} \cup H_2^{n+1}$ . Then, we have

$$(\mathcal{S}_2 \setminus G_2^{n+1}) \cap F_2^n \cap E_2^0 = H_2^{n+1} \cap F_2^n \cap E_2^0.$$

But furthermore, since  $E_2^0$  is an atom,  $F_2^n \cap E_2^0$  must also be an atom, and we obtain

$$P_2(G_2^{n+1} \cap F_2^n \cap E_2^0) = 0 \quad \text{or} \quad P_2(H_2^{n+1} \cap F_2^n \cap E_2^0) = 0.$$

Set  $F_2^{n+1} = H_2^{n+1}$  in the former case and  $F_2^{n+1} = G_2^{n+1}$  in the latter one. Then, we obtain  $P_2(F_2^{n+1} \cap F_2^n \cap E_2^0) = P_2(F_2^n \cap E_2^0) = P_2(E_2^0)$ . Note also  $F_2^{n+1}$  and  $F_2^n$  cannot be disjoint because  $P_2(E_2^0) > 0$ . By (iii), we must have  $F_2^{n+1} \subseteq F_2^n$  and obtain  $P_2(F_2^{n+1} \cap E_2^0) = P_2(E_2^0)$ .

Now, (ii) and the previous paragraph show that  $P_2(E_2^0) \leq P_2(F_2^n) = 1/2^n$  for all  $n \geq 0$ . This implies  $P_2(E_2^0) = 0$  and contradicts the assumption that  $E_2^0$  is an atom.  $\square$

**Lemma 15** *For all  $\lambda \in (0, 1]$  and  $\alpha, \beta, \gamma \in \mathcal{A}$ , there exist  $f, g, h \in \mathcal{F}_0$  and  $E_2 \in \Sigma_2$  such that*

- (i)  $\alpha = \Phi(f)$ ,  $\beta = \Phi(g)$ ,  $\gamma = \Phi(h)$  and  $\lambda = P_2(E_2)$ ,
- (ii)  $f$ ,  $g$  and  $h$  are ordinally independent from  $E_2$ .
- (iii)  $\Phi(f_{E_2}h) = \lambda\alpha + (1 - \lambda)\gamma$  and  $\Phi(g_{E_2}h) = \lambda\beta + (1 - \lambda)\gamma$ ,

*Proof.* Since  $P_2$  is nonatomic, there exists  $E_2 \in \Sigma_2$  such that  $\lambda = P_2(E_2)$ . Then, let  $\Pi_2$  be the partition of  $\mathcal{S}_2$  generated by  $E_2$ . Lemma 14 provides a  $\sigma$ -algebra  $\mathcal{B}_2$  in  $\Sigma_2$  satisfying

- (a) The restriction of  $P_2$  to  $\mathcal{B}_2$  is nonatomic,
- (b)  $\mathcal{B}_2$  is cardinally independent from  $\Pi_2$ .

By Proposition 1 and Lemma 9,  $\mathcal{B}_2$  is admissible and ordinally independent from  $\Pi_2$ . By (a), we can find marginally  $\mathcal{B}_2$ -measurable  $f, g, h \in \mathcal{F}_0$  such that  $\alpha = \Phi(f)$ ,  $\beta = \Phi(g)$  and  $\gamma = \Phi(h)$ . Finally, (ii) holds by construction, and (iii) follows from Corollary 2.  $\square$

A pair  $\{\alpha, \beta\}$  where  $\alpha, \beta \in \mathcal{A}$  is said to be *comonotonic* if there are no  $s_1, s'_1 \in \mathcal{S}_1$  such that  $\alpha(s_1) \succ_{\mathcal{A}} \alpha(s'_1)$  and  $\beta(s'_1) \succ_{\mathcal{A}} \beta(s_1)$ .

A binary relation  $\succsim_{\mathcal{A}}$  on  $\mathcal{A}$  is said to satisfy *Independence* if for all  $\alpha, \beta, \gamma \in \mathcal{A}$  and all  $\lambda \in (0, 1]$ ,  $\alpha \succsim_{\mathcal{A}} \beta$  if and only if  $\lambda\alpha + (1 - \lambda)\gamma \succsim_{\mathcal{A}} \lambda\beta + (1 - \lambda)\gamma$ . It is said to satisfy *Comonotonic Independence* if for all  $\alpha, \beta, \gamma \in \mathcal{A}$  such that the pairs  $\{\alpha, \gamma\}$  and  $\{\beta, \gamma\}$  are comonotonic and all  $\lambda \in (0, 1]$ ,  $\alpha \succsim_{\mathcal{A}} \beta$  if and only if  $\lambda\alpha + (1 - \lambda)\gamma \succsim_{\mathcal{A}} \lambda\beta + (1 - \lambda)\gamma$ .

