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*Can one be a fictionalist and a platonist at the same time?
Lessons from Leibniz*

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Abstract:

Drawing on Leibniz' thinking, this paper advocates two claims. Firstly, a continuous path can be drawn from the use of fictions inside mathematics, a widespread practice in the 16^e and 17^e centuries, to the use of mathematical entities as fictions outside mathematics (i.e. when it comes to employing them for describing the natural world). In the first case, fictitious entities are contrasted to other mathematical entities posited as “real”; in the second, all mathematical entities can be said to be fictitious compared to “real” things, taken in the sense of “existing in the natural world”. Secondly, I will show why the path drawn in the first section allows one – and even requires one if we follow Leibniz – to be both a fictionalist and a Platonist at the same time.

L'auteur:

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In memoriam Mic Detlefsen (1948-2019)

Introduction

Contemporary versions of mathematical fictionalism tend to forget (or simply ignore?) that there is a long and rich tradition of working with fictions in mathematics. In the period I will consider, this use was widespread and relatively independent of the philosophical positions endorsed by mathematicians on the nature of mathematical objects. It was a common practice when dealing with a large variety of entities such as negative, fractional or irrational numbers, imaginary roots, logarithms, points to infinity, angles, infinitely large or infinitely small quantities, etc¹. However, as some actors clearly saw, the fact that one can treat some entities in a fictional mode could be built into the very nature of what mathematical objects have to be. *A minima*, it would seem that one should give an account of the fact that mathematics is a type of knowledge in which one can reason with fictions.

Leibniz may have been one of the first philosophers to articulate such an account in the framework of a type of knowledge he called “blind” (or “symbolic” or “suppositive”) and which, he claims, was ubiquitous. Moreover, since this “blind” knowledge was secured in mathematics through various strategies (proofs, computations, exhibition of a model, etc.), Leibniz saw that the use of fictions in mathematics might give an interesting answer to the question posed more generally by the use of “ideal” objects in our knowledge (typically mathematical concepts themselves, but more generally any abstract property). In the same way that we can safely handle fictions in mathematics, we could safely handle mathematical concepts *as fictions* in the description of the natural world (how the two processes relate will be spelled out in section 2). Moreover, Leibniz showed that this relation was grounded in the way we perceive the world, for example when we perceive continuous transformations or uniform shapes.

The fact that we can treat mathematical concepts as fictions in the description of the natural world puts Leibniz in close connection to certain forms of what is nowadays called “mathematical fictionalism”. Now, the most interesting part of the story is that Leibniz was *also* a declared Platonist – in strong contrast, this time, to modern day fictionalists who tend to portray themselves as resolute anti-Platonists². The references to Plato or Platonists such as St Augustine were explicit. On numerous occasions, the philosopher claimed that mathematical

¹ Throughout this paper, I will speak rather loosely of “entity”. Since we will see “fictions” as a way to treat some expressions “as if” they refer to objects, this will be innocuous.

² As expressed by Mark Balaguer in the very first sentence of his entry on “Fictionalism in the Philosophy of Mathematics” for the *Stanford Encyclopedia of Philosophy*: “Mathematical fictionalism (...) is best thought of as a reaction to mathematical Platonism”.

truths exist independently of us, in a separate realm which he called “the region of ideas”, subsisting in God’s mind. In his parlance, the connexion between the two stances was expressed by saying that even if mathematical objects may be considered as fictions, they do not “reduce to fictions” or that they are not pure “chimera” (arbitrary fictions).

In this paper, I would like to come back to the three aspects mentioned above, because they offer an interesting entry into some puzzles in which contemporary philosophy of mathematics seems sometimes to lose itself. I will first describe how Leibniz related to the practice of introducing fictions in mathematics and reflected on it in terms of a certain type of knowledge, which he realized was very widespread. Then, I will focus on the way in which he drew a path between this use of fictions in mathematics and the use of mathematical objects as fictions outside mathematics. Finally, I will briefly expound the metaphysics behind these claims and in particular, the reasons why Leibniz thought that *all* mathematical objects could be said to be fictitious although he felt compelled to maintain a strict and explicit Platonism regarding them.

I. The practice of fictions in mathematics

Contrary to what has been thought for a long time, Leibniz’ use of fictions in mathematics was not prompted by the debates on the foundations of differential calculus which occurred at the turn of the XVIIIth century and in which the philosopher famously presented his infinitesimals as “fictions utiles”. Well-known claims from this period are such as the one expressed to Varignon in 1702, “I wrote some years ago to Mons. Bernoulli of Groningen that the infinite and infinitely small could be taken as fictions, similarly to imaginary roots, without that having to do a wrong to our calculus, these fictions being useful and founded in realities”³. In fact, when one turns to the exchange with Bernoulli mentioned to Varignon, one realizes that Leibniz refers to ideas already expressed more than twenty years earlier, when he was in Paris. More specifically, he refers to a treatise he had written at the time (but did not publish) under the title *Quadratura arithmetica circuli ellipseos et hyperbolae* (1676):

But between ourselves I would also add this, that I also wrote in the said unpublished manuscript [i.e. the *De Quadratura*] that it is possible to doubt whether there could be infinitely long straight lines that were nevertheless also in fact bounded. I wrote moreover that it suffices for calculus that they be taken as fictions [*fingantur*], like imaginary roots in algebra.⁴

It is important to note right away that although Leibniz explicitly speaks about “fictions” or uses the verb “to feign”, the exchange with Bernoulli also qualifies the very same entities as

³ GM IV 93–94. In this paper, I use the standard way of referring to Leibniz’ editions: GM = *Leibnizens mathematische Schriften*, ed. Pertz and Gerhardt, Berlin: A. Asher and Halle: H. W. Schmidt, 1849–1863; GP = *Die philosophischen Schriften*, ed. C. Gerhardt, Halle; H. W. Schmidt, 1875–1889; A = *Sämtliche Schriften und Briefe*, herausgegeben von der Berlin-Brandenburgischen Akademie der Wissenschaften und der Akademie der Wissenschaften zu Göttingen, Reihe 1–8, Darmstadt, Leipzig, Berlin, 1923-. L = *Leibniz. Philosophical Papers and Letters*, transl. Loemker, Dordrecht, Kluwer, 1989.

⁴ The remaining part of the passage is also interesting since it justifies the use of fictions in terms of paraphrases à la Archimedes and explains that there are no infinitesimals in nature (GM III, 522–524; A III 7, 855–858).

“imaginary”⁵. The synonymy between the two terms is something we will find again in many other passages I will quote in this paper.

In the last thirty years or so, Leibniz scholarship has amply confirmed that this practice of employing fictions began very early and was not a convenient way of escaping the vivid controversies surrounding the development of differential calculus at the turn of the century (or of hiding Leibniz’ own intimate convictions). Indeed, it can be found in numerous texts dating from even before the creation of the differential algorithm⁶. What may be less well known, however, is that this strategy applied to many more entities than infinitely large and infinitely small quantities. Moreover, it clearly took place in a very rich tradition of which I shall say a quick word to begin with.

The idea that one might reason with fictitious entities because they are useful in calculation or proofs was not an invention by Leibniz. A famous example is provided in the work of Michael Stifel who introduced negative numbers as *numeri ficti* in his *Arithmetica integra* (1544). More importantly, Stifel referred this use to a widespread practice amongst mathematicians when introducing square roots of non-square numbers or fractional numbers⁷. Indeed, such “numbers”, although largely used at the time, contradicted the definition of number as a multiplicity of (indivisible) units given by Euclid and other authorities. The term “fictitious numbers” can be found in many other authors, especially for designating negative numbers⁸ – but other names could be used, such as “imaginary”, “false”, *gedicht* (the German word to designate a poem), *songé* (the French word to designate a dream). This practice was seen as belonging to a more general setting in which one introduces entities by supposition, i.e. without knowing in advance if they satisfy a question or not. The most widespread case amongst arithmeticians was the practice of “false position” in which the number introduced at the beginning of the process was, once again, called “fictitious” (by opposition to the “true” number, which is the one found as a solution to the problem)⁹. It was also common to see this kind of supposition at play in the practice of Diophantine number theory and even, more generally, in algebra¹⁰. Indeed, this was a strong argument in favour of considering Diophantus as one of the fathers of the old art of analysis (on a par with Pappus, who described it in a geometrical context) and to claim, as Vieta and Descartes did, that the “new algebra” was a revival of this art, which the Ancients had kept secret.

As far as we can tell, Leibniz first encountered the issue of “fictions” when he arrived in Paris (1672). Be it in his studies on numerical series, when thinking about paradoxes on infinite collections or when computing specific quadratures for curves, he regularly stumbled upon the fact that an “infinite number” or multitude was a notion leading to contradiction. Infinitely small numbers, of the kind introduced by Wallis in his *Arithmetica infinitorum* (1655) as the inverse of an infinite number ∞ , naturally inherited this qualification – as did the geometrical

⁵ To Bernoulli, June 1698, GM III, 499.

⁶ See, in particular, the texts edited by R. Arthur: *G.W. Leibniz, The Labyrinth of the continuum: Writings on the continuum problem, 1672–1686*, New Haven: Yale University Press, 2001. Hereafter LLC.

