Increasing returns, externalities and equilibrium in Riesz spaces

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Increasing returns, externalities and equilibrium in Riesz spaces

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Abstract

This paper studies the appropriate pricing rule and its associated equilibrium concept when there are market imperfections in a Riesz space setting. We extend the notion of marginal pricing equilibria to situations with non convex production sets and external factors in an abstract vector lattice whose topological dual is a sublattice of its order dual. Our main result guarantees that a non-competitive equilibrium exists and it is related with first order condition for profit maximization at the time that it encompasses a wide range of economic situations since previous results in the literature become particular cases of it. Furthermore, we developed a new properness assumption that takes into account the non convexity of the production correspondences together with the presence of externalities which in some sense is a weakening of some known conditions in competitive economies.

Keywords: Riesz space, marginal pricing rule, non-competitive equilibrium,

σ-locally τ-uniform properness.

JEL classification: D51, C62

1 Introduction

This paper is part of a series of works on general equilibria in non-competitive economies in infinite dimensional spaces whose precedents are [6], [4], [13], [14] and [8]. With the exceptions of [6] and [13], in all papers a marginal pricing rule was explicitly defined and when non-convex production sets are considered. Furthermore, [13], [14] and [8] also consider externalities which is not only another market failure but commonly is a source of nonconvexity. The current paper extends the results obtained in [8] when the commodity space is a vector lattice or Riesz space.

In [8], we were motivated by both the economic and mathematical relevance of non-competitive equilibria results in Banach spaces. On the one hand, abstract Banach lattices covers many interesting topics like uncertainty, infinite time horizon, option pricing and commodity differentiation among others since complete normed lattices include all the sequence spaces l_p $(1 \leq p \leq \infty)$, all the Lebesgue spaces L_p $(1 \leq p \leq \infty)$ and the space of measures on a compact metric space K, $\mathcal{M}(K)$. Hence, that existence theorem encompasses specific contexts like (i) endogenous growth models *à la* Romer [26] or [20]; where external effects play a key role and the idea of sustained growth requires to consider infinitely many periods. Further, in both models production exhibits increasing returns to scale which is a special case of non-convexity; (ii) Product differentiation and international trade; where [19] bases his model on increasing returns and product differentiation and shows the correlation between market size and trade in differentiated products which is relevant with an infinite dimensional commodity space and (iii) financial markets with bankruptcy chains; where a typical form of externality is what can be generated through a "chain" of default values between agents holding "short" and "long" positions, possibly interrelated mutually. In these contexts, models of general equilibrium with incomplete markets with default or bankruptcy and a continuum of states of the world or with infinitely many periods fit as special cases of our model (see, for example, the special issue of Journal of Mathematical Economics, 1996).

On the other hand, the presence of increasing returns or more generally, non-convex production sets, demands a new concept for the producer behavior beyond profit maximization such as average pricing rule, voluntary trading or the marginal pricing rule. A proper mathematical treatment of the latter in a finite dimensional setting was first provided in [15] and then generalized in [11]: the marginal pricing rule in a vector y_j belonging to the set Y_j is formally the Clarke's normal cone to y_j at Y_j (see [9]). This rule associates, to each efficient production plan y_j at Y_j, the convex hull of the set of vectors that are perpendicular to y_j and those that are perpendicular to productions around y_j in a neighborhood of it. This pricing rule satisfies the necessary conditions for optimality, and it coincides with profit maximization when production sets are convex. When the commodity space is infinite dimensional, [4] proved that the marginal pricing rule establishes a larger set than that of Clarke's normal cone. The search for a new cone comes from the fact that the graph of the Clarke's normal cone is not closed for the relevant topologies. Then, [8] introduced a new extension satisfying closedness properties and considering both the norm and the weak-star topology on the Banach lattice commodity space together with order intervals instead of open balls to measure the "distance" to the production set.

The main economic motivation for passing from a topological vector lattice[1] to a Riesz space is to expand the theoretical scope of the analysis by

[1] Banach lattices are special cases of topological vector lattices or locally-solid Riesz spaces.

covering a larger set of economically interesting settings. For instance, even though the space $(\mathcal{M}(K), \|\cdot\|)$, which is usually considered to formalize commodity differentiation models, is a Banach lattice, all compatible topologies with the commodity-price pairing $\langle \mathcal{M}(K), C(K) \rangle$ are not locally solid. Hence, the existence theorem of [8] rules out models studied by [22] and [17] among others since there, the commodity space is equipped with the weak*-topology $\sigma(\mathcal{M}(K), C(K))$ so that $(\mathcal{M}(K), \sigma(\mathcal{M}(K), C(K)))$ is not a topological vector lattice. In the same way, models with strong intertemporal substitution properties in consumption as the one studied by [16] where the commodity-price pairing is $\langle M([0,1]), \mathrm{Lip}([0,1]) \rangle$[2], makes use of the compatible topology $\sigma\left(M([0,1]), \mathrm{Lip}([0,1]\right)$ which is not locally solid either, thus remaining outside the scope of the existence theorem of [8]. Besides these topological vector space based arguments, our existence theorem in [8] assumes that the positive cone of the commodity space has quasi-interior points for the norm topology. This is an unpleasant restriction since $\mathcal{M}(K)_+$ has no quasi-interior points unless K is countable when the topology is the norm one .

Thus, the consequence of such a generalization is an abstract Riesz space which encompasses the vast majority of economic situations. This is at the heart of the motivation of the existence theorems in Walrasian models. We go beyond by incorporating market failures which in turn implies the emergence of important technical challenges: (a) the formalization of the marginal pricing rule that not only must be consistent with first order condition for profit maximization but also has to encompass those cones used previously in the literature as particular cases, and (b) the properness assumption that has to be developed when production sets are non-convex at the time that it must be coherent with what has been proposed previously for the competitive economies in order to achieve truly inclusive results.

[2] $\mathrm{Lip}([0,1])$ is the space of Lipschitz functions

With respect to (a), we define a tangent cone whose dual defines the marginal pricing rule. We note that the economic idea in which we construct the tangent cone is analogous to the way it was constructed in the past: for computing prices, the owner of the j-th firm observes not only the current production plan y_j and the environment z, but also all production plans y'_j close to y_j that are consistent with the environments z' which are close to z. The main difference comes from the complexities derived from a larger space, in particular, the mathematical notion of *nearness*. Indeed, on the one hand, we consider weak-open neighborhoods of both z and y_j since weak topologies are appropriate for showing that the limit of prices satisfying the marginal pricing rule is still satisfying the marginal pricing rule under specific conditions. However, on the other hand, weak-open neighborhoods are too large in the sense that they are never bounded, so we also have to consider strong neighborhoods so that production plans and externalities are not too far from the reference points. As for (b) we show that the properness assumption must take into consideration not only the production plan y_j of $Y_j(z)$ but also production plans that belongs to a neighborhood of y_j at the time that they belong to $Y_j(z')$ for z' in a neighborhood of z. This is consistent with the idea of the tangent cone to a nonconvex set and both notions, the tangent cone and the properness condition, fit well and are consistent with previous results when the space (L, τ) is a locally solid Riesz space (or a topological vector lattice) and/or when technologies are smooth, i.e., when they are described by a differentiable transformation function. Even though, the properness condition may be seen as a natural generalization of the standard assumption in Riesz spaces when there are no external factors and when production sets are convex. In comparison with the literature on competitive equilibria we note that the marginal pricing rule does not coincide with the profit maximization behavior when production sets are convex unless we add a new restriction on the production sets that we termed *upper order bounded* or *UOB*: for every (consistent) externality z, each $y_j \in Y_j(z)$ is "almost" bounded from above for some multiple of a reference vector $e > 0$ in the sense that arbitrarily close to y_j there will be a vector ξ which is smaller than such ne for some n. We show that under condition UOB, the marginal pricing rule coincides with the profit maximization behavior and that properness condition with UOB hypothesis becomes quite close to both the E-properness assumption used by [12] and the M-properness condition developed by [29], two of the more general papers in the literature on Walrasian equilibrium in Riesz spaces. Even more, in some sense, which will be precised in Section 5.1, our definition is weaker than theirs.

The assumptions on consumption sets are analogous to those that we assumed when the commodity space had a locally solid topology ([8]) with the sole exception of assumption E-properness which we had not used then. However, E-properness is the natural generalization of the respective assumption made in [12]. Unlike what mostly assumed in competitive economies we do not impose that the consumption set is the positive cone L_+.

Concerning the method of our existence proof, we follow the idea of restricting the economy to a smaller commodity space, namely, the one of the order ideal generated by the e, $L(e)$. Under standard arguments, we get an equilibrium and then we show that such an equilibrium can be extended to the original economy under mild assumptions. Indeed, we extend equilibrium prices to the whole commodity space L by borrowing techniques originally developed by [24] and extended by [12]. Unlike many papers on competitive equilibria (e.g. [23], [25]) we cannot follow the Negishi approach since it requires the aggregate production plan to be efficient which is not necessarily the case when production sets are non-convex.

The paper is organized as follows: Section 2 deals with the mathematical structure. Section 3 presents the model and the Assumptions together with the specification of the new marginal pricing rule. In Section 4, we state an existence result in the intermediate economies which becomes the starting point for Section 5, where a general existence theorem is presented. In this section we also compare our results with previous ones both in competitive and non-competitive economies. In Section 6, we prove the existence theorem. With the exception of this section, all proofs are given in the homonymous section at the end.

2 Terminology and Notation

Let L be a Riesz space[3] (or vector lattice), i.e., an ordered vector space such that for any x, x' in L the set $\{x, x'\}$ has a supremum, $x \vee x'$, and an infimum, $x \wedge x'$, in L given the order $\geq$ in L. The cone $L_+ = \{x \in L : x \geq 0\}$ is called the positive cone of L. For $x, x' \in L$ with $x \leq x'$, we denote by $[x, x']$ the order interval $\{y \in L : x \leq y \leq x'\}$. Let $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x^+ + x^-$ be respectively the positive part, the negative part and the absolute value of x. A subset A of L is called solid if $|x'| \leq |x|$ and $x \in A$ implies $x' \in A$. We say that A is radial at $x \in A$ if for each $x' \in L$, there exists a real number λ_x, $0 < \lambda_x \leq 1$ such that $(1-\lambda)x + \lambda x' \in A$ for every λ with $0 \leq \lambda \leq \lambda_x$.

We denote by $L^\sim$ the order dual of L, i.e., the vector space consisting of all linear functionals on L which map order intervals of L to bounded subsets of $\mathbb{R}$, ordered by the relation $f \geq g$ whenever $f(x) \geq g(x)$ for all $x \in L_+$. Let f and g in $L^\sim$ and $x \in L_+$. Then the Riesz-Kantorovich formula states $f \vee g(x) = \sup\limits_{0 \leq x' \leq x} \{f(x') + g(x - x')\}$. If $x \in L_+$, $L(x) := \bigcup_{n \in \mathbb{N}} [-nx, nx]$ is the

[3] An ordered vector space is a (real) vector space endowed with a partial ordering $\geq$ such that $x \geq x'$ implies $x + y \geq x' + y$ for $x, x', y \in L$ and if $x \geq 0$ it entails $tx \geq 0$ for $t \in \mathbb{R}_{++}$. $x \leq x'$ means $x' \geq x$.

principal ideal in L generated by x.

We endow L with a Hausdorff locally convex linear topology τ[4] such that L_+ is closed and order intervals are bounded. Let L^* be the topological dual of L, i.e., the vector space of all τ-continuous linear functionals on L. For $x \in L$ and $\pi \in L^*$, $\pi \cdot x$ is the evaluation of x under π. Since every order interval of L is τ-bounded it follows that $L^* \subset L^\sim$. Throughout this paper we shall assume that L^* is a vector sublattice of $L^\sim$. Note that if τ is locally solid, then both L_+ is τ-close and L^* is a vector sublattice of $L^\sim$. There exists a base of τ-neighborhoods of 0, $\mathcal{V}_\tau(0)$, such that for each $W \in \mathcal{V}_\tau(0)$: (i) There exists $W' \in \mathcal{V}_\tau(0)$ such that $W' + W' \subset W$, (ii) W is radial, i.e., for every finite set N of L, there exists μ in $\mathbb{R}_+$ such that $N \subset \mu W$, (iii) W is circled, i.e., $\lambda W \subset W$ for $|\lambda| \leq 1$ and (iv) W is convex. ([27], 1.2, p. 14; p. 47). We shall also consider the weak topology $\sigma(L, L^*) = \sigma$ on L. This linear topology is also Hausdorff and locally convex. $\mathcal{V}_\sigma(0)$ is a base of weak neighborhoods of 0 with the same properties as $\mathcal{V}_\tau(0)$.