The following lemma is key in using A8 (resp. A8') to obtain Independence (resp. Comonotonic Independence). In fact, it only uses a weak version of A8. Consider first a collection  $\{f_i, i \in I\}$  of acts in  $\mathcal{F}$  and an event  $E_2 \in \Sigma_2$ . We say that  $\{f_i, i \in I\}$  is *ordinally (stochastically) independent from  $E_2$*  if there exists an admissible  $\sigma$ -algebra in  $\Sigma_2$  that is ordinally stochastically independent from  $E_2$  and makes  $f_i$  marginally measurable for all  $i \in I$ . Clearly, in this case, each  $f_i$  is ordinally independent from  $E_2$  for all  $i \in I$ , but the converse implication might fail. It turns out that Theorem 7 remains correct when one replaces A8 with the following weaker version of Subjective Independence:

**(A8<sup>-</sup>)** For all nonnull  $E_2 \in \Sigma_2$  and all  $f, g, h \in \mathcal{F}$  such that  $\{f, g, h\}$  is ordinally stochastically independent from  $E_2$ ,  $f \succsim g$  if and only if  $f_{E_2}h \succsim g_{E_2}h$ .

Indeed, the construction of the representation from Theorem 7 only uses A8<sup>-</sup>. (See Lemma 16.) To see why A8<sup>-</sup> is strictly weaker than A8, consider the case where  $\mathcal{S}_2 = [0, 1] \times [0, 1]$ , and  $\Sigma_2$  consists of all Borel subsets. Suppose that preferences  $\succsim_2$  have an SEU representation with respect to the independent product of two copies of the Lebesgue measure on  $[0, 1]$ . Consider also the admissible  $\sigma$ -algebras defined by  $\mathcal{B}_2^1 = \{A \times [0, 1], A \in \mathcal{B}\}$  and  $\mathcal{B}_2^2 = \{[0, 1] \times A, A \in \mathcal{B}\}$ , where  $\mathcal{B}$  denotes the  $\sigma$ -algebra of Borel subsets on  $[0, 1]$ . Clearly,  $\mathcal{B}_2^1$  and  $\mathcal{B}_2^2$  are ordinally independent from  $E_2 = [0, 1/2] \times [0, 1/2] \cup [1/2, 1] \times [1/2, 1]$ . But there can be no  $\sigma$ -algebra containing both  $\mathcal{B}_2^1$  and  $\mathcal{B}_2^2$  and ordinally independent from  $E_2$ , because this algebra should necessarily also contain  $E_2$  itself. In other words, if  $f, g, h \in \mathcal{F}$  are such that, say,  $f$  is  $\mathcal{B}_2^1$ -marginally measurable while  $g$  and  $h$  are  $\mathcal{B}_2^2$ -marginally measurable, then all three of  $f, g$  and  $h$  are ordinally independent from  $E_2$ , and A8 applies. But  $\{f, g, h\}$  cannot be ordinally independent from  $E_2$ , and A8<sup>-</sup> does not apply.

**Lemma 16** *Suppose  $\succsim$  satisfies Axioms A1–A4, A5', A6–A7 and A8'. Let  $\succsim_{\mathcal{A}}$  be the binary relation from Lemma 12. Then,*

- (i)  $\succsim_{\mathcal{A}}$  satisfies *Comonotonic Independence*,
- (ii) *If  $\succsim$  additionally satisfies Axiom A8, then  $\succsim_{\mathcal{A}}$  satisfies Independence.*

*Proof.* Fix  $\lambda \in (0, 1]$  and  $\alpha, \beta, \gamma \in \mathcal{A}$ . Let  $f, g, h \in \mathcal{F}$ ,  $\mathcal{B}_2$  and  $E_2 \in \Sigma_2$  be as in Lemma 15. Suppose first that the pairs  $\{\alpha, \gamma\}$  and  $\{\beta, \gamma\}$  are comonotonic. Then, the pairs  $\{f, h\}$  and  $\{g, h\}$  are marginally comonotonic. By A8', we obtain

$$f \succsim g \iff f_{E_2}h \succsim g_{E_2}h.$$

Then, Lemma 12 and items (i) and (iii) from Lemma 15 imply

$$\alpha \succsim_{\mathcal{A}} \beta \iff \lambda\alpha + (1-\lambda)\gamma \succsim_{\mathcal{A}} \lambda\beta + (1-\lambda)\gamma.$$

In the case where A8 holds, Independence follows from the extension of this argument to all acts  $\alpha, \beta$  and  $\gamma$  in  $\mathcal{A}$ .  $\square$