⁷ Liber I, cap. 11, p. 8; liber III, cap. 5, p. 248-249.

⁸ I refer the reader to my “Negatives as fictions” (forthcoming) for more details. I just list here some authors with the date of publication of their treatises: Clavius (1608), Jenish (1609), Samuel Alzias (1646), Caramuel (1670), Kersey (1673-74), Millet de Chasles (1674), Gottignies (1675), Renaldini (1682).

⁹ Examples are: Gemma Frisius (1540), Peletier (1549), Trenchant (1558), Forcadel (1565), Henrion (1621), Mersenne (1625).

¹⁰ Typical examples are: Gosselin (1578) and Alsted (1630).

“indivisibles” from the Cavalierian tradition, which the former were supposed to express in a numerical way. This reflection was already well developed in Leibniz before the invention of the differential algorithm in the autumn of 1675. Thus, one of the first occurrences of the term itself, *fictio*, was related to the fact that formal manipulations on series expressing the quadrature of the hyperbola would make it possible to prove that the whole area under the curve is equal to one of its part¹¹. Since Leibniz took Euclid’s axiom stating that “the whole is greater than the part” as an analytic truth¹², his conclusion was the following: “By this argument it is concluded that the infinite is not a whole, but only a fiction, since otherwise the part would be equal to the whole” (A VII 3, 468; October 1674, transl. Arthur). This had strong echoes in the consideration of the so called “Galileo’s paradox”, which Leibniz studied at the same time¹³. Yet it did not mean that one could not use infinite numbers in calculation, provided one was aware that it cannot designate genuine numbers and should be accompanied by relevant proofs in order to secure the results¹⁴.

It is remarkable that this reflection was clearly taken into the tradition I just sketched above, as is apparent in an early text in which Leibniz puts his conclusion on infinite not being a “whole”, in parallel with other type of “imaginary” objects in mathematics – and not only those occurring in the “Arithmetics of the infinites” *à la* Wallis or the “Geometry of the indivisibles” *à la* Cavalieri:

Therefore the infinite is nothing, and cannot be a whole having parts. An infinite is not larger or smaller or equal to another one, because there is no such thing as the magnitude of an infinite. But it does not mean that the Arithmetics of the infinites and the Geometry of the indivisibles fail, any more than surd roots, imaginary dimensions and numbers less than nothing fail. And the ratio of a point to a line, etc. Indeed there is also a certain ratio or truth about impossibilities and false things (A VII, 3, 69; end of 1672).¹⁵

Note in passing that Leibniz is here very explicit about the fact that sentences in which fictions occur are literally false – a basic claim of any “fictionalism”. This is not a problem, however, because it is possible, as he would repeat again and again (in particular against Cartesians), to express some truths about falsities or impossibilities. In later texts, he would often say that expressions involving fictions, such as his Law of continuity, are not “rigorously true” or that they are true “by tolerance” (i.e. provided one paraphrases them correctly).

Many documents from this period confirm that Leibniz called not only infinitely large or small quantities fictions, but negative (“less than nothing”) and irrational (“surd”) numbers, imaginary roots, geometrical dimensions greater than three, logarithmic elements (“ratio of a point to a line” or *rationculae*), etc. More generally, he could call fictions the entities manipulated as indeterminate in algebra or the fact that a number such as “12” could designate not a quantity, but a position (the second term in the first line, for example) – a notation Leibniz

¹¹ See R. Arthur, “Leibniz’s Syncategorematic Infinitesimals, Smooth Infinitesimal Analysis, and Second Order Differentials”, *Archive for History of Exact Sciences* 67(5), 2013, p. 555-557.

¹² See A VI, 2, 482-483, and still GM VII, 20.

¹³ See Eberhard Knobloch, “Galileo and Leibniz: Different approaches to infinity”, *Archive for History of Exact Sciences* 54, 1999, p. 87–99, and Sam Lewey, “Comparability of Infinities and Infinite Multitude in Galileo and Leibniz”, in N. Goethe, P. Beeley and D. Rabouin (eds.), *G.W. Leibniz: Interrelations between Mathematics and Philosophy*, Dordrecht, Springer, 2015, p. 157-187.

¹⁴ For some examples of this practice in the case of numerical series, see G. Ferraro, “True and Fictitious Quantities in Leibniz’s Theory of Series”, *Studia Leibnitiana* (32), no. 1, 2000, p. 43–67.

¹⁵ Except otherwise stated, translations are mine (with precious help of Richard Kennedy).

invented and that he called accordingly “fictitious numbers”. Both imaginary roots and indeterminate quantities of algebra were, in fact, interesting cases since they could be considered as “second order” fictions in their relation to negative numbers – the latter being already themselves considered as fictitious entities¹⁶. We will encounter this situation again later, but it should be noted right away: the hypothetical reasoning, which allows for the introduction of a fictitious entity “as if” it is of the same type than some other entity posited as “real”, often entails that the “reality” of the latter is no less hypothetical than the “fictionality” of the former. Further analyses could show that the “real” entity under consideration was in fact itself fictitious (for example, when a line segment sought in a problem of algebraized geometry happens to be a negative quantity) – or symmetrically, that the “fictitious” entity was “real” (as in the example of $\sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$)¹⁷.

In the vast repertoire of fictitious entities, imaginary roots constituted a particularly intriguing case. By working on Bombelli’s *Algebra* and in particular his discovery of the behaviour of what we now call “conjugate” complex numbers, Leibniz came to realize that some expressions involving imaginary roots are in fact “real”. His favourite example was the fact that $\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}$ is equal to $\sqrt{6}$ – a fact which was considered amazing by Huygens when Leibniz showed him the calculation. This already indicated that imaginary roots should not be considered as contradictory in and of themselves (since they could refer to “real” numbers or possess a *fundamentum in re*)¹⁸. In particular, they were useful in order to express the general rules for the resolution of equations, as had already been claimed by Albert Girard in his *L’invention nouvelle en l’algèbre* (1629). However, as Leibniz explains in a note from October 1675, geometrical constructions would be the best way to provide some “reality” to these objects, if we do not want them to be only “empty” fictions (*ne scilicet pro figmentis inanibus humanae mentis habeantur*)¹⁹.

But when Leibniz tried to determine how one could interpret these fictions geometrically in general (and not only in particular examples), he stumbled upon the fact that they pose specific problems. These are remarkable texts since Leibniz had already seen, in collaboration with Tschirnhaus (who may have inherited this idea from discussions with Wallis), that imaginary numbers could be represented geometrically on an axis orthogonal to that of real quantities²⁰. Yet, contrary to what authors from the 19th century would consider, he thought that this was not an acceptable solution because it amounted to introducing an addition between independent linear quantities, a definition which would contradict the usual (Euclidean and Cartesian) concept of addition as a juxtaposition of linear segments (or things which can be transformed into such a form). This did not mean that imaginary roots were not useful in a geometrical context, for example for indicating the relationship between the quadrature of the hyperbola and the quadrature of the circle (we would say between exponential and

¹⁶ This remark was made explicit by Wallis in his later exchange with Leibniz in 1699 (A III, 8, 42).

¹⁷ This example was already found by Bombelli and is mentioned by Leibniz in his letter to Malebranche, 4 August 1679 (A II, 1, 738-739).

¹⁸ O. Ottaviani, “Leibniz’s Imaginary Bridge: The Analogy between Pure Possibles and Imaginary Numbers in the Paris Writings”, *Oxford Studies in Early Modern Philosophy* 20, 2021, p. 133-168.

¹⁹ A VII, 2, 687.

²⁰ A VII, 2, 745.

trigonometric functions). It is, in fact, in this precise context that Leibniz made an explicit link between the use of fictions in mathematics and his idea of a “blind” knowledge:

This leads me to indicate with an example how we often reason with fictions on the example of true [things].

Whether a quantity like $\sqrt{-1}$ is just nothing at all, or in truth contains I don't know what, should be discussed with much care. Indeed, although it cannot be carried out, it can be understood in a way, not in itself, but by means of characters and analogy, an example of the kind of thought I call 'blind'. (...) It is sufficient, however, that its character be useful, for it expresses real things when joined with others (A VII, 7, 560).

In the part of the text I have skipped, Leibniz explains that this is what we do when introducing irrational “numbers” defined by the fact that, multiplied by themselves, they produce a genuine “number” (the same example that had already been used by Stifel). But note right away that both examples of “fictions” are different precisely because the second is directly interpretable in geometric terms (hence has a kind of “reality”, in Leibniz' own terms, that the other does not possess). This shows, as will be confirmed by other passages, that “fictions” here designates a way to relate to some expressions (not to be taken literally) rather than some specific kind of entity (as if there were “fictitious” entities in addition to “real” entities in mathematics).

