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# On the uniqueness of the optimal path in a discrete-time model à la Lucas (1988)

Stefano BOSI\*, Carmen CAMACHO†, Thai HA-HUY‡

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## Abstract

In a simple discrete-time version of the Lucas (1988) model, we prove, first, that the Balanced Growth Path (BGP) is optimal and, second, that the optimal solution is unique. After briefly discussing the results obtained in some of the continuous-time versions of the Lucas model, we address the issues of existence and uniqueness of the optimal solution in discrete time. Making use of the supermodularity of the value function, we prove that the optimal solution must be monotonically increasing. This feature does indeed suit the BGP, although nothing can ensure yet its optimality. After providing the exact expression of the BGP, we do prove both its optimality and its uniqueness.

**Keywords:** human capital, balanced growth path.

**JEL codes:** C61, D50, O40.

## 1 Introduction

Lucas' model (1988) has become a classic, flexible and easy-enough benchmark where economic growth depends on human capital, and where the trade-off between working and educating can describe many different scenarios. Because of its simplicity and versatility, it has served as a platform to study imbalance effects between output and capital, and describe the growth process more accurately by introducing, for instance, stochastic production functions and more sophisticated utility functions, more complete, even though complex, representations of the accumulation and evolution of human capital.<sup>1</sup> And, this, both

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<sup>1</sup>See, among others, Ladrón de Guevara et al. (1997), Gómez (2003), Boucekkine and Ruiz-Tamarit (2004), La Torre and Marsiglio (2010), Bucci et al. (2011), and Gorostiaga et al. (2013).

in continuous and discrete-time settings. One of the challenging issues of Lucas (1988) was to ensure the uniqueness of the optimal solution. It goes without saying that, even if we are not the first to address this issue, to the best of our knowledge, we are the first to prove the uniqueness of the optimal solution in a simple discrete-time version of the Lucas (1988).

The problem of the uniqueness of the optimal solution to the Lucas' model has been approached from many different angles in continuous-time settings. Early works focused on the phase diagram of the balanced path like Romer (1986); on numerical simulations like Mulligan and Sala-i-Martin (1993); on local stability like Benhabib and Perli (1994); on global dynamics like Xie (1993). Only recently, the pressing issue of uniqueness has been addressed by Boucekkine and Ruiz-Tamarit (2004), Ruiz-Tamarit (2008) and Hiraguchi (2009), who solve analytically the system of necessary optimal conditions associated to the Lucas' problem. However, to the best of our knowledge, only Bethmann (2007, 2013) addresses the optimality problem and raises the uniqueness issue when time is discrete. In both papers, the author provides with an explicit solution using a guess-and-verify method to obtain a value function that suits the maximization problem in the Lucas' model. Nevertheless, although he does obtain an optimal solution, he cannot prove that this solution is unique given that the value function is not necessarily unique. Our paper proves the uniqueness of the solution to a simple and workable version of the Lucas' model in discrete time, exploiting, first, the supermodularity of the value function to prove the monotonicity of any optimal solution, then, a direct method to prove uniqueness. Indeed, simplifying the Lucas' model by considering a linear technology with only one production factor, we prove not only that the Balanced Growth Path (BGP) is optimal, but also that any other alternative trajectory is inefficient.

The article is structured as follows. In the first section, we introduce the fundamentals and we write an equivalent agent's program involving an indirect utility. Then, in the second section, we derive the Balanced Growth Path and prove its optimality and uniqueness. The last section concludes. All the proofs are gathered in the Appendix.

## 2 A simple version of the Lucas' model

Let agents share the same preferences and endowments and, for simplicity, let the size of population be constant and equal to one. Hence, variable  $l_t$  will denote at the same time both the individual and the aggregate supply of labor services at period  $t$ . For simplicity, labor is the only factor required for the production of the unique final good, and technology is linear:  $y_t = Al_t$ . Besides, let us assume that all production is entirely consumed, that is  $c_t = y_t$ .