For the subset A of L we shall denote by $\tau - \mathrm{int}A$ and $\tau - \mathrm{cl}A$ (resp. $\sigma - \mathrm{int}A$ and $\sigma - \mathrm{cl}A$) the corresponding τ-interior and τ-closure of A (resp. the σ-interior and σ-closure of A). Let L^M be the product space given by the Cartesian product of M copies of the space L. We denote by $\prod_{L^M} \tau$ (resp. $\prod_{L^M} \sigma$) the product topology on L^M when the topology on each L is τ (resp. σ). Hence, $\left(L^M, \prod_{L^M} \tau\right)$ is a Hausdorff Riesz space while $\left(L^M, \prod_{L^M} \sigma\right)$ is a locally convex Hausdorff Riesz space.

Let $D : L^M \mapsto L$ be a correspondence. We say that D has τ-closed values if for every $x \in A$, $D(x)$ is a τ-closed subset of L.

For further details on infinite dimensional spaces and Riesz spaces, we refer to [1, 27].

[4]The assumption that τ is locally convex is quite standard and is at the heart of several separation theorems. However, we point out that all results along the paper can be proved without this particular assumption

3 The Model

We consider an economy whose commodity space is the Riesz space L. There are finite sets of consumers and producers I and J respectively with cardinality $|I|$ and $|J|$. We index each consumer by $i \in I$ and each producer by $j \in J$. An element $z = ((x_i)_{i\in I}, (y_j)_{j\in J})$ in $L^{|I|+|J|}$ is called an environment. Instead of consumption sets we have consumption correspondences, that is, the i-th consumer has a consumption correspondence $X_i : L^{I+J} \twoheadrightarrow 2^{L_+}$ depending on the environment z. Given z, $X_i(z) \subset L_+$ is her consumption set. Preferences also depend upon the actions of the other economic agents. We shall denote them by $\succeq_{i,z}$, the preference relation of consumer i given the environment z. This relation is assumed to be complete, reflexive and transitive. The relation of strict preference $x \succ_{i,z} x'$ is then defined by $x \succeq_{i,z} x'$ and not $x' \succeq_{i,z} x$. We do not assume that we can compare two commodity bundles if they do not share the same environment. Let $\omega_i \in L_+$ be the initial endowment of the i-th agent such that $\omega_i \in X_i(z)$ for all $z \in L^{|I|+|J|}$. Let us denote the total initial endowment of the economy by $\omega = \sum_{i\in I} \omega_i \neq 0$.

Each producer j has a production set which also depends upon the actions of the other agents. For each $j \in J$, $Y_j : L^{|I|+|J|} \twoheadrightarrow 2^L$ is the production correspondence. For the environment $z \in L^{I+J}$, $Y_j(z) \subset L$ is the set of all feasible production plans for the j-th producer. We denote the τ-boundary of $Y_j(z)$ by $\partial Y_j(z)$

The wealth function of the i-th consumer is given by a function $r_i : \mathbb{R}^{1+|J|} \to \mathbb{R}$. If $\pi \in L^*$ and $(y_j)_{j\in J} \in \prod_{j\in J} Y_j(z)$, her wealth is $r_i\left(\pi \cdot \omega_i, (\pi \cdot y_j)_{j\in J}\right)$. This definition is general enough as to encompass the private ownership economy case, i.e., when $r_i\left(\pi \cdot \omega_i, (\pi . y_j)_{j\in J}\right) = \pi \cdot \omega_i + \sum_{j\in J} \theta_{ij} \pi \cdot y_j$ for $\theta_{ij} \geq 0$ and $\sum_{i\in I} \theta_{ij} = 1$ for all $j \in J$.

For a given environment $z \in L^{|I|+|J|}$ and a given initial endowments $\omega' \in L_+$, we shall denote by $A(\omega', z)$ the set of attainable productions, that is,

$$A(\omega', z) := \left\{ (y'_j) \in \prod_{j \in J} \partial Y_j(z) : \sum_{j \in J} y'_j + \omega' \in L_+ \right\}$$

In order to consider only consistent situations, we define the set

$$Z := \left\{ z \in L^{|I|+|J|} : \forall i \in I,\ x_i \in X_i(z);\ \forall j \in J,\ y_j \in \partial Y_j(z) \right\}$$

The set of *weakly efficient attainable allocations* corresponding to a given total initial endowment $\omega' \in L_+$ is given by

$$A(\omega') := \left\{ z \in Z : \sum_{i \in I} x_i = \sum_{j \in J} y_j + \omega' \right\}$$

We assume that there exists a reference commodity bundle $e \in L_+ \setminus \{0\}$ which is *comparable* with ω in the sense that there is some $n_0 \in \mathbb{N}$ for which $\omega \leq n_0 e$ or, in other words, ω belongs to the order ideal $L(e)$. In previous works e has been suitably chosen depending on the commodity space L. For example, when $L = \mathbb{R}^L$ as in [7], e is the unit vector $(1, 1, ..., 1)$. When $L = L_\infty$, ([3], [4]) e is the constant function equal to 1 which obviously belongs to the $\|\cdot\|_\infty$-interior of L_∞^+. In our previous work, [8], e is a quasi-interior point of L_+ satisfying $\|e\| = 1$.

The above are examples of spaces where e is an interior or a quasi-interior point of L_+. This fits well in topological vector lattices so various economic models with infinitely many commodities are covered. However, these works rule out some economically important spaces such as the normed lattice $(M([0,1]), \|\cdot\|)$ (which is used for commodity differentiation analysis) since $\mathcal{M}_+([0,1])$ has no quasi-interior points. On the other hand, when equipping $\mathcal{M}([0,1])$ with the weak*-topology, the space is not a topological vector lattice (See [25]). We can say more: any topology compatible with the duality $\langle M[0,1], C[0,1]\rangle$ is not locally-solid. Another topology, $\sigma(M[0,1], \mathrm{Lip}[0,1])$, which is the one used to model intertemporal consumption by [16], is also not locally solid. See further comments in the introduction.

Actually, e plays a fundamental role both in the properness assumption, that we shall define later, and in the definition of the commodity space of the intermediate economies. Consequently, the fact that L is not a topological vector lattice does strengthens the relevance of the vector e.

3.1 Marginal pricing rule and equilibrium

The marginal pricing rule doctrine is not new. It comes at least from the thirties, when Hotelling argued that when the firms exhibit increasing returns to scale, prices should be proportional to marginal costs to reach a Pareto optimal allocation. Hotelling also paid attention to the fact that, in some cases, a firm or even an industry which adopts marginal cost pricing will run at a loss if there are high fixed costs. The deficit must then be financed from income taxes. In Hotelling's conception, the production sets are non-convex but with smooth boundaries. Hence, when the latter does not hold true, we replace the marginal cost condition by the first order necessary condition for profit maximization (see [11], [15]). This rule is called marginal pricing rule because when technologies are smooth, marginal productivities are proportional to the prices. This notion extends to very general frameworks (see, e.g., [10, 5, 4]) where the marginal pricing rule is defined by a notion of normal cone, the set of outward direction, which is the polar cone of the tangent cone, the set of inward or quasi-inward directions. In an finite dimensional space, the Clarke's normal cone as introduced in [11] and extended in [7] to encompass externalities, is the right concept and, in some sense, the smallest one for the existence of equilibrium.

However, things change when the commodity space is infinite dimensional since the graph of the Clarke's normal cone is not necesarily closed for the relevant topologies. Then, [4] introduced a new cone satisfying closedness properties and considering both the norm and the weak-star topology in the space L_∞. Later, [8] goes a step beyond by considering Banach lattices as commodity spaces and, again, it was necessary to formulate a new normal cone as the set where the producer sets her prices satisfying first order condition for profit maximization. The main difference is that order intervals are used, instead of open balls, to measure the "distance" to the production set in the definition of a quasi inward vector.

We move now to a definition of tangent and normal cones in a Riesz space. For $z \in Z$ and $y_j \in \partial Y_j(z)$ let us consider the following set

$$\mathscr{T}_{Y_j(z)}(y_j) = \bigcap_{\mathcal{C} \in \mathcal{V}_\tau} \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$$

where

$$\mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j) := \left\{ \nu \in L \;\middle|\; \begin{array}{c} \exists\ \eta > 0 \text{ such that } \forall\ r > 0, \\ \exists \quad U \in \mathcal{V}_\sigma(0) \text{ and } \varepsilon > 0 \ \ | \\ \forall \ \ z' \in \left(\{z\} + (\mathcal{C} \cap U)^{|I|+|J|} \right), \\ \forall \ \ y'_j \in (\{y_j\} + \mathcal{C} \cap U) \cap Y_j(z') \\ \text{and } \forall \ \ t \in (0, \varepsilon),\ \exists\ \xi \in r[-e,\ e] \text{ such that} \\ y'_j + t(\nu + \eta(y_j - y'_j) + \xi) \in Y_j(z') \end{array} \right\}$$

Then, the tangent cone at y_j in $Y_j(z)$ is $\hat{\mathscr{T}}_{Y_j(z)}(y_j) := \mathscr{T}_{Y_j(z)}(y_j) \cap L(e) - L_+$.

One of the properties of the tangent cone in [8] is the fact that when L is a Banach lattice and e a quasi-interior point of L_+, the restriction of the tangent cone to the order ideal $L(e)$ is enough to define the marginal pricing rule by polarity in $L(e)$. Indeed, every element in the tangent belongs to the closure of that restriction. This is a key step to get an extension from a marginal pricing rule in $L(e)$ to L. The big difference with the current case is the "size" of $L(e)$ relative to L. When e is a quasi-interior point of L_+, the former is a large subset of the latter. Hence, all activities are close to $L(e)$. This is no longer true when L_+ has no strictly positive vectors. In this case, $L(e)$ is small relative to L. These facts have been taken into account in the construction of the tangent cone above.

Proposition 3.1 *$\hat{\mathscr{T}}_{Y_j(z)}(y_j)$ is a cone.*

First, notice that every element ν of $\hat{\mathscr{T}}_{Y_j(z)}(y_j)$ is order bounded from above by a multiple of e. The way in which we constructed this cone is by taking into account neighborhoods of z and y_j in such a way that the vector $y'_j + t(\nu + \eta(y_j - y'_j))$ is close to the set $Y_j(z')$ for the order relation $r[-e, e]$. In addition, we take the cone $-L_+$ which is consistent with the free disposal property of technologies.

Second, from an economic point of view, the idea is very simply: for computing prices, the owner of the j-th firm observes not only the current production plan y_j and the environment z, but also all production plans y'_j close to y_j that are consistent with the environments z' which are close to z. This is the same economic idea as in the models of [7], [4] and [8]. The consideration of environments z' in a neighborhood of z is necessary since $Y_j(z)$ is not convex. The main difference with the quoted literature comes from the mathematical notion of *nearness*. Indeed, on the one hand, we consider weak-open neighborhoods of both z and y_j since weak topologies are appropriate for showing that the limit of prices satisfying the marginal pricing rule is still satisfying the marginal pricing rule under specific conditions. This result is implicitly used in Subsection 4 when we refer to Theorem 4.1 in [8]. However, on the other hand, weak-open neighborhoods are too large in the sense that they are never bounded, so we also have to consider strong neighborhoods so that production plans and externalities are not too far from the reference points.

Third, the *marginal pricing rule* is now formally defined: given $(y_j, z) \in \partial Y_j(z) \times Z$, the j-producer chooses prices in $\hat{\mathscr{N}}_{Y_j(z)}(y_j) := \left[\hat{\mathscr{T}}_{Y_j(z)}(y_j)\right]^\circ$. In words, to set prices according to the marginal pricing rule, the producer takes into account the fact that his production set depends on external effects in the sense that around the production plan, the shape of the production set changes when externalities vary. We note that this rule is equivalent to saying that the owner of the firm chooses the prices π for which y_j maximizes profits on $\{y_j\} + \hat{\mathscr{T}}_{Y_j(z)}(y_j)$. On the other hand, it is immediate that $\hat{\mathscr{N}}_{Y_j(z)}(y_j) \subset L^*_+$.