In later texts, Leibniz would gather all the examples of that sort under the idea that they constitute various ways of dealing with impossibility in mathematics (by introducing symbols which allow the entities at play to be manipulated “as if” they exist, “by means of character and analogy”). This general strategy had already been put forward by John Wallis in his *Arithmetica infinitorum*:

And therefore what arithmeticians usually do in other work, must also be done here; that is, where some ἀδύνατον is arrived at, which indeed must be assumed to be done, but nevertheless cannot actually be done, they consider some method of representing what is assumed to be done, though it may not be done in reality. And this indeed happens in all operations of arithmetic involving resolution, for example, in subtraction: if it is proposed that a larger number must be taken from a smaller, thus 3 from 2 or 2 from 1, since this can not be shown in reality, there are considered negative numbers, by means of which a supposed subtraction of this kind may be expressed, thus $2 - 3$, or $1 - 2$, or -1 (John Wallis, *Arithmetica infinitorum* 1655, transl. J. Stedall, *The Arithmetic of Infinitesimals: John Wallis 1656*, Springer, 2004, p. 169).

As can be seen in this passage, this approach was not presented as a novelty but as a common practice amongst arithmeticians. For readers of that time, the use of the Greek term ἀδύνατον was a clear reference to a much older tradition. Indeed, when describing the art of “analysis”, Pappus isolated a specific kind of problematic analysis which he described in the following way:

In the case of the problematic kind [scil. of analysis], we assume the proposition as something we know, then proceeding through its consequences, *as if* true, to something established, if the established thing is possible and obtainable, which is what mathematicians call ‘given’, the required thing will also be possible, and again the proof will be the reverse of the analysis; *but should we meet with something established to be impossible* [ἀδύνατον], *then the problem too will be impossible* (Pappus Book 7 of the *Collection*, transl. A. Jones, Berlin Heidelberg, Springer, 1986, p. 82-83; my emphasis).

In fact, according to Proclus' testimony, the latter case was a common strategy in order to devise proofs by *reductio ad absurdum* – in which case, we just have to reverse the analysis by starting with the impossible situation as hypothesis and derive the contradiction²¹. But another strategy, described by Pappus in the next sentence, was to *reconsider* the problem in order to see if some conditions could not be fixed so as to render the problem soluble. This was a typical example of what Greek mathematicians called a “diorism”²².

To place the use of fictions in this broader context is crucial for two reasons. First, it reminds us that although the term “fiction” is quite typical of early modern mathematics, the fact of encountering fictitious situations was already very common in ancient mathematics: parallel lines that do meet, circles that meet in more than two points, tangent that intersects a curve, magnitudes at the same time greater and smaller than a given one, etc. All of this was (and still is, in various forms!) part of the learning of basic mathematics. This was not without consequence on the apprehension of mathematical objects themselves²³. As had already been explained by Pappus, we perform analysis by proceeding “as if” a certain situation holds. In cases where we stumble upon an impossibility, the synthesis given by reversing the analysis into a *reductio* proof amounts likewise to feigning that an impossible situation holds. Note that there is no way to rule out the “impossible” situation from the start on the grounds that it is a non-denoting expression since it would simply prevent the proof from beginning. We have to proceed “as if” our strange diagram stands for genuine “lines” or “circles” or whatever objects are under consideration. More interestingly, we really acquire knowledge about the objects under consideration through the proof. What I realise at the end of Euclid III, 10, for example, is not simply that the proof was about a chimera: I really learned something *about circles*²⁴.

That this case was included in the general category of mathematical “fiction” is made explicit by Leibniz on several occasions, for example in his correspondence with Clarke (IV, §16-17). But it is worth recalling that he had already mentioned it in one of the first occurrences of the word *caecus* (“blind”). Interestingly enough, the context was his study of *casus perplexi* in Law, a typical case in which “fictions” had to be introduced (in the widespread practice of *fictio juris*). According to Leibniz, this kind of reasoning should be put on a par with what happens in mathematics, particularly in symbolic algebra, when demonstrating an impossibility²⁵.

The second reason why it is important to have this general context in mind is to confirm that fiction was not intended as designating a specific kind of entity. It was a way of dealing

²¹ *In Eucl.* 255–256.

²² Pappus describes it as “the preliminary distinction of when, how, and in how many ways the problem will be possible” (ibid.).

²³ After having argued – for reasons I cannot detail here – that diagrams should be considered as the proper object of mathematical discourse in Greek geometry, Reviel Netz stumbles upon the problem of *reductio proof* and their “impossible diagrams” (hence impossible “objects” in his reading). The only way out, according to him, is precisely to interpret them as *fictions* in a game of make-believe – he compares them to Cinderella’s carriage/pumpkin (R. Netz, *The Shaping of Deduction in Greek Mathematics. A Study in Cognitive History*, Cambridge University Press, Cambridge and New York, 1999, p. 54-55).

²⁴ In fact, *reductio* proofs are still an issue for a modern account of the traditional semantic of proofs, see K. Manders, “The Euclidean diagram”, in P. Mancosu (Ed.), *The philosophy of mathematical practice*, Oxford: Oxford University Press, 2008, p. 84. As noted by Manders, this has nothing specific to do with diagrams: a *reductio* proof about groups or modules would function in the same way by recourse to symbolic writings.

²⁵ A VI, 1, 235-236. *Disputatio de casibus perplexis in jure*, November 1666. See also, then years later, the remarks on Spinoza: A VI, 3, 277.

with expressions “as if” they refer to “real” objects. Accordingly, one can find under Leibniz’ pen all the possible outcomes: it could be that the fiction turned out to be an irreducible impossibility (but still remain a useful tool for calculation as is the case with infinite numbers), that the impossibility could be rephrased/reinterpreted so as to disappear in favour of a genuine/”real” entity (as in the case of negative numbers interpreted as debts or infinitesimals paraphrased in an Archimedean way in terms of finite quantities), that the impossibility was related to some conditions in the problem and that it was possible to modify them, or making them explicit, in order to make the impossibility vanish (as in the case of imaginary roots seen as solution to certain intersection problems in algebraic geometry) or, finally, that the fiction was, from the start, referring to a genuine/”real” entity (as in the case of expressions involving imaginary roots but designating in fact real numbers). It is crucial to keep in mind that the same type of expression (typically imaginary roots or infinitely small quantities) can enter into various categories and, thus, sometimes refer to a “real” object and sometimes not²⁶.

Note that the broad context of analysis in which one introduces an entity “as if” it exists concerns virtually *all* mathematical objects now. This is the first step on our continuous path. It amounts to what is sometimes called a *linguistic* version of fictionalism (in which one relates to sentences by *suspending* their referential import)²⁷. In this fictional mode, one can talk about “objects” although what one says has not to be taken as literally true. This had to be done when encountering an apparent impossibility, the initial situation occurring with negative numbers, infinitely small quantities or hypotheses in a *reductio*; but this was also at play *in any kind of analysis* in which it was not known in advance whether or not the result was even possible. As I indicated, this larger framework of the ancient art of “analysis” was the one *explicitly* emphasized by Wallis and Leibniz for the use of “fictions” in mathematics. Reflecting on the Cartesian criterion according to which one should reason only with “clear and distinct ideas”, the latter replied that it was too strong a demand. We often say true things about impossibilities (be they genuine or apparent impossibilities)²⁸. Moreover, he came to realize that this “suppositive” mode of knowing is, in fact, the one we most commonly use not only in mathematics, but in any complex knowledge in which we proceed “as if” the concepts at play were already fully understood. This is the very description he gave of “blind” or “symbolic” knowledge in his published text from 1684 *Meditationes de cognitione, veritate et ideis*, arriving at the conclusion that we use it not only “in algebra and arithmetic”, but “indeed virtually everywhere”²⁹.

It is important to keep in mind that this first kind of fictionalism did not reduce to a mere philosophical speculation about mathematics. Not only was it widespread in the mathematical practice of the time, but, as recalled at the beginning of this section, it was instrumental in Leibniz’ practice when dealing with infinitesimals. This example also confirms the linguistic nature of the kinds of fictions at play (we will encounter other uses of the term “fiction” in section 2 and 3, with more ontological import to them). Indeed, although Leibniz himself had

²⁶ For an example with infinitely small and infinitely large quantities, see A VI, 4, 521. The argument developed here is crucial in order to counter a view according to which fictions such as infinitesimals denote a fixed kind of object – as is assumed in interpretations inspired by non-standard analysis.

²⁷ On the distinction between linguistic and ontological fictionalism, see M. Eklund, “Fictionalism”, *The Stanford Encyclopedia of Philosophy* (Winter 2019 Edition), Edward N. Zalta (ed.), URL = <<https://plato.stanford.edu/archives/win2019/entries/fictionalism/>>.

²⁸ See Letter to Elisabeth, 1678 (A II, 1, 662).

²⁹ A VI, 4, 588.

reasons to doubt that infinitesimals could denote genuine mathematical entities and although he repeated, again and again, that one can always substitute them with expressions in which only finite quantities occur, he did not see this strategy as a way to convince his interlocutors that infinitesimals were dispensable. His claim was more likely that the use of fictions allowed mathematicians *not to enter* into this question (by contrast with any form of ontological fictionalism, in which an argument of dispensability is mandatory). The fact that some actors, like himself, did not believe in their existence, whereas others (including some of his strongest supporters, such as Johann Bernoulli or Fontenelle) thought of them as existing entities, was of no importance in practice, precisely because infinitesimals could be handled *in a fictional mode* without doing harm to the computations. In this setting, “fictionalism” is an *ontologically neutral* position. As Leibniz put in his response to Bayle: “mathematicians do not need all these metaphysical discussions, nor need they embarrass themselves over the real existence of points, indivisibles, infinitesimals, and infinites in any rigorous sense” – the continuation of the passage making it clear that what Leibniz had in mind is the fictional treatment of infinitesimals³⁰.