We identify each  $l \in \Delta$  with the act in  $\mathcal{A}$  constantly equal to  $l$ . We likewise identify each  $x \in \mathcal{X}$  with the degenerate lottery that yield  $x$  with probability 1. Moreover, a binary relation  $\succsim_{\mathcal{A}}$  on  $\mathcal{A}$  is said to satisfy *Continuity* if for all  $\alpha \in \mathcal{A}$  and  $x, y \in \mathcal{X}$  such that  $x \succ_{\mathcal{A}} \alpha \succ_{\mathcal{A}} y$ , there exist  $\lambda, \mu \in (0, 1)$  such that  $\mu x + (1-\mu)y \succ_{\mathcal{A}} \alpha \succ_{\mathcal{A}} \lambda x + (1-\lambda)y$ . Note that this version of Continuity is slightly weak and nonstandard but nonetheless sufficient for the Anscombe and Aumann (1963) representation.

**Lemma 17** *Suppose  $\succsim$  satisfies Axioms A1–A4, A5', A6–A7 and A8'. Let  $\succsim_{\mathcal{A}}$  be the binary relation from Lemma 12. Then,  $\succsim_{\mathcal{A}}$  satisfies Continuity.*

*Proof.* Fix  $\alpha \in \mathcal{A}$  and  $x, y \in \mathcal{X}$  such that  $x \succ_{\mathcal{A}} \alpha \succ_{\mathcal{A}} y$ . Then, we can find  $f \in \mathcal{F}_0$  such that  $\Phi(f) = \alpha$ . By Lemma 12, we have  $x \succ f \succ y$ .

Let  $\Lambda$  denote the set of all values  $P_2(E_2)$  for  $E_2 \in \Sigma_2$  such that  $f \succ x_{E_2}y$ . We will show the following: for all  $\lambda \in \Lambda$ , there exists  $\lambda' \in [0, 1]$  such that  $\lambda' > \lambda$  and  $\lambda' \in \Lambda$ . Indeed, suppose  $\lambda \in \Lambda$  and let  $E_2 \in \Sigma_2$  be such that  $P_2(E_2) = \lambda$  and  $f \succ x_{E_2}y$ . By A6, we can find a finite  $\Sigma_2$ -measurable partition  $\Pi_2$  of  $\mathcal{S}_2$  such that  $f \succ x_{F_2}(x_{E_2}y) = x_{E_2 \cup F_2}y$  for all  $F_2 \in \Pi_2$ . Now, suppose  $P_2(E_2 \cup F_2) = P_2(E_2)$  for all  $F_2 \in \Pi_2$ . Then,  $P_2(E_2 \cap F_2) = P_2(F_2)$  for all  $F_2 \in \Pi_2$ , and, by additivity,  $P_2(E_2) = 1$ . This implies  $f \succ x_{E_2}y \sim x$ , a contradiction. Hence, there exists  $F_2 \in \Pi_2$  such that  $P_2(E_2 \cup F_2) > P_2(E_2)$  and  $f \succ x_{E_2 \cup F_2}y$ . Define  $\lambda' = P_2(E_2 \cup F_2)$ . Then,  $\lambda' > \lambda$  and  $\lambda' \in \Lambda$ .

Furthermore, note that  $0 \in \Lambda$ . Then, the previous paragraph provides  $\lambda \in (0, 1) \cap \Lambda$ . Let  $E_2 \in \Sigma_2$  be such that  $P_2(E_2) = \lambda$  and  $f \succ x_{E_2}y$ . Then, by Lemma 12, we have  $\alpha \succ_{\mathcal{A}} \Phi[(x_{E_2}y) \circ \pi_2] = \lambda x + (1-\lambda)y$ . A symmetric argument shows the existence of  $\mu \in (0, 1)$  such that  $\mu x + (1-\mu)y \succ_{\mathcal{A}} \alpha$ .  $\square$

**Lemma 18** *Suppose  $\succsim$  satisfies Axioms A1–A8. Let  $u$  and  $P_2$  be as in Lemma 10. Then, there exists a unique countably additive probability measure  $P_1$  on  $(\mathcal{S}_1, \Sigma_1)$  such that, for all  $f, g \in \mathcal{F}_0$ ,*

$$f \succsim g \iff \int_{\mathcal{S}} u \circ f \, dP_1 \otimes P_2 \geq \int_{\mathcal{S}} u \circ g \, dP_1 \otimes P_2.$$

*Proof.* By Lemmata 12, 16 and 17,  $\succsim_{\mathcal{A}}$  satisfies all the axioms of the Schmeidler (1989) version of the Anscombe and Aumann (1963) theorem. Hence, there exists a unique (finitely additive) probability measure  $P_1$  on  $(\mathcal{S}_1, \Sigma_1)$  and a cardinally unique non-constant function  $v$  from  $\mathcal{X}$  to  $\mathbb{R}$  such that, for all  $\alpha, \beta \in \mathcal{A}$ ,

$$\alpha \succsim_{\mathcal{A}} \beta \iff \mathbb{E}_{P_1}[\mathbb{E}_{\alpha}[v]] \geq \mathbb{E}_{P_1}[\mathbb{E}_{\beta}[v]].$$