Finally, the set of *production equilibria* is

$$PE := \left\{(\pi, z) \in L^*_+ \times Z : \pi \in \cap_{j \in J} \hat{\mathscr{N}}_{Y_j(z)}(y_j)\right\}$$

We are now able to state the definition of a marginal pricing equilibrium.

Definition 3.2 *A marginal pricing equilibrium of the economy $\mathcal{E}$ is an element $(\bar{z} = (\bar{x}_i)_{i\in I}, (\bar{y}_j)_{j\in J}), \bar{\pi})$ in $Z \times L^*_+$ such that:*

1. *For all $i \in I$, $\bar{\pi} \cdot \bar{x}_i \leq r_i(\bar{\pi} \cdot \omega_i, (\bar{\pi} \cdot y_j)_{j\in J})$ and $\bar{\pi} \cdot x'_i > r_i(\bar{\pi} \cdot \omega_i, (\bar{\pi} \cdot y_j)_{j\in J})$ whenever $x'_i \succ_{i,\bar{z}} \bar{x}_i$.*

2. *For all $j \in J$, $\bar{\pi} \in \hat{\mathscr{N}}_{Y_j(\bar{z})}(\bar{y}_j) \setminus \{0\}$*

3. $\sum_{i\in I} \bar{x}_i = \sum_{j \in J} \bar{y}_j + \omega$

Condition 1. says that every consumer maximizes her preference under her budget constraint. Condition 2. says that every producer sets on the market the same equilibrium price vector $\bar{\pi}$ which satisfies the marginal pricing rule. Condition 3. says that all markets clear.

3.2 Assumptions

We now posit the following assumptions. Some of them became standard in the literature.

Assumption (C)

For all $i \in I$

1. X_i is a convex-valued correspondence with a $(\prod_{L^{|I|+|J|}} \sigma, \sigma)$-closed graph. Furthermore, for all $z \in L^{|I|+|J|}$, $0 \in X_i(z)$ and $X_i(z)$ is a solid subset of L_+.

2. For all $z \in L^{|I|+|J|}$, the half-line $\{\delta e : \delta > 0\}$ is included in $X_i(z)$. For all $x_i \in X_i(z)$ and for all $\delta > 0$ there exists a neighborhood $V \in \mathcal{V}_\sigma(0)$ such that $x_i + \delta e \in X_i(z')$ for all $z' \in \{z\} + V^{|I|+|J|}$.

3. For every $z \in L^{|I|+|J|}$, and all $x_i \in X_i(z)$ both sets $\{x'_i \in X_i(z) : \ x'_i \succeq_{i,z} x_i\}$ and $\{x'_i \in X_i(z) : \ x_i \succeq_{i,z} x'_i\}$ are τ-closed. For all $x'_i \in X_i(z)$ such that $x'_i \succ_{i,z} x_i$, for all $t \in (0,1)$, $tx'_i + (1-t)x_i \succ_{i,z} x_i$.

4. The set $G_i = \{(x'_i, x_i, z) \in L^2 \times L^{|I|+|J|} : (x'_i, x_i) \in X_i(z)^2, x'_i \succeq_{i,z} x_i\}$ is a $(\tau \times \sigma \times \prod_{L^{|I|+|J|}} \sigma)$-closed subset of $L^2 \times L^{|I|+|J|}$.

5. The wealth function $r_i : \mathbb{R}^{1+|J|} \to \mathbb{R}$ is continuous, increasing and homogeneous of degree one with respect to the price vector. Furthermore, for all $((v_i), (v_j)) \in \mathbb{R}^{|I|+|J|}$, $\sum_{i\in I} r_i(v_i, (v_j)_{j\in J}) = \sum_{i\in I} v_i + \sum_{j\in J} v_j$ and if $\sum_{i\in I} r_i(v_i, (v_j)_{j\in J}) > 0$ then $r_i(v_i, (v_j)_{j\in J}) > 0$ for all i.

6. Let $z = ((x_i),(y_j)) \in A(\omega) \cap L(e)^{|I|+|J|}$. Preferences are E-proper relative to $L(e)$ at $x_i \in X_i(z)$, i.e., there exists a τ-open convex cone with vertex 0 containing $e\, \Gamma_i$, a set $R_{x_i} \subset L$ radial at x_i and a sublattice $K_{x_i} \subset L(e)$ satisfying $x_i \in K_{x_i}$ and $K_{x_i} + L(e)_+ \subset K_{x_i}$ such that

$$\emptyset \neq (\{x'_i \in X_i(z) : x'_i \succ_{i,z} x_i\} + \Gamma_i) \cap K_{x_i} \cap R_{x_i} \subset \{x'_i \in X_i(z) : x'_i \succ_{i,z} x_i\} \tag{1}$$

and

$$\{x'_i \in X_i(z) : x'_i \succ_{i,z} x_i\} \cap R_{x_i} \subset (K_{x_i} + L_+) \tag{2}$$

Assumption (P) For every $j \in J$

1. $Y_j : L^{|I|+|J|} \twoheadrightarrow 2^L$ has a $(\prod_{L^{I+J}} \sigma, \sigma)$-closed graph.

2. For every $z \in L^{|I|+|J|}$, $Y_j(z) \cap L_+ = \{0\}$ and Y_j satisfies the free-disposal condition, that is, $Y_j(z) - L_+ = Y_j(z)$.

3. For all $z \in L^{|I|+|J|}$, for all $y_j \in \partial Y_j(z)$ and for all $\delta > 0$, there exists $V' \in \mathcal{V}_\sigma(0)$, such that $y_j - \delta e \in Y_j(z')$ for all $z' \in \{z\} + V'^{|I|+|J|}$.

4. For all $z \in L^{|I|+|J|}$, for all $y_j \in \partial Y_j(z)$, for all $\mathcal{C} \in \mathcal{V}_\tau(0)$ and for all $\delta > 0$ there exists $\hat{U} \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$ such that for all $z' \in \left(\{z\} + (\mathcal{C} \cap \hat{U})^{|I|+|J|}\right)$, for all $y'_j \in Y_j(z') \cap \left(\{y_j\} + \mathcal{C} \cap \hat{U}\right)$, for all $t \in (0,\varepsilon]$, $t(y_j - \delta e) + (1-t)y'_j \in Y_j(z')$.

5. Let $z = ((x_i),(y_j)) \in A(\omega) \cap L(e)^{|I|+|J|}$. The technology Y_j is σ-locally τ-uniformly-proper relative to $L(e)$ at $y_j \in Y_j(z)$, i.e., there exists a τ-open convex cone with vertex 0, Γ_j, containing the vector e and an open convex neighborhood $V_j \in \mathcal{V}_\sigma(0)$, such that for all $z' \in \left(\{z\} + V_j^{|I|+|J|}\right) \cap L(e)^{|I|+|J|}$

$$[(\{y_j\} + V_j) \cap Y_j(z') - \Gamma_j] \cap L(e) \cap (\{y_j\} + V_j) \subset Y_j(z') \tag{3}$$

Assumption B (Boundedness) For all $\omega' \geq \omega$ there exists $b \in \mathbb{R}_{++}$ such that, for all $z \in Z$, $A(\omega', z) \cap L(e)^J \subset [-be, be]^J$.

Assumption SA (Survival) For all $z \in Z$, for all $t \in \mathbb{R}_+$ and for all $(\pi, (y_j)) \in L^* \times A(\omega + te, z)$, if $\pi \in \cap_{j \in J} \hat{\mathscr{N}}_{Y_j(z)}(\ y_j) \setminus \{0\}$, it follows that $\pi \cdot \left(\sum_{j \in J} y_j + \omega + te\right) > 0$.

Remark 3.1 *Assumption (C) gathers standard conditions on the continuity and convexity of preferences and with respect to the convexity and closedness of the consumption set. Assumption C(1) states that $X_i(z)$ is solid which is necessary since we are not considering consumption sets equal to the positive cone L_+. We make use of this property in Subsection 4* [5]*. Assumption C(3) is a strong form of lower hemi-continuity that has been widely discussed in [13] and [8]. We refer the reader to these works. We point out that homogeneity of r_i with respect to price vector π is not necessary for getting equilibria but to deal with normalized prices.*

The properness assumptions are well known in the literature on existence of equilibria. The main consequence of this condition is to get a continuous extension of supporting prices in $L(e)$ to the whole space L. Regarding preferences we use the one of [12] so we refer to it for further discussion on this condition[6]*. Notice that* (1) *implies non-satiation on the attainable sets restricted to $L(e)$. Together with the convexity of preferences, we have local non-satiation on the subspace. This non-satiation in $L(e)$ rather than in L will be sufficient for our purposes since $L(e)$ includes the relevant commodity bundles in the economy.*

On the production side, assumptions on closedness and free disposal are standard. Assumption P(3) is a stronger form of lower hemi-continuity and we

[5] Here, we implicitly apply Theorem 4.1 of [8] to the economies $\mathcal{E}_{|L(e)}$ which requires solidity.

[6] We note that if the set V_x in the definition of E-properness in [12] is $\{x_i' \in X_i(z) : x_i' \succ_{i,z} x\} + \Gamma_i$, the inclusions (2.1) and (2.2) there, are exactly the same as (1) and (2) in C(6) since preferences are assumed to be convex.

refer the reader to [13] and [8] for details. We discuss now the remaining items of Assumption (P)

Remark 3.2 *If $Y_j(\cdot)$ satisfies Assumption P(4), we say that Y_j is $\prod_{L^{|I|+|J|}}\sigma$-locally star-shaped with respect to $y_j - \delta e$ whenever $y_j \in Y_j(z)$. If $Y_j(z)$ is convex for all z, Assumption P(4) is clearly a consequence of Assumption P(3) and, in turn, if $\varepsilon \geq 1$ in P(4) one has P(3) by implication. The next lemma shows the equivalence between Assumption P(4) and the fact that the null vector belongs to the tangent cone to $Y_j(z)$ at y_j, and P(4) is a consequence of free-disposal in finite dimensional commodity spaces.*

Lemma 3.3

1. *Condition P(4) is equivalent to $0 \in \mathscr{T}_{Y_j(z)}(y_j)$ under Assumptions P(1)-P(3).*

2. *Condition P(4) is a consequence of free disposal when L is finite dimensional*

Remark 3.3 *Properness condition means that if $y'_j \in Y_j(z')$ is close to the production vector $y_j \in Y_j(z)$ for the externality z' close to z, and we add to y_j the quantity e of inputs, then it is still producible if we add a vector small enough and the resultant vector is again both sufficiently near to y_j and belongs to $L(e)$. Thus, marginal rates of substitution with respect to e are bounded away from zero in a neighborhood of y_j.*

Unlike [24] and [12] the idea of "sufficiently near to y_j" is of topological nature instead of belonging to a radial set at y_j. Even though it is a stronger requirement, it is consistent with the idea of the tangent cone to a nonconvex set which not only focuses on y_j but also to productions around. By taking the intersection with a radial set, it may change the tangent cone because it shrinks the production vectors taken into account. An alternative way could be to impose *additional requirements on the size of the radial set but we did not follow this idea.*

When technologies are convex, we complement this assumption with condition UOB which is quite similar to the inclusion (2.4) in [12] and to the definition of M-properness in [29]. We shall define and discuss this Assumption on Section 5.1.