This last point makes this first kind of Leibnizian fictionalism very close to what Yablo has coined “figuralism” (based on the fact that one can take parts of sentences, sometimes in one and the same sentence, in a “figurative” way)³¹. Indeed, according to Yablo an “engaged” nominalist, who talks about mathematical objects “as they are postulated to be in the game”³², and a Platonist, who believes these objects to be “really there”, do not disagree in their ordinary talk about mathematical objects – precisely because the figurative way of speaking allows discussion without commitment to the reality of the objects under consideration. Moreover, Yablo claims that this kind of fictionalism makes it possible to tackle “an under-discussed problem in the philosophy of mathematics. How is it that mathematicians can happily communicate despite having different views of the nature, and even the existence, of mathematical objects?”. This is exactly the context we just mention for the debates on infinitesimals. But Yablo’s remark extends to less controversial objects on which mathematicians can have very different views (just think of the many various ways of interpreting what a “set” is)³³.

Moreover, Leibniz shared the view that a sentence involving fictions should be read in a *figurative* way. He stated this explicitly, for example, to Bodenhause in order to explain that he sometimes considered some letters occurring in a formula as designating fictitious entities (infinitesimals) and sometimes not. According to him, this could be compared with the reading of the Holy scriptures in which one has to practice a “figurative” reading in order to make sense of some passages³⁴. In the same manner, when talking about his famous “law of continuity”, which allowed amongst other things proceeding “as if” rest was a limit case of motion and points were infinitely small lines, he made it clear that it was only by what he called “a rhetorico-philosophical figure”³⁵.

³⁰ GP IV, 569; L 583-584.

³¹ S. Yablo, “Go Figure: A Path through Fictionalism”, *Midwest Studies In Philosophy*, 25, 2001, p. 72-102.

³² *Ibid.*, p. 84.

³³ *Ibid.*, p. 87.

³⁴ A III, 4, 524.

³⁵ GM V, 385.

II. From mathematical fictions to mathematical objects as fictions

In this section, I intend to show that Leibniz was also engaged in a stronger version of fictionalism than the one I described in the first section (“linguistic” fictionalism), namely the “ontological” one. A complete account of this position will have to wait for a clarification of Leibniz’ metaphysical views. But we can first take this “ontological” orientation as meaning that one could establish a link between the uses of fictions attached to mathematical practice and philosophical claims about what mathematical objects have to be in their relation to the natural world.

This continuity is put forward very clearly in a text from his sojourn in Paris, which will play a central role in my reconstruction and which I will henceforth quote at length:

The circle – as a polygon greater than any assignable, as if (*quasi*) that were possible – is a fictive entity, and so are other things of that kind. So when something is said about the circle we understand it to be true of any polygon such that there is some polygon in which the error is less than any assigned amount \underline{a} , and another polygon in which the error is less than any other definite assigned amount \underline{b} . However, there will not be a polygon in which this error is less than all assignable amounts \underline{a} and \underline{b} at once, even if it can be said that polygons somehow approach such an entity in order. And so if certain polygons are able to increase according to some law, and something is true of them the more they increase, our mind feign some ultimate polygon; and whatever it sees becoming more and more so in the individual polygons, it declares to be perfectly so in this ultimate one. And even though this ultimate polygon does not exist in the nature of things, one can still give an expression for it, for the sake of abbreviation of enunciations (*Numeri infiniti*. A VI, 3, 498/LLC 89 modified).

The reference here is clearly to Archimedean techniques for the measurement of the area or the perimeter of the circle, in which the only objects occurring in the proofs are finite polygons. In this sense, the circle is fictitious because it is meant as a limit case, i.e. a polygon with an infinite number of sides, although this object cannot exist (as we saw in the previous section, an infinite number is, according to Leibniz, a genuine impossibility). Moreover, the introduction of this fictitious entity (the polygon with an infinite number of sides) allows a more direct proof to be given (by “passing to the limit” in the series of inscribed polygons \underline{a} and that of circumscribed polygons \underline{b}) than the tedious “exhaustion proof” devised by Archimedes (who proved, by a double *reductio*, that the intended result holds because the supposition that it is either greater or lesser leads to contradiction when increasing the sides of the finite polygons). The introduction of the fiction as an expression (*expressio*) is innocuous since it acts just as a compendious way (*compendiosarum enuntiationum causa*) to abbreviate a complex proof in which the dubious entity would not occur – an “eliminativist” description of mathematical “fiction” Leibniz would repeat over and over in his texts on infinitesimals.

Such a description already extends the class of fictional entities considerably by encompassing more basic objects such as circles and “other things of that kind” (which were certainly not considered “impossible” in and of themselves!). As is well known, Leibniz did in fact consider *any* continuous curve³⁶ as a kind of infinitangular polygon and the very core of his differential approach was grounded on this identification. In this sense, many more objects than the one we came across in the first section could be treated as fictions. But there is more: if we had a way to interpret the other more basic objects, such as line segments constituting our

³⁶ We would specify that they need to be differentiable.

finite polygons, as referring directly to some physical entities, what the proof would “talk about” could be, at the end of the day, not only finite but concrete entities. This is a crucial step because it shows that the strategy which allows dealing with fictions *inside* mathematics (the circle seen as a limit case of finite polygons or an “infinitangular polygon”) immediately transfers *outside* mathematics, provided that the “real” objects are said to be so in mathematics because they serve to describe entities “existing in the real world”. When I paraphrase a proof on the circle seen as an “infinite polygon” in terms of finite ones (*à la* Archimedes), what I do is also to show that the circle as a limit case is a dispensable object when it comes to the description of the natural world. This was an important issue for Leibniz’ physical reflections of the time, since he disposed of rectilinear tendencies to motion (or *conatus*) considered as genuine physical entities and had to provide an account of other motions, such as circular motion, in terms of these basic entities³⁷.

This general strategy, in which the mathematical fiction is eliminated by a paraphrase mentioning only “real” entities, referring to things existing “in nature”, appears very clearly in later texts in which Leibniz deals with the kind of fictions we mentioned in section 1, such as negative numbers, and do not hesitate to say that they can be “exhibited in nature”³⁸. The line of argument is similar. As was already well known at the time, any algebraic problem in x in which one obtains a negative solution can be paraphrased into a problem formulated in $-x$ giving a positive result³⁹. This was the formalization of an ancient strategy (already existing in Indian mathematics) in which a problem concerning a certain amount of money was reinterpreted, when stumbling upon a negative solution, as asking for the positive amount of a debt (also expounded by Cardano in his *Ars Magna* from 1545). Another strategy, more recent at the time (although already present in astronomical computations) and devised by geometers such as Girard or Descartes, was to reinterpret a question concerning “negative” entities, typically line segments or angles, as dealing with the same object going “in the other direction”. Algebraic formalism allowed both cases to be considered under a uniform procedure to eliminate the fictitious entity and replace it by a “real” one. The possibility of this paraphrase being granted, one could then proceed with the fiction of negative numbers without harm. Accordingly, the strategy deployed inside mathematics to transform the fiction (in our case a “negative number”) into an object considered as “real” (a positive quantity) could be transferred outside (to something existing “in nature”), because the “real” object was said to be so in mathematics precisely because it was supposed to refer to some physical entities (an amount of money, a distance between concrete objects).

Yet, the extension of the realm of fictitious objects to ordinary entities such as the circle is not straightforward. How can we contrast, for example, the circle (as fiction) and the line segment (as real) if both are needed from the start in Euclid’s geometry (just think of the construction of the equilateral triangle in the very first proposition of the *Elements*)? But this is precisely a beautiful example of the “suppositive” nature of mathematical knowledge. As Leibniz explains on several occasions, any axiom which is not “identical” is, to his eyes,

³⁷ See A VI, 3, 492, in which Leibniz talks again of the circle, but also of the parabola “and other things of that kind” as *Entia fictitia*.

³⁸ A VI, 4, 520; GM VII, 69-70.

³⁹ Stevin, *Arithmétique*, Leiden, Plantin, 1585, p. 332. This strategy worked because mathematicians were usually interested in finding at least *one* solution to their algebraic problems. But, as Descartes explained in his *Géométrie*, one can also transform *all* the negative solutions to a problem so that they become positive ones (Descartes, *La Géométrie*, Engl. Transl. D. Smith and M. L. Lathan, Dover, 1954, p. 168).

demonstrable. But this does not prevent mathematicians from using non-identical axioms and definitions in order to structure their knowledge, typically by allowing the construction of line segments and circles as in Euclid's postulates⁴⁰. At the end of the day, however, one should always make sure that the objects introduced in this suppositive way (i.e. by stipulations) express possible things and one way to do so is to bring them back to objects already recognized as "real". The fact that this process could go to infinitum was no objection, according to Leibniz, since any reduction of this type, however provisory it may reveal itself to be, was progress toward more distinction in our knowledge⁴¹. If, for example, the admissibility of the more basic objects revealed itself to be dependent on conditions which were omitted in the first formulation, as appeared historically with the discovery of non-Euclidean geometry, this negative result (the objects assumed to be possible were not so *unconditionally*) would be at the same time a positive knowledge of the conditions under which these objects could be given⁴².