At any period  $t$ , each worker is endowed with one unit of labor, which she can spend either working or investing in human capital through education and health care. Accordingly, labor services are the product of the amount of the agent's human capital  $h_t$  and her working time  $u_t$ :  $l_t \equiv h_t u_t$ , with  $u_t \in [0, 1]$ . The remaining time  $1 - u_t$  is devoted to human capital accumulation according

to a simple law:

$$h_{t+1} - h_t \leq B(1 - u_t) h_t \quad (1)$$

for any  $t \geq 0$ . Note that we are assuming that human capital never wears-off with time.

The representative agent maximizes an intertemporal utility function where all utility comes from consumption, and where instantaneous utility is measured by a logarithmic function. Using that  $c_t = y_t = Al_t$ , the agent maximizes

$$\sum_{t=0}^{\infty} \beta^t \ln c_t = \frac{\ln A}{1 - \beta} + \sum_{t=0}^{\infty} \beta^t \ln l_t$$

which is equivalent to maximizing  $\sum_{t=0}^{\infty} \beta^t \ln l_t$  by choosing the sequence of working times  $(u_t)_{t=0}^{\infty}$ , under the law of motion for the accumulation of human capital in (1).

Thus, the agent solves the following equivalent program:

$$\max \sum_{t=0}^{\infty} \beta^t \ln (h_t u_t) \quad (2)$$

subject to  $h_{t+1} \leq h_t + B(1 - u_t) h_t \in \Gamma(h_t)$  with

$$\Gamma(h_t) \equiv \{h_{t+1} \text{ such that } h_t \leq h_{t+1} \leq (1 + B) h_t\}$$

Since

$$l_t \equiv h_t u_t \leq \frac{1 + B}{B} h_t - \frac{1}{B} h_{t+1}$$

we can introduce an indirect function  $V$ , defined as

$$V(h_t, h_{t+1}) \equiv \ln \left( \frac{1 + B}{B} h_t - \frac{1}{B} h_{t+1} \right) = \ln l_t \quad (3)$$

Eventually, program (2) can be rewritten in terms of function  $V$  as

$$\max \sum_{t=0}^{\infty} \beta^t V(h_t, h_{t+1}) \quad (4)$$

subject to  $h_{t+1} \in \Gamma(h_t)$  for any  $t$ .

### 3 The Balanced Growth Path

Observe that the cross-derivative of  $V$ ,  $V_{12}$ , is positive, that is

$$V_{12}(h_t, h_{t+1}) = \frac{1}{B} \frac{1 + B}{B} l_t^{-2} > 0$$

for any  $t \geq 0$ , proving that function  $V$  is supermodular.<sup>2</sup> Then, by Lemma 2.1 in Ha-Huy and Tran (2020), any optimal path to our problem is either strictly

<sup>2</sup>See Amir (1996) among others.

monotonic or constant. Additionally, recall that, according to (1), any sequence  $(h_t)_{t=0}^{\infty}$  satisfies that  $h_t \leq h_{t+1}$  for any  $t \geq 0$ . Then, along any optimal path, it must be that either  $h_t = h_0$  for every  $t \geq 0$ , or  $h_t < h_{t+1}$  for any  $t \geq 0$ . Let us show next that the optimal path  $(h_t)_{t=0}^{\infty}$  goes along the second option, that is, that any optimal path is strictly increasing and balanced.

**Proposition 1** *Any optimal path to program (4) satisfies the sequence of first-order necessary conditions:*

$$h_{t+1}u_{t+1} = \beta(1+B)u_t h_t \quad (5)$$

$$h_{t+1}/h_t = 1+B(1-u_t) \quad (6)$$

for any  $t \geq 0$ .

In order to ensure that  $u$  remains bounded along the optimal path, we require the following assumption

**Assumption 1** *The speed of human capital accumulation is bounded from below:*

$$B > \frac{1-\beta}{\beta} \quad (7)$$

The next proposition provides us with the explicit expression for the BGP

**Proposition 2** *Let Assumption 1 hold. The optimal solution described in (5)-(6) admits a BGP*

$$h_t = [\beta(1+B)]^t h_0 \quad (8)$$

$$l_t = [\beta(1+B)]^t h_0 u \quad (9)$$

$$c_t = [\beta(1+B)]^t A h_0 u \quad (10)$$

with

$$u_t = u = (1-\beta) \frac{1+B}{B} \in [0,1) \quad (11)$$

for any  $t \geq 0$ . The intertemporal utility along the BGP is given by

$$U = \frac{1}{1-\beta} \left( \ln c_0 + \frac{\beta}{1-\beta} \ln [\beta(1+B)] \right) \quad (12)$$

with  $c_0 = A h_0 u$ .