Note that both  $u$  and  $v$  represent the restriction of  $\succsim_{\mathcal{A}}$  to  $\Delta$ . Then, by the von Neumann and Morgenstern (1947) theorem,  $u$  and  $v$  are positive affine transformations of each other. Finally, by Lemma 12 and Formula (4), we have for all  $f, g \in \mathcal{F}_0$ ,

$$f \succsim g \iff \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ f(s_1, s_2) dP_2(s_2) dP_1(s_1) \geq \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ g(s_1, s_2) dP_2(s_2) dP_1(s_1).$$

The Fubini theorem then provides the representation. Finally, by Lemma 3 from Arrow (1970)[Chapter 2],  $P_1$  is countably additive.  $\square$

**Lemma 19** *Suppose  $\succsim$  satisfies Axioms A1–A4, A5', A6–A7 and A8'. Let  $u$  and  $P_2$  be as in Lemma 10. Then, there exists a unique capacity  $v_1$  on  $(\mathcal{S}_1, \Sigma_1)$  such that, for all  $f, g \in \mathcal{F}_0$ ,*

$$f \succsim g \iff \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ f(s_1, s_2) dP_2(s_2) dv_1(s_1) \geq \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ g(s_1, s_2) dP_2(s_2) dv_1(s_1).$$

*Proof.* The proof is similar to that of Lemma 18 with the Schmeidler (1989) theorem playing here the role the Anscombe and Aumann (1963) theorem played there.  $\square$

**Lemma 20** *Suppose  $\succsim$  satisfies Axioms A1–A8. Let  $(u, P_1, P_2)$  be as in Lemma 18. For all  $f \in \mathcal{F}$ , there exists  $f_2 \in \mathcal{F}_2$  such that  $f \sim f_2 \circ \pi_2$ .*

*Proof.* Fix  $f \in \mathcal{F}$ . Since  $f$  is finitely-valued, A1 implies the existence of  $x_1, x_0 \in \mathcal{X}$  such that  $x_1 \succsim f(s_1, s_2) \succsim x_0$  for all  $(s_1, s_2) \in \mathcal{S}$ . Then, fix  $s_1 \in \mathcal{S}_1$ . We have  $x_1 \succsim_2 f(s_1, s_2) \succsim_2 x_0$  for all  $s_2 \in \mathcal{S}_2$ . Hence, by iterated applications of A1 and A3, we obtain  $x_1 \succsim_2 f(s_1, \cdot) \succsim_2 x_0$ . As this holds for all  $s_1 \in \mathcal{S}_1$ , A7 further yields  $x_1 \succsim f \succsim x_0$ . If  $f \sim x_1$  or  $f \sim x_0$ , the result is straightforward. So we suppose  $x_1 \succ f \succ x_0$

Now, let  $P^-$  be the set of all values  $P_2(E_2)$  for  $E_2 \in \Sigma_2$  such that  $f \succ x_{1E_2}x_0$ . Suppose  $p \in P^-$  and  $q \in [0, 1]$  is such that  $q < p$ . Let  $E_2 \in \Sigma_2$  be such that  $P_2(E_2) = p$  and  $f \succ x_{1E_2}x_0$ . Then, by the nonatomicity of  $P_2$ , we can find  $F_2 \subseteq E_2$  such that  $P_2(F_2) = q$ . Since  $F_2 \subseteq E_2$ , A3 implies  $x_{1E_2}x_0 = x_{1E_2 \setminus F_2}(x_{1F_2}x_0) \succ x_{0E_2 \setminus F_2}(x_{1F_2}x_0) = x_{1F_2}x_0$ , hence  $f \succ x_{1F_2}x_0$ . Thus,  $q \in P^-$ . This shows that  $P^-$  is of the form  $[0, p^-]$  or  $[0, p^-)$ .