Remark 3.4 *Boundedness assumption says that feasible production vectors belonging to $L(e)$ are order bounded which, in turn, implies that the attainable set in $L(e)^{|I|+|J|}$ belongs to $[-\omega' - |J|be,\ \omega' + |J|be]$. For spaces whose positive cone has a non-empty interior, Assumption B is automatically satisfied if $A(\omega')$ is norm-bounded.*

4 The restricted economy in $L(e)$

The main reason of this section is to show the existence of a marginal pricing equilibrium in the restricted economy, that is, the one with commodity space $L(e)$ as a starting point to get an equilibrium in the original economy. In $L(e)$, the order interval $[-e, e]$ is absorbing and its gauge induces a norm topology on $L(e)$ with $\|e\|_e = 1$. We call it the $\|\cdot\|_e$-topology. Actually, $[-e, e]$ is the closed unit ball on $L(e)$ while $B^e_1 = \{x \in L(e) : \|x\|_e < 1\}$ is the open unit ball on $L(e)$. In general, for $\epsilon > 0$, $\epsilon[-e, e]$ and $B^e_\epsilon = \{x \in L(e) : \|x\|_e < \epsilon\}$ are respectively the closed and open balls of center 0 and radius ϵ on $L(e)$. We note that $[-e, e]$ is τ-bounded by the properties of τ. Consequently, there exists $\mathcal{C} \in \mathcal{V}_\tau(0)$ such that $B^e(0, 1) \subset \mathcal{C} \cap L(e)$. In turn, let $L(e)^*$ denote the $\|\cdot\|_e$-dual of $L(e)$. The relativization of the topologies τ and σ to $L(e)$ are weaker than the $\|\cdot\|_e$-topology. Furthermore, it is straightforward to check that the relativization of σ to $L(e)$ is a Hausdorff and locally convex topology such that all order intervals in $L(e)$ are bounded.

Let $L(e)_+ = L_+ \cap L(e)$. Clearly $L(e)_+$ is $\|\cdot\|_e$-closed in $L(e)$ and has a non-empty $\|\cdot\|_e$-interior which contains e. Furthermore, $\omega \in L(e)_+$ since ω is comparable to e. Let $Y_j^e : L(e)^{|I|+|J|} \twoheadrightarrow 2^{L(e)}$ be the restricted production correspondence such that for all $z \in L(e)^{|I|+|J|}$, $Y_j^e(z) = Y_j(z) \cap L(e)$. Since we are looking for an equilibrium in the economy $\mathcal{E}$ when the commodity space is $L(e)$, we need the definition of the tangent cone of [8] in $L(e)$

$$\hat{\mathcal{T}}_{Y_j^e(z)}(y_j) = \bigcap_{\rho>0} \hat{\mathcal{T}}^{\rho}_{Y_j^e(z)}(y_j)$$

where

$$\hat{\mathcal{T}}^{\rho}_{Y_j^e(z)}(y_j) := \left\{ \nu \in L(e) \;\middle|\; \begin{array}{c} \exists\ \eta > 0 \text{ such that } \forall\ r > 0, \\ \exists\ \ U \in \mathcal{V}_\sigma(0) \text{ and } \varepsilon > 0\ \mid \\ \forall\ \ z' \in \left(\{z\} + \left(B^e_\rho \cap U\right)^{|I|+|J|}\right), \\ \forall\ \ y'_j \in \left(\{y_j\} + B^e_\rho \cap U\right) \cap Y_j(z') \\ \text{and } \forall\ \ t \in (0,\varepsilon),\ \exists\ \xi \in r[-e,\ e] \text{ such that} \\ y'_j + t\left(\nu + \eta(y_j - y'_j) + \xi\right) \in Y_j(z') \cap L(e) \end{array} \right\}$$

Hence, the marginal pricing rule is the normal cone in $L(e)$

$$\hat{\mathcal{N}}_{Y_j^e(z)}(y_j) = \left[\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)\}\right]^\circ = \left\{p \in L(e)^* : p \cdot \nu \le 0\ \forall\ \nu \in \hat{\mathcal{T}}_{Y_j^e(z)}(y_j)\right\}.$$

We remark that $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$ is not the adaptation of $\hat{\mathscr{T}}_{Y_j(z)}(y_j)$ to $L(e)$. The former considers open balls $B^e(0,\rho)$ in $L(e)$ while the latter would take the neighborhoods $\mathcal{C} \cap L(e)$. Consequently, $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$ is equivalent to the definition of [4] with externalities (compare with Proposition 5.3 below). The following lemma provides important inclusions that we shall exploit later.

Lemma 4.1 *For all* $(z, y_j) \in L(e)^{|I|+|J|} \times L(e)$*,* *1.* $\mathscr{T}_{Y_j(z)}(y_j) \cap L(e) \subset \hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$

2. $(-\Gamma_j) \cap L(e) \subset \hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$

We consider an economy restricted to the commodity space $L(e)$ and its equilibria. Let $X_i^e : L(e)^{|I|+|J|} \twoheadrightarrow 2^{L(e)_+}$ be the restricted consumption correspondence such that for all $z \in L(e)^{|I|+|J|}$, $X_i^e(z) = X_i(z) \cap L(e)_+$. We suitably restrict the preference relation $\succeq_{i,z}$ to $X_i^e(z)$ by $\succeq_{i,z}^e$ for all $z \in L(e)^{|I|+|J|}$. We remark that $\partial Y_j^e(z) \subset \partial Y_j(z) \cap L(e)$, whence $Z^e \subset Z$ and then $A^e(\omega') = A(\omega') \cap Z^e \subset A(\omega')$. The revenue functions (r_i) are the same and

$$PE^e = \{(p, z) \in L(e)^* \times Z^e : p \in \cap_{j \in J} \hat{\mathcal{N}}_{Y_j^e(z)}(y_j)\}$$

is the production equilibria set. Hence, the economy $\mathcal{E}_{|L(e)}$ is fully described by $\big((X_i^e, \succeq_{i,z}^e, r_i)_{i \in I}, (Y_j^e)_{j \in J}, \omega\big)$.

Theorem 4.1 in [8] implies that $\mathcal{E}_{|L(e)}$ has an equilibrium $((\bar{x}_i)_{i \in I}, (\bar{y}_j)_{j \in J}, \bar{p}) \in L(e)^{|I|+|J|} \times L(e)_+^*$ since all assumptions hold. The only condition for which we need to provide a specific proof is (SA)

Lemma 4.2 *The economy $\mathcal{E}_{|L(e)}$ satisfies the Survival Assumption* (SA)*.*

5 Existence of equilibria

We are now ready to state our main result

Theorem 5.1 *Let $\mathcal{E}$ be an economy. Under Assumptions* (C)*,* (P)*,* (B) *and* (SA) *there exists a vector $(\bar{z}, \bar{\pi})$ in $Z \times \hat{\mathscr{N}}_{Y_j(\bar{z})}(\bar{y}_j) \setminus \{0\}$ which is a marginal pricing equilibrium of $\mathcal{E}$.*

Remark The proof of Theorem 5.1 provides an "additional information" on the equilibrium price, namely, $\bar{\pi}_{|L(e)} = \left[\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)\}\right]^\circ$. This is useful to compare with previous existence results.

5.1 The marginal pricing equilibria in special cases

The comparison of Theorem 5.1 with previous results in the literature is not obvious. Then, in order to perceive the scope of our existence result we analyze the marginal pricing rule in three particular circumstances that are commonly assumed in economic theory, namely, when technologies are convex, i.e., the competitive case; when the cone L_+ has strictly positive vectors and when technologies are smooth, i.e., production sets may be described by differentiable transformation functions. The analysis will show that Theorem 5.1 encompasses competitive existence results as Theorems 3.1 and 2.7 of [12] and [29] respectively (Proposition 5.2), non-competitive existence results as Theorem 5.1 in [8] (Proposition 5.3) and theorems on existence of equilibria under smooth technologies (Proposition 5.6).

Convex technologies and competitive equilibria

Competitive equilibria is related with a production sector whose technologies are convex and producers' behaviour is profit maximization. By comparing our properness condition with that of [12] and [29] we note that it is necessary to impose additional requirements on the structure of the production sets when $Y_j(z)$ is convex for every $z \in Z$.

Upper Order Bounded (UOB) For all $z \in Z$ and for all $y_j \in Y_j(z) \cap L(e)$, there exists a radial set at y_j, R_{y_j}, such that

$$Y_j(z) \cap R_{y_j} \subset \tau - \mathrm{cl}\,[Y_j(z) \cap L(e) - \Gamma_j - L_+] \cap \tau - \mathrm{cl}\,(L(e) - L_+) \qquad (4)$$

We claim that the above assumption means that for every $z \in Z$ and for every $y'_j \in Y_j(z)$, there exists ξ as close as we want to y'_j such that it is bounded from above (or *upper order bounded*) by a vector in $L(e)$: let $y'_j \in Y_j(z)$ then there exists $\lambda_{y_j} \leq 1$ such that $(1-\lambda)y_j + \lambda y'_j \in Y_j(z) \cap R_{y_j}$ for every $\lambda \in (0, \lambda_{y_j})$ since $Y_j(z) \cap R_{y_j}$ is convex. Let $W_j \in \mathcal{V}_\tau(0)$. By the inclusion in (4), there exists $u \in W_j$ such that $(1-\lambda)y_j + \lambda y'_j + \lambda u \in L(e) - L_+$. This implies that $y'_j + u$ belongs to $L(e) - L_+$ and the claimed is proved since $\xi := y'_j + u$.

Notice that (4) is close to Condition (2.4) of [12]; we recall it under our notation: Let $z \in Z$, for $y_j \in Y_j(z) \cap L(e)$ there exists a τ-open convex set F, there is a lattice $K_{y_j} \subset L(e)$ verifying $K_{y_j} - L(e)_+ \subset K_{y_j}$ and some subset R_{y_j} of L, radial at y_j, such that $y_j \in \tau - \mathrm{cl}\,(F) \cap K_{y_j}$ and

$$Y_j(z) \cap R_{y_j} \subset \tau - \mathrm{cl}\,(F) \cap \left(K_{y_j} - L_+\right) \tag{5}$$

In (4), we could take K_{y_j} as in (5) but for simplicity we left $L(e) = K_{y_j}$. In the way of comparison, we are taking $F = Y_j(z) \cap L(e) - \Gamma_j - L_+$ and we slightly weakened (5) by considering the τ-closure of $L(e) - L_+$ instead of $L(e) - L_+$.

On the other hand, (4) is also related to the M-properness condition of Tourky ([29]) that we transcript with our notation: $Y_j(z)$ is M-proper at $y_j \in Y_j(z)$ if there are convex sets $\hat{Y}_j(z)$ and $\hat{K}_{y_j}$ such that (i) $\hat{Y}_j(z) \cap \left(\hat{K}_{y_j} - L_+\right) = Y_j(z)$, (ii) $y_j - e \in \tau - \mathrm{int}\hat{Y}_j(z)$[7], (iii) $0 \in \hat{K}_{y_j}$ and (iv) if y_j, y'_j belong to $\hat{K}_{y_j}$ then $y_j \vee y'_j$ belongs to $\hat{K}_{y_j}$.

It is clear that both $Y_j(z) \cap L(e) - \Gamma_j - L_+$ and $L(e)$ satisfy the requirements of $\hat{Y}_j(z)$ and $\hat{K}_{y_j}$ respectively. Then, the two conditions are similar but $L(e)$ is contained in $\hat{K}_{y_j}$, which is stronger. On the contrary, (4) only requires an inclusion on $Y_j(z) \cap R_{y_j}$ which is smaller than on $Y_j(z)$.

[7] Actually, Tourky takes ω instead of e.

Proposition 5.2 *Let $z \in Z$ and $y_j \in \partial Y_j(z) \cap L(e)$. Assume that firm j has a convex technology and satisfies Assumptions (P), then:*

1. $\{\pi \in L^* : \pi \cdot y_j \geq \pi \cdot y'_j \ \forall \ y'_j \in Y_j(z)\} \subset \hat{\mathscr{N}}_{Y_j(z)}(y_j)$

2. *Furthermore, if Assumption UOB holds, then for every π_j in $\hat{\mathscr{N}}_{Y_j(z)}(y_j)$ such that $\sup\{\pi_j \cdot (-\gamma_j) : \gamma_j \in \Gamma_j\} \leq 0$, it follows that $\pi_j \cdot y'_j \leq \pi_j \cdot y_j$ for all $y'_j \in Y_j(z)$. Furthermore, for $\pi'_j \geq \pi_j$ such that $\pi'_{j|L(e)} = \pi_{j|L(e)}$, $\pi'_j.y'_j \leq \pi'_j.y_j$ for all $y'_j \in Y_j(z)$.*

The above proposition means, on the one hand, that profit maximization behavior is part of the pricing rule when technologies are convex and, on the other hand, it is the only choice if production sets satisfy condition UOB. As a consequence, we stress that the marginal pricing equilibria of the Theorem 5.1 is a competitive equilibria if production sets are convex and condition UOB holds since the equilibrium price $\bar{\pi}$ satisfies the conditions of Item 2. of the above proposition (See Section 6).