Observe that growth is balanced in the sense that all the model variables grow at the same constant rate  $h_{t+1}/h_t = l_{t+1}/l_t = c_{t+1}/c_t = \beta(1+B)$ .

We are now ready to provide with the main result of this paper. Proposition 3 below proves, first, that the BGP is optimal and, second, that the BGP is the unique optimal path.

**Proposition 3** *The balanced growth path  $(h_t, l_t)_{t=0}^{\infty}$  starting from  $(h_0, l_0)$  and evolving in time according to (8) and (9) with*

$$u_0 = u = (1 - \beta) \frac{1 + B}{B}$$

*is the unique optimal path.*

Hence, as announced, we have proved that our simple version of the Lucas' model in discrete time has an optimal solution, that this optimal solution is a Balanced Growth Path, and that this solution is unique. Furthermore, we have provided the explicit trajectory for capital and labor from  $t = 0$ .

## 4 Conclusions

The Lucas (1988) model has become a benchmark model to study the accumulation of human capital and perpetual growth.

Despite its major role, relative few papers have delved with the crucial and sensitive question of the uniqueness of the optimal solution. Indeed, most of the literature focuses on the BGP because of its practical properties. In this paper, we have presented a simple version of the Lucas' model in discrete time, which keeps intact the most important ingredients of the original model: the role of human capital in production and the competition for labor between the production sector and the human capital sector. Taking advantage of the supermodularity of the value function, we prove that all optimal solutions must be monotonically increasing and that the set of optimal necessary conditions admits a BGP. Finally, we show that the BGP is the unique optimal solution to our simplified Lucas' model.

## 5 Appendix

### Proof of Proposition 1

We maximize the Lagrangian function

$$\sum_{t=0}^{\infty} \beta^t \ln(Ah_t u_t) + \sum_{t=0}^{\infty} \lambda_t [h_t + B(1 - u_t)h_t - h_{t+1}]$$

with respect to the sequence  $(h_{t+1}, u_t, \lambda_t)_{t=0}^{\infty}$ . After eliminating the multipliers  $\lambda_t$ , we get the first-order conditions (5) and (6). ■

### Proof of Proposition 2

Computing  $h_{t+1}/h_t$  from (5) and replacing it in (6), we obtain

$$\frac{1}{u_{t+1}} = \frac{1}{\beta} \left( \frac{1}{u_t} - \frac{B}{1+B} \right) \tag{13}$$

for any  $t \geq 0$ . Setting  $u_{t+1} = u_t = u$  for any  $t \geq 0$ , we obtain the stationary state (11).

If  $u_{t+1} = u_t = u$ , then according to (5), we have that  $h_{t+1} = \beta(1+B)h_t$ , which, by induction yields (8). Since  $l_t \equiv h_t u_t$ , equation (5) also implies that  $l_{t+1} = \beta(1+B)l_t$  and, by induction, that  $l_t = [\beta(1+B)]^t l_0$  with  $l_0 = h_0 u$ . Since  $c_t = A l_t$ , we also get (10). Using (9), we can find the expression for overall welfare in (12). Indeed one can write that

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \ln l_t &= \sum_{t=0}^{\infty} \beta^t \ln \left( [\beta(1+B)]^t l_0 \right) = \ln l_0 \sum_{t=0}^{\infty} \beta^t + \ln [\beta(1+B)] \sum_{t=0}^{\infty} t \beta^t \\ &= \frac{\ln(h_0 u)}{1-\beta} + \frac{\beta \ln[\beta(1+B)]}{(1-\beta)^2} \end{aligned} \quad (14)$$

■

### Proof of Proposition 3

In this proof, we will first show that the BGP is optimal, and then that the optimal solution is unique.