This appears remarkably in the fact that the very same line segments that we take to be "real" in the measure of the circle can reveal themselves as fictitious in other texts in which Leibniz tackles the issue of their measurement (as is precisely the case when they are used in physics). Indeed, in order to define a measure of line segments, we need to fix a scale, which amounts to fixing a given line segment as the unit. When the measure of a given line segment cannot be expressed with the given unit, two situations may occur: either, one can choose a new unit which measures both and re-express them in terms of this new unit; or one cannot. This latter case is that of "incommensurability" between magnitudes. This has been characterized since Euclid by the fact that the algorithm to determine their greatest common measure goes to infinity⁴³. The ancient strategy devised to circumvent this problem, and still dominant in Leibniz' time in physics, was to deal not with the quantities themselves, but with ratios. In early modern times, it appeared more and more clearly that it was also possible to work directly with fictitious quantities (the ancestor of our real numbers). We could, for example, introduce an "infinitely small" fictitious common measure: we just have to posit this ideal object by showing that the error obtained by truncating the result of the algorithm can be rendered as small as one wishes (this is similar to what we do in the practice of measurement in natural sciences when we fix a certain level of accuracy, provided one could fix a more precise one). This practice was very common at the time in the elaboration of logarithm tables (for example by fixing a certain measure for the "whole sine" and its "minimal" subdivisions). Another fiction would be to identify the whole infinite series of partial quotients given by the algorithm as a new type of "number". More generally, one could identify an incommensurable magnitude with a series of terms converging to it (as is the case in the famous series Leibniz obtained for expressing $\pi/4$). These various strategies are explained by Leibniz in many texts such as his *De Magnitudine et Mensura (On Magnitude and Measure)*, in which he explicitly mentions the role of these

⁴⁰ See A VI, 4, 704.

⁴¹ See A VI, 3, 436. One may compare this to the modern strategy in which one proves the "relative consistency" of a theory T by showing that it can be reduced to a theory S, of which the consistency has been assumed.

⁴² More importantly, the whole process would be neutral with regard to the relation to "real" objects. An example is provided in contemporary physics by the fact that the formulation of General Relativity in a (pseudo) Riemannian framework leads to the specification of *physical* conditions allowing physical phenomena to be treated in a Newtonian/Euclidean way under some circumstances. Hence, we do not completely eliminate the Euclidean objects: we reinterpret them as "real" under some physical conditions (typically when objects are considered at a certain scale).

⁴³ See *Elements* X, 2.

“fictions”⁴⁴. This remark is of tremendous importance since it shows that the “direct” reference to “real” entities can be mediated by various mechanisms in which they might appear, after analysis, as being themselves fictitious – this is the very meaning of what a “blind” knowledge consist of. The same kind of reasoning would apply when trying to show what it means for a perfectly straight line segment to be instantiated “in nature” (more about this below).

For now, note that Leibniz’ views are certainly close to a form of ontological fictionalism *à la Field* which intends to show that an important part of mathematics (the one which can be “applied”) is just a convenient way of talking of physical entities which could be described otherwise without using them. In the recent literature, such a strategy has been coined, that of “nominalization” and the theory obtained without recourse to the mathematical apparatus is therefore called “nominalistic”⁴⁵. However, Leibniz would have disagreed with this terminology for a reason which is central to our reflection: what these strategies show is that one could dispense with some portion of mathematical discourse from an ontological point of view, but not that what is obtained is “nominalistic” in the usual philosophical sense of not referring to abstract objects⁴⁶. Indeed, physical entities such as magnitudes, forces, masses or motions seem to be, on the face of it, abstract objects. The reason why modern day fictionalists conceive of them as “nominalistic” entities is related to the fact that they often belong to a tradition in which it is simply *assumed* that physical notions are themselves ways of talking of *concreta* (typically regions of space time and relations between them). But this last assumption is far from being obvious, as testified, if needed, by the numerous discussions surrounding the various trends of contemporary “nominalism” in philosophy. At any rate, the “eliminativist” strategy making it possible to dispense with mathematical objects is neutral with regard to the question of the ontological status of physical entities⁴⁷.

Be it as it may, this is the point where Leibniz would have pushed his nominalist interlocutor a step further and ask how we are supposed to access “concrete” entities. Interestingly enough, this is precisely one of the issues he tackles in the continuation of the passage from *Numeri infiniti*:

Even though these entities are fictitious, geometry nevertheless exhibits real truths which can also be expressed in other ways without them. But these fictitious entities are excellent abbreviations for expressions, and for this reason extremely useful. For entities of this kind, i.e. polygons whose sides do not appear distinctly, are made apparent to us by the imagination, whence there arises in us afterwards the suspicion of an entity having no sides. But what if that image does not represent any polygons at all? Then the image presented to the mind is a perfect circle. Here there is a surprising and subtle difficulty. For even if the image is false, the entity in it is nevertheless true; and so it follows that in the mind there is a perfect circle, or rather, there is a real image⁴⁸.

⁴⁴ GM VII, 39.

⁴⁵ Hartry Field, *Science Without Numbers: A Defense of Nominalism*, 2nd ed., Oxford University Press, 2016.

⁴⁶ Field has a loose understanding of “nominalism” as simply meaning the denial of (mathematical) “Platonism” (itself understood as a belief in the existence of mathematical entities), see Field (2016), Preface to the Second edition”, p. 3.

⁴⁷ For example, Field believes that space is constituted “substantially” by points (see note 67 below). But his “eliminativist strategy” is, in fact, completely *independent* of this belief – although this is a crucial ground for calling it “nominalistic”.

⁴⁸ A VI, 3, 499/LLC 89.

This is a very rich and difficult text and I have no space to enter into all of its subtleties. What I would like to emphasize is the following: the very fact that my perception does not allow me to make certain distinctions after a certain level of resolution is what provides me access, *from my perception*, to a polygon for which I cannot distinguish the sides. One just has to look at a chiliogon to see that it appears as a circular form in the sense that it is not possible for a human eye to distinguish its sides. This image is “false” in the sense that the material object has sides (this may be shown by changing the scale of resolution of the image). But still, what we perceive at the initial scale in the sense of “the image presented to the mind” *is* a circular form. We literally cannot “see” the polygon as such since we cannot distinguish its sides. Moreover, the “abstractness” of the perceived form is attested by the fact that one can increase the number of edges without making a perceptive difference (this had already been Descartes’ argument in order to support the claim that this image was not a “clear and distinct” idea). One could also replace the polygon under consideration by another one in which the apices have been rotated. In all of these cases, it will be perceived as *the same object*, while being materially different.

This is something Leibniz would repeat in many texts and which amounts, I would claim, to an important shift in the meaning of what an “abstract object” consists of (compared to previous traditions). According to the philosopher, abstraction is not something we get only by detecting similarities between objects or by neglecting differences in an intentional way – as was (and is still) often assumed in the empiricist tradition. Abstraction is something we also get by not detecting differences *in a non-intentional way*. More importantly, this structure is inscribed at the very core of our perceptive access to the world. It amounts to a form of “abstraction” which is not of an empiricist kind (as Leibniz argues in detail in his exchange with Locke in the *Nouveaux Essais sur l’entendement humain*). For now, let us simply note that it indicates that modern day “fictionalisms”, even taken in an “ontological” spirit, will not be sufficient for providing an anti-Platonist argument if they limit themselves to showing that one can interpret mathematical discourse in terms of “physical” objects (whatever strategy they follow to insure this reinterpretation). Indeed, it then has to be shown that the objects to which we relate mathematical objects are not themselves abstract⁴⁹.

But Leibniz’ claim is even stronger. What *grounds* the “direct” reference of some mathematical objects, according to him, is *precisely* the fact that in physics and in perception we access abstract objects. What we refer to, for example, when we use line segments to denote tendencies to rectilinear motion is a collection of indiscernible phenomena (more about this question of indiscernibility in the next section), because we cannot make a sensible difference (at our scale) between perfectly rectilinear motion and motions fluctuating in an insensible manner⁵⁰ – what he would later call, in the preface to the *Nouveaux essais sur l’entendement humain*, “les petites perceptions”. According to contemporary physics, this situation is ubiquitous (all physical objects perceived at our scale as stable are fluctuating at a microscopic level). This has radical consequences in the debate about fictionalism, which are too often left unnoticed.

⁴⁹ One modern “solution” is to characterize abstract objects as non-spatiotemporal and causally inert entities, but this is clearly *ad hoc* since physical entities are then “concrete” only by definition.

⁵⁰ Field claims that some geometrical theories such as Euclid’s, or Hilbert’s reformulation thereof, are “physical” (Field 2016, p. 27) and this claim plays a crucial role in his argument. But he does not explain how the objects of such theories can be concrete (except by *assuming* that points of space time are “concrete” entities). In Leibniz’ conception, what grounds the direct reference is precisely the fact that their physical counterpart is itself abstract – as would be, more generally, any perceptual form.