Spaces with quasi-interior points in L_+

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We provide now some properties of the marginal pricing rule when e is a quasi-interior point of L_+. The following result shows that all positive continuous linear functionals in $\hat{\mathscr{N}}_{Y_j(z)}(y_j)$, whose restriction to $L(e)$ belong to the polar of $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$ is also a subset of $\mathscr{T}_{Y_j(z)}(y_j)$. This fact is important because it implies that equilibrium prices here coincide with equilibrium prices of [8] when L is a Banach lattice.

Lemma 5.3 *Let $(y_j, z) \in L(e) \times L(e)^{|I|+|J|}$ and $\pi \in \hat{\mathscr{N}}_{Y_j(z)}(y_j)$. Under Assumption (P), if e is a quasi-interior point of L_+ and $\pi_{|L(e)} \in [\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)]^\circ$, then $\pi \in [\mathscr{T}_{Y_j(z)}(y_j)]^\circ$.*

Since it is always the case that $[\mathscr{T}_{Y_j(z)}(y_j)]^\circ \subset \hat{\mathscr{N}}_{Y_j(z)}(y_j)$, the above lemma shows the converse inclusion of the latter. Hence, a marginal pricing equilibrium with the cone $[\mathscr{T}_{Y_j(z)}(y_j)]^\circ$ is a marginal pricing equilibrium with the cone $\hat{\mathscr{N}}_{Y_j(z)}(y_j)$ provided that $\pi_{|L(e)} \in [\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)]^\circ$. Under this remark we get the following immediate result:

Proposition 5.4 *Let $(L, \|\cdot\|)$ be a Banach lattice such that e is quasi-interior point of L_+. The marginal pricing equilibrium given by Theorem 5.1 is a marginal pricing equilibrium in the sense of* [8][8]. *If* $\mathrm{int}L_+ \neq \emptyset$ *and e belongs to* $\mathrm{int}L_+$*, then $\hat{\mathscr{N}}_{Y_j(z)}(y_j)$ is the normal cone of* [4] *with externalities and if $L = \mathbb{R}^l$, $\hat{\mathscr{N}}_{Y_j(z)}(y_j)$ is the marginal pricing rule of* [7].

We close this section by showing that Condition UOB holds when e is a quasi-interior point of L_+ and technologies are convex. Hence, Assumption UOB is only necessary when the commodity space is a vector lattice what agrees with the literature.

Lemma 5.5 *Let (L, τ) be a topological vector lattice such that e is a quasi-interior point of L_+. If Y_j is a convex-valued production correspondence satisfying σ-locally τ-uniformly properness relative to $L(e)$ at $y_j \in Y_j(z)$, then condition* UOB *holds.*

Smooth technologies

Smooth technologies are a common assumption in economics. Indeed, the first results on equilibria with increasing returns are in this framework (see [21] and [2]) and even those with infinitely many commodities ([28]). It means that the production set is defined by an inequality involving a differentiable transformation function, that is, for all $z \in Z$ the technology is $Y_j(z) = \{\zeta_j \in$

[8] We remark that in the current paper, τ is the strong topology unlike in [8] where τ denotes the weak one. Proposition 5.4 takes the particular case where τ is the norm topology.

$L : f_j(\zeta_j, z) \leq 0\}$ where $f_j : L \times L^{|I|+|J|} \to \mathbb{R}$ is a differentiable mapping. Let us consider the following assumption:

Assumption SB (Smooth boundary)

1. f_j is continuous for the $\sigma \times \prod_{L^{|I|+|J|}} \sigma$-topology on $L \times L^{|I|+|J|}$.

2. $f_j(\cdot, z)$ is Gateaux differentiable on L.

3. $\nabla_1 f_j(\zeta_j, z)$ belongs to $L^*_+ \setminus \{0\}$ if $\zeta_j \in \partial Y_j(z)$ where $\nabla_1 f_j(\zeta_j, z)$ is the gradient of f_j with respect to ζ_j.

4. $\nabla_1 f_j(\zeta_j, z) \cdot e > 0$ for every $\zeta_j \in \partial Y_j(z)$.

5. Let H be a $\sigma(L, L^*)$-equicontinuous subset of L^*[9]. For all $(\zeta_j, z) \in \partial Y_j(z) \times L^{|I|+|J|}$ and for all $\beta > 0$ there exists $U' \in \mathcal{V}_\sigma(0)$ such that $\nabla_1 f_j(\zeta'_j, z') - \nabla_1 f_j(\zeta_j, z) \in \beta H$ for all $(\zeta'_j, z') \in (\{\zeta_j\} + U') \times \big(\{z\} + U'^{|I|+|J|}\big)$.

Proposition 5.6 *If the production set $Y_j(\cdot)$ is described by a transformation function f_j satisfying Assumption* SB*, then*

$$\hat{\mathscr{N}}_{Y_j(z)}(\zeta_j) \subset \{\nu \in \ \tau - \mathrm{cl}L(e) : \nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0\}^\circ$$

This means that for every $\pi_j \in \hat{\mathscr{N}}_{Y_j(z)}(y_j)$ there exists $\lambda_j > 0$ such that the restriction of π_j to $\tau - \mathrm{cl}L(e)$ equals the restriction of $\lambda_j \nabla_1 f_j(\zeta_j, z)$ to $\tau - \mathrm{cl}L(e)$. Of course, when e is a quasi-interior point of L_+, L equals $\tau - \mathrm{cl}L(e)$ so one can say that $\hat{\mathscr{N}}_{Y_j(z)}(\zeta_j)$ equals $\{\lambda \nabla_1 f_j(\zeta_j, z) : \lambda > 0\}^\circ$ which is the Clarke´s normal cone by Lemma 3.1 in [8]

[9] That is, for every real number $a > 0$ there exists $U \in \mathcal{V}_\sigma(0)$ such that $g(U) \subset (-a, a)$ for all $g \in H$.

6 Proof of the Existence Theorem

Let $((\bar{x}_i),(\bar{y}_j),\bar{p})$ be an equilibrium of $\mathcal{E}_{|L(e)}$, which exists from the result in Section 4. We shall prove that there exists a positive linear and continuous extension of $\bar{p}$ to the whole of L, $\bar{\pi}$, such that $((\bar{x}_i),(\bar{y}_j),\bar{\pi})$ is a marginal pricing equilibrium of the economy $\mathcal{E}$. Throughout the proof, we make use of the following Lemma whose first part comes from [24] and the second one is due to [12].

Lemma 6.1 (Lemma 2, [24]; Lemma 2.1, [12]) *Let K be a linear subspace of L. Let A and B be convex subsets of L with A τ-open, $B \subset K$ and such that $A \cap B \neq \emptyset$. Let f be any linear functional on K satisfying for some $x \in \tau - \mathrm{cl}(A) \cap B$, $f \cdot x \leq f \cdot x'$, for all $x' \in A \cap B$. Then,*

1. *There exist f_1 and f_2 with $f_1 \in L^*$ and f_2 a linear functional on L such that $f = f_{1|K} + f_{2|K}$ and*

$$f_1 \cdot x \leq f_1 \cdot x', \forall x' \in A \text{ and } f_2 \cdot x \leq f_2 \cdot x', \forall\ x' \in B$$

2. *If $K_+ = K \cap L_+$ and $B + K_+ \subset B$ then $f_{2|K} \geq 0$, $f_{1|K} \leq f$ and $f \cdot (x' - x) = f_1 \cdot (x' - x)$ for each $x' \leq x$, $x' \in B$.*

We now prove that $\bar{\pi}$ is marginal pricing equilibria vector through the following claims.

Claim 6.2 *For each $j \in J$, there exists $\pi_j \in L_+^*$ such that $\pi_{j|L(e)} = \bar{p}$.*

Proof. From Lemma 4.1. (2), $(-\Gamma_j) \cap L(e) \subset \hat{\mathcal{T}}_{Y_j^e(\bar{z})}(\bar{y}_j)$. Furthermore, $(-\Gamma_j) \cap L(e) \neq \emptyset$ since $-e$ belongs to the intersection, for all $\zeta \in (-\Gamma_j) \cap L(e)$, $-\bar{p} \cdot \zeta \geq 0$ since $\bar{p} \in \hat{\mathcal{N}}_{Y_j^e(z)}(\bar{y}_j)$ and clearly 0 belongs to $L(e) \cap \tau - \mathrm{cl}\,(-\Gamma_j)$. By Lemma 6.1. (1)[10], there exist $\hat{\pi}_j$ and $\hat{\pi}'_j$ such that $\hat{\pi}_j$ is τ-continuous on L, $\bar{p} = \hat{\pi}_{j|L(e)} +$

[10] We are taking $A = -\Gamma_j$, $B = K = L(e)$ and $x = 0$.

$\hat{\pi}'_{j|L(e)}$, $-\hat{\pi}_j \cdot \zeta \geq 0$ for all $\zeta \in -\Gamma_j$ and $-\hat{\pi}'_j \cdot x \geq 0$ for all $x \in L(e)$. From the last assertion, we deduce that $\hat{\pi}'_{j|L(e)} = 0$ since $L(e)$ is a linear subspace, so $\bar{p} = \hat{\pi}_{j|L(e)}$.

We note that $\pi_j := \hat{\pi}_j^+ = \hat{\pi}_j \vee\ 0$ is continuous since L^* is a sub-lattice of $L^\sim$ by assumption. Since $\bar{p} \in L(e)^*_+$ we prove that $\pi_{j|L(e)} = \bar{p}$. Indeed, let $x \in L(e)_+$, by the Riesz-Kantorovich formula, $\pi_j \cdot x = \sup\left\{\pi_j \cdot \widetilde{x} : 0 \leq \widetilde{x} \leq x\right\} = \sup\left\{\bar{p} \cdot \widetilde{x} : 0 \leq \widetilde{x} \leq x\right\}$ and since $0 \leq \tilde{x} \leq x$ implies $\tilde{x} \in L(e)_+$, $\bar{p} \in L(e)^*_+$ implies that $\pi_j \cdot x = \bar{p} \cdot x$. Since for all $\zeta \in L(e)$, one has $\zeta = \zeta^+ - \zeta^-$ and both belong to $L(e)_+$, one deduces that $\bar{p} \cdot \zeta = \pi_j \cdot \zeta$. So π_j is a continuous extension of $\bar{p}$ belonging to L^*_+.

■

Claim 6.3 *For all $i \in I$, there exists $\pi_i \in L^*$ such that $\pi_{i|L(e)} \leq \bar{p}$ and $\pi_i \cdot \bar{x}_i \leq \pi_i \cdot x'_i$ for all $x'_i \in \tau - \mathrm{cl}\left(\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i\right)$ and $\pi_i \cdot (\bar{x}_i - \zeta_i) = \bar{p} \cdot (\bar{x}_i - \zeta_i)$ for all $\zeta_i \in K_{\bar{x}_i}$ such that $\zeta_i \leq \bar{x}_i$*

Proof.

By equilibrium conditions in $\mathcal{E}_{|L(e)}$ and (1) we deduce $\bar{p} \cdot \bar{x}_i < \bar{p} \cdot x_i$ for all $x_i \in (\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i) \cap K_{\bar{x}_i} \cap R_{\bar{x}_i}$. Since $R_{\bar{x}_i}$ is radial, this implies that $\bar{p} \cdot \bar{x}_i < \bar{p} \cdot x_i$ for all $x_i \in (\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i) \cap K_{\bar{x}_i}$. Indeed, take $x_i \in (\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i) \cap K_{\bar{x}_i}$. There exists $\lambda_{\bar{x}_i} > 0$ such that $(1-\lambda)\bar{x}_i + \lambda x_i$ belongs to $K_{\bar{x}_i} \cap R_{\bar{x}_i}$ for all $0 \leq \lambda \leq \lambda_{\bar{x}_i}$ since $K_{\bar{x}_i}$ is convex[11]. Since $x_i = x''_i + \xi$ for some $\xi \in \Gamma_i$ and $x''_i \in \{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\}$, it follows that $(1-\lambda)\bar{x}_i + \lambda x_i = (1-\lambda)\bar{x}_i + \lambda x''_i + \lambda\xi \in \{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i$ by Assumption C(3). Hence, $(1-\lambda)\bar{x}_i + \lambda x_i$ belongs to $(\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i) \cap K_{\bar{x}_i} \cap R_{\bar{x}_i}$ and by the above result, $\bar{p} \cdot [(1-\lambda)\bar{x}_i + \lambda x_i] > \bar{p} \cdot \bar{x}_i$ so $\bar{p} \cdot x_i > \bar{p} \cdot \bar{x}_i$.