But furthermore suppose  $p \in P^-$ . Let  $E_2 \in \Sigma_2$  such that  $P_2(E_2) = p$  and  $f \succ x_{1E_2}x_0$ . By A6, there exists a finite  $\Sigma_2$ -measurable partition  $\Pi_2$  of  $\mathcal{S}_2$  such that  $f \succ x_{1F_2}(x_{1E_2}x_0) = x_{1E_2 \cup F_2}x_0$  for all  $F_2 \in \Pi_2$ . Now, suppose  $P_2(E_2 \cup F_2) = P_2(E_2)$  for all  $F_2 \in \Pi_2$ . Then,  $P_2(E_2 \cap F_2) = P_2(F_2)$  for all  $F_2 \in \Pi_2$ , and, by additivity,  $P_2(E_2) = 1$ . This implies  $f \succ x_{1E_2}x_0 \sim x_1$ , a contradiction. Hence, there exists  $F_2 \in \Pi_2$  such that  $P_2(E_2 \cup F_2) > P_2(E_2)$  and  $f \succ x_{1E_2 \cup F_2}x_0$ . Define  $p' = P_2(E_2 \cup F_2)$ . Then,  $p' > p$  and  $p' \in \Lambda$ . This shows  $P^-$  is of the form  $[0, p^-)$ . A symmetric argument shows that the set  $P^+$  of all values  $P_2(E_2)$  for  $E_2 \in \Sigma_2$  such that  $x_{1E_2}x_0 \succ f$  is of the form  $(p^+, 1]$ .

Clearly, we must have  $p^- \leq p^+$ . By nonatomicity, there exists  $E_2 \in \Sigma_2$  such that  $p^- \leq P_2(E_2) \leq p^+$ . Then, we must have  $f \sim x_{1E_2}x_0$ .  $\square$

**Lemma 21** Suppose  $\succsim$  satisfies Axioms A1–A8. Let  $(u, P_1, P_2)$  be as in Lemma 18. For all  $f \in \mathcal{F}$  and  $f_2 \in \mathcal{F}_2$  such that  $f \sim f_2 \circ \pi_2$ , we have  $\int_{\mathcal{S}} u \circ f dP_1 \otimes P_2 = \int_{\mathcal{S}_2} u \circ f_2 dP_2$ .

*Proof.* Fix  $f \in \mathcal{F}$  and  $f_2 \in \mathcal{F}_2$  such that  $f \sim f_2 \circ \pi_2$ . Fix also  $N \geq 1$ . Since  $f$  is finitely-valued, A1 implies the existence of  $x_1, x_0 \in \mathcal{X}$  such that  $x_1 \succsim f(s_1, s_2) \succsim x_0$  for all  $(s_1, s_2) \in \mathcal{S}$ . By applying positive affine transformations if necessary, we may suppose  $u(x_1) < 1$  and  $u(x_0) = 0$  without loss of generality. Then, by Lemma 18, we have  $1 > u \circ f(s_1, s_2) \geq 0$  for all  $(s_1, s_2) \in \mathcal{S}$ . Furthermore, let  $\Pi_1 = \{E_1^1, \dots, E_1^N\}$  be the collection of subsets in  $\Sigma_1$  defined for all  $n \in [1 \dots N]$  by

$$E_1^n = \left\{ s_1 \in \mathcal{S}_1, \frac{n-1}{N} \leq \int_{\mathcal{S}_2} u \circ f(s_1, \cdot) dP_2 < \frac{n}{N} \right\}$$

Note that  $\Pi_1$  forms a measurable finite partition of  $\mathcal{S}_1$  possibly with empty subsets. Then, by the monotonicity of the integral, we further obtain

$$\sum_{n=1}^N P_1(E_1^n) \cdot \frac{n-1}{N} \leq \int_{\mathcal{S}} u \circ f dP_1 \otimes P_2 \leq \sum_{n=1}^N P_1(E_1^n) \cdot \frac{n}{N}. \quad (5)$$

Meanwhile, fix any  $n \in [0 \dots N]$ . By the nonatomicity of  $P_2$ , there exists  $E_2^n \in \Sigma_2$  such that

$$\frac{n}{N} = P_2(E_2^n) = \int_{\mathcal{S}_2} u \circ f_2^n dP_2,$$

where  $f_2^n = x_{1E_2^n} x_0 \in \mathcal{F}_2$ . Then, we define  $f_0, g_0 \in \mathcal{F}_0$  in the following way:

$$f_0 = \sum_{n=1}^N \mathbf{1}_{E_1^n} f_2^{n-1} \quad \text{and} \quad g_0 = \sum_{n=1}^N \mathbf{1}_{E_1^n} f_2^n$$

Now, fix  $s_1 \in \mathcal{S}_1$  and let  $n \in [1 \dots N]$  be such that  $s_1 \in E_1^n$ . Then,  $f_0(s_1, \cdot) = f_2^{n-1}$  and  $g_0(s_1, \cdot) = f_2^n$  and therefore

$$\int_{\mathcal{S}_2} u \circ f_0(s_1, \cdot) dP_2 = \frac{n-1}{N} \leq \int_{\mathcal{S}_2} u \circ f(s_1, \cdot) dP_2 \leq \frac{n}{N} = \int_{\mathcal{S}_2} u \circ g_0(s_1, \cdot) dP_2.$$