Indeed, for many contemporary “fictionalists”, one should contrast sentences like “the marble tile is on the table” and “3 is a prime number” because the second is supposed to deal with an abstract entity contrary to the first, which is supposed to be “concrete”. But for Leibniz, there is no such thing as a “marble tile” in the real world. The recourse to the word “tile” is already a convenient fiction and refers to an abstract entity no less than the number three would (if they are considered as denoting terms)⁵¹. Even if we mean the sentence “the marble tile is on the table” to refer to *that* particular tile, which we can point to with our finger and which occupies a specific region of space-time, what we point to is still a collection of perceptive facts and, more importantly, of imperceptible variations which we bring together under the expression “that tile”. There are certainly differences between these various statements, in particular when it comes to what makes them true or false – or to what contemporary philosophers would call their “modal profile” – but not on the issue of abstraction. Or, better, if there is one, it should be argued for⁵².

For sure, the fact that one can dispense with mathematical objects ruins one form of “mathematical Platonism”. If mathematical objects are just “ways of speaking” of physical and perceptual facts, the idea that they constitute independent objects of their own seems to be endangered, to say the least. I will come back to Leibniz’ reply to this objection in the final section. But, at any rate, if the deep motivation for rejecting Platonism is the denial of the existence of abstract objects *in general*, the so-called “nominalization” is simply of no help. It just amounts to replacing abstract objects by others and we can wonder what we gain by denying to mathematical entities what we allow secretly for other entities (typically, as Field does, the objects of Hilbert’s axiomatization of geometry, which he considers as “physical”). More than that, Leibniz has very strong arguments to claim that there is *no way* to dispense with abstract objects at the level of our perceptive access to the world, so that the empiricist strategy seems doomed to fail here. I cannot enter into all of his arguments, but let us note that the example of the circle transfers immediately to the perception of any continuous phenomenon. Indeed, according to Leibniz, continuity is based on the fact that one can divide an object into parts in many possible ways *without making a sensible difference*. This image is false if we do not want to admit that reality is itself completely indeterminate, but it is a “real” image in the sense that, like in the case of the chiliogon, we *really* cannot distinguish all the parts into which the phenomena are actually divided⁵³.

This is where Leibniz developed another interesting argument going, this time, the other way round. Let say we grant the nominalist that there are no such things as abstract entities “in the real world”. If we are doomed to describe the world with abstract concepts, as I just argued, we certainly need a way to guarantee that this use of abstract concepts is grounded in other “real” entities and is of no harm to our knowledge. I will come back to the first part of the justification in the next section. As regard the second part, Leibniz saw that the path he drew between fictions in mathematics to mathematics as fictions (entering into our description of the natural the world) could be reversed so as to secure the use of abstract concepts in general.

⁵¹ On this example of the “marble tile” (“quarreau de marbre”) as a “fiction of the mind”, see A II, 2, 121.

⁵² True enough, some versions of “nominalism”, such as that of Goodman, allow for a reference to abstract entities (such as abstract particulars, the mereological sum of which are then supposed to constitute *concreta*). But this meaning of “nominalism” is known to be heterodox and testifies to the difficulties in which one enters as soon as one questions, as did Goodman, the status of the perceptual *concreta*.

⁵³ See the letter to Sophie quote below p. +++.

He tackles this issue explicitly in a fascinating text devoted to “the abstract and the concrete” written in the 1680s. The question was that of the “reality” one can confer to abstract properties. If we characterize an individual, say Seneca, by some properties such as the fact that he is rich, wise, etc., a question naturally arises concerning the status of these properties, in particular when predicated of other individuals, say Socrates. If we claim that each of these properties is concrete in as much as it enters into relation with a specific set of other properties in concrete individuals, then the two properties will simply *not* be identical. Because Seneca was rich, for example, the kind of wisdom he must have developed was intrinsically different from the kind of wisdom to be found in Socrates. If we maintain that there is nonetheless a “level” (in their “formal aspect”, as Leibniz puts it) at which both properties can still be said to be “identical”, then we just make them behave *like mathematical objects* of which there are several identical exemplars, which can be distinguished when they enter into distinct relations (for example when we write “ $2 + 2$ ” or when we make two identical circles intersect). But this analogy is misguided for a very deep reason: since our exemplars are supposed to be concrete instances, they would be, contrary to mathematical objects, inseparable from the circumstances in which they occur. For example, we would have to say at the same time that the wisdom in Seneca is identical to the wisdom in Socrates *and* that the latter is destroyed when Socrates dies – which makes it *ipso facto* non-identical to the former. Note that there would be no way to reply that the instances of abstract properties, although identical, were distinguished merely *externally*, by their spatio-temporal location (as in the definition of indiscernibles differing only *solo numero*). Indeed, time and space are prototypical of *abstract* structures – as Leibniz argued at length in his correspondence with Clarke – and, in any case, there is precisely no “real” entity differing only *solo numero* from another one (on the pain of violating the principle of sufficient reason). At the end of the day, the question at stake is that of the principle of indiscernibles, as we will see more clearly in the next section.

For now, what interests us is Leibniz’ answer to the preceding dilemma. It amounts to saying that although abstract properties are fictions (because there are only non-identical individual accidents in the real world), they can be used without harm *on the model of what we do in mathematics*:

However, one can use such notions to facilitate reasoning, as we use imaginary roots in algebra, and perhaps infinite and infinitely small lines in geometry. And if someone were to say that a numerically identical wisdom, or a numerically identical heat, is in one subject and in another, he would be refuted by the fact that it is equally well asserted that the wisdom of the one dies when the wisdom of the other still remains⁵⁴.

In other words, we postulate identical exemplars of abstract properties, in the same way that we introduce fictions in mathematics, although such entities are strictly speaking impossible (if somebody objects that there “really” are identical exemplars, he would be refuted by the argument based on the destruction of one of the exemplars)⁵⁵. Leibniz proposes here a remarkable twist: the very same reasoning which can be employed to show that mathematical objects could be “dispensable” without loss can hence be reversed to show that one can talk

⁵⁴ A VI, 4, 991.

⁵⁵ The general strategy described by Leibniz could be related to what we do, for example, when we introduce abstract predicates in logic (i.e. when we postulate that one and the same predicate *F* can “apply” to several individuals like *a* and *b*) – a strategy copied from mathematics which revealed itself so fruitful in the development of modern logic, but is too often wrongly considered as being ontologically neutral.

about abstract objects *in physics and in perception* without harm⁵⁶. In this sense, if mathematical objects are “dispensable”, mathematics as “objective” knowledge might not be so. Indeed, the soundness of the use of fictions (such as “abstract properties”) is, at the end of the day, grounded inside mathematical knowledge itself⁵⁷. This, as we will see in the next section, was at the core of Leibniz’ defence of a strong form of Platonism.

III. *From a metaphysical point of view*

In the preceding sections, we have seen that there is a smooth path running from the use of fictions in mathematics to the use of mathematics as a stock of fictitious expressions useful for describing the world. Yet, to reach full-fledged fictionalism, it would remain to show that *any* mathematical object is a fiction in the sense of not being a “real” entity. The linguistic version of fictionalism already allows us to say that any mathematical object can be *taken* as a fiction in the “suppositive” regime of knowledge, which we use “almost everywhere”. But in that case, the fictitious mode of speaking is always contrasted with some entities which are considered as “real” inside mathematics (although their domain might vary and/or be very narrow). What about the ontological version? In particular, can we claim, and in what sense, that any mathematical entity *is* a fiction? In the first argument we came across on our path (the one taken from *Numeri infiniti*), this was not the case: the circle was still said to be fictitious *by difference* with “real” mathematical entities such as polygons made of line segments. But we also saw that this strategy was provisory. In the same way that the circle can be treated as a fiction, when it comes to measuring its perimeter, line segments could also be treated as fictions when considered as perfectly straight lines or as measured by a general notion of “number”.

To push things further, we cannot escape envisaging the “big” metaphysical question here: which entities should we consider as “real” from an ontological point of view? Due to the limited space, I will not be able to tackle this issue in a fully argumentative way and will limit myself to recall some elements of Leibniz’ metaphysics in a rather dogmatic fashion. My only excuse will be that all of these theses are well known and easily accessible. What I intend to show is that Leibniz defended the strongest (“metaphysical”) version of fictionalism in the sense that he claimed that *no* mathematical object (and more generally no abstract object) whatsoever can be “real” *stricto sensu*. However, far from being an objection, this was considered by him as an argument *in favour* of Platonism. Let us try to understand why.

Before entering into this development, let me start with a quick terminological remark. We saw that Leibniz had no problem calling genuine mathematical objects, such as circles or parabolas, “fictitious”. Yet, this is not a term he favoured much, presumably because the term “fiction” was associated not only with non-existence, but also with a kind of arbitrariness. By contrast, Leibniz was convinced very early on that mathematical essences exist in a specific

⁵⁶ This has strong echoes in Field’s strategy where he uses a mathematical theory (Hilbert’s axiomatization of geometry) as if it referred to physical entities, allowing *ipso facto* some predicates such as “congruent” to apply identically in all their physical instances.