(1) The BGP for human capital,  $(h_t)_{t=0}^{\infty}$ , satisfies equations (5) and (6) with  $u_t = u$  for any  $t$  according to (11). Along this BGP, and always according to (5) and (6), the optimal first-order condition of  $V$  must be verified, that is

$$V_2(h_t, h_{t+1}) + \beta V_1(h_{t+1}, h_{t+2}) = 0 \quad (15)$$

for any  $t \geq 0$ , where  $V_1$  and  $V_2$  denote the partial derivatives of (3) with respect to  $h_t$  and  $h_{t+1}$ . Moreover,

$$V_1(h_t, h_{t+1}) = \frac{1+B}{B l_t} \text{ and } V_2(h_t, h_{t+1}) = -\frac{1}{B l_t} = -\frac{1}{B u h_t}$$

where

$$l_t = \frac{1+B}{B} h_t - \frac{1}{B} h_{t+1}$$

This implies

$$\lim_{t \rightarrow \infty} \beta^t V_2(h_t, h_{t+1}) h_{t+1} = -\lim_{t \rightarrow \infty} \left( \beta^t \frac{1}{B u} \frac{h_{t+1}}{h_t} \right) = -\frac{\beta}{1-\beta} \lim_{t \rightarrow \infty} \beta^t = 0 \quad (16)$$

since  $h_{t+1}/h_t = \beta(1+B)$ .

Let us compare the BGP solution for human capital,  $(h_t)_{t=0}^{\infty}$ , with any other feasible path  $(h'_t)_{t=0}^{\infty}$  starting from  $h_0$ . We notice that  $\ln l_t - \ln l'_t \geq (l_t - l'_t)/l_t$  because of the concavity of  $\ln$ , where  $\ln l_t = V(h_t, h_{t+1})$ . Then, we can obtain the difference in the value function at time  $t$  associated to these two paths:

$$\begin{aligned} &V(h_t, h_{t+1}) - V(h'_t, h'_{t+1}) = \ln l_t - \ln l'_t \\ &\geq \frac{l_t - l'_t}{l_t} = \frac{1+B}{B l_t} (h_t - h'_t) - \frac{1}{B l_t} (h_{t+1} - h'_{t+1}) \\ &= V_1(h_t, h_{t+1}) (h_t - h'_t) + V_2(h_t, h_{t+1}) (h_{t+1} - h'_{t+1}) \end{aligned}$$

Aggregating these differences in time, we can prove that the BGP dominates  $(h'_t)_{t=0}^\infty$

$$\begin{aligned}
& \sum_{t=0}^{\infty} \beta^t V(h_t, h_{t+1}) - \sum_{t=0}^{\infty} \beta^t V(h'_t, h'_{t+1}) \\
&= \lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t [V(h_t, h_{t+1}) - V(h'_t, h'_{t+1})] \\
&\geq \lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t [V_1(h_t, h_{t+1})(h_t - h'_t) + V_2(h_t, h_{t+1})(h_{t+1} - h'_{t+1})] \\
&= V_1(h_0, h_1)(h_0 - h'_0) + \beta \lim_{T \rightarrow \infty} \sum_{t=0}^{T-1} \beta^t V_1(h_{t+1}, h_{t+2})(h_{t+1} - h'_{t+1}) \\
&\quad + \lim_{T \rightarrow \infty} \sum_{t=0}^{T-1} \beta^t V_2(h_t, h_{t+1})(h_{t+1} - h'_{t+1}) \\
&\quad + \lim_{T \rightarrow \infty} \beta^T V_2(h_T, h_{T+1})(h_{T+1} - h'_{T+1}) \\
&= \lim_{T \rightarrow \infty} \sum_{t=0}^{T-1} \beta^t [V_2(h_t, h_{t+1}) + \beta V_1(h_{t+1}, h_{t+2})] (h_{t+1} - h'_{t+1}) \\
&\quad + \lim_{T \rightarrow \infty} \beta^T V_2(h_T, h_{T+1}) h_{T+1} - \lim_{T \rightarrow \infty} \beta^T V_2(h_T, h_{T+1}) h'_{T+1} \\
&= - \lim_{T \rightarrow \infty} \beta^T V_2(h_T, h_{T+1}) h'_{T+1} \geq 0
\end{aligned}$$

because  $h_0 = h'_0$ ,  $V_2(h_T, h_{T+1}) < 0$  and because, along the BGP, the first-order condition in (15) holds, that is

$$V_2(h_t, h_{t+1}) + \beta V_1(h_{t+1}, h_{t+2}) = 0$$

and  $\lim_{T \rightarrow \infty} \beta^T V_2(h_T, h_{T+1}) h_{T+1} = 0$  according to (16).