Lemma 10 then yields  $g_0(s_1, \cdot) \succsim_2 f(s_1, \cdot) \succsim_2 f_0(s_1, \cdot)$ , and this holds for all  $s_1 \in \mathcal{S}_1$ . Then, A7 further yields  $g_0 \succsim f \succsim f_0$ . Since  $f \sim f_2 \circ \pi_2$ , we obtain  $g_0 \succsim f_2 \circ \pi_2 \succsim f_0$ . By Lemma 18, we have:

$$\sum_{n=1}^N P_1(E_1^n) \cdot \frac{n-1}{N} \leq \int_{\mathcal{S}_2} u \circ f_2 dP_2 \leq \sum_{n=1}^N P_1(E_1^n) \cdot \frac{n}{N}. \quad (6)$$

Combining Formulae (5) and (6) gives

$$\left| \int_{\mathcal{S}} u \circ f dP_1 \otimes P_2 - \int_{\mathcal{S}_2} u \circ f_2 dP_2 \right| \leq \sum_{n=1}^N P_1(E_1^n) \cdot \frac{n}{N} - \sum_{n=1}^N P_1(E_1^n) \cdot \frac{n-1}{N} = \frac{1}{N}.$$

We finally obtain the result by taking the limit as  $N$  goes to  $\infty$ .  $\square$

**Lemma 22** Suppose  $\succsim$  satisfies Axioms A1–A8. Let  $(u, P_1, P_2)$  be as in Lemma 18. For all  $f, g \in \mathcal{F}$ ,

$$f \succsim g \iff \int_{\mathcal{S}} u \circ f \, dP_1 \otimes P_2 \geq \int_{\mathcal{S}} u \circ g \, dP_1 \otimes P_2.$$

*Proof.* Let  $f, g \in \mathcal{F}$ . By Lemma 20, we can find  $f_2, g_2 \in \mathcal{F}_2$  such that  $f \sim f_2 \circ \pi_2$  and  $g \sim g_2 \circ \pi_2$ . Hence, we have

$$f \succsim g \iff f_2 \circ \pi_2 \succsim g_2 \circ \pi_2 \iff \int_{\mathcal{S}_2} u \circ f_2 \, dP_2 \geq \int_{\mathcal{S}_2} u \circ g_2 \, dP_2,$$

where the second equivalence is by the representation from Lemma 18 and because  $f_2 \circ \pi_2$  and  $g_2 \circ \pi_2$  both lie in  $\mathcal{F}_0$  since they are constant on subsets of the form  $\mathcal{S}_1 \times E_2 \in \Sigma_0$  for  $E_2 \in \Sigma_2$ . Then, the result follows readily from Lemma 21.  $\square$

**Lemma 23** Suppose  $\succsim$  satisfies Axioms A1–A4, A5', A6–A7 and A8'. Let  $(u, v_1, P_2)$  be as in Lemma 19. For all  $f, g \in \mathcal{F}$ ,

$$f \succsim g \iff \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ f(s_1, s_2) \, dP_2(s_2) \, dv_1(s_1) \geq \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ g(s_1, s_2) \, dP_2(s_2) \, dv_1(s_1).$$

*Proof.* The proof is similar to that of Lemma 22.  $\square$

**Lemma 24** Suppose  $(u, P_1, P_2)$  provides a representation of  $\succsim$  as in Theorem 7. Let  $f, g \in \mathcal{F}$  and  $E_2 \in \Sigma_2$ . If  $f$  and  $g$  are cardinally independent from  $E_2$ , then

$$\mathbb{E}_{P_1 \otimes P_2}[u \circ (f_{E_2} g)] = P_2(E_2) \cdot \mathbb{E}_{P_1 \otimes P_2}[u \circ f] + P_2(E_2^c) \cdot \mathbb{E}_{P_1 \otimes P_2}[u \circ g].$$

*Proof.* For all  $h \in \mathcal{F}$ , let  $V(h) := \mathbb{E}_{P_1 \otimes P_2}[u \circ h]$ . Let also  $p = P_2(E_2)$  and  $q = P_2(E_2^c)$ . Then, by the Fubini theorem and the additivity of the integral, we have

$$V(f_{E_2} g) = \int_{\mathcal{S}_1} \left\{ \int_{E_2} u[f(s_1, s_2)] \, dP_2(s_2) + \int_{E_2^c} u[g(s_1, s_2)] \, dP_2(s_2) \right\} dP_1(s_1).$$

But since  $f$  and  $g$  are cardinally independent from  $E_2$ , this formula simplifies into

$$V(f_{E_2} g) = \int_{\mathcal{S}_1} \left\{ p \cdot \int_{\mathcal{S}_2} u[f(s_1, s_2)] \, dP_2(s_2) + q \cdot \int_{\mathcal{S}_2} u[g(s_1, s_2)] \, dP_2(s_2) \right\} dP_1(s_1).$$