⁵⁷ I cannot enter into the detail of all of Leibniz’ arguments here, but let me just mention that one of them is related to the possibility of using equivalence classes in our description of the world. That this discourse is modelled on mathematics is made explicit by Leibniz when explaining that we access the idea of space by the notion of equivalence of places, which is modelled on the Euclidean way of dealing with ratios (*Correspondence with Clarke* V, § 47, GP VII, 400). This kind of back and forth movement is erased in the way a fictionalist like Field talks of Euclidean Geometry as being “originally” conceived as a theory of *physical* space (p. 27) – a puzzling claim I must say.

realm of their own and obey some rules which escape any human decision or creation. In the 1680s, his favourite term was “imaginary” (both for mathematical objects and natural objects which have no substantial unity like the marble tile), but still he had no difficulty in associating this imaginary character to a “fiction de l’esprit”⁵⁸. In later texts, he would instead call mathematical objects “ideal” – precisely to emphasize the fact that the fictitious nature was compatible with Platonism. Although it sounds to our ears as a different orientation, it is important to keep in mind that this was not the case. This should be clear from the fact that Leibniz had no problem in calling his infinitesimals, the prototypical case of “useful fictions” at the time, “ideal”⁵⁹.

Be it as it may, this terminological evolution had no impact on Leibniz’ fictionalism taken in the following sense: mathematical objects are not real *stricto sensu*. The only things that really exist are simple substances (later to be called “monads”). In this sense, physical or sensible phenomena are not real either and the problem is not related to the fact that mathematical entities “do not exist in the natural world” (i.e. have no space-time location and are not taken into causal interactions). However, “real” can still be used *lato sensu* in order to designate anything that can be grounded on “real” things (or have a *fundamentum in re*). This is why Leibniz had no problem talking about “real” phenomena (as opposed to “imaginary” ones) or, as we saw, to deal with some mathematical objects as “real”. This broad sense of “real” is innocuous as long as one keeps in mind that the only genuine *res* are simple substances. But it remains to explain how the “real” entities taken *lato sensu* are grounded on “real” entities taken *stricto sensu* (simple substances). The answer to this question relates to the fact that simple substances are not inert entities but come equipped with an intrinsic activity or “perception” (taken in a very broad sense). Accordingly, the world is not something “outside” these substances (there is nothing other than simple substances), nor a simple substance of its own. What God did when creating the world was precisely to choose the system of monadic activities which was the most harmonious. Consequently, any existing substance is a “point of view” on this “world” (i.e. system of monadic activities) and comes with a “region” it perceives more distinctly. The crucial point is that although this knowledge is related to the way in which human beings “perceive the world”, it is not a creation of their minds. It is based on a *structure* which grounds the convergence of all monadic activities, i.e. (clear and distinct) perceptions. Beyond the vividness and variety which characterize our perceptions of the “real world”, as opposed to dreams, a very important criterion is therefore their regularity and their convergence, that Leibniz sometimes calls their “congruence”⁶⁰.

The “congruence” of phenomena is what mathematics expresses when describing the world. Yet laws of nature, even framed in mathematical language, do not reduce to their mathematical expression because some choices have often to be made amongst various models. Leibniz’ favourite example is that of the conservation of force, which he opposes to the Cartesian model in which only the quantity of motion is preserved. Both models are sound and plausible from a mathematical point of view. Yet only one of them describes accurately the entire range of physical phenomena related to motion. For this reason, mathematics, although it can express real phenomena and can be “founded in reality” in this sense, also expresses

⁵⁸ A II, 2, 121.

⁵⁹ To Varignon, 2 feb. 1702 (GM IV 92).

⁶⁰ See “On the method of distinguishing real from imaginary phenomena” (L 363).

more. It deals with *possible* structures and a complete account would have to explain the ontological nature of these *possibilia*.

Before entering into this question, let me quickly complete the preceding argument and bring to mind why mathematical objects *cannot* be genuine “substances”. As early as his sojourn in Paris (1672-1676), Leibniz had come to realize that one characteristic property of substances, that of being “complete” (one could say “completely determined”), could not apply to mathematical objects. This went with the fact that simple substances are individuated – the grounds of what Leibniz would later call the “principle of the indiscernibles”. Leibniz’ arguments are mainly theological: it would be incompatible with God’s wisdom to choose a world containing non-individuated entities because it would amount to arbitrarily choosing a world indiscernible from another one (i.e. in which at least one entity could be replaced by another which is indiscernible from it). But it may be worth recalling that the problem occurred initially also from epistemological considerations. One of Leibniz’ starting points was that a mathematical object being given, say a square, it is not possible to determine how it was produced (by joining triangles, by joining rectangles or by transporting an already existing square, etc.)⁶¹. The main difficulty arose from the fact that physical processes, expressed with mathematical tools, would have to be compatible with this regime of indiscernibility. From the point of view of the result or “effect” (here a square seen as possibly referring to a physical situation), all of the ways of producing it (its “causes”) would be equivalent and this will introduce a form of indeterminacy in the natural processes themselves⁶². What is crucial for us in this reasoning is the negative thesis, that is to say that mathematical objects can never be “real” in a strong sense because they are not perfectly determined. A triangle, for example, is not a thing in the sense in which Socrates is: it is a collection of “similar” entities, which can be obtained by various constructions/transformations. The fact that we can *identify* these various results is crucial to mathematical theory. More generally, a mathematical object can be characterized by a kind of indiscernibility (we would say “equivalence relation”) with respect to some transformation or property and this transfers to any use of these objects in physics⁶³. In this sense, *any* mathematical object is a fiction in the most common sense of the term: not necessarily an impossible notion, but something that cannot be a “real” thing (or, better still, not really be a “thing”, since it is not completely determinate)⁶⁴.

Note that this completes our argument about Leibniz’ fictionalism on a crucial point. Because most of our examples so far have been related to geometry, one may have wondered if this held for all of mathematics, and especially for basic arithmetic. Now that we have reached

⁶¹ See *Meditatio de principio individui* (April 1676), A VI, 3, 490-491.

⁶² This is a fact Field stumbles upon when using Hilbert “representation theorem”, since it states, as he recalls, that physical phenomena (in this case, points or regions of space) can be represented in a unique way by using “real numbers” *up to Euclidean transformation* (p. 50). What it means is that our mathematical representation will be true of any other versions of the world in which everything has been changed by any of these transformations (Field may discard this possible objection by talking in terms of “passive transformations”, i.e. choice of coordinates, but this is only a matter of interpretation and “from the point of view” of the mathematical structure, all the invariants can also be interpreted as transformations acting on space-time points). This remark also applies to Field’s treatment of scalar fields as invariant under specified linear transformations.

⁶³ See, for example, A VI, 4, 168, where Leibniz defines mathematical objects as types of “indiscernibles”. This, as is well known, is at the core of the main criticism against Newtonian physics in which space is considered as real or “absolute”. Leibniz objects that this amounts to introducing indeterminacy in the world since God could have chosen another disposition of things (for example, “by changing east into west”) without it making any difference.

⁶⁴ On these two meanings of *fictio*, see A VI 4, 570.

the metaphysical grounds of the fictitious character of mathematical entities, the answer should be obvious. Precisely because there are no such things in the world as differing *solo numero*, the very idea of number can occur only on the assumption of fictitious entities such as collections of indiscernible objects (which themselves can never be “real”)⁶⁵.

Now, according to Leibniz, mathematics accurately expresses physical and perceptual phenomena *precisely* because the latter are characterized by the same kind of indeterminacy than the former. As he writes to the Electress Sophie:

It is our imperfection and the shortcomings of our senses that make us conceive physical things as mathematical entities, in which there is indeterminacy. And it can be demonstrated that there is no line or shape in nature which gives exactly and keeps uniformly through the least space or time the properties of a straight or circular line, or of some other line of which a finite mind can grasp the definition. In bodies, whatever shape they might be, the mind can conceive and draw through it using the imagination any line that one wants to imagine, just as one can join the centres of spheres by imaginary straight lines, and conceive axes and circles in a sphere which does not have any real axes and circles. But nature cannot do this, and the divine wisdom does not will to trace exactly these shapes of limited essence, which presuppose something indeterminate and consequently imperfect in the works of God. (GP VII, 563, transl. L. Strickland)

At the end of the day, from a strict nominalist perspective, it seems thus that we have only fictions (mathematical objects) referring to fictions (physical things and perceptual facts involving some indeterminacy). It is because they are indeterminate or “incomplete” (i.e. not “real”) that physical and perceptual phenomena can be described by mathematics and this process is grounded in the “shortcomings of our senses”⁶⁶. This opens the way to the danger of what Leibniz sometimes calls “ultra-nominalism” (in which any truth would just be a matter of stipulations/conventions).