[11] Notice that $K_{\bar{x}_i}$ is convex since it is a sublattice of $L(e)$ and $K_{\bar{x}_i} + L(e)_+ \subset K_{\bar{x}_i}$. Indeed, let ξ and ξ' belong to $K_{\bar{x}_i}$, hence in $L(e)$. For every $t \in (0,1)$, $t\xi + (1-t)\xi' \geq \xi \wedge \xi'$ and thus $\xi \wedge \xi' + [t\xi + (1-t)\xi' - \xi \wedge \xi]$ belongs to $K_{\bar{x}_i} + L(e)_+$ which implies that $t\xi + (1-t)\xi'$ belongs to $K_{\bar{x}_i}$.

By Lemma 6.1 (2), we deduce the existence of $\pi_i \in L^*$ such that $\pi_{i|L(e)} \leq \bar{p}$ and $\pi_i \cdot \bar{x}_i \leq \pi_i \cdot x_i$ for all $x_i \in \tau - \mathrm{cl}\left(\{x_i' \in X_i(z) : x_i' \succ_{i,z} \bar{x}_i\} + \Gamma_i\right)$. Furthermore, $\pi_i \cdot (\bar{x}_i - \zeta_i) = \bar{p} \cdot (\bar{x}_i - \zeta_i)$ for all $\zeta_i \in K_{\bar{x}_i}$ such that $\zeta_i \leq \bar{x}_i$.

■

Claim 6.4 *Let $\bar{\pi} = (\vee_{i\in I}\pi_i) \vee (\vee_j \pi_j)$. Then $\bar{\pi} \in L^*_+$ and $\bar{\pi}_{|L(e)} = \bar{p}$.*

Proof. $\bar{\pi} \in L^*$ since L^* is a sublattice of $L^\sim$. Furthermore, $\bar{\pi} > 0$ since $\pi_j > 0$ for all j. Let ζ be any vector in $L(e)$. Then ζ^+ and ζ^- belong to $L(e)$ since this is an ideal. On the one hand, $\bar{\pi} \cdot \zeta_j^+ \geq \pi_j \cdot \zeta_j^+ = \bar{p} \cdot \zeta_j^+$. On the other hand, from the Riesz-Kantorovich formula,

$$\bar{\pi} \cdot \zeta^+ = \sup\left\{\sum_{i\in I} \pi_i \cdot \zeta_i^+ + \sum_{j\in I} \pi_j \cdot \zeta_j^+ : \sum_{i\in I} \zeta_i^+ + \sum_{j\in I} \zeta_j^+ = \zeta^+\right\} \leq \bar{p} \cdot \zeta^+$$

Hence, $\bar{\pi} \cdot \zeta_j^+ = \bar{p} \cdot \zeta_j^+$. Analogously, $\bar{\pi} \cdot \zeta^- = \bar{p} \cdot \zeta^-$. Thus, $\bar{\pi} \cdot \zeta = \bar{p} \cdot \zeta$ and the proof is complete.

■

Claim 6.5 $\bar{\pi} \in \bigcap_{j\in J} \hat{\mathcal{N}}_{Y_j(\bar{z})}(\bar{y}_j)$.

Proof. The proof is a direct consequence of Lemma 4.1 1. and Claim 6.4 together with the fact that $\bar{p} \in \hat{\mathcal{N}}_{Y_j^e(\bar{z})}(\bar{y}_j) = \left[\hat{\mathcal{T}}_{Y_j^e(\bar{z})}(\bar{y}_j)\right]^\circ$ ■

Claim 6.6 $\bar{\pi} \cdot \bar{x}_i = r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j))$ *and if* $x_i \succ_{i,\bar{z}} \bar{x}_i$ *then* $\bar{\pi} \cdot x_i > r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j))$

Proof. By the Survival Assumption, $\bar{\pi} \cdot \left(\sum_{j\in J} \bar{y}_j + \omega\right) > 0$. From Assumption C(5), $\sum_{i\in I} r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j)) > 0$ and $r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j)) > 0$ for all $i \in I$. Furthermore, $\bar{\pi} \cdot \bar{x}_i = r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j))$ follows since $((\bar{x}_i)_{i\in I}, (\bar{y}_j)_{j\in J}, \bar{p})$ is an equilibrium in $\mathcal{E}_{|L(e)}$ and $\bar{\pi}_{|L(e)} = \bar{p}$.

Let $x_i \in X_i(\bar{z})$ such that $x_i \succ_{i,\bar{z}} \bar{x}_i$. Since $R_{\bar{x}_i}$ is radial at $\bar{x}_i$ and $X_i(\bar{z})$ is convex, there exists $\bar{\lambda} \in (0,1)$ such that for $\lambda \in (0,\bar{\lambda}]$, $\widetilde{x}_i = (1-\lambda)\bar{x}_i + \lambda x_i \in X_i(\bar{z}) \cap R_{\bar{x}_i}$ and by Assumption C(3), $\widetilde{x}_i \succ_{i,\bar{z}} \bar{x}_i$. Due to E-properness relative to $L(e)$ (C (6.1)), $\widetilde{x}_i \in K_{\bar{x}_i} + L_+$, that is, there exists $\zeta_i \in K_{\bar{x}_i}$ such that $\zeta_i \leq \widetilde{x}_i$. Let $\zeta'_i = \bar{x}_i \wedge \zeta_i$. $\zeta'_i \in K_{\bar{x}_i}$ since $K_{\bar{x}_i}$ is a sublattice and $\widetilde{x}_i \in \tau - \operatorname{cl}\left(\{x'_i \in X_i(z) : x'_i \succ_{i,z} \bar{x}_i\} + \Gamma_i\right)$ by (2) of E-properness. It follows by Claim 6.3 that $\pi_i \cdot (\bar{x}_i - \zeta'_i) \leq \pi_i \cdot (\widetilde{x}_i - \zeta'_i)$ and $\pi_i \cdot (\bar{x}_i - \zeta'_i) = \bar{p} \cdot (x'_i - \zeta'_i)$. By Claim 6.4, $\bar{\pi} \cdot (\bar{x}_i - \zeta'_i) = \bar{p} \cdot (\bar{x}_i - \zeta'_i)$. Since $\bar{\pi} \geq \pi_i$, we get

$$\begin{aligned}\bar{\pi} \cdot \left(\widetilde{x}_i - \bar{x}_i\right) + \bar{p} \cdot (\bar{x}_i - \zeta'_i) = \bar{\pi} \cdot \left(\widetilde{x}_i - \bar{x}_i\right) + \bar{\pi} \cdot (\bar{x}_i - \zeta'_i) = \bar{\pi} \cdot \left(\widetilde{x}_i - \zeta'_i\right) \\ \geq \pi_i \cdot \left(\widetilde{x}_i - \zeta'_i\right) \geq \pi_i \cdot (\bar{x}_i - \zeta'_i) = \bar{p} \cdot (\bar{x}_i - \zeta'_i)\end{aligned}$$

Consequently, $\bar{\pi} \cdot \widetilde{x}_i \geq \bar{p} \cdot \bar{x}_i \bar{\pi} \cdot \bar{x}_i$ whence, $\bar{\pi} \cdot x_i \geq \bar{\pi} \cdot \bar{x}_i$.

Let us suppose that $\bar{\pi} \cdot x_i = \bar{\pi} \cdot \bar{x}_i$. Since $\bar{\pi} \cdot \bar{x}_i > 0$, $\bar{\pi} \cdot (\gamma x_i) < \bar{\pi} \cdot \bar{x}_i$ for all $\gamma \in (0,1)$. Notice that $(\bar{x}_i, x_i, \bar{z}) \notin G_i$ and γx_i belongs to $X_i(z)$ because this set is convex and 0 belongs to it. Hence, Assumption C(4) implies that for γ close enough to 1, $(\bar{x}_i, \gamma x_i, \bar{z}) \notin G_i$ and thus, $\gamma x_i \succ_{i,\bar{z}} \bar{x}_i$. By the previous result, $\bar{\pi} \cdot (\gamma x_i) \geq \bar{\pi} \cdot \bar{x}_i$ which contradicts the above converse inequality. Consequently, $\bar{\pi} \cdot x_i > r_i(\bar{\pi} \cdot \omega, (\bar{\pi} \cdot \bar{y}_j))$ and the proof is complete.

■

Proofs

Proposition 3.1

It suffices to show that $\mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$ is a cone since $L(e)$ is a cone. Let $\mathcal{C} \in \mathcal{V}_\tau(0)$, $\kappa > 0$ and $\nu \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j) \subset \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$. Then there exists $\eta > 0$ such that for all $r > 0$ there exists $U \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$ such that for all $z' \in$

$\left(\{z\} + (\mathcal{C} \cap U)^{|I|+|J|}\right)$, for all $y'_j \in (\{y_j\} + \mathcal{C} \cap U) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$ there exists $\xi \in r[-e, e]$ such that $y'_j + t(\nu + \eta(y_j - y'_j) + \xi) \in Y_j(z')$. Let U and ε be the parameter and the open neighborhood associated for ν to $\frac{r}{\kappa}$. Hence, for all z', for all y'_j and for all $t \in \left(0, \frac{\varepsilon}{\kappa}\right)$, there exists $\xi \in \frac{r}{\kappa}[-e, e]$ such that $y'_j + \kappa t(\nu + \eta(y_j - y'_j) + \xi) = y'_j + t(\kappa\nu + \kappa\eta(y_j - y'_j) + \kappa\xi) \in Y_j(z')$. Then, $\kappa\nu \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$.

Lemma 3.3

1. Let $z \in Z$ and $y_j \in \partial Y_j(z)$. Let us assume that Condition P(4) is satisfied. Let $\mathcal{C} \in \mathcal{V}_\tau(0)$ and $\eta = 1$. Let $r > 0$ and choose $\delta > 0$ smaller than r. Let $\hat{U}$ be the σ-open set and $\varepsilon > 0$ as given by Assumption P(4). Then, for all $z' \in \left(\{z\} + (\mathcal{C} \cap \hat{U})^{|I|+|J|}\right)$, for all $y'_j \in \left(\{y_j\} + \mathcal{C} \cap \hat{U}\right) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$, the vector $t(y_j - \delta e) + (1-t)y'_j = y'_j + t(y_j - y'_j - \delta e) \in Y_j(z')$, which means $0 \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$. Since this is true for all neighborhood $\mathcal{C}$, we conclude that $0 \in \mathscr{T}_{Y_j(z)}(y_j)$.

   For the converse, let $z \in Z$, $y_j \in \partial Y_j(z)$ and let us assume that $0 \in \mathscr{T}_{Y_j(z)}(y_j)$. Let $\mathcal{C} \in \mathcal{V}_\tau(0)$ and $\delta > 0$. So, using the free-disposal Assumption on $Y_j(z')$ and the fact that $0 \in \mathscr{T}_{Y_j(z)}(y_j)$ (Lemma 3.3 (1)) there exists $\eta > 0$ and for $r = \delta\eta > 0$, there exists $U \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$, such that for all $z' \in \left(\{z\} + (\mathcal{C} \cap U)^{|I|+|J|}\right)$, for all $y'_j \in (\{y\} + \mathcal{C} \cap U) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$

$$y'_j + t(\eta(y_j - y'_j) - re) = (1 - t\eta)y'_j + t\eta(y_j - \delta e) \in Y_j(z')$$

   So Assumption P(4) is satisfied with $\hat{U} = U$ and $\varepsilon' = \eta\varepsilon$.