Additivity and the Fubini theorem yield  $V(f_{E_2} g) = p \cdot V(f) + q \cdot V(g)$ .  $\square$

**Lemma 25** Suppose  $(u, v_1, P_2)$  provides a representation of  $\succsim$  as in Theorem 8 and denote by  $V$  the representing functional from  $\mathcal{F}$  to  $\mathbb{R}$ . Let  $f, g \in \mathcal{F}$  and  $E_2 \in \Sigma_2$ . Suppose  $f$  and  $g$  are marginally comonotonic. If  $f$  and  $g$  are cardinally independent from  $E_2$ , then

$$V(f_{E_2} g) = P_2(E_2) \cdot V(f) + P_2(E_2^c) \cdot V(g).$$

*Proof.* Let  $p = P_2(E_2)$  and  $q = P_2(E_2^c)$ . By the additivity of the inner integral, we have

$$V(f_{E_2}g) = \int_{\mathcal{S}_1} \left\{ \int_{E_2} u[f(s_1, s_2)] dP_2(s_2) + \int_{E_2^c} u[g(s_1, s_2)] dP_2(s_2) \right\} dv_1(s_1).$$

But since  $f$  and  $g$  are cardinally independent from  $E_2$ , this formula simplifies into

$$V(f_{E_2}g) = \int_{\mathcal{S}_1} \left\{ p \cdot \int_{\mathcal{S}_2} u[f(s_1, s_2)] dP_2(s_2) + q \cdot \int_{\mathcal{S}_2} u[g(s_1, s_2)] dP_2(s_2) \right\} dv_1(s_1).$$

Note now that, since  $f$  and  $g$  are marginally comonotonic, the functions  $F_1$  and  $G_1$  from  $\mathcal{S}_1$  to  $\mathbb{R}$  defined by

$$F_1(s_1) = p \cdot \int_{\mathcal{S}_2} u[f(s_1, s_2)] dP_2(s_2) \quad \text{and} \quad G_1(s_1) = q \cdot \int_{\mathcal{S}_2} u[g(s_1, s_2)] dP_2(s_2)$$

for all  $s_1 \in \mathcal{S}_1$  are comonotonic. So by the comonotonic additivity and positive homogeneity properties of the Choquet integral, we obtain

$$V(f_{E_2}g) = p \cdot \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u[f(s_1, s_2)] dP_2(s_2) dv_1(s_1) + q \cdot \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u[g(s_1, s_2)] dP_2(s_2) dv_1(s_1),$$

which establishes  $V(f_{E_2}g) = P_2(E_2) \cdot V(f) + P_2(E_2^c) \cdot V(g)$ .  $\square$

*Proof of Theorem 7. Sufficiency of the axioms.* Lemma 22 shows the sufficiency of the axioms for the representation.

*Necessity of the axioms.* The necessity of A1—A5 follows from standard arguments. A7 is a consequence of the Fubini theorem.

As for A6, fix  $f, g, h \in \mathcal{F}$  such that  $f \succ g$ . Let  $\Pi = \{E^1, \dots, E^N\}$  be a finite and  $\Sigma$ -measurable partition of  $\mathcal{S}$  such that  $f$  and  $g$  are constant on each subset in  $\Pi$ . The values of  $f$  and  $g$  on each  $E^n$  are respectively denoted by  $x_n$  and  $y_n$ . Moreover, since  $h \in \mathcal{F}$  is finitely-valued, there are best and worst consequences in the range of  $h$ , which we respectively denote by  $z_1 \in \mathcal{X}$  and  $z_0 \in \mathcal{X}$ . Finally, for all  $k \in \mathcal{F}$ , let  $V(k) := \mathbb{E}_{P_1 \otimes P_2}[u \circ k]$ . Then, let  $\epsilon > 0$  be such that  $V(f) > V(g) + \epsilon$ .

Now, let  $I$  be the set of all  $n \in [1 \dots N]$  such that  $x_n \succ z_0$ . Let also  $J$  be the set of all  $n \in [1 \dots N]$  such that  $z_1 \succ y_n$ . By the nonatomicity of  $P_2$ , we can find a finite and  $\Sigma_2$ -measurable partition  $\Pi_2$  of  $\mathcal{S}_2$  such that, for all  $E_2 \in \Pi_2$ , all  $n \in I$  and all  $m \in J$

$$0 < P_2(E_2) \leq \frac{\epsilon}{u(x_n) - u(z_0)} \quad \text{and} \quad 0 < P_2(E_2) \leq \frac{\epsilon}{u(z_1) - u(y_m)}.$$

Then, for all  $E_2 \in \Pi_2$  and all  $n, m \in [1 \dots N]$ , we have

$$[u(x_n) - u(z_0)] \cdot P_2(E_2) \leq \epsilon \quad \text{and} \quad [u(z_1) - u(y_m)] \cdot P_2(E_2) \leq \epsilon.$$