Leibniz encountered this problem very early on when discussing some sceptical and nominalistic arguments and it is precisely on this occasion that he elaborated his “Platonist” reply. In an exchange with Foucher from 1675, he objects that even a radical sceptical view on the “outside world” would have to retain hypothetical truths, precisely because they “affirm, not that something does exist outside us, but only what would happen if something existed there”. According to him, this already saved from doubt all of mathematics, as it is based on such truths. But this, in fact, is sufficient to establish that there are truths “outside us and independent of us”. Indeed if *possibilia* are not and cannot be “real” entities, in the sense of complete substances, they are not “chimera which we create, since all that we do consists in recognizing them, in spite of ourselves and in a constant manner”⁶⁷. Their independent being is

⁶⁵ The “solution” to this problem which consists of expressing numbers by “logical” means usually hides the difficulty by putting “identity” as a primitive of the language. We simply *postulate* that it makes sense to state that two things are “identical” and then proceed by showing that the concept of numbers can be derived from this postulation. From a strict nominalistic perspective, however, it would remain to show that “identical” is a “concrete” property of genuine “things”, a fact which is far from being obvious (and which, for Leibniz, is false: no genuine “things” can be perfectly identical).

⁶⁶ This is the argument Leibniz repeats again and again to Electress Sophie and her daughter Sophie-Charlotte to convince them that one has to posit a different realm of “real unities” in order to ground this endless process of multiplicities referring to multiplicities *ad infinitum*. As Leibniz explains (A II, 2, 639), these multiplicities are some “êtres de fiction” because no multitude can be conceived, according to him, without constituting unities.

⁶⁷ Same argument in a letter to the same written twenty years later GP IV, 491-492.

therefore grounded on the fact that our knowledge of them is structured and converges⁶⁸. Yet, this convergence should not be interpreted as “attributed solely to the nature of the human mind”, it should also be taken “in the sense that phenomena or experiences confirm them when some appearance (...) strikes our senses”⁶⁹.

We find again the idea that we access abstract objects and their regularity directly at the level in which we access “real phenomena”. But, and this is a crucial point, there is no way to *ground* the regularity at play in mathematics directly on the regularity at play in the phenomenal world for a reason I already mentioned: it is often the case that we dispose of several ways of describing the same set of phenomena mathematically⁷⁰. Hence, we need at the same time the regularity between the phenomenal world and the mathematical world to be of the same kind (or mathematics will not be able to accurately express natural regularities) *and* to be independent (or there will never be any *choice* in the way of expressing the laws of nature by mathematics – a fact which is contradicted by the very existence of a history of mathematical physics)⁷¹. Accordingly, there is an objectivity in the regularity provided in the various models which has to be independent of the natural regularity they might *possibly* express. This, according to Leibniz, is enough to ground a realism of mathematical truths or *possibilia*. They exist in a realm of their own, independently of the fact that we can “apply” them or not to describe our world. In modern terminology, objectivity is all we need here⁷².

⁶⁸ Leibniz does not give details about this convergence, but note that it can take various forms from the fact that several cultures have independently discovered properties of the circle (his example here) to the fact that these properties can also be recovered from independent mathematical theories. All of this certainly makes mathematical entities behave as genuine object, existing independent of us.

⁶⁹ GP I, 370/L 152. Leibniz would repeat the same position in the *Nouveaux Essais* (IV, xi, § 10), in which he makes his belief in another “realm, “the sphere of eternal truth” clearer as he made its Platonist origin clearer.

⁷⁰ The most intriguing example is when we dispose of equivalent ways of presenting one and the same physical theory such as is the case with the Newtonian and the Hamiltonian formalism in classical mechanics, see M. Vorms, *Qu'est-ce qu'une théorie scientifique ?* Paris, Vuibert, 2011. But there are also cases in which one and the same theory can describe various set of phenomena with *incompatible* mathematical formalisms (for examples of this sort, some of them directly inspired by Leibniz, see Mark Wilson, *Wandering Significance*, Oxford, Clarendon Press, 2006).

⁷¹ As we saw, Field claims that Hilbert’s geometry can be interpreted directly as a theory of physical space provided one endorses a “substantivalist” view on space (Field 2016, p. 35). To the objection that this strategy amounts to putting the abstract mathematical structure in disguise in the physical structure, he responds that the latter is less rich than the former – a claim on which Leibniz would have agreed. Moreover, its adoption is “all an empirical postulate, subject to revision by experience” (p. 34). But this last claim raises a series of difficulties regarding the status of “falsified” physical theories (as is precisely the case with Euclidean geometry seen as a physical theory, cf. p. 27). What it means, for example, is that we have examples of “physical theories”, which were considered to deal with “concrete” entities, although a new theory tells us that such entities *do not exist* (think for example of Ptolemean epicycles). Accordingly, we will have mathematical theories (in our example, the mathematics needed in Ptolemean astronomy) formalizing such falsified physical theories, i.e. dealing with *non-existing* entities. The conclusion here should be – although this is certainly not Field’s! – that mathematical discourse is *not* a way of speaking of existing things, but of *possibly* existing things. If we add that the preceding argument applies not only to past, but also to future physical theories, this amounts to Leibniz’ position according to which mathematics deals with *possibilia*.

⁷² Field acknowledges the existence of this type of Platonism, but simply says that it will not be the target of his Anti-Platonist move (Field 2016, p. 2). I hope to have shown that it should have been part of the discussion, especially after recognizing the problem related to the historicity of natural sciences as formulated in the previous note. Moreover, there are strong arguments to maintain that a Platonism based on objectivity is closer to Plato’s claim than a Platonism based on the existence of mathematical objects (since Plato took mathematical objects, as he explained in the famous metaphor of the “line” from the *Republic*, to be hypothetical by nature).

But there is more: since our ordinary experience already involves, according to Leibniz, abstract objects, the only way to guarantee that these objects are genuine ones or “real” phenomena (and not “empty fictions”) is precisely to have a norm to grant our knowledge of them as genuine object (by contrast to pure chimera). In other words, we need a way to detect and express regularity in the phenomenal world. In this regard, if we deny that objectivity grounds access to genuine mathematical objects (whatever they may be), we may simply end up having no way to guarantee access to any “object” at all – because physical objects are of the same kind (in the sense that they are fictitious *abstracta*), but with *less* guarantee⁷³. Fictionalism is not only compatible with Platonism, it leads to it, according to Leibniz, as the only way to escape scepticism about our knowledge of objects *tout court*.

Conclusion

There is sometimes an objection made to “mathematical fictionalism” that it is mere philosophical speculation with no link to the practice of the mathematician. In their practice, it is believed, mathematicians tend to be “naïve” Platonists and act as if confronted with genuine objects, which they have to discover and to describe (even if they can create ingenious means to express them). But everything here is in the “as if”. Moreover, as argued by Yablo, the very fact that mathematicians do not have to care much about the “reality” of their objects, or can disagree about it without harm, has something to tell us about the type of knowledge they are practicing.

I hope to have shown first that the use of fictions can be related to an ancient practice in mathematics and that it amounts to a form of discourse, which remains *neutral* as regard the ontological commitments at play⁷⁴. In the same way, that one learns to read a *reductio proof* or a hypothetical reasoning as a kind of game in which one is asked to assume some possibly monstrous entity, in the same way one is accustomed to considering *any* mathematical entity as waiting for further conceptual analysis. One reason this attitude is widespread amongst mathematicians is that it corresponds to two important features of their most ordinary practice: first, the fact that objects apprehended under a certain form appear often, after some time (long or short – sometimes as brief as a ten line proof), as “really” being something else; second, the fact that mathematicians often disagree, for various reasons, on the “real” nature of the object under consideration – although they do not disagree on the fact that they are working with “the same” object on the pain of having no disagreement at all. Both attitudes are deeply intertwined via the history of mathematics, for example when a contemporary mathematician considers that what everything Euclid stated was true, although it must be understood that he was dealing with under the names “numbers” and “magnitudes” were “really” various kinds of “sets” (or whatever other entity assumed as being the “real” one).

What we have seen in this paper is that there is a smooth path running from this linguistic form of fictionalism, embedded in practice, in which one acts “as if” some objects exist, to stronger versions of a more metaphysical flavour. The crucial point is that as soon as one has a way to dispense of a fictitious object in mathematics, one also has a way to dispense of it in the

⁷³ To take our example again: if there are no circles, it seems that there will not be planets or trajectories of planets in Ptolemean astronomy either. The same applies, of course, for ellipses in Newtonian astronomy.

⁷⁴ It is easy to show that such a fictionalism is still at play in contemporary mathematics. Look, for example, at any definition of “V”, the “universe of sets”, in a handbook of Set Theory.

description of the natural world. But what I find particularly fruitful in Leibniz' approach is his casting of doubt on the fact that as soon as one has identified a way to dispense with some fictitious objects, this is enough to guarantee that we have reached some "real" ground. His practice as a mathematician may have helped him to see that this was not the case. What is often the case is more likely that we end up with fictions referring to fictions referring to fictions, etc. This was, according to him, a key issue in order to vindicate a form of Platonism – since mathematical objectivity is, at the end of the day, one of the rare ways, if not the only one, by which we access genuine objects through the interplay of fictions referring to fictions (possibly *ad infinitum*). This might explain why our mathematician acting "as if" mathematical objects exist may indeed be a sincere Platonist and a sincere fictionalist at the same time.