2. Let $z \in L^{|I|+|J|}$ and $y_j \in \partial Y_j(z)$. Since L is finite dimensional, the two topologies τ and σ coincide with the standard norm topology and L_+

has a nonempty interior. Let $\mathcal{C}$ be a neighborhood of 0 and $\delta > 0$. Let us consider the neighborhood of 0, $\hat{U} = \{-\delta e\} + \mathrm{int}L_+$. Thus, for all $z' \in \left(\{z\} + (\mathcal{C} \cap \hat{U})^{|I|+|J|}\right)$, for all $y'_j \in Y_j(z') \cap \left(\{y_j\} + (\mathcal{C} \cap \hat{U})\right)$ and for all $t \in (0,1)$, the vector $t(y_j - \delta e) + (1-t)y'_j = y'_j - t(\, y'_j - y_j + \delta e)$ belongs to $Y_j(z')$ from the free-disposal assumption since $y'_j - y_j + \delta e$ belongs to L_+.

Lemma 4.1

1. Let $\nu \in \mathscr{T}_{Y_j(z)}(y_j) \cap L(e)$ then $\nu \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j) \cap L(e)$ for all $\mathcal{C} \in \mathcal{V}_\tau(0)$. Let $\rho > 0$. There exists some $\mathcal{C}' \in \mathcal{V}_\tau(0)$ such that $B^e_\rho \subset \mathcal{C}'$ because order intervals are τ-bounded. Hence, since $\nu \in \mathscr{T}^{\mathcal{C}'}_{Y_j(z)}(y_j)$ there exists $\eta > 0$ such that for all $r > 0$ there exists $U \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$ such that for all $z' \in \left(\{z\} + (B^e_\rho \cap U^e)^{|I|+|J|}\right) \subset \left(\{z\} + (\mathcal{C}' \cap U)^{|I|+|J|}\right)$, for all $y'_j \in (\{y_j\} + B^e \cap U^e) \cap Y_j(z') \subset (\{y_j\} + \mathcal{C}' \cap U) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$, there exists $\xi \in r[-e,\ e]$ such that $y'_j + t(\nu + \eta(y_j - y'_j) + \xi) \in Y_j(z') \cap L(e)$. Hence, $\nu \in \hat{\mathcal{T}}^{\rho}_{Y^e_j(z)}(y_j)$ and since it is true for all $\rho > 0$, $\nu \in \hat{\mathcal{T}}_{Y^e_j(z)}(y_j)$.

2. By Lemma 3.3 (1), $0 \in \mathscr{T}_{Y_j(z)}(y_j)$ and then in $\hat{\mathcal{T}}_{Y^e_j(z)}(y_j) = \bigcap_{\rho>0} \hat{\mathcal{T}}^{\rho}_{Y^e_j(z)}(y_j)$ by Lemma 4.1 (1). Let $\rho > 0$ and let $\eta > 0$ be the parameter associated to it in the definition of $\hat{\mathcal{T}}^{\rho}_{Y_j(z)}(y_j)$. Let $r > 0$ and let $U \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$ be the neighborhood and the parameter associated to r. Let $V'_j \in \mathcal{V}_\sigma(0)$ such that $V'_j + V'_j + V'_j \subset V_j$ where V_j is the neighborhood coming from the properness condition for firm j at $y_j \in Y_j(z)$.

Let $U' = U \cap V'_j$, then for all $z' \in \left(\{z\} + \left(B^e_\rho \cap U'\right)^{|I|+|J|}\right)$, for all $y'_j \in \left(\{y_j\} + B^e_\rho \cap U'\right) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$, there exists $\xi \in r[-e, e]$ such that $y'_j + t(\eta(y_j - y'_j) + \xi) \in Y_j(z') \cap L(e)$. We note that $\eta(y_j - y'_j) + \xi \in n_0[-e, e]$ for some $n_0 \in \mathbb{N}$. Since $n_0[-e, e]$ is τ-bounded, it is also σ-bounded and therefore there exists $\lambda > 0$ such that $n_0[-e, e] \subset \lambda V'_j$. Then, by choosing $\varepsilon' < \min\left\{\varepsilon, \frac{1}{\lambda}\right\}$, we have $t(\eta(y_j - y'_j) + \xi) \in V'_j$ for all $t \in (0, \varepsilon')$ since V'_j is circled.

Let $\zeta \in (-\Gamma_j) \cap L(e)$. There exists $\varepsilon'' > 0$ such that $t\zeta \in V'_j$ for all $t \in (0, \varepsilon'')$. Consequently, for all $0 < t < \varepsilon''' < \min\{\varepsilon', \varepsilon''\}$ we have, $y'_j + t(\eta(y_j - y'_j) + \xi + \zeta) \in \left(\{y_j\} + V'_j + V'_j + V'_j\right) \cap L(e) \subset \{y_j\} + V_j$. Since $y'_j + t(\eta(y_j - y'_j) + \xi) \in Y_j(z') \cap \left(\{y_j\} + V'_j + V'_j\right) \subset Y_j(z') \cap (\{y_j\} + V_j)$, we deduce by σ-locally τ-uniform properness at y_j relative to $L(e)$ that $y'_j + t(\zeta + \eta(y_j - y'_j) + \xi) \in Y_j(z') \cap L(e)$ which implies that $\zeta \in \mathcal{T}^{\rho}_{Y^e_j(z)}(y_j)$. Since this holds for all $\rho > 0$, we have the desired result.

Lemma 4.2

We want to prove that for all $(p, z, t) \in S^e \times Z^e \times \mathbb{R}_+$, if $(y_j) \in A^e(\omega + te, z)$ it follows that $p \cdot \left(\sum_{j \in J} y_j + \omega + te\right) > 0$. By Claims 6.2 and 6.5, there exists a continuous, positive, linear functional π which extends p to the whole space L and $\pi \in \cap_{j \in J} \hat{\mathscr{N}}_{Y_j(z)}(y_j)$. Hence, $(\pi, z, t) \in L^*_+ \times Z \times \mathbb{R}_+$ and $(y_j) \in A(\omega + te, z)$. Since $\sum_{j \in J} y_j + \omega + te$ belongs to $L(e)_+$ we get by Assumption SA, $0 < \pi \cdot \left(\sum_{j \in J} y_j + \omega + te\right) = p \cdot \left(\sum_{j \in J} y_j + \omega + te\right)$ and the lemma is proved.

Proposition 5.2

Let

$$PM(y_j, z) = \{\pi \in L^* : \pi \cdot y_j \geq \pi \cdot y'_j \ \forall\ y'_j \in Y_j(z)\}$$

be the profit maximization rule.

1. Let $\pi \in PM(y_j, z)$. By free disposal it follows that $\pi \geq 0$. If $\nu \in \mathscr{T}_{Y_j(z)}(y_j)$ then, for $r > 0$ there exists $\varepsilon > 0$ such that for $t \in (0, \varepsilon)$, $y_j + t(\nu - re) \in Y_j(z)$. Consequently $\pi \cdot (y_j + t(\nu - re)) \leq \pi \cdot y_j$ which implies $\pi \cdot \nu \leq 0$ since r is arbitrary. Consequently, for every $\nu' \in \mathscr{T}_{Y_j(z)}(y_j) \cap L(e) - L_+$ we get $\pi \cdot \nu' \leq 0$ whence, $PM(y_j, z) \subset \hat{\mathscr{N}}_{Y_j(z)}(y_j)$.

2. We first prove that for all $\pi \in \hat{\mathscr{N}}_{Y_j(z)}(y_j)$, $\pi \cdot y'_j \leq \pi \cdot y_j$ for all $y'_j \in Y_j(z) \cap L(e)$. Indeed, let $\zeta_j - y_j \in (Y_j(z) - \{y_j\})$. We shall prove that for all $\mathcal{C} \in \mathcal{V}_\tau(0)$, $\zeta_j - y_j \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$. Let $\mathcal{C} \in \mathcal{V}_\tau(0)$, $\eta = 1$, $r > 0$, $0 < \delta < r$, $\varepsilon = 1$. By Assumption P(3), there exists $V' \in \mathcal{V}_\sigma(0)$ such that $\zeta_j - \delta e \in Y_j(z')$ for all $z' \in \{z\} + V'^{|I|+|J|}$. Let $z' \in \left(\{z\} + (\mathcal{C} \cap V')^{|I|+|J|}\right)$. Let $U = V'$, then for $y'_j \in (\{y_j\} + \mathcal{C} \cap V') \cap Y_j(z')$ and $t \in (0,1)$, $t(\zeta_j - \delta e) + (1-t)y'_j \in Y_j(z')$ since $Y_j(z')$ is convex. But this means that $y'_j + t(\zeta_j - \delta e - y_j + (y_j - y'_j)) \in Y_j(z')$. Since $-\delta e \in r[-e,\ e]$, we deduce that $\zeta_j - y_j \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$. Since this is true for all $\mathcal{C} \in \mathcal{V}_\tau(0)$, $\zeta_j - y_j \in \bigcap_{\mathcal{C} \in \mathcal{V}_\tau(0)} \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$ and thus $(Y_j(z) - \{y_j\}) \cap L(e) \subset \bigcap_{\mathcal{C} \in \mathcal{V}_\tau(0)} \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j) \cap L(e) \subset \hat{\mathscr{T}}_{Y_j(z)}(y_j)$. Since y_j belongs to $L(e)$ this implies that for every $\pi \in \hat{\mathscr{N}}_{Y_j(z)}(y_j)$, $\pi \cdot y'_j \leq \pi \cdot y_j$ for all $y'_j \in Y_j(z) \cap L(e)$.

   We proceed now to show that for every π_j in $\hat{\mathscr{N}}_{Y_j(z)}(y_j)$ such that $\pi_j \cdot \gamma_j \geq 0$ for all $\gamma_j \in \Gamma_j$, it follows that $\pi_j \cdot y'_j \leq \pi_j \cdot y_j$ for all $y'_j \in Y_j(z)$. Let $y'_j \in Y_j(z)$. There exists $0 < \lambda_{y_j} \leq 1$ such that for every $0 \leq \lambda \leq \lambda_{y_j}$, $(1-\lambda)y_j + \lambda y'_j \in R_{y_j}$ whence in $Y_j(z) \cap R_{y_j}$ since $Y_j(z)$ is convex. Let W be a τ-neighborhood of 0. By Assumption (UOB) there exist $w \in W$, $\zeta_j \in Y_j(z) \cap L(e)$ and $\xi \in \Gamma_j$ such that $(1-\lambda)y_j + \lambda y'_j + \lambda w \leq \zeta_j - \xi$. By the above result, $\pi_j \cdot (\zeta_j - \xi) \leq \pi_j \cdot y_j$ since $\pi_j \cdot (-\xi) \leq 0$. Hence, $\pi_j \cdot \big((1-t)y_j + ty'_j + tw\big) \leq \pi_j \cdot y_j$ which implies thus $\pi_j \cdot y'_j \leq \pi_j \cdot y_j$ since W is an arbitrary neighborhood of 0. Then we proved that $\pi_j \in PM(y_j, z)$.

   We finish with (2). Let π'_j as indicated in the Proposition. Let $y'_j \in Y_j(z)$. From Assumption (UOB), for every W in $\mathcal{V}_\tau(0)$ there exists $w \in W$ such that $y'_j + w \leq \zeta$ for some $\zeta \in L(e)$. Without any loss of generality we may assume that $\zeta \geq y_j$. Hence,

   $$\pi'_j \cdot (y'_j + w - y_j) + \pi_j \cdot (y_j - \zeta) = \pi'_j \cdot (y'_j + w - y_j) + \pi'_j \cdot (y_j - \zeta) = \pi'_j \cdot (y'_j + w - \zeta) \leq \pi_j \cdot (y'_j + w - \zeta) + \leq \pi_j \cdot (y_j + w - \zeta).$$

This implies $\pi'_j.y'_j \leq \pi'_j.y_j$ since W is an arbitrary neighborhood of 0.