Put differently, we have for all  $s \in \mathcal{S}$  and  $E_2 \in \Pi_2$

$$P_2(E_2) \cdot u(z_0) \geq P_2(E_2) \cdot u \circ f(s) - \epsilon \quad \text{and} \quad P_2(E_2) \cdot u(z_1) \leq P_2(E_2) \cdot u \circ g(s) + \epsilon.$$

The first of these inequalities implies for all  $s_1 \in \mathcal{S}_1$  and  $E_2 \in \Pi_2$

$$\int_{\mathcal{S}_2} u \circ (z_0 E_2 f)(s_1, \cdot) dP_2 = P_2(E_2) \cdot u(z_0) + \int_{E_2^c} u \circ f(s_1, \cdot) dP_2 \geq \int_{\mathcal{S}_2} u \circ f(s_1, \cdot) dP_2.$$

By the monotonicity of the integral, we obtain  $V(z_0 E_2 f) \geq V(f) - \epsilon > V(g)$  for all  $E_2 \in \Pi_2$ . Proceeding similarly, we obtain  $V(z_1 E_2 g) \leq V(g) + \epsilon < V(f)$  for all  $E_2 \in \Pi_2$ . Finally, since  $z_1$  and  $z_0$  are best and worst consequences in the range of  $h$ , the monotonicity of  $V$  implies  $V(h_{E_2} f) > V(g)$  and  $V(h_{E_2} g) < V(f)$  for all  $E_2 \in \Pi_2$ . The result then follows readily from the representation.

As for A8, fix a nonnull  $E_2 \in \Sigma_2$ , and  $f, g, h \in \mathcal{F}$  that are ordinally independent from  $E_2$ . Then, there exists a  $\sigma$ -algebra  $\mathcal{B}_2$  in  $\Sigma_2$  that is admissible, ordinally independent from  $E_2$  and makes  $f$  marginally measurable. By Proposition 1,  $\mathcal{B}_2$  is also cardinally independent from  $E_2$ . Hence,  $f$  is cardinally independent from  $E_2$ . Likewise,  $g$  and  $h$  are cardinally independent from  $E_2$ . Then, we have

$$\begin{aligned} f_{E_2} h \succsim g_{E_2} h &\iff \mathbb{E}_{P_1 \otimes P_2}[u \circ (f_{E_2} h)] \geq \mathbb{E}_{P_1 \otimes P_2}[u \circ (g_{E_2} h)] \\ &\iff \mathbb{E}_{P_1 \otimes P_2}[u \circ f] \geq \mathbb{E}_{P_1 \otimes P_2}[u \circ g] \\ &\iff f \succsim g, \end{aligned}$$

where the first and last equivalences are by the representation, and the middle one is by Lemma 24 and because  $E_2$  is nonnull.

*Uniqueness of the representation.* Suppose each of the terms  $(u, P_1, P_2)$  and  $(u', P'_1, P'_2)$  provides a representation as in Theorem 7. Then, each of the pairs  $(u, P_2)$  and  $(u', P'_2)$  provides a representation as in Lemma 10. By uniqueness in this lemma,  $P_2 = P'_2$ , and  $u$  and  $u'$  are positive affine transformations of each other. But furthermore let  $\succsim_{\mathcal{A}}$  be defined as in Lemma 12. Then, each of the pairs  $(u, P_1)$  and  $(u', P'_1)$  provides a representation as in Lemma 18. By uniqueness, we have  $P_1 = P'_1$ .  $\square$

Proof of Theorem 8. *Sufficiency of the axioms.* Lemma 23 shows the sufficiency of the axioms for the representation.

*Necessity of the axioms.* The necessity of A1—A4 and A5' follows from standard arguments. A7 is a consequence of the monotonicity of the Choquet integral. The proof of A6 is similar to that of Theorem 7. Finally, the proof of A8' is similar to that of Theorem 7 but uses Lemma 25 instead of Lemma 24.

*Uniqueness of the representation.* Suppose each of the terms  $(u, v_1, P_2)$  and  $(u', v'_1, P'_2)$  provides a representation as in Theorem 8. Then, each of the pairs  $(u, P_2)$  and  $(u', P'_2)$  provides a representation as in Lemma 10. By uniqueness in this lemma,

$P_2 = P'_2$ , and  $u$  and  $u'$  are positive affine transformations of each other. But furthermore let  $\succsim_{\mathcal{A}}$  be defined as in Lemma 12. Then, each of the pairs  $(u, v_1)$  and  $(u', v'_1)$  provides a representation as in Lemma 19. By uniqueness, we have  $v_1 = v'_1$ .  
 $\square$

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