Therefore, we have proven that  $\sum_{t=0}^{\infty} \beta^t V(h_t, h_{t+1}) \geq \sum_{t=0}^{\infty} \beta^t V(h'_t, h'_{t+1})$  so that the BGP "dominates" any other other feasible path. In other words, the BGP  $(h_t)_{t=0}^\infty$  is optimal.

(2) In order to prove the uniqueness of the optimal path  $(h_t)_{t=0}^\infty$ , the BGP, consider an alternative optimal path  $(h'_t)_{t=0}^\infty$ . We want to prove that  $h_t = h'_t$  for any  $t \geq 0$ .

Assume that, to the contrary,  $(h_t)_{t=0}^\infty \neq (h'_t)_{t=0}^\infty$  with  $h_0 = h'_0$ . Let  $(c_t)_{t=0}^\infty$  and  $(c'_t)_{t=0}^\infty$  denote the consumption paths associated to the optimal paths  $(h_t)_{t=0}^\infty$  and  $(h'_t)_{t=0}^\infty$ . Then,  $(c_t)_{t=0}^\infty \neq (c'_t)_{t=0}^\infty$  because, otherwise,

$$c_t = A \left( \frac{1+B}{B} h_t - \frac{1}{B} h_{t+1} \right) = c'_t = A \left( \frac{1+B}{B} h'_t - \frac{1}{B} h'_{t+1} \right)$$

for any  $t \geq 0$  and, since  $h_0 = h'_0$ , by induction, we would have that  $h_t = h'_t$  for any  $t \geq 0$ , which would be a contradiction.

Let us define an average consumption path as  $c_t^\lambda \equiv \lambda c'_t + (1 - \lambda) c_t$  with  $\lambda \in (0, 1)$ , and let us prove next that this path is feasible.

Define  $h_t^\lambda \equiv \lambda h'_t + (1 - \lambda) h_t$  for any  $t$ . We observe that since  $h_t \leq h_{t+1} \leq (1 + B) h_t$  and  $h'_t \leq h'_{t+1} \leq (1 + B) h'_t$ , we have that  $h_t^\lambda \leq h_{t+1}^\lambda \leq (1 + B) h_t^\lambda$  for any  $t \geq 0$ . Thus, the sequence  $(h_t^\lambda)_{t=0}^\infty$  is feasible. Let  $u_t^\lambda$  be defined as

$$u_t^\lambda \equiv \frac{1 + B}{B} - \frac{1}{B} \frac{h_{t+1}^\lambda}{h_t^\lambda}$$

$h_t^\lambda \leq h_{t+1}^\lambda \leq (1 + B) h_t^\lambda$  implies that  $0 \leq u_t^\lambda \leq 1$ . Moreover,  $c_t^\lambda = A h_t^\lambda u_t^\lambda$  for every  $t$ . Then, the path  $(c_t^\lambda)_{t=0}^\infty$  is feasible.

Since  $c_t \neq c'_t$  for some  $t$ , we have that  $\ln c_t^\lambda > \lambda \ln c'_t + (1 - \lambda) \ln c_t$  for some  $t$ , because of the strict concavity of  $\ln$ . We can then compute the associated overall welfare associated to both paths, multiplying by  $\beta^t$  and computing the infinite sum of all the per-period utilities. We obtain that  $\sum_{t=0}^\infty \beta^t \ln c_t^\lambda > \lambda \sum_{t=0}^\infty \beta^t \ln c'_t + (1 - \lambda) \sum_{t=0}^\infty \beta^t \ln c_t = \sum_{t=0}^\infty \beta^t \ln c_t$ . Hence,  $\sum_{t=0}^\infty \beta^t \ln c_t^\lambda > \sum_{t=0}^\infty \beta^t \ln c_t$ . This would imply that  $(c_t)_{t=0}^\infty$  is no longer optimal, which is a contradiction.

We conclude that  $(h)_{t=0}^\infty = (h')_{t=0}^\infty$ , demonstrating that the BGP is the unique optimal path. ■

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