Lemma 5.3

Let $\pi \in L^* \setminus \{0\}$ such that $\pi_{|L(e)} \in [\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)]^\circ$and let $\nu \in \mathscr{T}_{Y_j(z)}(y_j)$. If we prove that $\mathscr{T}_{Y_j(z)}(y_j)$ is a subset of the τ-closure of $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$, then $\pi \cdot \nu \leq 0$ which implies that $\pi \in \left[\mathscr{T}_{Y_j(z)}(y_j)\right]^\circ$ and the proof is complete. We prove now the above assertion by showing that for all $\beta > 0$ and for all $W \in \mathcal{V}_\tau(0)$, there exists $w \in W$ such that $\nu - \beta e + w \in \hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$. Let $W' \in \mathcal{V}_\tau(0)$ such that $-\beta e + W' \subset -\Gamma_j$. Let $\rho > 0$. There exists $\mathcal{C} \in \mathcal{C}_\tau(0)$ such that $[-\rho e, \rho e] \subset \mathcal{C}$. Since $\nu \in \mathscr{T}_{Y_j(z)}(y_j) \subset \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(y_j)$, there exists $\eta > 0$ such that for all $r > 0$, there exists $U \in \mathcal{V}_\sigma(0)$ and $\varepsilon > 0$ such that for all $z' \in \left(\{z\} + (\mathcal{C} \cap U)^{|I|+|J|}\right)$ for all $y'_j \in (\{y_j\} + \mathcal{C} \cap U) \cap Y_j(z')$ and for all $t \in (0, \varepsilon)$, there exists $\xi \in r[-e,\ e]$ such that

$$y'_j + t(\nu + \eta(y_j - y'_j) + \xi) \in Y_j(z') \tag{6}$$

Let $U' \in \mathcal{V}_\tau(0)$ such that $U' + U' + U' + U' + U' \subset V_j \cap U$ where V_j comes from Properness Assumption P (5). Since $L(e)$ is τ dense in L, there exists $w \in W \cap W' \cap U'$ such that $\nu + w \in L(e)$. Furthermore, $t(-\beta e + w) \in -\Gamma_j$ for all $t > 0$. Since U' is radial, there exists $\alpha > 0$ such that $\nu - \beta e \in \alpha U'$ and from σ-boundedness of order intervals, there exists $\delta > 0$ such that $-r[e, e] \subset \delta U'$. Let $\varepsilon' > 0$ strictly smaller than 1, ε, $\frac{1}{\alpha}$, $\frac{1}{\eta}$ and $\frac{1}{\delta}$. Hence, by invoking (6) above, we deduce that for all $z' \in \left(\{z\} + (\mathcal{C} \cap U')^{|I|+|J|}\right)$ for all $y'_j \in (\{y_j\} + \mathcal{C} \cap U') \cap Y_j(z')$ and for all $t \in (0, \varepsilon')$, there exists $\xi' \in r[-e,\ e]$ such that

$$\begin{aligned} y'_j + t(\nu + \eta(y_j - y'_j) + \xi') + t(-\beta e + w) &= y'_j + t(\nu - \beta e + w + \eta(y_j - y'_j) + \xi') \\ &\subset [(\{y_j\} + U') \cap Y_j(z') - \Gamma_j] \cap (\{y_j\} + U' + U' + U' + U' + U') \\ &\subset [(\{y_j\} + V_j) \cap Y_j(z') - \Gamma_j] \cap (\{y_j\} + V_j) \end{aligned}$$

So, if $y'_j \in L(e)$

$$y'_j+t(\nu+\eta(y_j-y'_j)+\xi)+t(-\beta e+w) \in [(\{y_j\}+V_j)\cap Y_j(z')-\Gamma_j]\cap(\{y_j\}+V_j)\cap L(e)$$

Hence, by σ-locally τ-uniformly properness relative to $L(e)$ at y_j, we conclude that

$$y'_j + t(\nu - \beta e + w + \eta(y_j - y'_j) + \xi) \in Y_j^e(z')$$

which means that $\nu - \beta e + w \in \hat{\mathcal{T}}^{\rho}_{Y_j^e(z)}(y_j)$. Since $\hat{\mathcal{T}}_{Y_j^e(z)}(y_j) = \bigcap_{\rho>0} \hat{\mathcal{T}}^{\rho}_{Y_j^e(z)}(y_j)$, we get that $\nu - \beta e + w \in \hat{\mathcal{T}}_{Y_j^e(z)}(y_j)$.

Proposition 5.4

The results follow since $\mathcal{V}_\tau(0) = \{B_\rho : \rho > 0\}$. Hence, the normal cone defined in [8] coincides with $\left[\mathscr{T}_{Y_j(z)}(y_j)\right]^\circ$. From Lemma 5.3, the equilibrium given by Theorem 5 is a marginal pricing equilibrium in the sense of [8]. The proof of the remaining items is straightforward.

Lemma 5.5

It suffices to show that $Y_j(z) \subset \tau\text{−cl}\,[Y_j(z) \cap L(e) - \Gamma_j]$ since $\tau\text{−cl}\,(L(e) - L_+) = L$. Let $y_j \in Y_j(z)$ and let V_j the 0-neighborhood from the definition of σ-locally τ-uniformly-properness relative to $L(e)$ at $y_j \in Y_j(z)$. Let W_j and W'_j be two neighborhoods in $\mathcal{V}_\tau(0)$ such that $W'_j + W'_j \subset W_j \cap V_j$ and $\{-e\} + W'_j \subset -\Gamma_j$. Since W'_j is absorbing and circled, there exists $0 < \delta_0 < 1$ such that for all $\delta \in (0, \delta_0)$, $-\delta e$ belongs to W'_j. Since $L(e)$ is τ-dense in L there exists a vector u in W'_j such that $y_j - \delta e + \delta u$ belongs to $L(e)$. Furthermore, since $-\delta e + \delta u$ belongs to $-\Gamma_j$ we conclude that $y_j - \delta e + \delta u \in \left[\left(\{y_j\} + W'_j + W'_j\right) \cap (Y_j(z) - \Gamma_j)\right] \cap L(e) \cap \left(\{y_j\} + W'_j + W'_j\right)$. Hence, by Assumption P (5) on properness, it follows that $y_j - \delta e + \delta u \in Y_j(z) \cap L(e) \subset \tau - \text{cl}\,[Y_j(z) \cap L(e) - \Gamma_j - L_+]$. Since $W'_j +$

$W_j' \subset W_j$, we have proved that $(\{y_j\}+W_j)\cap\tau-\mathrm{cl}\,[Y_j(z)\cap L(e)-\Gamma_j-L_+]\neq\emptyset$. This ends the proof.

Proposition 5.6

We show first that $\{\nu\in L:\nabla_1 f_j(\zeta_j,z)\cdot\nu\leq 0\}\subset\mathscr{T}_{Y_j(z)}(\zeta_j)$. By SB (2) and (3), $\nabla_1 f_j(\zeta_j,z)\in L_+^*\setminus\{0\}$ and $\nabla_1 f_j(\zeta_j,z)\cdot e>0$ respectively. Let $\nu\in L$ such that $\nabla_1 f_j(\zeta_j,z)\cdot\nu\leq 0$, let $\mathcal{C}\in\mathcal{V}_\tau(0)$, $\eta=1$ and $r>0$. Since H is equicontinuous, there exists $U\in\mathcal{V}_\sigma(0)$ such that $\bigcup_{g\in H} g(U)\subset(-1,1)$ ([27], Theorem 4.1 (c), p. 83). Since U is absorbing, there exists $\alpha>0$ such that $\nu-\frac{r}{2}e\in\alpha U$. Let $\beta>0$ such that $\beta<\frac{r\nabla_1 f_j(\zeta_j,z)\cdot e}{4(\alpha+1)}$. By SB (5), there exists $U'\in\mathcal{V}_\sigma(0)$ such that for all $(\zeta_j',z')\in(\{\zeta_j\}+U')\times\left(\{z\}+U'^{|I|+|J|}\right)$, $\nabla_1 f_j(\zeta_j',z')-\nabla_1 f_j(\zeta_j,z)\in\beta H$

Let $U''=\left\{\zeta_j'\in L\mid\nabla_1 f_j(\zeta_j,z)\cdot\zeta_j'<\frac{r\nabla_1 f_j(\zeta_j,z)\cdot e}{4}\right\}$ be a weak neighborhood of 0. We claim that there exists $U'''\in\mathcal{V}_\sigma(0)$ such that $U'''+U'''\subset U'$ and $U'''\subset U\cap U''$. Indeed, let $U_0\in\mathcal{V}_\sigma(0)$ such that $U_0\subset U\cap U'\cap U''$. Since σ is a locally convex topology, there exists $U_0'\in\mathcal{V}_\sigma(0)$ such that $U_0'=\frac{1}{2}U_0'+\frac{1}{2}U_0'\subset U_0$. By denoting $U'''=\frac{1}{2}U_0'$, we prove the claim.

We note that U''' absorbs $\nu-\frac{r}{2}e$, i.e., there exists $\gamma>0$ such that $\nu-\frac{r}{2}e\in\gamma U'''$. Let $\varepsilon>0$ such that $\varepsilon<\frac{1}{\gamma+1}$. From the Mean Value Theorem for topological vector spaces ([18], Theorem 1, p. 2) for all $z'\in\left(\{z\}+(\mathcal{C}\cap U''')^{|I|+|J|}\right)$, for all $\zeta_j'\in(\{\zeta_j\}+\mathcal{C}\cap U''')\cap Y_j(z')$ and for all $t\in(0,\varepsilon)$, there exists $\theta\in(0,1)$ such that $\zeta_j''=\zeta_j'+\theta t\left(\nu+\zeta_j-\zeta_j'-\frac{r}{2}e\right)$ belongs to $\left[\zeta_j',\zeta_j'+t\left(\nu+\zeta_j-\zeta_j'-\frac{r}{2}e\right)\right]$ and satisfies

$$f\left(\zeta_j'+t\left(\nu+\zeta_j-\zeta_j'-\frac{r}{2}e\right),z'\right)=f(\zeta_j',z')+t\nabla_1 f_j(\zeta_j'',z')\cdot\left(\nu+\zeta_j-\zeta_j'-\frac{r}{2}e\right)$$

We remark that $\xi_j'\in\{\xi_j\}+U'''$ implies $\xi_j-\xi_j'\in U'''$ since U''' is circled. From our choice of ε, it follows that $\zeta_j''\in\{\zeta_j\}+U'''+U'''\subset\{\zeta_j\}+U'$. Hence,

$\nabla_1 f_j(\zeta_j'', z') - \nabla_1 f_j(\zeta_j, z) \in \beta H$. It follows that

$$\nabla_1 f_j(\zeta_j'', z') \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right) =$$
$$(\nabla_1 f_j(\zeta_j'', z') - \nabla_1 f_j(\zeta_j, z)) \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right) + \nabla_1 f_j(\zeta_j, z) \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right)$$

From above we get, $\nu + \zeta_j - \zeta_j' - \frac{r}{2}e \in \alpha U + U''' \subset \alpha U + U = (\alpha + 1)U$ since U is convex[12]. Consequently, $(\nabla_1 f_j(\zeta_j'', z') - \nabla_1 f_j(\zeta_j, z)) \cdot (\nu + \zeta_j - \zeta_j' - \frac{r}{2}e) < \frac{r\nabla_1 f_j(\zeta_j, z)\cdot e}{4}$. In addition, $\nabla_1 f_j(\zeta_j, z) \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right) \leq \frac{r\nabla_1 f_j(\zeta_j, z)\cdot e}{4} - \frac{r\nabla_1 f_j(\zeta_j, z)\cdot e}{2}$. Consequently, $\nabla_1 f_j(\zeta_j'', z') \cdot \left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right) \leq 0$. Since $f(\zeta_j', z') \leq 0$, we get $f\left(\zeta_j' + t\left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right), z'\right) \leq 0$, that is $\zeta_j' + t\left(\nu + \zeta_j - \zeta_j' - \frac{r}{2}e\right) \in Y_j(z')$. Hence, $\nu \in \mathscr{T}^{\mathcal{C}}_{Y_j(z)}(\zeta_j)$. Since this is true for all $\mathcal{C} \in \mathcal{V}_\tau(0)$, we have that $\nu \in \mathscr{T}_{Y_j(z)}(\zeta_j)$. Thus, $\{\nu \in L(e) : \nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0\} \subset \hat{\mathscr{T}}_{Y_j(z)}(\zeta_j)$ whence, $\hat{\mathscr{N}}_{Y_j(z)}(\zeta_j) \subset \{\nu \in L(e) : \nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0\}^\circ = \{\nu \in \tau - \mathrm{cl}L(e) : \nabla_1 f_j(\zeta_j, z) \cdot \nu \leq 0\}^\circ$.

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